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2018 Mathematical Association of Victoria Annual conference proceedings

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THE MATHEMATICAL ASSOCIATION OF VICTORIA

Foreward

Presenting at a conference such as this is a demanding task on its own. Presenters must judge what will interest a particular group of teachers of mathematics, along with teacher educators, administrators, and other interested stakeholders. They must choose a topic relevant to the theme of the conference, reflecting on their personal experience of practice, or on their research into the practice of others, and prepare a meaningful account of the salient aspects to share within a limited amount of time. All of these require a great deal of thought, from conceptualisation through to the final product of preparing the actual presentation so as to engage the audience as fully as possible. Preparing a paper for publication in the proceedings, so that audience members and other readers can have the opportunity to share in the stories, innovative ideas, and novel insights drawn from actual practice, takes the commitment and effort to whole new level. Working within parameters of page length, APA writing style, publication deadlines, etc., all place serious demands on teachers who are already very busy as professionals, not to mention their lives outside of education. I acknowledge the generosity of all authors in submitting papers, and then responding promptly and graciously to all my requests for changes and explanations, based largely on reviewers' comments and evaluations.

In my career as a teacher of adults returning to study, as well as vocational mathematics students, followed by research into how mathematics is used at work, some comments stand out:

- I was never any good at maths when I was at school/ I felt really stupid
- I will always remember the teacher who taught me .... and gave me the confidence in maths
- Maths is boring!

As a collection, the papers in these proceedings address all of these comments — explicitly as well as implicitly. Among all teachers, it appears that people teaching mathematics at any level do have a considerable impact on the personal, social, and intellectual well being of those being taught — not only in the immediate classroom, but also in contexts beyond the classroom in time and space, as students go out into the world as citizens, parents, carers, friends, mentors, etc.; also as workers in a rapidly changing, increasingly digitised world. The teachers and teacher educators who have contributed to these 2018 proceedings clearly demonstrate a strong commitment to providing a mathematics education that will have a positive impact on students, now and in the future. They are also strongly committed to supporting teachers in their own schools and neighbourhoods; as well as more broadly through the publication of these proceedings.

THE PAPERS

I have tried to group the papers by their overall themes as they appeared to me — clearly a subjective choice. The first group may be appropriate to all levels of mathematics education, even though they have arisen from observations of, and reflections on, particular levels of practice. The second group is generally more focused on the primary years, and the third group on secondary schooling.

Bernadette Mercieca describes **Flipping the Mathematics Classroom** as a way of structuring learning whereby the students view content videos and/or complete online activities at home, as opposed to following the traditional structure. She claims that this allows more time for collaborative and real life problem solving in the classroom, and that this may be especially beneficial to students whose first language is not English. Stacey Lamb, in her short paper **Challenge, Persist, and Share**, argues that, with the right amount of challenge, a positive mindset and the platform to share their learning and engage in ideas and strategies, students can learn at a high level. Narcisa Corcaci's paper, **The Art of Enrichment**, shows how capable students may be extended through the strategy of transforming existing word questions to gradually extend them to new contexts or to deepen them in the original context. In their paper, **STEM resources from reSolve: Maths by Inquiry**, Kaye Stacey and Lucy Bates introduce a set of resources, spanning levels F-10, which demonstrate three approaches to STEM: (a) teaching about the process of mathematical modelling, (b) using real world contexts to teach about target content, and (c) employing integrated units to demonstrate mathematics in use in other subjects. Here, their focus is on the secondary level. According to Daisy O'Bryan, in **Acquiring the Habit**

of Digital Innovation, making a change to incorporate learning technologies into their regular mathematics lessons can give rise to a cognitive conflict for some teachers. She stresses the importance of understanding both teachers' habits and their approaches toward innovation in teaching and learning, and how these may be mediated to produce new learning outcomes in preparing students for a digital world. Paul Staniscia draws on his own research in the paper **Evaluating the Impact of Evidence Based Practices on the Teaching and Assessing Mathematical Proficiency**. He focuses on the importance of developing in students the qualities of a positive disposition towards using mathematics, and the persistence required to become proficient. Also important for teachers is the ongoing processes of evaluation of their own practices, through students' responses to tasks and to questions about their tasks, to ultimately benefit all stakeholders. In their paper, **Sustaining and Scaling up Research-Based Professional Learning for Mathematics Teachers**, Rob Proffitt-White, Merrilyn Goos, and Anne Bennison discuss their Mathematics Capability Cluster (MCC) model which brings together primary and secondary school teachers and principals to analyse student performance data, and create diagnostic tasks that reveal students' mathematical understanding. Their goal is promote teaching practices that address both students' learning difficulties and ways to embed critical and creative thinking into all classrooms.

In the second group are four papers addressing themes of greater relevance to primary education, but which may be adapted to higher levels of school as well as to adult mathematics education. Cassandra Lowry focuses on stimulating the curiosity of young children once they enter more formalised schooling, by asking different kinds of questions from those typically offered in mathematics. Her paper, **How long is a 30-centimetre ruler? Rethinking familiar classroom practices to help revitalise the curiosity of students**, suggests alternative but relatively straightforward ways of introducing problems in order to encourage a more active, engaging, and rewarding approach to learning. On a related theme, James Russo's paper, **The Challenges of Teaching with Challenging Tasks: Developing Prompts**, supports the development of student reasoning, together with critical and creative mathematical thinking, in the early and middle years of primary school, through unpacking enabling and extending prompts. The paper, **A Matter of Time**, by Margaret Thomas and Philip Clarkson, draws on Thomas's doctoral research, arguing that there is more to the teaching and learning of time than the reading of clocks and calendars. They offer a Framework for the Learning and Teaching of Time, identifying the four major Components of time and the relationships between these, which can also form the basis of an assessment tool for students in the middle years of primary school. Finally in this group, James Russo and Toby Russo ask: **What Makes a "Good" Mathematical Game?** Although mathematical games are often a valuable pedagogical tool, the problem for teachers is to decide which games to introduce in their classrooms. To address this question, and to support teachers, they present five principles of good mathematical games.

The third group includes papers addressing more theoretical aspects of mathematics, although not necessarily exclusively to secondary education. Lorraine Day and Derek Hurrell, in their paper **Process over Product: It's More than an Equation**, argue that developing number and algebra together provides opportunities for searching for patterns, conjecturing, justifying, and generalising mathematical relationships. This allows students to focus on the process of mathematics and noticing the structure of arithmetic, rather than just the product of arriving at a correct answer. Robert Money and John Widmer's paper, **Excel-ent Maths**, argues that digital skills are of central importance in 21st century workplaces. They demonstrate the use of *Excel* spreadsheets, commonly in use in many workplaces, and offer examples to support teachers to embed this technology into their mathematics curriculum at the junior secondary level. Problem solving and mathematical puzzles have been part of the long history of mathematics. In the paper, **Problem Solving: What did you Learn?**, Pumadevi Sivasubramaniam used recreational problems, better known as mathematical puzzles, to explore what pre-service teachers learnt from their experience of solving them. The data collected was then used to outline a series of questions for future instruction, guiding the individual to explore the underlying rules to solve the puzzles rather than merely solving them (cf. Day & Hurrell). Similarly, in his short paper, **Numerating the Community: What is Mathematics?**, Karim Noura shares puzzles and problems he has collected over time, together with teaching ideas, of interest to parents, educators, and students. The paper by Terence Mills and Jonathan Ratcliffe, **Mathematics in "The Greek Anthology,"** describes some ancient problems which may not be so well-known in

mathematical circles. They suggest that these problems could be used to enrich mathematics in the classroom, or to make innovative connections with subjects such as classical studies or ancient history, underlining the inter-disciplinary nature of mathematics (cf. Stacey & Bates). Heading towards more analytic mathematics, Steve Hu's paper, **Algorithms for Partial Fraction Decomposition**, offers some alternative ways to quickly resolve a few types of proper algebraic fractions which, when using traditional methods such as the process of solving a set of simultaneous equations, often involve very complicated calculations.

CONCLUSION

From a personal perspective, attending MAV conferences from my earliest teaching days in secondary schools, stimulated the interest in continuing my studies in Mathematics Education, and eventually gave me the confidence to publish my first journal article in *Vinculum*, the Mathematics Association Journal, in 1993. I believe that many authors of the 2018 proceedings may also be encouraged, perhaps, to go on to further study, to submit journal articles and book chapters, and/or to join a team of textbook authors. And, of course, to return to the MAV Annual Conference in 2019!

In the books and journals that I have edited, my philosophy has always been to support both authors and readers so that intended meanings are as clear as possible, while respecting the integrity of the author's voice. I am extremely grateful to the reviewers for their willingness to accept this task, and to do so within the short space of time available, showing deep concern for supporting the authors themselves, as well as mathematics teachers and their students in general. I also wish to thank Ann Downton and the MAV conference organising committee, together with MAV staff Jacqui Diamond and Louise Gray, who have worked tirelessly to ensure that this conference and the proceedings contribute to improving the teaching and learning of mathematics. Clearly, the teachers as authors in the 2018 proceedings are indeed creating impact!

Gail FitzSimons
Chief Editor - The University of Melbourne

THE REVIEW PROCESS

The Editors received 24 papers for reviewing. 2 Keynote papers, 6 Double blind (4 accepted, 2 reclassified), 11 papers for Peer review (2 were rejected) and 5 Summary papers (1 reclassified). These papers were reviewed by a combination of external reviewers and the editorial team.

In total, 22 papers are published in the Mathematical Association of Victoria's 55th Annual Conference Proceedings. A total of 11 reviewers (including Editor) assisted in the process, all of whom provided thoughtful feedback and were outstanding in responding quickly to requests.

Contents

Keynotes

Bringing about success in mathematics for students in the margins: what makes for good practice

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Being successful in mathematics is about more than "tricks" and "gimmicks" to help students engage with concepts and processes. In considering the development, and implementation, of a quality school mathematics program, there are many levels of practice that need to be considered. Drawing on the outcomes of a large national project, this presentation explores a comprehensive model that promotes quality mathematics learning in some of the most marginalized communities in the Australian educational landscape. The learnings from this project have relevance to quality learning environments across all mathematics classrooms. Teachers' work has considerable potential to create impact on the learning for all students.

WHAT IS MEANT BY "TEACHING IN THE MARGINS"

Within the Australian educational landscape there are many sites where students are at greater risk of being successful, or not, depending on a number of key factors. While NAPLAN results only give a snapshot of who experiences success in mathematics at a particular point in time, and by implication, who does not have success, there are very observable patterns in outcomes. The ICSEA (Index of Community Socio Education Advantage) measure correlates strongly which achievement where those students whose ICSEA score is high, then there is a greater chance that they will have a high achievement score in Numeracy. The converse is also the case. Where students are more likely to experience failure or poor performance in mathematics is often associated with a range of social, cultural, cognitive, and language issues that teachers need to negotiate in their teaching of mathematics.

Teaching mathematics per se cannot be separated from the context within which it is taught. Various terms have been adopted by programs, theorists, researchers and policy makers to refer to what learners bring to the learning context. Such terms can include cultural backpack (Kincade-Blake, 2009), habitus (Bourdieu, 1977), identity (1998); and subjectivity (Kollosche, 2016). Such terms are used to discuss the knowledges, skills and dispositions that learners bring into the classroom and how these impact on the learning of mathematics. In some cases, the differences between the home knowledges/languages and those valued in school mathematics may be minimal and thus make learning mathematics a relatively easy task. The ease in learning comes about not so much as some innate ability but the synergy between what cultural and language attributes the learners bring to mathematics and how closely these align with the practices adopted in the teaching of school mathematics. Using Bourdieu's (1977) framing, the habitus of the learners aligns with the practices of mathematics. By contrast, for those learners who have significant difference between what they bring to school mathematics and what is valued in the practices of school mathematics can make the learning of mathematics a much more challenging process. The challenge is not in the mathematics per se but through the differences in the pedagogical process through which mathematics is taught. For these learners to be successful in mathematics, they must reconstitute their primary (home) habitus so align with the valued practices of the field of mathematics. As a result, there is a significant amount of cognitive labour for the marginalised learners if they are to become successful in the study of school mathematics.

Consider the role of language. For speakers of another language other than the language of instruction, in this case, standard Australian English (SAE), there are many ways in which miscommunication and misunderstandings can arise. As I have argued many times, a good example of this is when teaching the concept that *two halves make a whole*. For some learners, this linguistic interaction can be misread or misinterpreted to mean two halves make a *hole*. When using doughnuts to illustrate this concept, it is not surprising that this misinterpretation can make sense to learners. I have observed interactions where students have misheard the term *prism* as prison; *odd* numbers have been (mis)interpreted to mean "strange" or "weird" numbers; *square* to mean old fashioned; and numerous other examples. My point here is to indicate how the language of mathematics can create challenges for learning, but more particularly for learners whose language is not that of SAE. Not only do they need to learn the language of mathematics per se, but also the language through which the learning is mediated.

Teaching in the margins refers to those cases where learning occurs in contexts that are not part of the hegemonic discourses and practices associate with mathematics. Mathematics has long been seen as part of the masculinist and classist discourses of the dominant discourses within education. Teaching in the margins thus requires teachers to acknowledge marginalised status of learners whose home practices do not align with the dominant discourses of school mathematics and, in so doing, find ways to create learning opportunities for their students that take into account the backgrounds of their learners.

The most at risk group of learners within the Australian educational landscape are Indigenous learners who live in remote contexts. Their geographical isolation creates challenges for incoming teachers who are often in their first year of teaching and their on-going professional learning. The geographical contexts are quite different from urban settings so what might work as a practical application (such as train timetables) in urban settings have no relevance in remote areas. The culture and languages that the learners bring to the mathematics classrooms are often vastly different from those of urban settings. Most remote Indigenous learners speak at least one home language which is often the language of community so fluency in SAE is not the norm. This means that for many remote Indigenous learning coming to learning mathematics is learning in a foreign language.

In the remainder of this paper, I discuss some of the findings from a large national study in which numeracy practices in remote schools were studied. The schools involved in the study were seen to be successful in the teaching of school mathematics. From the 39 case studies, there were some practices that were observed across many of the schools, but it is without doubt that the schools were all different so there is no one common approach that can be seen to be effective. This is not surprising given the diversity of schools and contexts. However, there were principles, or norms, that can be extrapolated from the data. These will be discussed in the later sections of this paper.

REMOTE NUMERACY PROJECT

The Remote Numeracy Project was funded through the Australian Research Council Discovery scheme (DP) to undertake 32 case studies but due to some significant cost savings, the final project involved 39 schools. Some case studies (4), however, were not published. The distribution of schools by state and type of school can be seen in Table One below. The schools included small, one teacher schools through to multicampus schools; primary only through to secondary and schools that catered from early years to senior; boarding schools; and a VET college. The range of schools meant that most types of schools were included in the study.

	Government	Catholic	Independent	Total
WA	11	3	7	21
QLD	5			5
SA	4			4
NSW	5			5
NT			1	1
TOTAL	**25**	**3**	**8**	**36**

Table 1. Distribution of Schools (Published Reports Only

The case studies are ethnographic in form and were developed through site visits to each school. Data were collected via interviews with members of the leadership team, teachers and local workers at the school; observations of classrooms; profiling of lessons, and collection of school artefacts. Collectively these were used to develop individual case studies for each site. A positive, strength-based report is generated in consultation with the school, and once approved, was uploaded to a website for sharing (and celebrating) the successes of the schools. The Remote Numeracy website (Jorgensen, 2018) is hosted at the University of Canberra and can be freely accessed.

The meta-analysis across the schools was undertaken for this report. Trends across the data were undertaken through the application of a software package – NVivo – into which all interviews were coded and analysed using grounded theory. This has enabled the identification of key trends across the data set. Two further analyses were conducted using

Leximancer and a separate NVivo of the published case studies to confirm the trends reported here were valid. Across the three analyses there was a very strong confirmation of the themes/coding.

KEY FINDINGS

Unsurprisingly, there is no unifying approach across the states, or schools. However, there are some features that are appearing in many cases that are noteworthy. While there are examples of practices that would appear to be diametrically opposed such as problem based/investigative group work with the highly structured worksheets of "direct instruction," there is a unifying philosophy behind the teachers' intent with the adoption of these practices. First is that they sought to identify the entry level of the students (through assessment for learning practices) and then to develop targeted strategies to meet the needs of the individual students (differentiation). Rather than focus on describing practices per se, the project has identified the norms that appear to underpin the practices.

To make sense of the multiple levels of practice observed across the study, three levels were developed – envisioned, enabling and enacted. Schools need to have a strong and well-articulated vision. They then put practices in place to enable the vision to be enacted by the staff at the school. Different schools had different emphases in their case studies. Each of these levels of analysis and examples are provided in this report. While this is represented in a nested manner, it is the case that each of the levels of practices interact with the other, thus suggesting a much more dynamic model.

