DETECTION AND REMEDIATION OF DECIMAL MISCONCEPTIONS

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This paper will report on the findings from a large-scale longitudinal study of decimal misconceptions by school students in Melbourne. Over three thousand students from 12 schools were involved and they completed nearly ten thousand tests. Nearly 60% of the one thousand students who were first tested in primary school (Years 4 to 6) were tracked to their secondary school. More than 600 students completed 5 or more tests over 3 years. This very large dataset has provided information about which decimal misconceptions are more prevalent at which year levels and which misconceptions are hardest to leave. Suggestions for teaching to overcome decimal misconceptions are then presented.

Inspecting a mathematics textbook might give the impression that, in order to be competent with decimal numbers, all that students need to do is remember a few rules for placing the decimal point, and otherwise perform operations with decimals as if they were whole numbers. Yet many students do not make this apparently small step with ease. Many studies have indicated that students have difficulty with decimals. For example, the Third International Mathematics and Science (Repeat) Study conducted in 1999 showed that internationally about only 50% of 13-year-old students could select the smallest decimal number from a list of five. Many students are also not aware of the property of *number density*: whereas there is no whole number between 3 and 4, there is always a decimal number between any two unequal decimals (or common fractions). For example, Bana, Farrell and McIntosh (1997) found that 23% of the 14-year-old Australian students in their sample thought (incorrectly) that there were no decimals between 1.52 and 1.53. The cause of these problems is that, whereas the rules for decimal operations are easy, understanding what a decimal number means is not.

Nor do these problems disappear with age; Putt (1995) found various difficulties with decimals in his sample of pre-service teachers, as did Stacey, Helme, Steinle, Baturo, Irwin and Bana (2001).

This paper will report some of the findings from a large-scale longitudinal study of decimal misconceptions conducted from 1995 to 1999 in Melbourne. Over three thousand students completed nearly ten thousand tests with an average of 8 months between tests. Nearly 60% of the one thousand students who were first tested in primary school (Years 4 to 6) were tracked to their secondary school and more than 600 students completed 5, 6 or 7 tests during this study. This very large dataset has provided information about which decimal misconceptions are more prevalent at which year levels and which misconceptions are hardest to leave.

What are misconceptions?

Confrey (1990) reviewed the literature on misconceptions in the three fields of science, mathematics and programming. She noted the varied terms that were in use in these fields: alternative conceptions, student conceptions, pre-conceptions, conceptual primitives, private concepts, alternative frameworks, systematic errors, critical barriers to learning, and naïve theories. She commented that the dominant perspective was that:

in learning certain key concepts in the curriculum, students were transforming in an active way what was told to them and those transformations often led to serious misconceptions. Misconceptions were documented to be surprising, pervasive, and resilient. Connections between misconceptions, language, and informal knowledge were proposed. (p. 19)

Graeber and Johnson (1991) noted these characteristics of misconceptions (p. 3-15): *self-evident* (one doesn't feel the need to prove them), *coercive* (one is compelled to use them in an initial response) and *widespread* among both naïve learners and more academically able students.
The suggestion that misconceptions are due to students actively constructing their own ideas is a recurrent theme in the literature. For example, Resnick, Nesher, Leonard, Magone, Omanson and Peled (1989) stated

In making these inferences and interpretations, children are very likely to make at least temporary errors. Errorful rules, on this view, are intrinsic to all learning- at least as a temporary phenomenon- because they are a natural result of children's efforts to interpret what they are told and to go beyond the cases actually presented. Several analyses…have shown that these errorful rules are intelligent constructions based on what is more often incomplete than incorrect knowledge. Errorful rules, then, cannot be avoided in instruction. (p. 26)

So, if it is true that such errorful rules or misconceptions cannot be avoided, teachers need to be aware of their existence and ensure that their students do not persist with these misconceptions for long periods of time. This longitudinal study found that some students retained the same misconception for several years. In particular, the misconceptions that are associated with interpreting the decimal portion as a whole number are the hardest for students to leave; these are referred to as *whole number thinking*, *reciprocal thinking* and *negative thinking* and are discussed later. Of the students who completed tests that indicated such thinking, about one in six completed tests in a similar way more than two years later (Steinle, 2004).

How we can diagnose decimal misconceptions?