THREE LEVELS OF PRACTICE

From the analysis, it became clear that there are three distinct levels of practice that impact on effective teaching. While the initial project intended to explore classroom practices, as the project progressed, this was extended to include issues around leadership, practices that mediated between the vision and leadership of the school and the practices at the level of the classroom. For the purposes of the study, the three levels should be seen to impact on each other rather than to be seen as distinct and discrete levels of practice. The three levels are - envisioned practices, enabled practices and enacted practices. They each have distinct practices associated with the levels. The three level model developed from the data can been seen in Figure 1 below.

Figure 1. Model of practices.

ENVISIONED PRACTICES

Many of the schools in the study were very clear about the culture of the school that they sought to develop or had developed and sought to maintain, and sustain beyond their time at the school. Vision and leadership were an important characteristic of this level of practice. Some of the practices associated with the envisioned practices included:

- Articulate and lead the rollout of a school-wide approach to the desired culture and vision for the school
- A supportive leadership team to work with staff to enable the effective management of the school culture – both in terms of the culture of the school, and the mathematics learning culture
- Working relationships with community to share the visions of both the community and the school.
- Change is slow if it is to be effective. Being prepared to evolve a positive culture over an extended period of time and to ensure that the culture is embedded so that it endures changes in staff is critical. Communities and families are often change-weary of leaders coming to make their personal mark for personal gain, rather than for the gains of students and community.
- Sharing vision and working with staff and community is an important factor for success.
- Middle leadership was a strong theme emerging from the school data – this level of leadership mediates the vision of the school, and supports teachers to enact the vision of the school.

ENABLED PRACTICES

To enable staff and students to meet the goals of the school and thrive in the classrooms, schools have employed a wide range of practices to enable teachers to be able to enact the vision of the schools. These practices sought to implement the vision of the school and to ensure that teachers were given quality opportunities to develop as teachers while aligning with the values and approaches of the school.

- Employment of quality local staff to work alongside teachers. Investment of time and resources were evident of local people who often took a strong role in the classroom and were an invaluable resource within the school
- Quality professional learning for teachers – most of the schools were staffed by graduate teachers who were often in their first remote position. So considerable support was made available to induct these teachers into remote education, and to provide on-going support in their development as teachers of numeracy/mathematics.
- Numeracy Coaches were a feature at many of the schools. These people's role varied depending on the context and needs of the teachers but included sharing the vision of the school and supporting teachers to enact the vision; providing in-class support for teachers, from planning lessons to providing feedback (middle leadership).

ENACTED PRACTICES

At the level of the classroom, there was an extensive range of quality practices that were articulated and observed. These included:

- Being explicit about the intent of learning, how lessons will be organised and what is expected of the students;
- Differentiating learning to enable identification of students learning needs through assessment for learning practices and then to build quality learning experiences that meets and extends the needs of each learner;
- Recognising language as a key variable in learning, providing appropriate scaffolding in language (home and SAE) to build bridges between the home and school, and provide entry into school mathematics;
- High expectations – of both students and staff – across social and mathematical norms. Students should be provided with age-appropriate learning outcomes (e.g. algebra for secondary students) and then quality teaching practices to scaffold learners to achieve those learning intents;

- Focus on mathematics – mathematics was a priority for learning. The mathematics that was being taught was age-appropriate so that students were being exposed to levels of mathematics that could be expected in regional/urban settings (as per National Curriculum descriptors). It became the task of the teachers to provide appropriate scaffolding for students to enable them to reach these levels of learning. High mathematical expectations were reinforced;
- Culturally responsive pedagogy was evident where many strategies were developed to cater for culture of the students. Most obvious were strategies used to build language (of mathematics and the home language as well); and to have strategies that were cognisant of issues of "shame" within the classroom. There has only been one class to date that incorporated the more overt aspects of culture (e.g. art) but other teachers have sought to draw on the everyday activities that the students undertake (e.g. fishing, trips to town);
- Creating a sense of numeracy for life. Most remote communities had different numeracy practices from those found urban settings (and validated in most curriculum documents. Teachers have developed many strategies to create opportunities for students to see the purpose of mathematics/numeracy in their lives; and to build on the numeracy practices that students bring to the school setting; and
- Pacing of lessons, or parts of lessons, was often quick so as to engage learners, and prepare them emotionally as well as mathematically for the mathematics lessons. Using a quick pace engaged the learners. Humour was often part of the lesson as well, again to engage learners in a non-threatening manner.

These three levels provided a way of thinking about the observed practices across the sites. The descriptions of these practices can be found in more detail at the Remote Numeracy site hosted at the University of Canberra. While the practices are very descriptive, there are some common principles that underpin each of the levels of practice. In the remainder of the paper, I will discuss the Enacted Mathematical Norms. These norms underpin the practices that were observed at the level of practice across the school sites.

ENACTED MATHEMATICAL NORMS

What has emerged from the study are clear norms (or principles) that underpin the diversity of practices. These norms are seen to be principles that can underpin effective teaching practices in classrooms in the margins, but equally can be applied to quality classrooms in any setting. While norms have been developed for each of the three levels, in the remainder of the paper, I will discuss the Enacted Mathematical Norms as these align strongly with the theme of this conference.

ALL STUDENTS CAN LEARN MATHEMATICS – TO HIGH LEVELS

Having high expectations of learners requires teachers to believe that students can learn mathematics to high levels. Assuming that students cannot learn mathematics means that a deficit model of learning is in operation and that there is a very strong chance that an impoverished curriculum will be offered to learners, thereby locking them out of learning. If the norm is that all students can learn high levels of mathematics, then the task of teaching is one to find ways in which this can be achieved. Developing good strategies for scaffolding learning becomes the mode of operation, thinking and planning that will lead to success for learners.

EMBEDDING MATHEMATICS IS CRITICAL FOR UNDERSTANDING – EMBEDDING IN THE BRAIN AS WELL AS EMBEDDING IN CONTEXTS

Providing opportunities to embed learning mathematics is important. The "embedding " comes in a number of forms. First teachers need to find contexts that are meaningful and appropriate for learners so that they can make sense of the mathematics. For remote learners the geographical, cultural and linguistic contexts are important considerations for teachers as they plan their teaching experiences.

A second consideration of embedding is the opportunity for learners to embed their learning so that it comes to be fluent and with some degree of automaticity. For many remote learners, many of the mathematical ways of thinking that are

part of the standard Australian curriculum are quite different from their ways of seeing and being in the world. For example, in some remote communities the counting system may be "one, two, three, big mob". This is not to say that the learners do not discern differences in quantity but rather that their counting system is different from that of school mathematics. Within this context then, the teachers' role is to support students to embed number sense and place value in the students' school world by providing opportunities for this to occur. This may require frequent opportunities for learners to experience the different counting systems so as to embed the school counting system into their mathematical habitus.

MATHEMATICS IS AS MUCH ABOUT LANGUAGE AS IT IS MATHEMATICAL CONCEPTS

For learners who speak a language different from SAE, coming to learn mathematics is about the learning the language of mathematics as well as the mathematical concepts and processes. Mathematics has a particular language or discourse. There are particular grammatical structures such as wording of problems, and the use of syntax that are unique to school mathematics. Similarly, there are particular words that have nuanced meanings in the context of school mathematics that may be very different from the out-of-school worlds of learners. An example of this is when a problem is posed and learner are asked to "find x". Here they may draw on their everyday understandings of "find " and look for an x not dissimilar to the commonly used book of Where's Wally and learners are required to find representations of the Wally character. The term "find x" has a very particular meaning in school mathematics. The lexical density of this very contracted statement literally means "to calculate the value of the x" but is expressed in this very tight format. For learners coming to engage with word problems can find this use of language quite confusing, particularly when they speak a language other than SAE.

TRANSPARENCY IN LEARNING AND TEACHING MATHEMATICS ENABLES STUDENTS TO ACCESS THE "SECRET KNOWLEDGE" OF SCHOOL MATHEMATICS

Making the intent of learning clear to learners enables them to engage with the intended learning rather than for them to try to guess what the teacher is wanting from them. Consider a lesson on fractions where students may be using a lot of equipment and resources. There is a lot of things happening in a lesson and learners are required to abstract the fraction concept from the plethora of other experiences. When learners are informed of the intent of the lesson, then they can focus on this aspect of the lesson and maximise learning time.

Where the intent of the lesson or learning experience is not known to the learners, there is consider scope for the learners to misinterpret what is happening and their learning is tangential to the intent of the experience. This creates opportunities for (mis)understandings to develop and learners become excluded from what the teacher had intended. For many, this exclusion process is often interpreted by learners that there are secrets and tricks in mathematics as they cannot access what is actually intended.

MATHEMATICS LESSONS SHOULD ENGAGE LEARNERS AT THEIR LEVELS OF UNDERSTANDING, AND THEN EXTEND LEARNING INTO NEW LEVELS

Learning experiences need to be targeted at the level of the learners. This may require some form of assessment so that teachers can gauge the current levels of understanding. Targetting the learning for the individual students enables them to engage with the learning and move forward. The extension into new levels of understanding is a integral part of the planning process so that it ensure movement forward.

In many instances the targeted learning can be similar activities for the class/group so that learners are not grouped by achievement, and in so doing, creates divisions within a classroom. Careful planning around activities can allow diversity in starting points and exit points. In a classroom, students can be given the same activity but has a unique starting point depending on the individual child. For example, in number work, the learners may be given an activity where they explore number properties (doubling, adding 50, etc) but some learners may start with 10, others in the 100s, others with fractions/decimals. This process requires careful planning and assessment prior to teaching

CONCLUDING COMMENTS

The data from this project is very rich and complex. The development of the tri-level model of practice has enabled an interactive understanding of the complexity of practice and how it is important to consider the three levels of practice. Given the vast diversity of practices within each level, a further theorising of the levels was called for in order to make better sense of what underpinned the practice. This gave rise to the development of norms, or principles, for each level. In this paper, the enacted norms were discussed. It is this level of norms that may be useful for classroom teachers to consider as they develop their mathematics teaching, particularly when considering the needs of those learners who have traditionally been marginalised in their study of school mathematics. A more complete account of these norms can be found in the final report of the study (Jorgensen, 2018).

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Making early explorations count

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Young children are naturally curious and want to explore their world in order to make meaning about the experiences, objects, and ideas that they encounter. Mathematics and the concepts inherent to it, can provide them with the language and skills to support explorations, document their discoveries, and to communicate and share their findings to an audience. Knowing, and using these skills enables numeracy, and literacy, which then act as the foundation for problem posing, problem solving contexts, and all future investigations that contribute to satisfying a curious mind. What "counts" as mathematical knowledge is outlined in the Australian National Curriculum, and in the preschool years the Early Years Learning Framework sets the scene for how young children might embark on such investigations as confident and competent learners. This paper considers the relevance of mathematics in the early years and highlights some of the ways that early childhood teachers (birth to 8 years of age) can create authentic contexts for learning the early mathematical skills and knowledge relevant in contemporary times. This includes the use of new technologies. Yet, what is different about learning in the 21st century is not that it is digital, but that it is multimodal. A range of resources are needed for deeper conceptual understandings to occur. Two examples from a research project are used to illustrate the ways in which new technologies can be used in collaboration with different resources and activities to enhance the understanding and use of number.

INTRODUCTION

Mathematics is an integral part of our everyday lives. When we use the skills and knowledge of mathematics in authentic contexts we are becoming *numerate*. For example, when we go shopping we use number to figure out what we need to buy, and we need expertise with money to find special deals and to pay for our purchases at the end of the experience. Children go shopping with adults and observe the everyday transactions of money, or cards, being exchanged for goods and their concepts about the experiences build and are modified with each visit. In play-based learning scenarios in preschool contexts, their understandings about number and money will also be extended as they create shops and assume the role of customers and/ or shop workers, who "buy and sell" items found in our various retail outlets. Children should have a wide variety of experiences to support their understandings about number and money in these contexts. They will come to understand that number is used, in many ways, for example, to identify a quantity (e.g., 7 cars), to signify order (e.g., the 5th car) and as a label (e.g., a car registration plate YLA041).

BECOMING NUMERATE WITH NEW TECHNOLOGIES

The advent of new technologies, such as the calculator, laptops, and more recently the iPad and smart phones, should have revolutionised the teaching of mathematics, in the same way that these technologies have impacted on our everyday lives and in other professions (e.g., architecture and medicine). The use of new technologies should have meant that rather than focus on the mechanical aspects of mathematics (e.g., adding up), we can spend more time using the language and resources of mathematics available for authentic investigations. For me, knowing *how* to add is important, but knowing *when* to add is even more important. There have been pockets of innovation in the use of new technologies in mathematics that have enabled us to discover new ways of thinking about how we become numerate in the 21st century (e.g., Yelland, 2005, 2007). Additionally, the use of new technologies to collect what is called "big data," should have also transformed the way we do mathematics. This is not only about collecting and reacting to data as children work on particular mathematics activities, but it has also become important for children to learn how to represent, organize and create, and interpret tables and graphs. Further, new technologies have also had an impact on the types of problems that students experience in mathematics. Yet, what is different about learning in the 21st century is not that its digital, but that it is multimodal. We have a range of resources in different modalities that can support children's deep learning of mathematics.

Additionally, in our rapidly changing world, learners require skills that enable them to flourish in novel contexts. Yelland, Diezmann, and Butler (2014, p. 2) have noted that "Computational skills alone are insufficient for the solution

of novel (non-routine) problems." They suggested that in order to "solve novel problems, students need to have a broad mathematical knowledge base, to employ thinking skills and to have a positive attitude towards mathematics" (p. 2). This requires us to view mathematics not as a set of procedures to be learned, or a body of knowledge to be mastered, but rather as a way of being, a literacy for experiencing and describing the world, making sense of it, and a language for communicating and sharing our ideas.

EARLY MATHEMATICAL EXPLORATIONS

Parents, teachers, and caregivers all have important roles in creating contexts to support a child's confidence and competence in mathematics. Children have opportunities to acquire and use their mathematical understandings as they observe and explore in a variety of places, and this can be documented with technologies (digital & non-digital). They are curious and will think about and test their ideas; they will also want to share their findings or ask questions about what they are observing. Early mathematical experiences can happen in their homes, educational centres, school, the park, the zoo, the beach, or at a birthday party. Planned experiences, such as weighing objects, are structured to provide opportunities for promoting mathematical understandings. Additionally, spontaneous experiences also provide valuable opportunities for exploring mathematical ideas and enriching young children's meaning making about their world. For example, a child learns about *Number* when they observe cars and their number plates, or about *Space* when asked which route we might use to go to the playground. As we play and work with young children, we have lots of opportunities to encourage mathematical understandings in explorations and interactions that are rich and varied. A young child needs a range of early mathematical experiences to develop the necessary concepts, processes, and attitudes for problem solving. During problem solving and problem posing experiences, a child may encounter or need to use a variety of representations.

CONTEXTS FOR EARLY MATHEMATICAL LEARNING

Any parent or person working with young children will notice that they enjoy being curious about their world. They are constantly looking to explore the objects that they encounter and communicate what they notice and can do in their interactions with others. In the preschool years from birth to five years of age, the foundations for mathematical understandings are established. In their everyday lives, activities and experiences which are informal and play based are contexts for being mathematical. Parents and educators can use intentional teaching pedagogies to consolidate specific language or to emphasise the relationships that are fundamental to mathematical thinking and understandings. For example, this might be when reading a book like the classic story of *The Hungry Caterpillar* by Eric Carle (also available as a digital book), where there are opportunities to create contexts for understanding mathematical concepts and processes. With number, this might be: *counting* the *number* things that the caterpillar likes to eat, and to realise that they not only are linked in a *sequence* of numbers, but also related to the *days* of the *week*. The caterpillar eats *one* apple on *Monday* and *two* pears on *Tuesday*… In the context of space we might promote the positional and relational terms that are integral to describing the location of items. For example, we can highlight that the caterpillar starts its life as an egg that has been laid *on* a leaf. It then hatches *out* from *inside* the egg and starts on a journey. The children might relate this to a journey they make – such as from home to school. The children can also study the *symmetry* of patterns like those on a butterfly and shapes in our environment.

It is during these early explorations that the beginning mathematical processes and early thinking skills can be established. They require specific language to provide young learners with the ability to describe and communicate their ideas about their world effectively. These early processes are *Describing Attributes, Matching, Comparing, Ordering, Sorting,* and *Patterning*. In play-based programs and in everyday activities the use of these skills and the associated language can help the children to describe what they observe and interact with. They are not hierarchical, but the process of being able to create and understand patterns can be viewed as being the essence of what mathematics is all about.

For example, *Patterning* is basically the repetition of a sequence of items or events. A child can create a patterned necklace by threading red, yellow, and blue beads in order, and repeating the sequence. A game may involve a clapping pattern, such as loud, soft, loud, soft, and continue. Patterns can be visual, auditory, and tactile, and are not always static. Visual patterns include horizontal, vertical, diagonal, and circular patterns. Patterning activities include looking for

patterns, copying a pattern, making a pattern, extending a pattern, and finding the missing element in a pattern. These basic processes underpin most of what we do later in mathematics, and can be regarded, along with the initial language of position and relationships, to form a strong foundation for later confidence and competence in school mathematics.

THE FOUNDATION CURRICULUM

In Australia, Mathematics is one of the eight learning areas in the national curriculum (ACARA, n.d.). The curriculum consists of three content strands; number and algebra, measurement and geometry, and statistics and probability. These are the "what" of the subject. Then the proficiency strands are the "how:" understandings (concepts), fluency, problem solving, and reasoning.

To use number as an example, it is evident that young children encounter numbers in their environment, in songs, rhymes, and books every day. They are fascinated by numbers, and seem to enjoy counting on a regular basis. Caregivers can support children's initial attempts of counting in many ways, but specifically by conversing with them as they experience many opportunities to count. In the pre-school years, children use number and recognize their number names and numerals in many scenarios. For example, while playing, a child might create a party for five friends, or need four specific blocks to complete their building construction. They will see letter boxes with the numeral *25* carved or glued on. Children's interest in number can be followed up by highlighting the use of numbers in daily life and by including numbers in conversations with children. For example, by saying: *Your friend, Mary is the same age as you – she is 3 years old*; by asking: *Will, how old is your baby brother? What number is your house in your address? How many days are there in a week, or in a month?* These are all important prerequisite experiences which will help children to extend their concept of number as they enter school for the first time, usually when they are 5 or 6 years old.

SUPPORT FOR NUMERACY: RECOGNISING AND SEQUENCING NUMERALS

We worked in one school where teachers identified eleven children who, after six months of being in Preparatory class, were unable to; recognize and draw the numerals to 10, indicate the sequence in which they occurred, or respond correctly when asked to identify the number before, or after, a specific numeral. These were foundational conceptual knowledge for the first year of compulsory schooling in the National Curriculum for mathematics (Australian Curriculum and Assessment Reporting Authority, n.d.).

We selected number-based Apps that we felt would enable the children to acquire the knowledge which is considered to be the foundation for later mathematical understandings. We then worked with the children for an hour a week, over six weeks, in order for them to acquire this knowledge and put it into action. They were allocated into two groups by the teachers; each group had a half hour session with us each week. They did traditional activities around early number, for example using three dimensional materials, two dimensional pictures (counting items), worksheets, and number rhymes at other times during the week.

The Apps used were:

- *Bugs and Numbers* (numeral recognition, sequencing, counting).
- *Bugs and Buttons* (using numbers in games)
- *Insects* (1:1 correspondence, matching how many?).
- *Tally Tots* (recognizing numerals, sequencing, counting to 20).
- *Intro to Math by Montessorium* (tracing and recognizing numerals to 10)
- *Eye Math* (recognizing numerals and filling in missing numerals).

In the first week we started using *Bugs and Numbers* with each group.
Bugs and Numbers contains six games:

1. **Circus identification** – Recognition of (matching) numerals. In this game the children are given a number/numeral

(and hear the number name). They then have to find (by tapping) all the numerals that pass by them that are pegged on two parallel lines. The game gets faster the more answers they answer accurately, so they are encouraged to move more quickly.

2. **Arcade – recognition of left and right** – the players have to follow the oral instructions to press either right or left in order to navigate a bug/ant forwards through obstacles. A score is provided in the upper left corner. A final score is provided when the ant crashes into an obstacle.

3. **Junk yard** – the player moves items to uncover hidden ladybugs. The number of bugs starts with three, and increases as the successful numbers are uncovered. A tally of the score is provided in the left hand corner.

4. **Diner -** tap and count – bugs appear on a turn table and, as the player touches them, the number is said out loud and a white numeral appears on the screen. The player then matches that number to a choice of two numerals on the left and, as the game goes on, the choice of numerals is extended to three

5. **Gallery** – color by numbers. Pictures of bees/ladybirds and snails are all that are provided – children have to match numerals and colors. and complete the pictures.

6. **Matching shapes** – this starts with numerals and the player has to drag like ones together as they are told the names of the numerals. The game then progresses to shapes, then real world (fruit) items, followed by robots in two parts, and then to animals (caricatures of bees, bugs, etc.) that have to be matched and assembled.

The observation notes we made indicated that the sequence of events and interactions with the students were based on *questioning* and *scaffolding* them as they attempted to play the games. In the first session the children were able to play some of the games with assistance, because they needed some help with *recognizing the numerals*, and *counting how many* in order to *select the correct numeral*. They were able to play the *matching game* and the *Arcade* game without assistance. Their enjoyment was shown on their faces and they were very engaged with the graphics and sounds of this well-produced game. At the end of the 30 minute session all the children, except two, were able to count to 10, recognize randomly chosen numerals, and point to a numeral that came before or after another one.

In weeks two to five the children consolidated their understanding about the numbers from 0 to 10 in various activities using the other five apps. In each of the sessions a preservice teacher (Dean) *scaffolded* and *questioned* the children about what they were doing so that he could consider what level of concept attainment they had achieved. Dean's observations indicated that he thought a structured approach was preferable to letting the children choose their own games. He was able to support the children more effectively in this way, catering for individual needs:

> A considered plan to integrate iPad use in the math program would map the progression of focus skills across multiple Apps, so that there was a progression path from one App to another in building on those skills.
> (Dean, Reflections, June 18)

In the final session each child was questioned to ascertain their level of understanding and ability to use numerals up to 10 and to know their sequence, and their capacity to reproduce each numeral. All of the children except two demonstrated that they could now do these things. The two (boys) who could not demonstrate the concepts were identified by their teacher as having specific learning difficulties that spanned all curriculum areas. Nonetheless, they could say the numbers in sequence when prompted, and were able to trace over the numerals while the App advised them of its number name.

In this way, it was apparent that the dedicated short and focused activity sessions, incorporating the use of the tablet, in addition to their daily mathematics activities, had a positive impact on the conceptual understandings and skill levels of nine of the eleven children in the group. They had not mastered these understandings in the previous six months of schooling when the content was taught only with traditional materials. Focused small group work, which is a common pedagogical technique in Preparatory classes, with a variety of materials, might have achieved this sooner. We believe that, ideally, the use of multimodal experiences that incorporate iPads, with manipulative materials, games, and writing could have the potential to achieve this earlier on in the school year for these children.

EXTENSION NUMERACY (REPRESENTING DATA IN GRAPHS)

While the group of eleven children had experienced difficulty with recognizing and using the numbers from 0 to 10, others in the class had "mastered" them and moved on to larger numbers, from 11 – 20. The National curriculum (ACARA, 2012) suggests that they should be able to count to 20, recognize the numerals, connect the number names to quantities, compare and order collections, *subitise* small collections, and represent addition stories.

We wanted to extend the tasks to include opportunities to use the numbers with language and to link to other areas of the curriculum in practical projects that we hoped would engage the children in active learning and applying their knowledge. Accordingly, we provided contexts for learning that included using number in stories (comic and visual formats), graphs, and in the creation of eBooks that incorporated number. The Apps used were, *Scribble Press*, *MadPad*, and *Comic Life*.

Scribble Press also enabled us to introduce the concept of collecting and analysing *data* to the children, and then follow this up with the presentation of data in the form of graphs. With the children, we took photographs of the fourteen of them, and then embarked on data collection to find out various things about them (e.g., What was their favourite farm animal? and Were they left or right handed?).

> I modelled a bar graph for them by drawing the vertical and horizontal axes and placing pictures at the bottom. I then asked K (the student) if she would like to draw the lines on a new page. This strategy worked effectively as she performed this with ease. I then showed K and A (students) where the pictures of the animals were, and, in turn, they navigated and moved the pictures to the correct positions. They then interviewed the children to ask which was their favourite animal…. A held the iPad and showed the children their options, while K talked to the children. After each one they would sit on the floor to place the photo above the corresponding animal. They took turns and navigated the iPads successfully without any direction needed.
> (Amy, Reflections, June)

When they had finished interviewing the children, Amy asked the pair which animal had the "most amount of people" and was therefore the favourite animal. They immediately knew it was the pig, and, when asked, one of the students said it was because it had the tallest line. The boy student then told Amy that six people liked pigs the best. After this both children accurately counted the number who chose the other animals (horses, dogs, cows, ducks, and sheep).

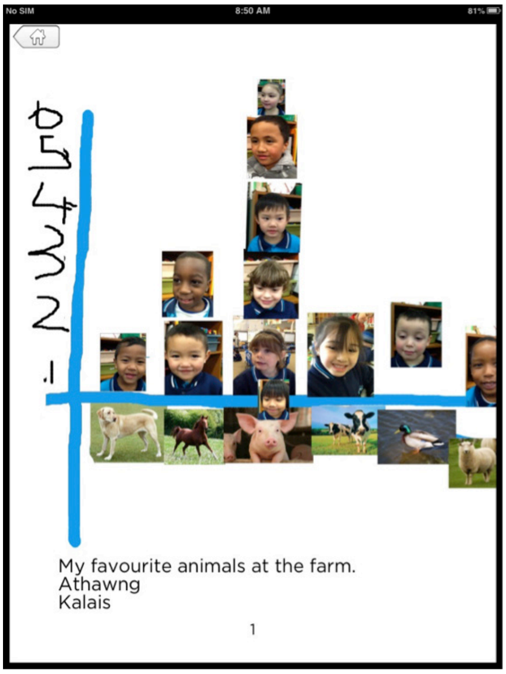

Figure 1. Our favourite farm animals.

> This activity so most valuable for the children's learning. It saw them gain an understanding of graphs and the purpose for them, something which they had not been exposed to before. The social and linguistic aspects of this lesson were evident from the start. K and A (students) worked collaboratively, and their confidence grew with each person they interviewed.
> (Amy, Reflections, June)

The children were going to a farm on the following Friday. Being able to use this as a basis for the activity was a valuable experience for the children. It linked the two tasks together and it made the activity purposeful for their learning.

SUMMARY

I have previously suggested (Yelland, 2007) that the goal of using and understanding mathematical concepts and processes is so that we can use them to become numerate. We need to be functional in our everyday lives and, increasingly, in contemporary times we communicate in multimodal ways that reflect our understandings about concepts. Young children need to be fluent in the use of the foundational skills of numeracy (and literacy) in order to be able to explore and investigate their life worlds, and to share their findings in a variety of different, and appropriate formats. This is why it is imperative that their mathematical explorations, beginning at birth, are positive and beneficial. This paper has highlighted the importance of mathematics in the early years and illustrated some of the essential components that are related to laying the foundations for these skills, dispositions, and knowledge.

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Double blind peer reviewed papers

Process over product: It's more than an equation

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Developing number and algebra together provide opportunities for searching for patterns, conjecturing, justifying, and generalising mathematical relationships. It allows the focus to be on the process of mathematics and noticing the structure of arithmetic, rather than the product of arriving at a correct answer. Two of the big ideas in mathematics are multiplicative thinking and algebraic reasoning. By noticing the structure of multiplicative situations, students will be in a position to reason algebraically, and the process of reasoning algebraically will allow students to appreciate the value of thinking multiplicatively rather than additively.

INTRODUCTION

Improving the quality of mathematics teaching and learning is an issue of discussion around the world (Cobb & Jackson, 2011), and yet, how to support it is not as well researched as it could be (Cohen, Moffitt, & Goldin, 2007). One area where there is substantial research is around the impact that teachers have on student academic success (Chetty, Friedman, & Rockoff, 2014). How such success is accomplished has encouraged further research into how students learn mathematics (Daro, Mosher, & Corcoran, 2011).

The connections between how students learn mathematics and the way in which teachers teach are as complex as the practice of teaching (Hattie, 2015). The practice of teaching asks teachers to make informed choices on a daily basis. These choices should always be focussed on developing mathematically powerful classrooms (Schoenfeld, 2014). It is difficult to have a mathematically powerful classroom if the classroom is one that is predicated solely on finding answers as opposed to one that values noticing the process or structure so that students can reach the answer to the immediate problem, but also generalise what they have learned in order to be able to solve future problems.

Generalisation, or the noticing of structure, is fundamental to mathematical success (Mason, Graham, & Johnston-Wilder, 2005) and needs to be incorporated at all levels of the teaching and learning of mathematics. Mason et al. (2005) wrote that, when they first come to school, young children are able to generalise. While generalising is quite innate, children need to practise, strengthen, and extend this natural ability. Teachers need to ask students explicit questions about what they notice, the patterns they can see, and how they are making sense of the mathematics (Day, 2017). Whether it be learning to trust the count or developing proportional reasoning, an understanding of structure is highly desirable. For the sake of this article we will explore noticing structure in terms of developing multiplicative and algebraic reasoning.

Empson, Levi, and Carpenter (2011) suggested that children seeing mathematics as a collection of procedures, rather than a process of noticing the structure of number and operations, causes major issues later in schooling. This is, we propose, very pertinent in the areas of multiplicative thinking and algebraic reasoning. The role that multiplicative thinking plays as a foundational concept underpinning the development of further mathematical ideas has been well documented (Hurst & Hurrell, 2016; Mulligan & Watson, 1998; Siemon, Izard, Breed, & Virgona, 2006). Brown and Quinn (2006) linked an ability to think multiplicatively to success with algebraic reasoning while Jacobs, Franke, Carpenter, Levi, and Battey (2007) contended that noticing number relationships and relational thinking are key indicators of success in algebraic reasoning. Multiplicative thinking and algebraic reasoning are clearly linked and we contend that this linkage is about noticing mathematical structure and the process of generalising.

MULTIPLICATIVE THINKING

Multiplicative thinking is not easy to teach or to learn. Whereas most students enter school with informal knowledge that supports counting and early additive thinking (Sophian & Madrid, 2003) students need to re-conceptualise their understanding about number to understand multiplicative relationships (Wright, 2011). Multiplicative thinking is distinctly different from additive thinking even though it is constructed by children following on from their additive thinking processes (Clark & Kamii, 1996). We would contend that multiplicative thinking is not just about a procedural

ability with multiplication, it is a deeper conceptual understanding of how multiplication acts on numbers, an understanding of the process or the structure of multiplication, and not just the product.

One of the best ways we can facilitate an exploration of structure is through implementing good tasks. Let us take for example the use of a multiplicative array to look at the structure and processes behind how we reach a product when completing a multiplication algorithm. It is difficult to explore abstract ideas such as the distributive property without having materials to manipulate. The materials, in this case the multiplicative array, promote the cultivation of the concept, and then the capacity to test any developing ideas, in order to create understanding (Hurst & Hurrell, 2016). The use of materials fits with Bruner's (1966) work on learning concepts through following a progression. Bruner's progression starts with the enactive stage (which requires concrete materials), moves to the iconic stage where pictorial representations are employed and finishes at the symbolic stage, the stage of abstraction. More contemporary researchers employ the same basic structure of learning in prosecuting the CRA (Concrete-Representation-Abstract) approach (Agrawal & Morin, 2016).

A multiplicative array is a representation of objects (we have found that, at least initially, it is preferable to use tiles which can tessellate) in which the multiplier and the multiplicand are exchangeable (Figure 1). Arrays are seen as a powerful way in which to represent multiplication (Young-Loveridge & Mills, 2009). Multiplicative arrays have the potential to allow students, among other things, to visualise factors, commutativity, associativity, and distributivity (Wright 2011). In this article we will keep the focus on distributivity.

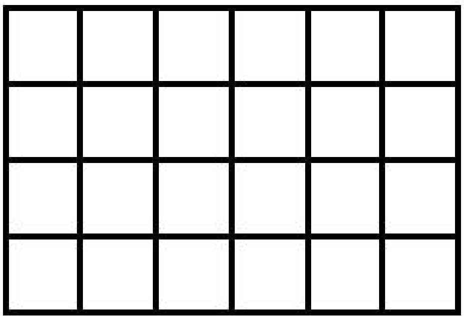

Figure 1. A multiplicative array showing 4 x 6.

Through the manipulation of a variety of arrays, the array should be established and accepted as a legitimate representation of single digit by single digit multiplication. Once this has occurred the array can then be extended into two-digit by one-digit multiplication. For example we can construct an array to represent 7 x 12 (Figure 2).

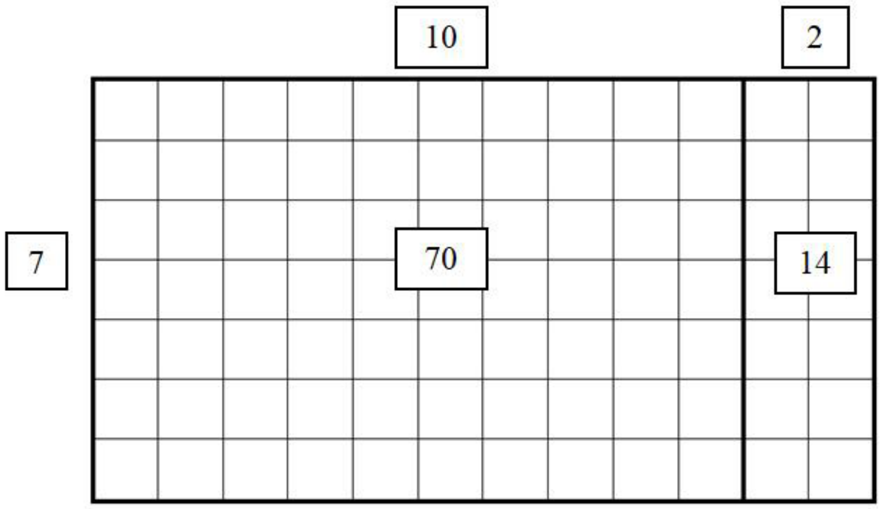

Figure 2. A partitioned 7 x 12 array.