The task of ordering decimals has been shown by various researchers around the world to be a most revealing one; see for example, Resnick et al. (1989). Not only does it indicate which students can complete the task successfully, a student's incorrect choice can provide an indication of how they may be thinking about decimals. Two typical comparison items are:
Circle the larger number in the pair (4.8, 4.75)
Circle the larger number in the pair (4.3, 4.65).
At a superficial glance, both of the items appear to be assessing the same knowledge of decimals and place value. However, a high percentage of students will get one right and the other wrong. Closer examination reveals that students with the first item correct and the second item incorrect have chosen 4.8 and 4.3 and these are the *shorter* decimals in each pair (i.e. they have fewer digits after the decimal point). Likewise, students who chose 4.75 (incorrect) and 4.65 (correct) have chosen the *longer* decimal in each pair (i.e. they have more digits after the decimal point). Hence, the two main incorrect behaviours exhibited by students, when asked to compare a set of decimal numbers, are:
Longer-is-larger (L) behaviour: choosing the decimal with the most digits after the decimal point as the largest, and
Shorter-is-larger (S) behaviour: choosing the decimal with the fewest digits after the decimal point as the largest.
While these may seem incredibly naïve groupings, it will be demonstrated that students do exhibit such behaviours.

Table 1 contains ten comparison items from the 30-item Decimal Comparison Test used in the longitudinal study. To allow for easier comparison by the reader, the larger (correct) number is listed first in this table, but you need to vary this if you use these items to test your students. A student exhibiting *Longer-is-larger* (L) behaviour will choose incorrectly for the first five items and then correctly for the next five. Note that this is not because they suddenly remembered how to correctly compare decimals, or because they made a "few careless errors". A student exhibiting *Shorter-is-larger* (S) behaviour will choose the opposite responses and hence be correct on the first five items and then incorrect on the next five. Test papers with these 10 items correct were allocated the code A, and papers which did not match either A, L or S patterns were allocated the code U to indicate Unclassified. For research purposes, an allowance of just one deviation (within each group of five items) was regarded as still meeting high criteria, and yet allowing for the occasional careless choice.

Table 1: Expected responses by students with particular behaviours

Details of items (larger listed first)			**A***	**L***	**S***	**U**
Type 1 items	4.8	4.63	√	X	√	None of A, L or S
	0.5	0.36	√	X	√	
	0.8	0.75	√	X	√	
	0.37	0.216	√	X	√	
	3.92	3.4813	√	X	√	
Type 2 items	5.736	5.62	√	√	X	
	0.75	0.5	√	√	X	
	0.426	0.3	√	√	X	
	2.8325	2.516	√	√	X	
	7.942	7.63	√	√	X	

** One deviation (per type) from the expected responses is allowed for research classification*

A reader with a healthy quota of scepticism should have some alarm bells ringing by now. The above descriptions of various behaviours may seem quite ludicrous and the next figure is provided to convince you that students do, in fact, exhibit L and S behaviours. During 1997, 3531 tests were completed and exactly 1200 tests contained no errors (i.e. 30 correct out of 30). The remaining 2331 tests were analysed according to the scores on the two groups of five items in Table 1; Figure 1 contains the number of tests which were allocated to each of the 36 possibilities (i.e. scores of 0 to 5 on both Types 1 and 2).

Figure 1 indicates clearly that there are three common outcomes. Not surprisingly, a considerable number of students answer all 10 items correctly; in fact the column at the back corner (5,5) in Figure 1 would be considerably taller than all other columns if the additional 1200 tests with full score were included in this figure. The two tall columns at the left and right sides correspond to S and L behaviours, respectively, and the low columns elsewhere demonstrate that students, on the whole, are not making careless errors on this test. The diagram shows that students tend to get all or none of the items in a group correct. These are not careless errors, but systematic errors which can be predicted once we are aware of these misconceptions.

Note that the reasoning or thinking which leads a student to exhibit one of these observed behaviours has not yet been discussed. At this point, suffice it to say that the responses that a student makes on the remaining test items provides further clues about how that student interprets decimal notation. For example, one of the ways of thinking behind L behaviour is referred to as *whole number thinking*. Such a student considers a decimal number as two separate whole numbers separated by a dot, and therefore 4.10 is thought to be the next number after 4.9.