What has been created is a visual representation of what is happening when multiplying 7 x 12, a representation which also encourages an understanding of the magnitude of numbers. Once the model has been used to develop two-by one-digit multiplication, it is then a reasonable step to move to a representation of two-by two-digit multiplication (Figure 3). This can still be done as a concrete model using tiles, but it does start to be less efficient. This loss of efficiency gives impetus for the student to move away from manipulative materials to the more efficacious, pictorial representations. One of the strengths of the pictorial representation of the array is that it illustrates a problem which is quite common when students start multiplying double digit by double digit numbers, that is not addressing all of the different parts which need to be multiplied. Students who have not used an array model may assume that 14×12 can be calculated by $10 \times 10 + 4 \times 3$. Using the array model and identifying the areas that are created, helps the students see why this is not the case.

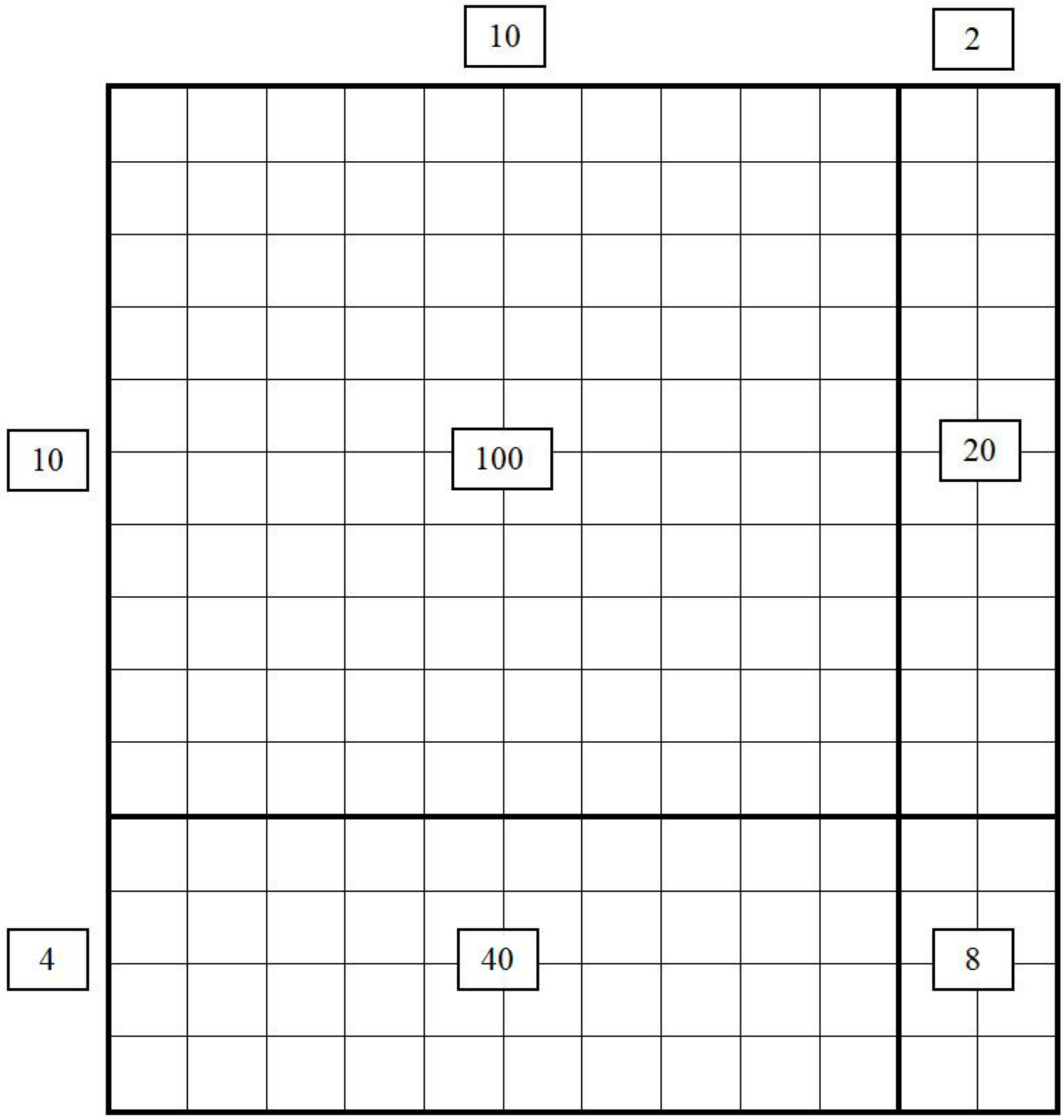

Figure 3. A partitioned 14 x 12 array.

It would now seem appropriate to move from the pictorial representation and to introduce an algorithm. By having the array representation at hand (Figure 3), a direct comparison can be made between this representation and a more abstract representation (Figure 4). The initial use of a non-standard algorithm may be a better way to illustrate to what the abbreviated notation really refers (e.g., the six in 168 refers to 60) than beginning with the standard algorithm (Figure 5).

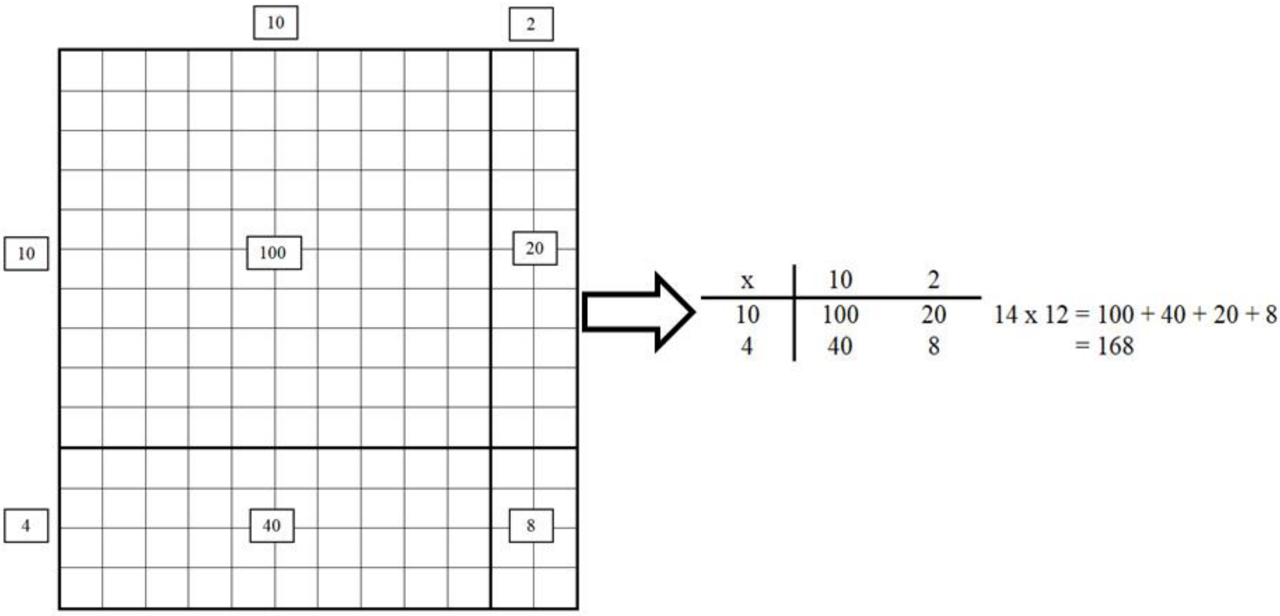

Figure 4. Moving from the pictorial representation of 14 x 12 to the abstract representation.

x	10	2
10	100	20
4	40	8

	1	4
x	1	2
	2	8
1	4	0
1	6	8

Figure 5. Illustrating the links between the non-standard and standard algorithms.

We can teach the algorithm without looking at the structure, and this would sit firmly in the area of computation, the goal being to arrive at the answer. By switching the focus about how we get to the answer, the task becomes about the structure. The answer, although still very important, is no longer the only goal, understanding how we reach the answer becomes the primary goal.

ALGEBRAIC REASONING

Kaput and Blanton (2005) defined "Algebraic reasoning [as] a process in which students generalise mathematical ideas from a particular set of instances, establish those through the discourse of argumentation and express them in increasingly formal and age appropriate ways" (p. 99). As with multiplicative thinking, in fact all mathematical thinking, truly focussing on algebraic reasoning is not simply about finding answers to problems, but about providing opportunities for discovering patterns, conjecturing, and generalising mathematical relationships (Siemon, Beswick, Brady, Clark, Faragher, & Warren, 2015). This is a process approach rather than a product approach.

Confusion about the four operations is often demonstrated by students making procedural errors, quite often by trying to use the structures that apply only to addition in other operations. For example, a common method used to add 45 + 18 is to add the tens, add the ones and then add the answers together 45 + 18 = (40 + 10) + (5 + 8) = 63. If this strategy is used for multiplication 45 x 18 = (40 x 10) + (5 x 8) provides the incorrect answer of 440. Schifter (2018) suggested that "students who make the errors … may be thinking of the correct addition strategy as the way *numbers* work, rather than how *addition* works" (p. 29).

A focus on the behaviour of the operations allows students to see that an operation is not just a procedure but also a mathematical object in its own right (Sfard, 1991). Once the operations are seen as objects with their own set of characteristics the students will be able to recognise and apply them to solve other problems. In order to do this students need to be encouraged to notice the structural properties of the operations and to explain in general terms the strategies they use when calculating answers (Schifter, 2018).

One of the important aids to noticing structure in the operations is utilising a CRA approach. Just as with multiplicative thinking, being able to visualise the ways that operations work allows students to explain how and why general strategies must work (Day, 2017). For example, if students are investigating what happens to a sum when one of the addends is increased by one, they may make a model, as in Figure 6. By noticing what changes and what stays the same in each case, students can come up with a conjecture that they can then test using diagrammatic representations, eventually arriving at a generalisation. The generalisation may be expressed in words or in symbols $a + (b + 1) = (a + 1) + b = (a + b) + 1$.

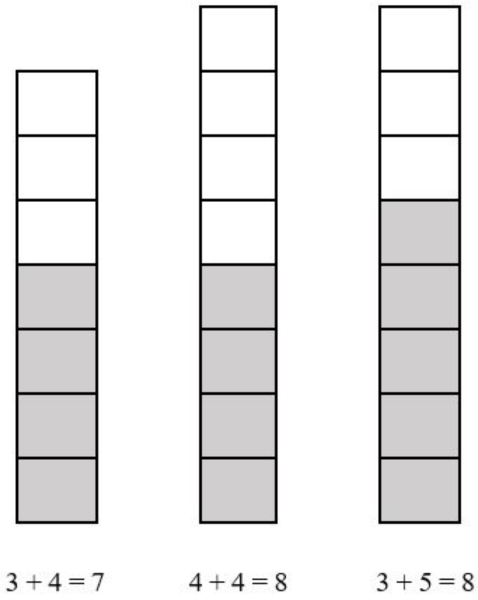

Figure 6. Representations of increasing the addend by one.

Students could then investigate what happens to the product of a number if you increase a factor by one, they may construct arrays, as in Figure 7.

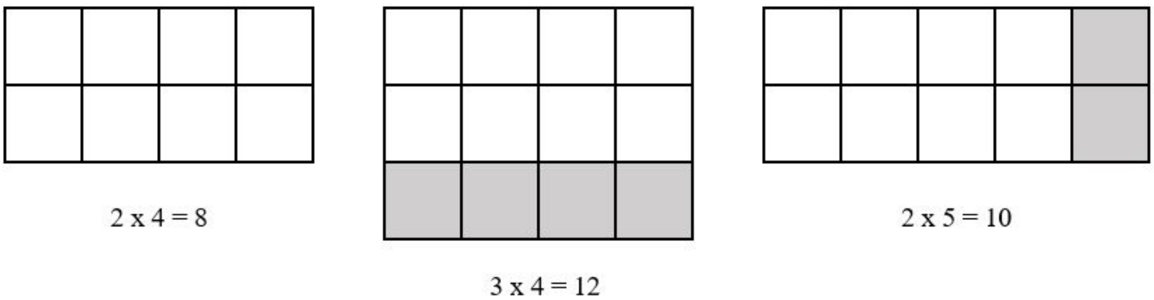

Figure 7. Arrays demonstrating increasing a factor by one.

The concrete representation allows students to physically see that increasing the number of groups (the first factor) affects the product by the number in each group (the second factor). They may have to look at several cases to recognise that this is what is happening. Students can be asked to represent the situations using other representations such as a story and a picture to illustrate their story (Schifter, 2018). Once again, by noticing what changes and what stays the same in

each case, students can arrive at a conjecture to test, finally generalising the result. The generalisation may be in words or symbols a x $b = ab$, $(a + 1)b = ab + b$ and $a(b + 1) = ab + a$, depending on the developmental stage of the students.

Importantly, students should have the opportunity to compare the differences between the different operations. In the cases mentioned above, students should have the opportunity to explain how multiplication differs from addition, otherwise the students may just see the two investigations as unrelated (Russell, Schifter, Kasman, Bastable, & Higgins, 2017; Schifter, 2018). Russell et al. (2017) created a teaching model based on five phases of investigation:

- Noticing regularity
- Articulating a claim
- Investigating through representations
- Constructing arguments
- Comparing and contrasting operations

This model can be used in a variety of situations to draw students' attention to the structure of arithmetic in order to use the structure to reason algebraically.

CONCLUSION

The argument here should not be about a choice between arithmetic fluency and an understanding of the underlying structure of the mathematics. Both can be achieved through the judicious use of quality tasks, and an attention to the processes used to reach an answer. As teachers it is important that we push the students to look at what is happening and why. Doing so helps us, and the students, to employ the Proficiency Strands as articulated in the Australian Curriculum.

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Flipping the mathematics classroom

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The flipped Mathematics classroom is a way of structuring learning whereby the student views content videos and/or completes online activities at home, allowing more time for collaborative and real life problem solving in the classroom. Research has shown that students who engage in flipped classrooms show significant gains in their learning. This paper will explore some of these research findings as well as detail the particular benefits for English as a Second Language students.

INTRODUCTION

Interest in using the flipped classroom as a way of teaching Mathematics has generated topical discussion in a variety of secondary school and tertiary contexts globally. With the emergence of ever increasingly sophisticated forms of technology and the large body of information that senior Mathematics courses are required to cover, it is not surprising that many practitioners are looking at new ways of maximizing classroom time. Further, students (born after 1980) not only ubiquitously use technology socially but expect it to be used in their learning (O'Flaherty & Phillips, 2015). Bishop and Verleger (2013) define the flipped classroom as an educational technique that consists of two parts: interactive group learning activities inside the classroom, and direct computer-based individual instruction outside the classroom. Hussey, Fleck, and Richmond (2014) define it more technically as "a synchronous set of learning activities where classroom based face-to-face interactions with teachers and peers is complemented asynchronously by out of class tasks" (p. 25). Ozamoli and Asiksoy (2016) and Bergmann and Sams (2014) are representative of those who focus on the fact that the flipped classroom is an active, student-centred approach that was formed to increase the quality of face-to-face time spent in classrooms. In this context, flipped learning can be considered as part of the contemporary move from the teacher being the "the sage on the stage" to the "guide on the side" (Morrison, 2014, p. 1). Even when a teacher is a very good sage on the stage, there are always some students who will be disengaged and some who will be marking time. Bergmann and Sams (2014) highlight the fact that content delivery in a direct instruction model is a lower order learning activity (involving remembering and understanding) in Bloom's Taxonomy, as illustrated in Figure 1.

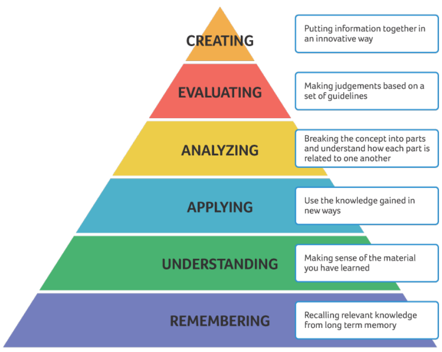

Figure 1. Blooms Taxonomy (retrieved from educationaltechnology.net)

In traditional classrooms, teachers tend to spend a significant amount of time at these lower levels as they try to teach heavily laden course content. Bergmann and Sams (2014) suggest that flipping the classroom can flip Bloom's Taxonomy: "Offloading the lower end of Bloom's Taxonomy from class time allows more class time to delve into the higher end of Bloom's Taxonomy" (p. 33). Bloom's Inverted Taxonomy is illustrated in Figure 2:

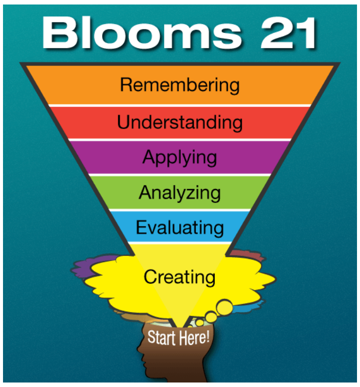

Figure 2. Inverted Bloom's Taxonomy (Retrieved from shelleywright.wordpress)

The advantages of flipping the Mathematics classroom and its particular relevance for English as a Second Language (EAL) students is the subject of this paper, as well as the practical ways of doing so.

THE ADVANTAGES OF THE MATHEMATICS FLIPPED CLASSROOM

Essentially, the flipped classroom reverses the role of homework and classwork. Lo and Chen's (2017) meta-analysis of 21 journal studies related to the flipped classroom in K-12 and higher education contexts found that there was an overall significant effect in favour of the flipped classroom compared with the traditional classroom for mathematics education (Hedges' $g = 0.298$, 95% CI). Rather than listening to key mathematical ideas being explained in the classroom by their teacher and taking notes, with a small amount of practice time, and then struggling through pages of exercises at home, students watch videos about theory, take notes and/or do online quizzes at home, and then utilise their class time in more collaborative and creative ways. This might include probing more deeply into course content with the support of their teacher and peers, individual problem solving and tackling real world problems in small group work. In effect, students have more opportunities to think for themselves and actively engage with mathematical content (Clarke, 2015; Gilboym, Heinerichs, & Pazzaglia, 2015; Hussey, Fleck, & Richmond, 2014).

Students in a flipped classroom environment can take ownership of their learning by deciding when and for how long they will learn, as well as receiving immediate support and guidance in class. As such, flipping the classroom has the advantage of supporting the needs of a diverse range of learners (Hamdan, McKnight, McKnight, & Arfstrom, 2013). Those who grasp the material easily can move quickly through the set material at home without having to sit through extended periods of direct instruction in the classroom. Students who struggle, as well as EAL students, can watch the videos as many times as they choose, as well as pausing the videos and consulting language dictionaries where necessary. This is something that would be more difficult in a traditional classroom, where the teacher may have moved onto the next point whilst the dictionary is being consulted. There is also significant value for students who are absent from a class in being able to access the key learning points from the lesson they have missed, and for all learners to have ongoing access to the online videos/learning activities as they prepare for assessments. For EAL learners, the flipped classroom approach also maximizes the amount of time students speak English in class and minimizes the amount of teacher talk time. Kim, Park, Jang, Nam, and Nam's (2017) research with such learners in Korea, found that students "interacted more deeply and cohesively than the students in the traditional classroom." They found that students were able to speak at greater length about their ideas and interact with each other more confidently. The reason for this, they suggest, is that they had more time to process the content of a lesson — because they had previewed it at home — and thus were able to utilise their reasoning skills more effectively and achieve "deeper information processing." Although their samples were relatively small, these are very encouraging outcomes for second language learners.

More specific detail about how to implement flipped learning in a senior mathematics classroom is provided by Lo and Hew (2017), in their comparative study of using a flipped classroom approach with both Form 6 (Year 12) underperforming students (Study 1) and Form 6 high-performing students (Study 2) in Hong Kong. They found significant learning gains in both groups of students. The program that was used is presented in Table 1.

Session	Video lecture (out-of-class)	Face-to-face lesson
1	Mid-point of two points; Distance between two points and slope of straight line	Transformation of point Advanced problems
2	Equation of straight line; x and y intercepts of straight line; Interception point of straight lines	Perpendicular lines Advanced problems
3	Slope of the equation of straight line; Line perpendicular to straight line; Perpendicular bisector of two points	Concept of locus Real-world problems

Table 1. Overview of the Class Schedule of the Remedial Program in Study 1

It is interesting to note in this schedule that the lower order learning skills of understanding linear algebra are covered by the out-of-class videos, whilst the higher order skills of tackling advanced and real-word problems take priority in the classroom. Time is still spent on some content areas (perpendicular lines and concept of focus), but content does not dominate. The results of the pre- and post-tests used to determine the learning gains in this study are presented in Table 2.

	Mean	Std deviation
Pre-test	2.77	1.79
Post-test	5.85	2.41

Table 2. Pre-test and Post-test Results for the Remedial Program, Study 1

There was a significant difference between the pre and post-test scores. Students reported, "We can review the videos where necessary" (Student 1) and "I find learning in groups better since my classmate can answer my questions immediately when I don't understand" (Student 7) in relation to the classroom activity (Lo & Hew, 2017, p. 228).

The program for high performing students is presented in Table 3.

Session	Video lecture (out-of-class)	Face-to-face lesson
1	Review of sequences; Introduction to arithmetic sequences	Advanced problems
2	Introduction to geometric sequences	Advanced problems
3	Distinguishing between arithmetic sequences and geometric sequences	Real-world problems
4	Summation of an arithmetic sequence	Real-world problems
5	Sum to the first n terms of a geometric sequence	Real-world problems
6	Sum to infinity of a geometric sequence	Real-world problems

Table 3. Overview of the Class Schedule of the Training Program in Study 2

A similar pattern of content delivery out of class and advanced and real-world problems in face to face lessons can be observed. The results of the pre-and post-tests used to determine the learning gains in this study, are presented in Table 4.

	Mean	Std deviation
Pre-test	2.00	1.77
Post-test	8.08	3.03

Table 4. Pre-test and Post-test Results for the Program, Study 2

A paired *t*-test showed that there was significant difference between the pre-test mean ($n = 24$, M = 2.00, SD = 1.77) and the post-test mean ($n = 24$, M = 8.08, SD = 3.03), $t(23) = 9.43$, $p < .0001$. Students in this group reported, "I am more motivated to learn in the flipped classroom" whilst 87.5% of students agreed or strongly agreed that "the flipped classroom has improved my learning of Mathematics" (Lo & Hew, 2017, p. 230). Whilst this study did not have a control group who used a conventional classroom approach, its findings provide a useful perspective on the way the flipped classroom can work for different groups of senior mathematics students and assist them in furthering their learning.

WAYS TO FLIP YOUR MATHEMATICS CLASSROOM

There are a variety of ways in which a mathematics classroom can be flipped. One example is provided by Fulton and Arbor (2013) who describe how they flipped their mathematics classroom in Byron High School, Minnesota, , by setting up online Moodle course sites for each mathematics class (pre-algebra through calculus 1) and embedding video lessons, lecture notes, homework solutions, and links to extra resources in each course site. The teachers felt it was important to generate their own material specifically targeted at their own students, and were provided with release time to generate these materials through a local grant.

For those who do not have the time or facilities to create their own materials, a particular way of using a flipped classroom in mathematics is through the use of a product such as *Khan Academy*, an online source of thousands of videos and interactive activities. The advantage of using a product such as this is that teacher preparation time is reduced, although the personal element of a teacher-prepared video is clearly not as great. Zengin (2017) used a mixed methodology to investigate the use of Khan Academy with 28 students in the department of Mathematics at a state university in Turkey. He found that the use of Khan Academy coupled with other mathematical software assisted his students to gain a greater understanding of the concept of double integrals that he was teaching. His students indicated that Khan Academy helped them to "visualise the concepts, promoted retention of knowledge and fostered easier learning of the concepts" (p. 96). Further, it helped his students come to class prepared. This is an important point for EAL learners, who can come to class with the basic language and concepts of the flipped lesson they have engaged with on Khan Academy, helping to scaffold their learning. Zengin's students also indicated that viewing videos and doing interactive quizzes on Khan Academy helped to learn mathematics in a meaningful way, rather just memorizing pages of a textbook. The nature of these quizzes is that students are given immediate feedback on their answers and supported with extra material if their responses are incorrect. The teacher is also provided with a record of the activities students have engaged with and their responses. Gilboy, Heinerichs, and Pazzaglia (2015), for example, found that they were able to design teaching strategies using the results of pre-class online Khan Academy modules, drawing on Bloom's taxonomy to ensure they were leading their students to higher levels of thinking. This is an excellent form of formative assessment, providing teachers with key points to target in the classroom or in subsequent flipped lessons.

Clearly, there are challenges in using a flipped classroom. O'Flaherty and Phillips (2015), in their scoping review of the use of flipped classrooms in higher education, found that the whilst there is increased evidence of student satisfaction and an increase in course grades with the use of a flipped learning model, there may be many educators who lack an understanding of how to do this successfully. There is no single pedagogical model for them to follow and the time that might be needed to set up teacher-created videos may be impractical. This is where the use of Khan Academy or similar

online material might ease the transition to a flipped classroom, allowing time for teachers and students to access the effectiveness of the model, and to determine whether there is a need to move to home-grown material or stay with the online options. Lo and Hew (2017) also emphasise the importance of carefully scaffolding the transition to a flipped classroom. This includes explaining the rationale of the flipped classroom approach and its potential benefits to both students and parents, and guiding students through introductory tasks. With Khan Academy, this might mean allowing time in class to introduce the videos and allowing students to experience doing the quizzes. Lo and Hew also point to the importance of teachers being discerning in their selection of online material for flipping, particularly if it is not their self-created material, as some material can be too difficult for students or not closely enough related to their course. Using short quizzes at the start of the classroom lesson, based on the previous evening's flipped material, is another way they suggest to motivate students to use this material. In addition, a fully flipped classroom might not necessarily be the best way for teachers to begin. Flipping some lessons and using the scaffolded approaches discussed could be a way of engaging students and ensuring that the majority of a class completes the required activities.

Overall, there would appear to be considerable benefit in exploring flipped classrooms for senior Mathematics, given the substantial body of research that points to the increased learning gains arising from it. The fact that there are also particular advantages for EAL learners in the multicultural classrooms which are increasingly the norm, should also be an imperative to at least experiment with this form of learning.

ENDNOTES

1. Explain Everything is an interactive online whiteboard where teachers can create their own videos inserted into the whiteboard screen.

2. The Khan Academy founded by Salman Khan has grown into an 80-person organization that aims at providing a free world-class education for anyone, anywhere (Khan Academy, 2016). Khan Academy can be accessed at www.khanacademy.org

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Mathematics in The Greek Anthology

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The Greek Anthology is a large collection of very short pieces of writing from Ancient Greece which were compiled over several centuries. A translation in English is available in 16 books spread over 5 volumes. The writings in this anthology include inscriptions copied from buildings, poems, and epigrams. However, book XIV is quite different from the other books. Sprinkled throughout book XIV are 45 elementary mathematical problems. Now Pythagoras, Euclid, and Archimedes are household names in the history of mathematics in ancient Greece. However, The Greek Anthology is not so well-known in mathematical circles. The purpose of this paper is to describe these ancient problems. Perhaps they can be used to enrich mathematics in the classroom, or to make innovative connections with subjects such as classical studies or ancient history

INTRODUCTION

The Greek Anthology is not simply a collection of works, but a history of redactions, of the collecting and refining of other prior collections of epigrams: short, sharp poetic statements. Most of it is composed of occasional poetry – a poetic style originally for religious and civil events such as festivals and the installation of monuments. The anthology's history begins with the sophist Meleager of Gadara, who, in around 60 BCE, took it upon himself to make a collection of forty-six of the best epigrammatic poets the Greek world had to offer under the title *Stephanos*, meaning a garland of flowers. Philippus of Thessalonica (c. 2nd c. CE) in his own redaction and supplementing of the text changed its name to *anthologia*, a word of similar meaning (Cameron, 1993, especially pp. 33-35; Gow & Page, 1965, 1968). The name stuck and is why we still refer to collections of poetry today as anthologies.

Other collectors repeated the process into the Byzantine Era. The compiler of *The Greek Anthology* in the fifteen books we have today was Constantine Cephalas, who likely lived during the Byzantine "Renaissance" of the tenth century emperor Constantine Porphyrogenitus (Cameron, 1993; Gow & Page, 1965). However, the first version of the Anthology that reached Europe, that of the fourteenth century monk Maximus Planudes which was published in Florence in 1494, omitted a great deal of this material. It included only some seven books. Luckily a manuscript of fifteen books was found in the Palatine Library at Heidelberg a century later, but was not published until 1776 (Cameron, 1993, pp. 178-200). A final sixteenth book of miscellanea later came to be added.

BOOK XIV

Book XIV of *The Greek Anthology* comprises 150 epigrams, 45 of which are arithmetical problems although Sir Thomas Heath counts the number of problems as 46 (Heath, 1921, p. 442). Epigrams 116-146 are "arithmetical epigrams" attributed to the grammarian Metrodorus in the original Greek (Paton, 1981, p. 84). Paton, the translator, suggests that all 45 of the mathematical problems in book 14 of *The Greek Anthology* were collected by Metrodorus "if we may judge by the style" and suggests that Metrodorus was "probably" a contemporary of Constantine the Great (A.D. 306-337) (Paton, 1981, p. 25). According to Heath, he was "probably of the time of the Emperors Anastasius I (A.D. 491-518) and Justin I (A.D. 518-527)" (Heath, 1921, p. 448). Midonick simply lists him simply as "c. AD 500" (Midonick, 1968, p. 51). However, as there is nothing known about this figure outside his name's association with this collection of epigrams and attempts to date their vocabulary, it seems almost impossible to fix anything about him and when he lived.

But how did the mathematical puzzles of Book XIV end up in a poetry collection? The Cephalas collection was an anthology of many other epigrammatic anthologies. Alan Cameron (1993) went as far as to call it the "definitive Byzantine collection". Much of Book XIV is composed of two similar varieties of puzzle: *ainigmata* (riddles, enigmas) (nos. 19-33, 35-47, 52-64, 101-111, 112-115, 147) and ancient oracles opaquely predicting the future (nos. 34, 65-100, 148-150). The short and often poetic nature of both of these forms likely encouraged the idea that such material might also be subsumed into the Cephalas collection. Dana Richards (1985, 123) may be trivially correct that most of these puzzles "are rarely challenging and often repetitive," though she herself is "surprised" at the clever use of punning in a

number of the lateral thinking puzzles that involve letter addition and deletion (e.g., nos. 16, 20, 31, 35). Sadly, however, none of these translates well into English.

But what is most interesting is that the riddles and other puzzles included in Book XIV were not simply popular entertainments committed to paper by a passing ancient folklorist. As has been noted, they often include Greek high cultural epic literary references, such as Heracles' task of cleaning the Augean stables rendered mathematical (no. 4), or riddles where the answers are the eye of the Cyclops from the Odyssey (no.100), Achilles' mother Thetis (no. 27), and the Argonauts' ship the Argo (no. 59). These symbols signalled membership in the small, exclusive educated elite in the Ancient and Mediaeval worlds: “Such mythical characters encoding mathematics ensures that the epigram is accessible to those Greeks who were being (or already) educated” (Leventhal, 2015, p. 210). They are the equivalent of how, today, cultural group membership often involves “getting” the reference to popular books, films, television programs, and music.

But what was it to “get the reference”? An entry level example is no. 64 in Book XIV of the *Anthology* - the “Riddle of the Sphynx” - perhaps the most famous Ancient Greek riddle. It is very likely that many educated Greeks would have known from an early age, as children today often still do, that the answer to what walks on four legs in the morning, two in the afternoon, and three in the evening is a human being. This would perhaps be a good place for mathematics teachers to start with these puzzles if they planned to utilise them in the classroom.

SELECTED PROBLEMS

This section contains a sample of problems selected from Book XIV, with our solutions and comments. All problems are quoted directly from Paton (1918).

Problem 1: On a statue of Pallas. I, Pallas, am of beaten gold, but the gold is a gift from lusty poets. Charisius gave half the gold, Thespis one-eighth, Solon one-tenth, and Themison one-twentieth, but the remaining 9 talents and the workmanship are the gift of Aristodicus. (p. 27)

The problem does not state explicitly what is required to be done. One must infer the requirements from the problem: find the weight of the statue in talents. Let talents be the total weight of gold in the statue of Pallas. The information in the problem leads to the equation $x = x\left(\frac{1}{2}+\frac{1}{8}+\frac{1}{10}+\frac{1}{20}\right)+9$.

Solving this equation leads us to conclude that the total amount of gold in the statue is 40 talents of which Charisius donated 20, Thespis 5, Solon 4, Themison 2, and Aristodicus 9. The problem introduces us to weights and measures in ancient Greece. The statue of Pallas was expensive, and some poets were apparently well-off.

Problem 2: The Graces were carrying baskets of apples, and in each was the same number. The nine Muses met them and asked them for some apples, and they gave the same number to each Muse, and the nine and three each had the same number. Tell me how many they gave and how they all had the same number. (pp. 49-50)

This problem states explicitly what is required to find. We are introduced to the world of Graces and Muses. The problem does not tell us explicitly how many Graces were carrying the apples, but there is a hint in the phrase ‘the nine and three’ that there were three Graces, which is in line with tradition (Hesiod, *Theogony*, 907). So there were 12 individuals, each of whom ended up with the same number of apples. The total number of apples is therefore a multiple of 12: there is not a unique solution.