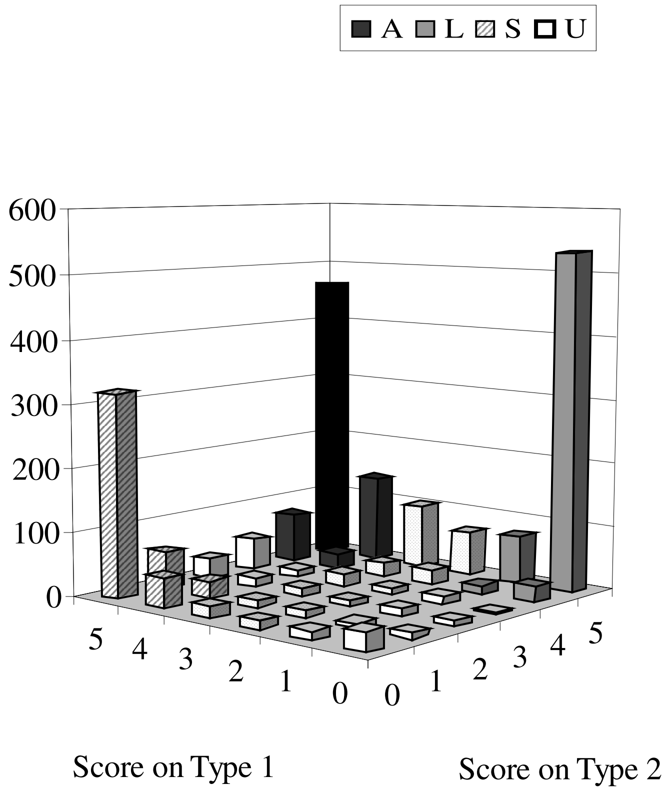


Figure 1. Actual distribution of scores on Types 1 and 2 based on data from 1997 (2331 tests with a maximum score of 29/30).

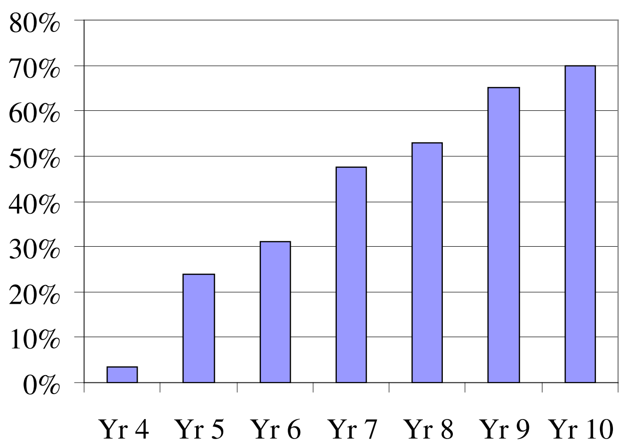
Three ways of thinking which lead to S behaviour are now described. Firstly, a student using *reciprocal thinking* will choose 0.3 as larger than 0.4, making a loose or strong analogy with their knowledge that 1/3 is larger than 1/4. A student using *negative thinking* will similarly choose 0.3 as larger than 0.4 but an interview might reveal that the student has confused (or made an analogy of) decimals with negative numbers and made their choice by considering that -3 is larger than -4. Thirdly, a student using *denominator focussed thinking* will correctly choose 0.4 is larger than 0.3, but then incorrectly choose 4.3 is larger than 4.65 by reasoning that any number with one decimal place involves only tenths and must be larger than a number with two decimal places (which contains hundredths).

It is important to note that not all of the students exhibiting A behaviour (see Table 1) choose correctly on the remaining test items; those that do are referred to as having demonstrated *expertise* on this task of decimal comparison. In particular, students with *money thinking* are able to answer the ten items in Table 1 correctly, but then have difficulty choosing the larger number from 17.35 and 17.353 as these may both be considered to be $17.35 and hence equal. Full details of the 30-item test and the procedure for allocating various codes (intended to represent 12 ways of thinking) to students' completed test papers are provided in Steinle and Stacey (1998).

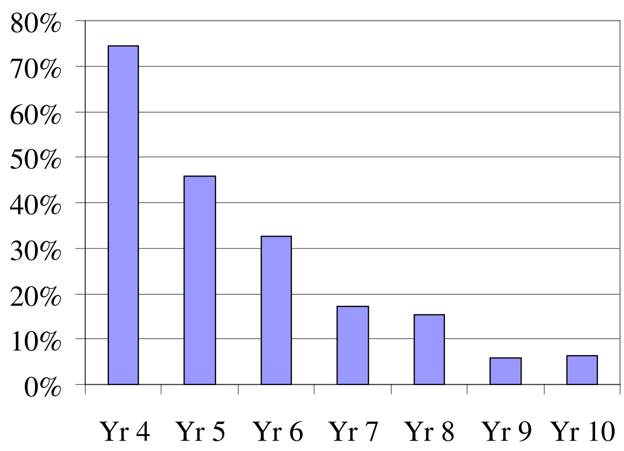
How many students with misconceptions in different year levels?