Problem 3: Make me a crown weighing sixty minae, mixing gold and bronze, and with them tin and much-wrought iron. Let the gold and bronze together form two-thirds, the gold and tin together three-fourths, and the gold and iron three-fifths. Tell me how much gold you must put in, how much brass, how much tin, and how much iron, so as to make the whole crown weigh sixty minae. (p. 51)

A problem for simultaneous equations! Let g minae represent the amount of gold, b minae represent the amount of bronze etc. Then the data in the problem lead to these equations:

$$g+b=40;\ g+t=45;\ g+i=36;\ g+b+t+i=60$$

the solution of which is

$$t=14\tfrac{1}{2};\ g=30\tfrac{1}{2};\ b=9\tfrac{1}{2};\ i=5\tfrac{1}{2}.$$

Evidently a system of four equations in four unknowns did not perturb mathematicians in Ancient Greece. How did the author expect the reader to solve this problem? Perhaps by trial and error; if so, the solution suggests that a considerable effort would have to go into finding the solution. How would our secondary students attack this problem, without any prior lessons in solving simultaneous equations, and without a CAS calculator? The crown would be heavy, perhaps it was to be a crown for a statue.

Problem 4: This tomb holds Diophantus. Ah, what a marvel! And the tomb tells scientifically the measure of his life. God vouchsafed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled for him the light of marriage after a seventh, and in the fifth year after his marriage He granted him a son, Alas! Late begotten and miserable child, when he had reached the measure of half his father's life, the chill grave took him. After consoling his grief by this science of numbers for four years, he reached the end of his life. (p. 93)

This problem is the most famous in the collection among mathematicians. The question, which is not stated explicitly, is asking the reader to develop a timeline of the life of Diophantus, one of the most distinguished mathematicians in Ancient Greece. The events in his life are presented chronologically in the statement of the problem. One can deduce from this problem that Diophantus died at the age of 84. Reading Heath (2009, p. 3), one might infer that this is an historical fact because we know little about the life of Diophantus. However, after reading all the problems in the *Anthology*, one might conclude that the problem is simply set in an interesting context, as we might do in trying to make our problems appealing to students.

LINKS TO THE AUSTRALIAN CURRICULUM

It is often said that learning Latin is one way to learn about English grammar. Similarly, solving problems in *The Greek Anthology* assists with solving mathematical problems in a contemporary setting. The Australian Curriculum summarises literacy issues in mathematics as follows.

> In the Australian Curriculum: Mathematics, students learn the vocabulary associated with number, space, measurement and mathematical concepts and processes. This vocabulary includes synonyms, technical terminology, passive voice and common words with specific meanings in a mathematical context. Students develop the ability to create and interpret a range of texts typical of mathematics ranging from calendars and maps to complex data displays. Students use literacy to understand and interpret word problems and instructions that contain the particular language features of mathematics. They use literacy to pose and answer questions, engage in mathematical problem-solving, and to discuss, produce and explain solutions. (ACARA, 2017)

Problems from *The Greek Anthology* introduce the reader to "the vocabulary associated with number, space, measurement" in Ancient Greece. The reader is required "to understand and interpret word problems and instructions that contain the particular language features of mathematics" of Ancient Greece. We are letting ancient history into the mathematics lesson (Barbin et al., 2009). Word problems are eternal in mathematics education.

Finally, *The Greek Anthology* shows us that writing mathematical problems can be a literary art form. Could you construct a mathematical epigram?

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What makes a ‘good’ mathematical game?

James Russo, Monash University and Toby Russo, Bell Primary School

Most primary school teachers would concur that mathematical games are a valuable pedagogical tool to deploy in the primary classroom; however, not all mathematical games are likely to be equally valuable. How might teachers decide which games to introduce in their classrooms? In this paper we attempt to support teachers to address this question through presenting five principles of good mathematical games. Rather than operating as definitive criteria, our intention for presenting these principles is to stimulate critical discussion and to guide teacher decision-making. Examples of mathematical games that we believe appropriately capture each principle are provided.

INTRODUCTION

As primary mathematics educators, we both have a passion for designing mathematical games. Much of the time, this game design process can feel as much of an art as a science, as inspiration often strikes seemingly randomly, through casual conversation, playing with maths equipment, or doing a completely unrelated activity. However, as Gough (2004) notes, mathematical games do tend to cycle around specific mechanics and dynamics; so, whether we are consciously aware of it or not, most newly created games are likely to be derivative of games that we have previously played, read about, or used in our classrooms. Reflecting on this observation of Gough’s as well as our own process has led us to ask the question: What makes a “good” mathematical game?

To answer this question, we present five principles of good mathematical games. Under each principle, we present an example of a game we feel effectively illuminates the particular principle under consideration. The five principles are:

1. Students are engaged.
2. Skill and luck are balanced.
3. Mathematics is central.
4. Flexibility for learning and teaching.
5. Facilitates home-school connections.

Our intention in presenting these principles is to generate discussion, and to encourage teachers to reflect on the games they use in their classrooms and their rationale for doing so. You may have additional factors not discussed in the current paper which you may wish to consider for including a particular game. For example, perhaps a student in the classroom was responsible for developing the game, and therefore the class is particularly motivated to play it. Alternatively, you may have reasons for disregarding a particular principle. For example, when evaluating a particular game, you may decide to ignore Principle 5, reasoning that facilitating home-school connections is not relevant because the game you want to play requires specialised equipment only available in a classroom environment.

From our perspective, the most important consideration is that teachers are active when making choices about which games their students should play, and have given some thought as to why they believe the game chosen has particular value. We present these five principles as a starting point for this critical reflection. Interested readers may wish to refer to an article where these five principles are unpacked in more depth (see Russo, Russo, & Bragg, 2018).

PRINCIPLE 1: STUDENTS ARE ENGAGED

Some of the strongest evidence for the utility of mathematical games as a pedagogical tool stems from the fact that they tend to generate more on-task behaviour and mathematical dialogue than other non-game based activities (Bragg, 2012). Indeed, anecdotally, many teachers employ mathematical games largely because students enjoy playing them, and because they tend to support students to engage in learning mathematics (Rutherford, 2015).

We would suggest that Principle 1 is so central to any benefits potentially derivable from playing mathematical games that, if students are struggling to engage in the game and would rather be doing something else in the mathematics class, then, in fact, they should be doing something else. Games that students are not motivated to play are highly unlikely to generate positive learning outcomes compared to alternative activities.

PROBABILITY FOOTBALL

One example of a game that many students find highly engaging and enjoyable is *Probability Football*. Developed by the second author to explore conditional probability with his Year 5 and Year 6 students, the game has become a staple fixture in his classroom, particularly during September (see Figure 1).

How to Play

Materials: game board (an AFL football field), a ten-sided die and a counter.

This game is for 2 players.

1. Decide which player is red (kicking right) and which player is blue (kicking left). The counter (or "ball") begins in the middle of the game board. The highest roll of the die determines which player starts in possession (simulating a "centre bounce").

2. The controlling player (Player 1) must choose to which adjacent section they want to kick the ball: there is a red (or blue) arrow pointing to the adjacent sections and a matching probability for each (or "likelihood of success").

3. To simulate a kick, Player 1 rolls the die. The number rolled determines if it was a successful play. For example, if the probability is 80%, a roll of 1-8 means the play is successful and they continue playing from the next section; however a roll of 0 or 9 means the play was unsuccessful and the other team (Player 2) takes possession from this point. Player 2 would then roll the die and follow the same process.

4. When a player is in a section with a box around the probabilities, they can choose to kick for goal. Success means they score a goal (and the ball begins in the middle with a "centre bounce"), failure means it is a behind (and the other team takes a kick in).

5. Play continues until the end of the given timeframe (e.g., 20 minutes), with the winner the player with the highest score (with a goal worth six points and a behind worth one point).

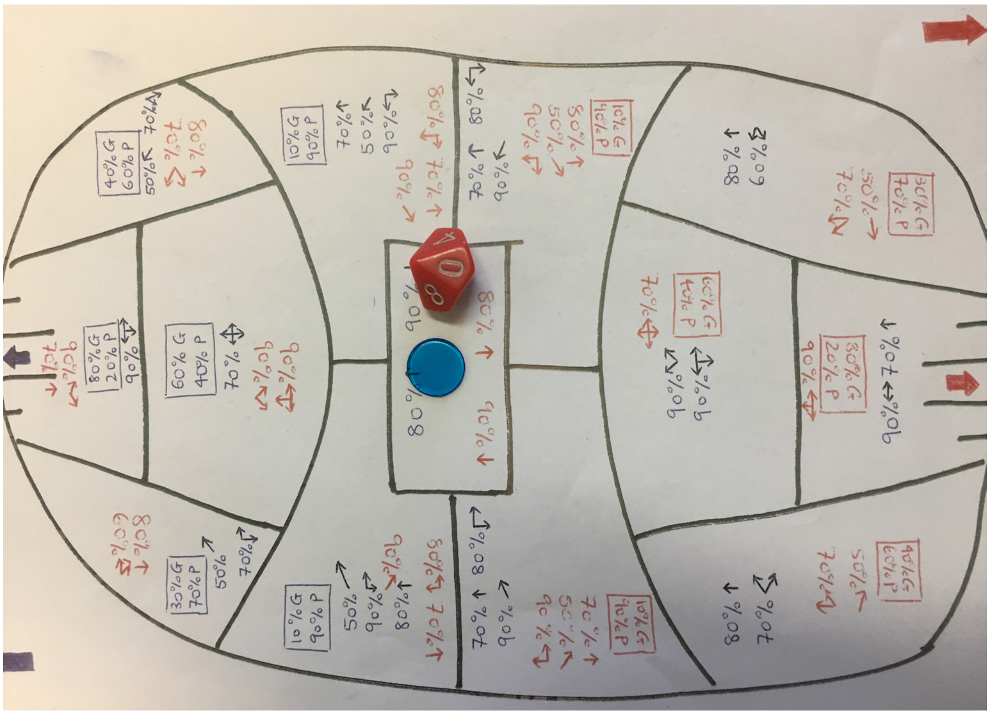

Figure 1. Probability Football game board.

PRINCIPLE 2: SKILL AND LUCK ARE BALANCED

Mathematical games should balance skill and luck.

In part to support prolonged student engagement in a mathematical game, we have found that it is important for the game to balance skill and luck. Without some component of luck, more mathematically-able students are at risk of dominating. This can be de-motivating for all learners. By contrast, without some component of skill, it is arguable that the activity even warrants being referred to as a game at all (Gough, 1999). Sometimes some form of turn-taking or role-reversal is central to ensuring that skill and luck are sufficiently balanced.

From an equity point of view, it is vital that all students are given opportunities to lose games, as well as to win. We would contend that losing games gracefully is integral to the developing characters of young people, and has powerful social consequences. Being a “good loser” or a “good winner” makes it more likely that other children will want to play games with you in the future, which is likely to be a latent objective for participants in any game. In addition, so-called failure provides a powerful opportunity for learning; a common sense claim increasingly substantiated by evidence (Kapur, 2014; Warshauer, 2015).

SKIP-COUNTING BINGO

The first author has often used a version of Bingo, called Skip-Counting Bingo, as an engaging game to play with his Year 2 students, partly because it effectively balances skill and luck to maintain student engagement.

How to Play

Materials: 100-chart, 6-sided die

This game is for 2 to 5 players.

1. To begin, children each choose three Bingo numbers in turn, and mark these numbers on their 100-chart. Note that players must choose numbers greater than 10 (or 20 to extend the range of numbers in the skip-counting sequence).
2. One of the children rolls the 6-sided die. Together, children begin counting by whatever the number rolled, using the 100-chart to keep track. For example, if they roll a four, they would begin counting by 4s from zero: 4, 8, 12, 16, 20. etc.
3. Children stop counting when they encounter a bingo number. In the game shown in Figure 2, children would stop counting at 28 if a four was rolled, as this is the first Bingo number encountered.
4. The die is rolled again, and a new counting sequence is explored. For example, if a 3 is rolled, the group would be counting by 3s from zero. They would again stop when they encountered a bingo number (42 in Figure 2).
5. Play continues until one of the players removes all their numbers and shouts “Bingo!”.

PRINCIPLE 3: MATHEMATICS IS CENTRAL

In order for a game to be of value in the mathematics classroom, it must have important mathematical ideas at its core. Badham (1997) suggested that the teacher should ensure that games are used purposefully, and that a chosen game matches the specific mathematical objective under focus.

The first author’s default approach when designing games for early primary school students is to use games to provide problem-based practice with a particular skill or concept (e.g., 10s facts; skip-counting sequences). The strategic aspect of the game serves to keep the game interesting through providing a compelling purpose for this practice. Conversely, Gough (1999) argues that games can operate as a context for learning a new concept, rather than simply practising a previously known skill. This is often the approach adopted by the second author when teaching upper primary students. For example, he has used *Colour in Fractions* (Clarke & Roche, 2010) to expose students to the notion of adding fractions with related denominators, prior to any formal instruction in the idea. Rather than being an explicit focus of game play, this concept emerges out of the game dynamics.

Figure 2. Three children begin a game of Skip-Counting Bingo.

HOW CLOSE CAN YOU GET?

We first came across this game in an older edition of *Australian Primary Mathematics Classroom*, where Vale (1999) was using it as a context for discussing mental computation strategies. We particularly like it because we have found it challenging to source worthwhile activities that focus on subtraction as the "difference between" two numbers.

How to Play

Materials: Playing cards.

This game is for 2 to 5 players. Remove the tens and picture cards from a deck of cards, leaving ace to 9 for the game.

1. Shuffle the cards. Deal each player 4 cards, face down.
2. Turn up 2 more cards. The first card goes in the tens place and the second in the ones place to form the target number. For example, a 6 then ace becomes 61.
3. Players turn up their four cards and arrange them into two 2-digit numbers, so that the difference between their two numbers is as close to the target number as possible.
4. To score, each player then finds the difference between his or her result and the target number. For example, if the target number was 61, and a player had A, 5, 3, & 9, the best she could do would be 95 – 31 = 64. Her score would be 64 – 61 = 3 for that round.

5. Note that you can go over or under the target number.

6. For the next round, turn up two new cards from the deck to form the next target number. Players can choose to use their same four cards or deal out four new cards.

7. At the end of five rounds the player with the lowest total score wins.

[Sourced from Vale (1999). Originally appeared in MAV/AAMT–Home Mathematics (1988).]

PRINCIPLE 4: FLEXIBILITY FOR LEARNING AND TEACHING

Good mathematical games should be flexible enough to cater for learners of different abilities. To some extent, this can be achieved through the balancing of skill and luck as a key aspect of the game dynamic. Consequently, many games have some degree of differentiation inherent in them, which allow learners of different abilities to play against each other, and to apply developmentally-appropriate strategies.

For example, consider the game Skip-Counting Bingo previously described. If this game was to be played by upper primary students, they would be more inclined to rely on their knowledge of multiples and factors, rather than on the skip-counting patterns used by lower primary students. This approach might allow such students to begin to calculate which numbers they are most likely to land on *before even playing the game*, rather than relying on insights that only emerge through repeated game play. Adopting this more sophisticated approach is likely to improve their chances of success in the game. However, the fact that the game literally relies on the "luck of the dice" still offers an opportunity for students who adopt less sophisticated strategies to be successful.

Other games more explicitly lend themselves to differentiation through directly modifying the game mechanics, including playing equipment (e.g., the dice used). Making such modifications may allow a teacher to group students with peers of similar-ability, thereby allowing the group to focus on a carefully targeted learning area. For example, if playing Multiple Mysteries (Russo & Russo, 2017), a teacher may group students together whom they determine would benefit from practice with a particular pattern of multiples (e.g., 3s, 7s, 8s).

MULTIPLE MYSTERIES

We like this game for its simplicity and versatility. It rewards both calculated risk-taking and an understanding of patterns and rules for identifying multiples.

How to Play

Materials: Playing cards, calculator.

This game is for 2 to 4 players. Remove the tens and picture cards from a deck of cards. You may choose to leave the Jacks and Jokers in the deck (representing zeroes).

1. The teacher, or the players, choose a target multiple (e.g., 3, 5, 8).

2. Five cards are dealt to each player, and the first player begins their turn.

3. The objective on any given turn is to use some or all of these five cards to make a multiple of the particular target that is the focus for that round. The more digits in the number you create, the more cards you use, and the more points you will score. For example, if the target multiple was 8, using the cards 4 and 8 to make the number 48 would earn you two points; whereas using the cards 2, 4 and 8 to make the number 248 would earn you three points.

4. The player then "banks" these cards in their own bank, and picks up some additional cards from the deck to ensure they have five cards in their hand at the beginning of the next turn.

5. On any given play, any opponent has the option of challenging the player's multiple through using a calculator and dividing the created number by the target multiple (e.g., 248 divided by 8). If the challenger is correct, the cards used

to create the number are "banked" instead by the challenger. If the challenger is incorrect, they must pay a one-card penalty to the player who was challenged.

6. Play continues in turn. Once all the cards in the deck are used, the "banked" cards are counted up. The player who "banked" the most cards wins the game.