Figure 2 contains our best estimates for the percentage of students in each year level who exhibit various behaviours. By far the majority of students, when first encountering decimal numbers in Grades 4 and 5, exhibit L behaviour (see Figure 2b). As students move into older grades, fewer and fewer exhibit L behaviour although about 1 in 5 secondary school students exhibit such behaviour at some time in their years in secondary school. There is a corresponding increase in the percentage of students who are able to complete the test with very few errors. Figure 2a shows our best estimates for the prevalence of expertise in each year level after careful analysis to remove sampling effects. About 30% of students in Grade 6 and 70% of students in Grade 10 were considered to be demonstrating expertise on this test.

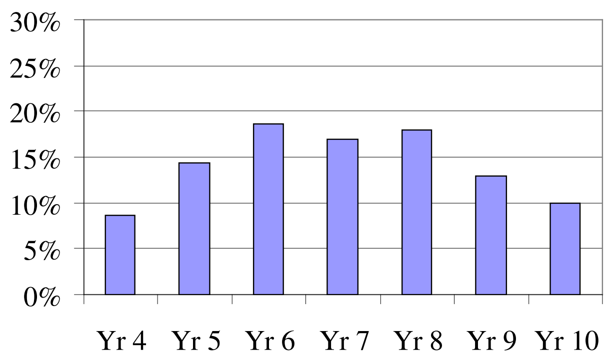
Figure 2c indicates that between 10% and 20% of students in each year level exhibit S behaviour, however, the reasons for S behaviour are more complex than for L behaviour. As different students move in and out of S behaviour at different times, a more useful measure is the proportion of students who exhibit S behaviour at some time or another. About 1 in 3 students will exhibit S behaviour at some time while they are in primary school and about 1 in 4 students while they are in secondary school.



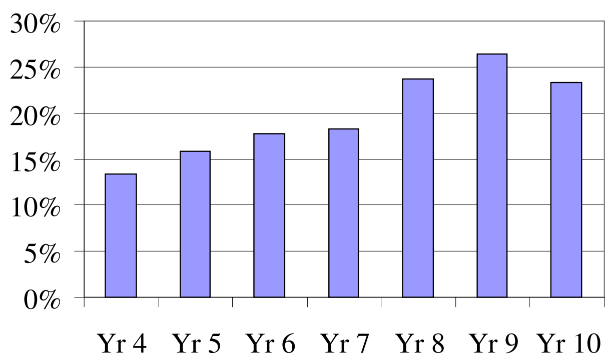
a) Prevalence of expertise* by year level



b) Prevalence of L behaviour by year level



c) Prevalence of S behaviour by year level



d) Prevalence of other behaviours by year level

Figure 2. Prevalence of various behaviours by year level (*expertise adjusted for sampling).

This study has found that some students with S behaviour exhibit unexpected tendencies. They become more likely to persist with this behaviour in older grades and they become less likely to move to expertise on their next test (on average 8 months later). We believe that this is due to interference of new teaching, in particular, the introduction of negative numbers and scientific notation (e.g., $0.0003 = 3 \times 10^{-4}$) in junior secondary school may increase the association of decimals with negative numbers; recall that *negative thinking* is one of the reasons behind S behaviour.

Figure 2d provides an estimate of the proportion of students in each year level who do not possess expertise and who do not exhibit either L or S behaviour. Included in this group are some students (about 10% of each year level) who can answer the ten items in Table 1, (i.e., exhibit A behaviour) but who are then unable to choose correctly when one decimal is a truncation of the other (such as 17.35 and 17.353). As mentioned above, such students may be using the analogy with money and then these numbers are both $17.35, or else they may use a strategy which often (but not always) provides the correct answer. For example, if a student follows the procedure of comparing digits in similar columns until they find a difference, then they would note that both numbers contain 17, then both contain the digit 3, then both contain the digit 5. At this point, their strategy has not provided them with a choice. They now try to compare the space at the end of the shorter number with the digit 3, and, in the absence of any meaning for the notation, they are left with no clear choice. Similarly, students who always round to two decimal places will not be able to order these numbers.