PRINCIPLE 5: FACILITATES HOME-SCHOOL CONNECTIONS

Mathematical games can be a tool for building positive connections between home and school environments (Rutherford, 2015). This can occur through bringing families into the school, as seen in family maths nights, but also by bringing maths "into the home." The second author is currently piloting a program where students play games in class, watch an instructional video of the game when they get home, and then play the game together with a carer or sibling as "homework."

There has been positive feedback about the program to date, and students have been highly motivated to play the games at home. They have taken on the role of "expert" when explaining the activities to family members, which has evidently helped to increase the confidence of some students. One student explained: *I taught the game to Mum without having to watch the video and then I had to teach her what multiples are!* The fun, competitive element of game play seemed to shift the perception around maths homework for a number of students. For example, another student, who is often a reluctant participant in maths activities and rarely completes homework tasks, spoke enthusiastically about playing Caught Red-Handed (Russo, 2017): *I had so much fun playing the red-handed game at home on the weekend. I beat my sister three times in a row and she couldn't work out how!*

CAUGHT RED-HANDED

Based on the concept of 11s, we have found this game to be useful for playing with students from Year 1 to Year 6. Children particularly enjoy the game because prior practice often results in them having a genuine competitive edge over novice adults.

How to Play

Materials: 20-sided die or spinner. Students can create their own game board by writing out the numbers 1 to 20. The game requires one red pen and a black/blue pen for each player.

This game is for 2 players.

1. Circle the number 20 with a red pen and roll a 20-sided die to determine the two other "red" numbers to circle.

2. A game of rock-paper-scissors decides who begins the game.

3. On their turn, each player is allowed to cross out 1 or 2 numbers. Players must cross out the numbers in order (i.e., 1, 2, 3, etc.).

4. The goal of the game is to cross out at least two of the three "red" numbers.

5. Variant: Play the game again (revenge match) with the red numbers in the same place, but this time allow players to cross out 1, 2 or 3 numbers. The loser of the previous game can decide whether they wish to play first or second.

CONCLUDING THOUGHTS

This paper has presented five principles of good mathematical games and provided some practical examples of games we believe illustrate these principles. As noted at the outset, rather than attempting to present definitive criteria of what denotes a good game, our intention is to stimulate critical discussion and to guide teacher decision-making. We encourage classroom teachers to use this paper as a springboard, and take the discussion of what constitutes a good mathematical game into the staffroom.

Finally, we do not want to leave readers with the impression that the only important role a teacher plays in a games-based mathematics lesson is in choosing an educationally-rich game. During both the exploration phase, when students are playing the game, and the post-game discussion, the teacher needs to play an active role in stimulating mathematical thinking (Badham, 1997). For example, the teacher may provide thoughtful prompts and questions that encourage students to:

- reason mathematically (Explain why that makes sense);
- justify decisions (Why did you make that move?);
- make predictions (What do you need to win?); and
- consider alternatives (What would you do differently next time?)

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Peer reviewed papers

The art of enrichment

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We all know that one of the most challenging tasks teachers face is creating rich experiences for their students (especially when attempting to extend the more capable ones). Spending hours searching and preparing material that will fit this purpose is not really a viable solution nor creating a new course or looking for a new textbook. In this paper I will present my way of extending capable students without any (or very little) special preparation. The alternative title of this paper is "How to turn an ordinary question into an extraordinary one" as this is the strategy I endeavour to use with my Year 7-10 students whenever the opportunity arises. It all starts with an ordinary question we can find in any textbook. What we do with it is what makes all the difference.

INTRODUCTION

Don't get me wrong … most mathematics questions are meaningful already the way they are. I am referring to the worded questions in particular. I just want to show my students that there are questions behind answers. Someone once said that "Solving a maths problem is like eating a piece of cake. You can gobble it up or …you can enjoy every bit of it." I am the "enjoy every bit of it" type. And while I do that, I also use some High Impact Teaching Strategies (DET, 2017), namely:

- Explicit teaching
- Worked examples
- Collaborative learning
- Multiple exposure
- Questioning
- Metacognitive strategies
- Differentiated teaching

As a teacher I need a good challenge as much as my students do. My own challenge in this case is to exploit any opportunity of challenging my students. Worded questions constitute such a great source. Worded questions enable students to explore new real-life contexts, apply new skills in these new contexts, transfer knowledge and make new connections between topics. Tackling worded questions represents a very important step in developing students' understanding. When discussing and solving worded questions in class, explicit support is given to students toward developing their critical and creative thinking capabilities. Therefore I regard solving worded questions as an important component of my teaching. I incorporate them in my teaching as often as possible once new skills are explicitly taught.

THE CONTEXT

Most questions are like little riddles. Therefore, in order to solve a question, we have to understand it first. Understanding takes time (… as we enjoy every bit of it). I give students time to read and comprehend each question. I sometimes let them take turns explaining the question to one another. Always, as a class, we look for the most important pieces of information given. Numbers (details) are not important. Only the big picture is what really matters. These are all metacognitive strategies that we model at the same time. We spend a decent amount of time understanding the question. Moreover, we can turn the joy of solving / developing it into an extraordinary experience.

HOW?

Ordinary questions may be transformed as follows. Under the same context by:

1. Choosing different values or harder expressions.
2. Extending the set of allowable values (for example from N to Z, from Z to Q or from Q to R).
3. Changing some of the conditions (you may add conditions or get rid of some)
4. Generalising if possible.
5. Asking students to scaffold / simplify / organise / categorise.

Using a new context, derived from the original question, by:

6. Asking students to write their own question.
7. Asking students to critique questions created by their peers (look for unnecessary information, incomplete details given, questions that are impossible to solve, unclear context etc.).
8. Asking students to create a template for a solution to a type of question.

These examples are just a few of the many possibilities.

HERE ARE SOME EXAMPLES!

EXAMPLE 1 – YEAR 7 – FRACTIONS – OPERATIONS WITH FRACTIONS

There is 1 L of juice in a bottle. James drinks $\frac{1}{5}$ L in the morning and $\frac{1}{8}$ L in the afternoon. What fraction (of the original amount) is left?

SOLUTION:

Most students will get that:

$$1-\frac{1}{5}-\frac{1}{8}=\frac{40-8-5}{40}=\frac{27}{40}$$

Worded answer: After drinking $\frac{1}{5}$ L of juice in the morning and $\frac{1}{8}$ L of juice in the afternoon there is $\frac{27}{40}$ L left in the bottle so the fraction left is $\frac{27}{40}$.

At this stage I am not sure if my students understood the difference between fractions and quantities. I could have asked for units for their calculation but instead I make …..

1ST CHANGE:

I ask: What if I had 2 L of juice originally in the bottle?

Students start calculating: $2-\frac{1}{5}-\frac{1}{8}=\frac{80-8-5}{40}=\frac{67}{40}$.

How is that for an answer?

After drinking $\frac{1}{5}$ L of juice and another $\frac{1}{8}$ L of juice there is $\frac{67}{40}$ L left in the bottle so the fraction left is $\frac{67}{40}$ (worded answer).

In this new context, does the answer make sense?

Giving students one minute to write down in their books one reason why $\frac{67}{40}$ is wrong allows me time to evaluate their understanding. Some students may need to read the question again while others to write the original quantity as a fraction. However most students are able to recognise that $\frac{67}{40}$ is bigger than 1.

One student explains:

But $\frac{1}{5}$ L represents $\frac{1}{10}$ of 2 L and $\frac{1}{8}$ L represents $\frac{1}{16}$ of 2 L so the fraction left is $1-\frac{1}{10}-\frac{1}{16}=\frac{80-8-5}{80}=\frac{67}{80}$.

Another student says:

$\frac{1}{5}$ L is actually 200m L and $\frac{1}{8}$ L is actually 125m L so there is 2000 – 200 – 125 = 1675 mL left out of 2000m L which means $\frac{1675}{2000}=\frac{67}{80}$.

What about my solution?
(Not that it is needed any more)

Out of 2 L, James already had $\frac{1}{5}\text{L}+\frac{1}{8}\text{L}=\frac{13}{40}\text{L}$ and there is $\frac{67}{40}\text{L}$ left in the bottle.

$\frac{67}{40}\text{L}$ out of 2 L means $\frac{67}{40}\div 2=\frac{67}{80}$.

2ND, 3RD OR 4TH CHANGE …

To be solved in class or set for homework: What if the original amount of juice was 5 L? Or 10 L? Or *n*L? or 0.5L? By the way ….is this possible? What is the minimum amount of juice possible in the bottle that makes this question valid? I already see a function, an expression and domain ….but they are just in Year 7, aren't they? I also often encourage students to:

- Place these new related questions posed into categories: easy, medium, and hard.
- Think of other questions that could be answered
- Change a different question following similar ways
- Try some of the questions written by peers for homework … etc.

Creating a new question by modifying the original question is a skill that must be modelled by teacher. I set similar tasks as homework only once, because, as a class we have made sufficient attempts for students to feel challenged but not overwhelmed by writing (and solving) their own questions. My aim is to assist students to become more curious and inquisitive.

EXAMPLE 2- YEAR 7 – FRACTIONS

James was given a cake for his birthday. He ate half of it that day and, every day after that, he ate half of what was left from the previous day. What fraction has he eaten after 4th day? If James continues to eat from his cake following this pattern, when will he be finishing the cake?

SOLUTION:

(I helped here with the setting out)

After day	Fraction eaten	Fraction left
1	$\frac{1}{2}$	$1-\frac{1}{2}=\frac{1}{2}$
2	$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$	$1-\frac{3}{4}=\frac{1}{4}$
3	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$	$1-\frac{7}{8}=\frac{1}{8}$
4	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}$	$1-\frac{15}{16}=\frac{1}{16}$

Table 1. After first 4 days.

Worded answer: After 4th day James has eaten $\frac{15}{16}$ of his cake and has $\frac{1}{16}$ of the cake left.

He will never finish the cake as there is always some cake left in the fridge since he is eating only half (theoretically).

I DEFINITELY CANNOT LET THIS OPPORTUNITY SLIP AWAY…THEREFORE I ASK

What is the fraction left after 5th day?

By now everyone knows what to do:

5	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}=\frac{31}{32}$	$1-\frac{31}{32}=\frac{1}{32}$

Table 2. After day 5.

What is the fraction left after 6th day?

No trick question here:

6	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}=\frac{63}{64}$	$1-\frac{63}{64}=\frac{1}{64}$

Table 3. After day 6.

How can I stop here when I am so close ….?

My next question was …

What is the fraction left after *n*th day?

They look for a pattern:

After 1 day $\frac{1}{2}$

After 2 days $\frac{1}{4}$

After 3 days $\frac{1}{8}$

After 4 days $\frac{1}{16}$

After 5 days $\frac{1}{32}$

After 6 days $\frac{1}{64}$

How are these numbers related: 2, 4, 8, 16, 32 …?

Powers of 2 .

Show me.

2^1, 2^2, 2^3, 2^4, 2^5 and 2^6.

What if we have "*n*" in the first column?

Therefore 2^n

After a few minutes, with great class and group discussions and some unsuccessful attempts, the students figured out the correct answer.

$$\frac{1}{2^n}$$

Worded answer: After n(th) day there is still $\frac{1}{2^n}$ of the cake left (theoretically).

Have I pushed too far …at the end of the day they are only in Year 7?

Maybe …. but it was worth it …

EXAMPLE 3 – YEAR 7 – RATES AND RATIOS

A snail slithers 2 mm every 5 seconds. How long will it take to slither 1 m?

(Greenwood et al, 2016, p. 603)

Just a matter of finding how many 2mm are in 1 m.

Some possible changes:

1. A snail slithers 2 mm every 5 seconds. How long will it take to slither 0.5 km?
2. A snail slithers 2 m in 3 hours at a constant rate. How long will it take to slither 5 mm?
3. A snail slithers 2 m in 5 hours at a constant rate. How long will it take to slither 1 mm?
4. A snail slithers 3 mm every 8 seconds. How long will it take to slither 1 m?
5. A snail slithers 2 mm every 5 seconds. After slithering for 5 seconds it stops for 4 seconds. How long will it take to slither 1 m?
6. A snail slithers 2 mm every 5 seconds while a second one slithers 2 m in 5 hours at a constant rate. Which one is faster?

7. A snail slithers 2 m in 5 hours at a constant rate. A second snail follows the path of the first one 10 minutes later. They meet each other after another 30 minutes. What was the constant rate at which the second snail slithers?

Ask students to:

- Place these new related questions posed into categories: easy, medium and hard.
- Think of other questions that could be answered; maybe harder
- Change a different question following similar ways
- Try some of the questions written by peers for homework … etc.

EXAMPLE 4 - YEAR 9 – LINEAR EQUATIONS

Two 20 cm candles are lit in the same time. One candle burns out in 4 hours. The other candle burns out in 5 hours. If both candles burn at a constant rate, how long it takes for one candle to be twice as high as the other one?

You must write and solve a linear equation in order to find the answer.

Possible prompts:

- Read the question and explain it to your partner (with your book closed).
- Draw a picture.
- Identify the candle that will be twice as long as the other one (second one in this case).
- Identify the unknown for our equation: t = time (in number of minutes) taken for the second candle to become twice as long as the first one .
- Emphasis on units.
- Working given in context (and the answer).

Solution:

Let t = time in min when second candle is twice the size of the first

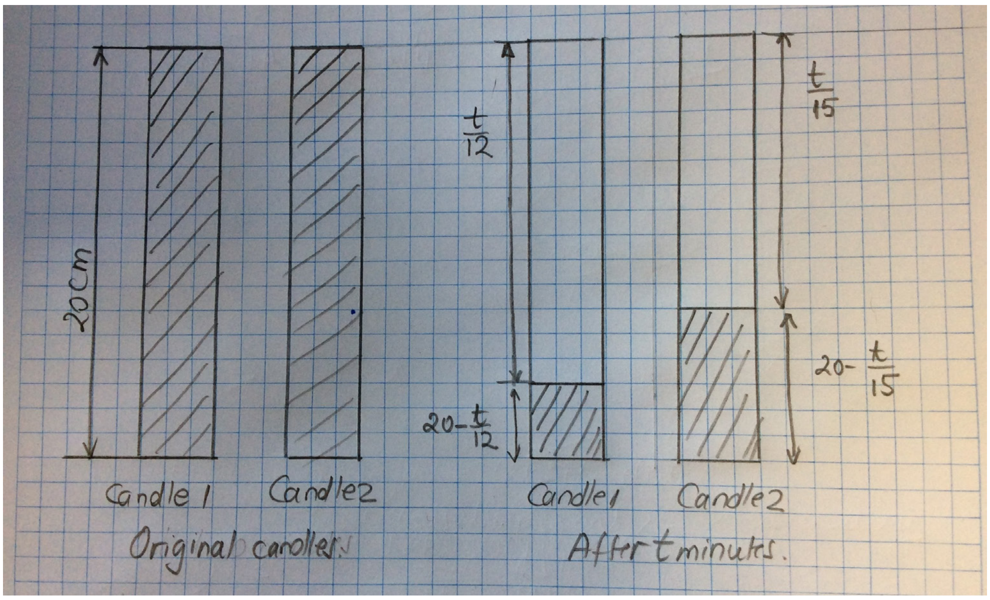

Figure 1: Original vs after *t* Minutes

In mathematical terms:

Height (cm)	Time (minutes)
20	240
h_1	t

Table 4. Finding the Fraction of the First Candle Burnt after *t* Minutes

$h_1 = \frac{20t}{240} = \frac{t}{12}$ (the height burnt off from the first candle after *t* minutes)

The height of the first candle is $20 - \frac{t}{12}$ given in cm

Similarly:

$h_2 = 20 - \frac{t}{15}$ is the height of the second candle after *t* minutes

The equation is: $20 - \frac{t}{15} = 2\left(20 - \frac{t}{12}\right)$

Solving the equation we find $t = 200$ minutes

Worded answer: After 200 minutes (3h and 20 minutes) the second candle is twice as high as the first one.

And now let's make some changes…

- Solve this question using a different method (graphing, trial and error, & algorithms come to mind).
- Write your own question similar to the one given (candles having same length but different burning times).
- Identify the piece of information that is not necessary in the original question.
- How can you make it more difficult? (choosing candles of different lengths, for example).
- Without giving the lengths of the candles, write a question that can be solved with two original candles having different lengths (that type of question seemed to challenge the most my students).

CONCLUSION

In the introduction I identified seven High Impact Teaching Strategies that are modelled through this type of class activity. There are also general capabilities (critical and creative thinking) that are developed through such tasks. The benefits of exposing students to various ways of modifying and enriching original questions are far greater than we first imagine. The use of whole class discussion, different questioning techniques, exploration of various options, and critique of the question itself are examples of higher order thinking opportunities in which students can take an active role. This is enrichment at its best. It should ensure that students can tackle investigations or other problem solving tasks when needed.

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Algorithms for partial fraction decomposition

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There are some traditional methods to resolve proper fractions as a sum of several fractions with simpler denominators. This process is called partial fraction decomposition. Finding the coefficients in the process of decomposition relies on solving a set of simultaneous equations in the traditional methods. The process of solving the set of simultaneous equations often involves very complicated calculations. This article will provide alternative ways to quickly resolve a few types of proper algebraic fractions. More general forms of algebraic fractions can be reduced to those forms.