When an algorithm fails to provide a definite solution the students may resort to a latent misconception such as basing their choice on the relative number of digits (i.e., L or S behaviours). This suggests that rather than students moving away from these misconceptions, misconceptions are retained and are often hidden behind algorithms and procedures, which, to an observer, incorrectly indicate understanding. In other words, some students are receiving teaching that is covering over rather than overcoming misconceptions.

Good news: it is possible to remove these misconceptions

Various studies have been reported which indicate successful interventions. For example, Peled (2003) identified students using *denominator focussed thinking* (one of the S behaviours mentioned above) and provided teaching which helped these students with ordering common fractions with different denominators. For example, these students typically believed that as 1/8 is smaller than 1/4, so then 5/8 must also be smaller than 1/4. Once this misconception was challenged and resolved, the students were provided with other examples such as 3/10 and 65/100 which they had previously ordered incorrectly as 0.3 > 0.65 (as tenths are larger than hundredths). Successful transfer from common fractions to decimal fractions resulted.

Helme and Stacey (2000a, 2000b) report on successful intervention; even a small amount of class time led to a reduction in the number of students with misconceptions accompanied by an increase in expertise and this improved understanding was evident some months later. In another study, Stacey, Helme, Archer and Condon (2001) compared MAB with LAB (Linear Arithmetic Blocks, described below) and found that this new material was better for teaching decimals for various reasons. As well as the fact that MAB was very closely tied to whole numbers for some students, Stacey et al. determined that a one-dimensional model (LAB) was easier for students than the three-dimensional model (i.e., MAB).

Seminal work by Swan (1983) proves that providing students with teaching that generates "cognitive conflict" can result in better outcomes for the students than "positive only" teaching. Various studies involving the use of constructivist computer games have been reported; see, for example, McIntosh, Stacey, Tromp and Lightfoot (2000), while Tromp (1999) comments on the usefulness of the Number Between Game in helping students with the positioning of decimals on a number line. Fuglestad (1998) reported on the use of spreadsheets to assist students to overcome misconceptions. The CD-ROM *Teaching and Learning about Decimals* (Steinle, Stacey and Chambers, 2002) contains many ideas and activities for teaching decimals, as well as templates for some of the materials which will be discussed later.

Implications for teaching

One of the findings from this longitudinal study is that many students persist with misconceptions from one grade to another as they move through school. Hence, it is clear that the normal teaching that many students receive is inadequate to remove such misconceptions. Hiebert's (1987) comment is pertinent:
Introducing decimal addition by saying "These are just like whole number problems after you line up the decimal points" is a quick way to get good performance in the short run but it is counterproductive in the long run. The findings from research encourage us to adopt a long-term view and take the time to develop meaning at the outset. (p. 22)
Furthermore, as noted by Swan (1990), teachers need to reconsider their "implicit beliefs":
Most common mathematical texts and teaching practices seem to be based on the implicit belief that repeated rehearsal of facts and skills somehow result in better conceptual understanding. (You understand decimals better if you learn to add them up successfully…) (p. 45)
There are a variety of models and teaching ideas presented below. The models vary from *concrete* (Linear Arithmetic Blocks or LAB) to *symbolic* (number expanders and number slide). Use of language and context is discussed and the issues of ragged decimals, number density, rounding and significant figures.

Careful use of language and the importance of "one"

We recommend that teachers and textbook writers consider the language they use. For example, it is likely that most adults recognize that students need to appreciate the difference between the *letters* (a to z) and the *words* created from various combinations of letters. Yet students (and textbooks) often don't distinguish between the *digits* (0 to 9) and the *numbers* created from various combinations of digits. We need to use clear language in the classroom which will help students to know when we are focussing on either the "forest" or a particular "tree".
Furthermore, we believe that it should be emphasized that the purpose of the decimal point is to "show where the ones column is". The *ones* are the building blocks of the whole numbers; the next place value column to the left is created by grouping ten *ones* together, and this process is repeated to get more place value columns further to the left. To create the place value columns to the right of the *ones* column, we partition *one* into ten equal pieces and this process is repeated to get more place value columns further to the right. Students who see the decimal point as a "separator between the whole number columns and decimal columns" have trouble imagining how a digit can jump over this wall and are likely to write 30.50 when asked to multiply 3.5 by ten. As will be discussed later, a *number slide* reminds us that 3.5 will become 35 when multiplied by 10.

We detected variation in students' abilities to select the larger decimal when the numbers were larger or smaller than one. For example, an item such as 4.8 and 4.63 was more likely to be answered correctly by students with L behaviour than other similar items (like 0.8 and 0.63); see Steinle and Stacey (2003). This suggests that we should introduce decimals by considering numbers which are larger than one, such as 1.2 and 1.26, before moving to 0.2 and 0.26. It is also likely that students will benefit from being introduced to common fractions which are larger than one (such as $1^2/_3$) rather than less than one ($^2/_3$). This is because the act of partitioning a *one* into smaller pieces destroys the original *one*, so that it no longer exists as a useful reference; yet $^2/_3$ only makes sense if we know the size of the *one*. For this reason, we suggest keeping (at least) a single copy of the *one* available as a reference point when common fractions or decimals are introduced.

LAB and the number-line

LAB (Linear Arithmetic Blocks) is a good physical or concrete model to make decimal numbers. Figure 3a shows 0.2, 0.26 and 0.3 made out of pieces of pipe which represent tenths and hundredths. Note that 0.26 is made with 2 tenths and 6 hundredths (0.26 = 2 tenths + 6 hundredths) which emphasizes the additive structure of our base ten numeration system. Teachers (and textbook writers) need to emphasize this additive structure as this awareness makes a comparison of 4.3 and 4.37 almost trivial. Students who are encouraged to "add zeros" and then compare 30 with 37 are not receiving any lasting teaching; such teaching is likely to result in only short-term gains. Indeed, discussions of 30 and 37 may reinforce the misconceptions that involve students treating the decimal portion of a number as a whole number. This study found that these misconceptions are the hardest for students to leave.

a) comparing 2 tenths (0.2) with 2 tenths + 6 hundredths (0.26) and with 3 tenths (0.3)

b) comparing 2 tenths (0.2) with 13 hundredths (0.13)

Figure 3. Modelling decimals with LAB.

The use of a number line is strongly recommended for classroom use, as it is possible to incorporate the various sets of numbers that tend to be dealt with separately in the curriculum; for example, whole numbers, common fractions, negative numbers and decimals. The number line appears to be one of the few models that are useful for discussing the density of decimals. LAB can be used to build number lines, whilst also strongly embodying the base 10 structure. This is what makes it different from an ordinary number line, which has no evident base ten structure.

Students need to be assisted with the concept of re-unitising; 3 tenths is equal to 30 hundredths. LAB demonstrates this clearly, as well as assisting with the ideas behind the rounding process. Figure 3a shows that 0.26 is closer to 0.3 than to 0.2, which is why 0.26 rounded to the nearest tenth is 0.3. Figure 3b shows that 0.2 is larger than 0.13, which would help students exhibiting L behaviour as they would have predicted that 0.13 was larger than 0.2.

Number expanders show various expansions

A number expander can be used to “show” that 682 is 6 hundreds + 8 tens + 2 ones, which emphasizes the additive structure mentioned above. Note that it is *not* a concrete model of the number; MAB is a useful concrete model for whole numbers while LAB (mentioned above) is useful for decimal numbers. Number expanders are, however, a powerful visual aid as shown in Figure 4. Figure 4a shows the fully expanded version of 3 174 682, (i.e., 7 digits in 7 place value columns) as well as two other versions: 3 million + 174 thousand + 682 ones, and 31746 hundreds + 82 ones. Teachers should emphasize the difference between these two questions:

What digit is in the hundreds column in 3 174 682? (answer: 6)

How many hundreds are there in 3 174 682? (answer: 31 746 hundreds and an extra 82 ones)

This last question emphasizes re-unitising (4 thousand = 40 hundreds) and is preparation for division problems such as: Divide 3 174 682 by 100: (answer: 31 746 remainder 82, or 31 746 $^{82}/_{100}$ or 31 746.82).

Similarly, the decimal number expander in Figure 4b shows that 3.145 is 3 ones + 145 thousandths as well as 3145 thousandths, and 31 tenths + 45 thousandths. Here are the corresponding questions from above:

What digit is in the tenths column in 3.145? (answer: 1)

How many tenths are there in 3.145? (answer: 31 tenths and an extra 45 thousandths)

Divide 3.145 by one tenth. (answer: 31.45)

a) Various expansions of 3174682 b) Various expansions of 3.145

Figure 4. Expansions with number expanders (whole number and decimal versions).

The difference between two number expanders in Figure 4 is that, for whole numbers, the ones column is always the right-most column, while for decimal numbers this is not the case. Hence, we need a marker for the ones column and this is the reason for a decimal point. In order that the expansions of 3.145 can be read clearly (for example as 3145 thousandths) it is important that the decimal point can be removed. We suggest that a piece of blu-tack, or something similar, is used as a decimal point in the ‘closed’ version of the number expander and that this should then be removed when the column names become visible.

Number slides model multiplication and division by 10, 100, 1000 etc.

A number slide is another powerful visual model, but again it does not replace the concrete models of MAB and LAB. It consists of a frame made from paper or cardboard on which is written the names of the place value columns and the decimal point. The digits which compose a number are written on the strip of paper which is woven through the frame. This usefulness of this model is to remind students about the effect of multiplying or dividing by powers of ten.

Figure 5 shows how a number slide indicates the result of multiplying 3.1 by 10 and by 100, and then the result of dividing 3.1 by 10 and by 100. The arrows indicate the direction that the strip of paper is moved. This model suggests that when a number is multiplied by 10, the *digits move into the next biggest column* rather than the usual rules of *add a zero* or *move the decimal point.* Similarly, when a number is divided by ten, the *digits move into the next smallest column*. In fact, the idea that the decimal point can be moved arbitrarily around the page is very confusing for students. This model reminds us that the decimal point is fixed (between the ones and tenths columns to mark the ones column) and that it is the digits that move to other columns.

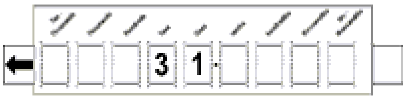
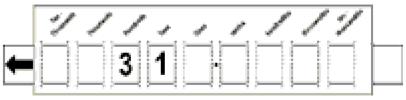
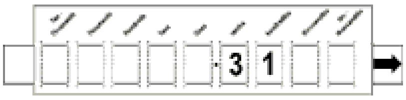
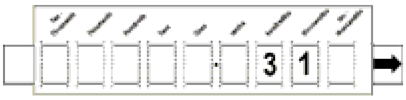
Number Slide	Usual Representation	Comment
	3.1	Original number
	31	After multiplying 3.1 by 10. Note the decimal point can be omitted, as there is no decimal portion.
	310	After multiplying 31 by another 10. Note the need for introducing a zero in the ones column.
	3.1	Back to original number
	0.31	After dividing 3.1 by 10. Note the additional zero in the ones column.
	0.031	After dividing 0.31 by another 10. Note the additional zero in the ones column as well as the tenths column.

Figure 4. Number Slide showing 3.1 multiplied (and then divided) by 10 and 100.

Ragged decimals and the issue of context

Many of the students with misconceptions are able to correctly choose the larger number when the decimals have the same number of digits after the decimal point; for example 0.4 / 0.3 and 1.85 / 1.84. Teachers will need to examine critically the texts and activities that they use in their classrooms, to determine if students with various misconceptions are able to answer correctly and hence go undetected. Note that the 10 items in Table 1 have pairs of numbers with unequal numbers of decimal places which are sometimes called *ragged* decimals. Students need to be exposed to ragged decimals to detect their misconceptions and to help them to realize that there is something that they do not know.

While using decimal numbers in a context may be considered as good teaching, a note of warning is given. The most common contexts are in measurement and are actually systems of units and subunits. For example, 64.37 m (i.e. one decimal number, one unit) can also be thought of as 64 metres and 37 centimetres (two whole numbers, two units). Note how the one number (64.37) has been split into two separate whole numbers (64 and 37). In another context, 64.37kg might be thought of as 64 kilograms and 370 grams. If we focus on the digit 3 in these numbers and listen to how it is read, we can see start to imagine how confusing this is for students:
Three (as in *point three seven* metres)
Thirty (as in *thirty seven* centimetres)
Three hundred (as in *three hundred and seventy* grams)
An emphasis on expanded notation (i.e., 6 tens + 4 ones + 3 tenths + 7 hundredths) provides a consistent reading of *3 tenths* in all contexts. Teachers need to be aware that the treatment of decimals in a measurement context does not necessarily teach about decimal numbers at all! Once the subunit has been identified in a given context, students revert to a system of two whole numbers, in which there is no need to partition into smaller and smaller amounts. Hence, students do not see the extension to further place value columns to the right or the fact that between any two (decimal) numbers there are an infinite number of decimals that can be written.
Brekke (1996) makes this comment regarding teaching of decimals within the context of money:

Teachers regularly claim that their pupils manage to solve arithmetic problems involving decimals correctly if money is introduced as a context to such problems. Thus they fail to see that the children do not understand decimal numbers in such cases, but rather that such understanding is not needed; it is possible to continue to work as if the numbers are whole, and change one hundred pence to one pound if necessary. It is doubtful whether a continued reference to money will be helpful, when it comes to developing understanding of decimal numbers; on the contrary, this can be a hindrance to the development of a robust decimal concept. (p. 138)

Brousseau (1997) also comments on how the over-generalisation of whole number properties is supported by teaching which works exclusively with decimals written to the same number of decimal places or with decimals always reflecting a money or measurement amount, such as millimeters,

But in fact, these school decimal numbers [i.e. of a fixed length] are really just whole natural numbers. In every measure there exists an indivisible submultiple, an atom, below which no further distinctions are made. Even if the definition claims that all units of size can be divided by ten, these divisions are never- in elementary teaching- pursued with impunity beyond what is useful or reasonable, even through the convenient fiction of the calculation of a division….Under these conditions, decimal numbers retain a discrete order, that of the natural numbers; many students using this definition will have difficulty in imagining a number between 10.849 and 10.850. (p. 125)

Hence, a “staged” treatment of decimals, in which decimals with the same number of decimal places are considered, is unlikely to provide students with suitable experiences for them to appreciate number density. Students’ incorrect beliefs about decimals are often exposed when they are asked to insert a number between two *apparently consecutive* numbers, for example, between 0.3 and 0.4, or between 0.52 and 0.53.
A useful non-measurement context to discuss decimals is the Dewey Decimal System for locating books in a library. A student looking for a book numbered 510.316 will need to know that it will be found after 510.31 and before 510.32. Hence, ordering of decimals can be explored in this context, but note that this decimal system does not have other measurement attributes; for example the books numbered 510.34, 510.35 and 510.36 are unlikely to be evenly spaced about the shelves.

Conclusion

Understanding misconceptions is one important step to improving instruction. They exist in part because of students' overriding need to make their own sense of the instruction that they receive and they can be overcome by teaching that pays attention to them. As Graeber and Johnson (1991) commented (p. 1-2),

It is helpful for teachers to know that misconceptions and buggy errors do exist, that errors resulting from misconceptions or systematic errors do not signal recalcitrance, ignorance, or the inability to learn; how such errors and misconceptions and the faulty reasoning they frequently signal can be exposed; that simple telling does not eradicate students' misconceptions or "bugs" and that there are instructional techniques that seem promising in helping students overcome or control the influence of misconceptions and systematic errors.

One of the findings of this longitudinal study is that, among the students who made some errors on the test, the students who were most likely to move to expertise on their next test were those who were *not* exhibiting L or S behaviours. Such students are possibly using a combination of different ideas, and may feel confused; appreciating that they have something to learn is the best state of mind for a non-expert. Such students may be more receptive to teaching than those with strongly held beliefs.

Teachers of students in junior secondary school also need to be aware that the introduction of negative numbers can interfere with students' thinking about decimals. Furthermore, the introduction of scientific (or exponential) notation and its use with small numbers (such as 1.3×10^{-6}) can reinforce the idea that decimals are some type of negative number (or at least in some way associated with them).

Teachers of students in middle and upper secondary school need to ensure that their students appreciate number density; in particular, that the decimals with three or four digits in the decimal portion (such as 0.3157) "live between" the decimals with only one or two digits (such as between 0.3 and 0.4, and between 0.31 and 0.32). Students without this understanding believe that 0.3157 is a "long way" from 0.3 (either much smaller or much larger, depending on their misconception) and then are unable to use approximation to check the reasonableness of their calculations. Furthermore, such students cannot make sense of the rules for rounding.

Teachers need to be aware that always rounding the result of a calculation to two decimal places can reinforce the belief that decimals form a discrete system and that there are no numbers between 4.31 and 4.32, for example. While it often appropriate to round the result of a calculation to two decimal places, or to consider, for example, only two significant figures due to limitations of the initial measurements, students seem to be confused about when this is reasonable or appropriate. For example, to convert 3/8 to a decimal we can determine the result of the division of 3 by 8 and will need to rewrite 3 as 3.000 in order to find the result of 0.375. Hence, it is sometimes appropriate to consider 3 ones to be the same as 30 tenths, 300 hundredths or 3000 thousandths (i.e., 3 = 3.0 = 3.00 = 3.000). Yet on other occasions, this is inappropriate as the presence of the zeros indicates the precision with which a measurement is taken. Such issues need to be discussed in classrooms.

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