INTRODUCTION

In the VCE curriculum, algebraic manipulation skills for partial fraction decomposition of rational functions are required in the topic Number Systems and Recursion of the Area of Study 2, Specialist Mathematics Units 1 & 2 (HREF2). Those skills are applied in anti-differentiation in Specialist Mathematics Units 3 & 4 (HREF2: Calculus, Area of Study 3). Questions of anti-differentiation by using partial fraction techniques are frequently assessed in VCE exam 1 of the subject Specialist Mathematics. I quote two questions from the past exams here.

Example 0.1
Question 2, 2017 Specialist Mathematics Written Exam 1-Victoria (HREF3)

Find $\int_1^{\sqrt{3}} \frac{1}{x(1+x^2)}dx$, expressing your answer in the form $\log_e\left(\sqrt{\frac{a}{b}}\right)$ where a and b are positive integers.

Example 0.2

Question 2, 2017 Specialist Mathematics Written Exam 1-NHT (HREF4)

Find the value of a given that $\int_{-2}^{a} \frac{8}{16-x^2}dx = \log_e(6), a \in (-2,4)$.

In answering Example 0.1, students used the methods taught in textbooks (Evans, Cracknell, Astruc, Lipson, & Jones, 2016, pp. 312-320) to present their work as following.

Let $\quad \frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$

Then $\quad \frac{1}{x(1+x^2)} = \frac{A(1+x^2)+x(Bx+c)}{x(1+x^2)}$

Therefore, $\quad 1 = A(1+x^2)+x(Bx+c)=(A+B)x^2+Cx+A$

Equating the corresponding coefficients gives the values $A = 1, B = -1$ and $C = 0$. Some students assigned three distinct values to x to get three equations; they could obtain the values of A, B and C by solving the equations simultaneously. According to the exam report (HREF5), this question tended to be answered well. However, a large number (38%) of students did not know how to use partial fraction, or could not use partial fraction correctly, or produced errors in solving equations. In fact, only 4% of candidates achieved full marks for this question.

As so many students struggled with this, we have to ask if there is a better way to teach partial fractions. This article will provide alternative ways to quickly resolve a few types of proper algebraic fractions. Using the methods presented here, the correct answers

$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$ and $\frac{8}{16-x^2} = -\frac{8}{(x-4)(x+4)} = -\frac{1}{8}\left(\frac{8}{(x-4)} - \frac{8}{(x+4)}\right) = -\frac{1}{x-4} + \frac{1}{x+4}$

can be obtained straight away. To decompose the fraction $\frac{3x+4}{(2x-3)(5x+6)}$, using traditional methods, is painstaking

for many students due to the complicated calculations involving fractions. Again, using the methods presented here, $\frac{3x+4}{(2x-3)(5x+6)}$ can be easily written as:

$$\frac{3x+4}{(2x-3)(5x+6)} = \frac{1}{27}\left(\frac{17}{2x-3} - \frac{2}{5x+6}\right).$$

Our algorithm for partial fraction decomposition is based on the traditional general theory which can be found in any textbook containing integration. HREF1 integrated the traditional theory into a comprehensive theorem and provided algorithms for decomposing a few types of fractions, although the methods are not necessarily easy. Our algorithm in Case 1 is a result of the case where the denominator of a proper fraction is a product of distinct linear factors. The algorithms in Case 2, Case 3, and Case 4 are original.

For theoretical completeness, the comprehensive theorem in HREF1 is summarised and quoted in the next section, Preliminary, although there are some changes to symbols and expressions. This theorem is not essential for understanding the algorithms in Cases 1-4. Hence, there is little consequence for the rest of this paper if the next section is skipped by the reader.

PRELIMINARY

For convenience, we assume that all polynomials and algebraic functions discussed here are defined over the rational field $\mathbb{Q}$. Denote the set of polynomials over $\mathbb{Q}$ by $\mathbb{Q}[x]$ and denote the degree of a polynomial $P(x)$ by $\partial^\circ(P(x))$.

Any algebraic function $\frac{P(x)}{Q(x)}, P(x), Q(x) \in \mathbb{Q}[x]$ can be expressed as a sum of a polynomial and a proper fraction. In the fraction part, the denominator is a product of linear and irreducible quadratic factors by the Fundamental Theorem of Algebra (Evans et al., 2016, p. 188). The result can be written as a theorem as following (HREF1).

Theorem 1

Let $P(x)$ and $Q(x)$ be non-zero polynomials over the rational field $\mathbb{Q}$. $Q(x)$ is a product of powers of distinct linear factors and irreducible quadratic factors:

$$Q(x) = \prod_{i=1}^{k} \left((Q_i(x))^{n_i} \right)$$

where $Q_i(x) \in \mathbb{Q}[x]$, for any $1 \le i \le k, k \in \mathbb{N}$, is a linear factor or an irreducible quadratic factor.

Then there are unique polynomials $I(x), A_{ij}(x) \in Q[x]$, with $\partial^\circ\left(A_{ij}(x)\right) < \partial^\circ\left(Q_i(x)\right)$, such that

$$\frac{P(x)}{Q(x)} = I(x) + \sum_{i}^{k} \sum_{j}^{n_i} \frac{A_{ij}(x)}{(Q_i(x))^j}.$$

If $\partial^\circ\left(P(x)\right) < \partial^\circ\left(Q(x)\right)$, then $I(x) = 0$.

For example, the algebraic fraction $\frac{2x^5 - 5x^4 + x^3 - 9x^2 - x - 9}{x^3 - 2x^2 - 2x - 3}$ can be expressed as the following.

$$\frac{2x^5 - 5x^4 + x^3 - 9x^2 - x - 9}{x^3 - 2x^2 - 2x - 3} = 2x^2 - x + 3 + \frac{x^2 + 2x}{(x-3)(x^2+x+1)}.$$

Furthermore, the proper fraction can be resolved as a sum of several simpler fractions:

$$\frac{x^2 + 2x}{(x-3)(x^2+x+1)} = \frac{15}{13(x-3)} - \frac{2x-5}{13(x^2+x+1)}.$$

A method called the Limit Method is illustrated in (HREF1) in determining the coefficients of each term. Traditionally, the process of resolving $\dfrac{x^2+2x}{(x-3)(x^2+x+1)}$ is as following.

Let $\dfrac{x^2+2x}{(x-3)(x^2+x+1)} = \dfrac{A}{x-3} + \dfrac{Bx+C}{x^2+x+1}$.

Then combine the terms on right-hand side of the equation into a single fraction and equate the numerators of both sides of the equation.

$$x^2+2x = A(x^2+x+1)+(x-3)(Bx+C)$$

Then three equations can be built up by assigning three different values to x, or by equating the corresponding coefficients of the polynomials in expanded form on both sides. The values of A, B, and C can be obtained by solving those three equations simultaneously.

The traditional process of decomposition works for all rational algebraic fractions. However, the calculation in the process demands a high level of numerical and algebraic skill in some situations. Alternative ways to resolve proper fractions quickly for a few types of algebraic fractions will be shown in this paper. More general forms of algebraic fractions can be reduced to those forms.

CASE STUDIES

Case 1

Resolving $\dfrac{P(x)}{Q(x)}, P(x), Q(x) \in \mathbb{Q}[x]$ where $\partial^\circ\big(P(x)\big) < \partial^\circ\big(Q(x)\big)$, $P(x)$ and $Q(x)$ are coprime, and $Q(x)$ is the product of distinct linear factors, by substitution. We may assume that the polynomial $Q(x)$ is monic without losing generality.

We will illustrate this method by resolving the fraction $\dfrac{P(x)}{Q(x)} = \dfrac{2x^2+x+3}{(x+1)(x-2)(x+3)}$.

Example 1.1

As we know by Theorem 1, this proper fraction can be written as following.

$$\frac{2x^2+x+3}{(x+1)(x-2)(x+3)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+3}.$$

The value of A can be found by substituting $x = -1$, the root of $x + 1 = 0$, into $\dfrac{2x^2+x+3}{(x-2)(x+3)}$:

$$A = \frac{2(-1)^2+(-1)+3}{(-1-2)(-1+3)} = \frac{4}{-6} = -\frac{2}{3}.$$

Similarly, we have

$$B = \frac{2(2)^2+(2)+3}{(2+1)(2+3)} = \frac{13}{15}, \text{ and}$$

$$C = \frac{2(-3)^2+(-3)+3}{(-3+1)(-3-2)} = \frac{18}{10} = \frac{9}{5}.$$

Therefore, $\dfrac{2x^2+x+3}{(x+1)(x-2)(x+3)} = \dfrac{-2}{3(x+1)} + \dfrac{13}{15(x-2)} + \dfrac{9}{5(x+3)}$

This method is similar to the method mentioned in (HREF1), but avoided using differentiation.

Case 2

Resolving $\dfrac{px+q}{(ax+b)(cx+d)}$ by cross-multiplication.

$$\frac{px+q}{(ax+b)(cx+d)}=\frac{1}{ad-bc}\left(\frac{aq-bp}{(ax+b)}-\frac{cq-dp}{(cx+d)}\right)$$

Example 2.1

$$\frac{P(x)}{Q(x)}=\frac{3x+4}{(2x-3)(5x+6)}.$$

- Find the difference of the cross products of the coefficients of factor $(2x-3)$ and factor $(5x+6)$: $2\times 6-(-3)\times 5=27$.
- Find the difference of the cross products of the coefficients of factor $(2x-3)$ and factor $(3x+4)$: $2\times 4-(-3)\times 3=17$.
- Find the difference of the cross products of the coefficients of factor $(5x+6)$ and factor $(3x+4)$: $5\times 4-6\times 3=2$.

Therefore, $\dfrac{3x+4}{(2x-3)(5x+6)}=\dfrac{1}{27}\left(\dfrac{17}{2x-3}-\dfrac{2}{5x+6}\right)$.

Case 3

Resolving $\dfrac{1}{(x-a)(x-b)^2}:\dfrac{1}{(x-a)(x-b)^2}=\dfrac{1}{(b-a)^2}\left(\dfrac{1}{(x-a)}-\dfrac{1}{(x-b)}\right)+\dfrac{1}{(b-a)(x-b)^2}$

We will use three examples to illustrate this process of decomposition of fractions in the required form or other forms that can be reduced to this form.

Example 3.1

$$\frac{1}{(x+5)(x-7)^2}=\frac{1}{(7+5)^2}\left(\frac{1}{x+5}-\frac{1}{x-7}\right)+\frac{1}{(7+5)(x-7)^2}$$

$$=\frac{1}{144}\left(\frac{1}{x+5}-\frac{1}{x-7}\right)+\frac{1}{12(x-7)^2}$$

Noting that $(7+5)^2$ is obtained by substituting the root of $x+5=0$ into $(x-7)^2$ and $(7+5)$ is obtained by substituting the root of $x-7=0$ into $(x+5)$.

If the numerator of the fraction is a linear or a quadratic factor then the fraction can be reduced to the required form by this method.

Example 3.2

$$\frac{2x+3}{(x-2)(x+6)^2}=\frac{2(x-2+2)+3}{(x-2)(x+6)^2}$$
$$=\frac{2(x-2)+7}{(x-2)(x+6)^2}$$
$$=\frac{2}{(x+6)^2}+\frac{7}{(x-2)(x+6)^2}$$
$$=\frac{2}{(x+6)^2}+\frac{7}{(2+6)^2}\left(\frac{1}{(x-2)}-\frac{1}{(x+6)}\right)+\frac{7}{(-6-2)(x+6)^2}$$
$$=\frac{7}{64}\left(\frac{1}{(x-2)}-\frac{1}{(x+6)}\right)+\frac{9}{8(x+6)^2}$$

Example 3.3

$$\frac{2x^2+x+3}{(x-3)(x+5)^2}=\frac{2(x+5)^2-20x-50+x+3}{(x-3)(x+5)^2}$$
$$=\frac{2(x+5)^2-19(x-3)-104}{(x-3)(x+5)^2}$$
$$=\frac{2}{x-3}-\frac{19}{(x+5)^2}-\frac{104}{(x-3)(x+5)^2}$$
$$=\frac{2}{x-3}-\frac{19}{(x+5)^2}-\frac{104}{(3+5)^2}\left(\frac{1}{x-3}-\frac{1}{x+5}\right)-\frac{104}{(-5-3)(x+5)^2}$$
$$=\frac{3}{8(x-3)}-\frac{13}{8(x+5)}-\frac{6}{(x+5)^2}$$

Case 4

Resolving $\frac{1}{(x-a)(x^2+b)}:\frac{1}{(x-a)(x^2+b)}=\frac{1}{a^2+b}\left(\frac{1}{x-a}-\frac{x+a}{x^2+b}\right)$

Example 4.1

$$\frac{1}{(x-3)(x^2+5)}=\frac{1}{3^2+5}\left(\frac{1}{x-3}-\frac{x+3}{x^2+5}\right)$$
$$=\frac{1}{14}\left(\frac{1}{x-3}-\frac{x+3}{x^2+5}\right)$$

Example 4.2

$$\frac{3x+4}{(x+6)(x^2+4)}=\frac{3(x+6)-14}{(x+6)(x^2+4)}$$
$$=\frac{3}{x^2+4}-\frac{14}{6^2+4}\left(\frac{1}{x+6}-\frac{x-6}{x^2+4}\right)$$
$$=\frac{7x+18}{x^2+4}-\frac{7}{20(x+6)}$$

Example 4.3

$$\frac{2x^2+2x+3}{(x+3)(x^2+9)}=\frac{2(x^2+9)+2(x+3)-21}{(x+3)(x^2+9)}$$
$$=\frac{2}{x+3}+\frac{2}{x^2+9}-\frac{21}{(x+3)(x^2+9)}$$
$$=\frac{2}{x+3}+\frac{2}{x^2+9}-\frac{21}{(3^2+9)}\left(\frac{1}{x+3}-\frac{x-3}{x^2+9}\right)$$
$$=\frac{5}{6(x+3)}+\frac{7x-9}{6(x^2+9)}$$

Example 4.4

Resolve $\frac{1}{(x+3)((x-2)^2+4)}$ into partial fractions.

Let $u=x-2$.

Then $\frac{1}{(x+3)((x-2)^2+4)}=\frac{1}{(u+5)(u^2+4)}$
$$=\frac{1}{(5^2+4)}\left(\frac{1}{u+5}-\frac{u-5}{u^2+4}\right)$$
$$=\frac{1}{29}\left(\frac{1}{x+3}-\frac{x-7}{(x-2)^2+4}\right)$$

DISCUSSION AND EXTENSION

There are other types of proper fractions which can be resolved by using formulas or reduced to one of the forms listed above. For example,

a. $\frac{1}{(x-p)^2(x-q)^2}$ can be expressed as

$$\frac{1}{(x-p)^2(p-q)^2}-\frac{2}{(x-p)(p-q)^3}+\frac{1}{(q-p)^2(x-q)^2}-\frac{2}{(q-p)^3(x-q)}.$$

A method for resolving the more general form $\frac{f(x)}{(x-\alpha)^m(x-\beta)^n}$ is introduced in Kim and Lee (2016).

b. $\frac{2x+1}{(x+3)^2(x-4)^2}=\frac{2(x+3)-5}{(x+3)^2(x-4)^2}=\frac{2}{(x+3)(x-4)^2}-\frac{5}{(x+3)^2(x-4)^2}$, while $\frac{2}{(x+3)(x-4)^2}$ and $\frac{5}{(x+3)^2(x-4)^2}$ can be resolved quickly using one of the skills shown above.

c. $\frac{2x^2+4x+1}{(x+3)^2(x-4)^2}=\frac{2(x-3)^2-8(x+3)+25}{(x+3)^2(x-4)^2}=\frac{2}{(x-4)^2}-\frac{8}{(x+3)(x-4)^2}+\frac{25}{(x+3)^2(x-4)^2}.$

However, the calculation may not be easier than the traditional method. The traditional method is a universal method in determining the coefficients of the partial fraction decomposition

$$\frac{P(x)}{Q(x)}=I(x)+\sum_{i}^{k}\sum_{j}^{n_i}\frac{A_{ij}(x)}{(Q_i(x))^j}.$$

although it is not always the easiest or best method.

In classroom teaching, I suggest the teacher introduce three examples such as $\frac{x+5}{(2x-1)(x+2)}$, $\frac{1}{(x-2)(x+3)^2}$ and $\frac{1}{(x-3)(x^2+4)}$ for decomposition using the algorithms from Case 2 to Case 4. The skills for the simplest cases are likely to be sufficient to answer questions in VCE exams. However, the teacher might set further tasks to encourage and guide students to explore the extended cases.

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How long is a 30-centimetre ruler? Rethinking familiar classroom practices to help revitalise the curiosity of students

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Children are naturally inquisitive. They will explore the world around them, ask questions, pose problems, experiment, take risks and make discoveries. Yet, too often when they come to school, students spend most of their time trying to work through problems we give them, using procedures we tell them to use. As teachers, we then question students' lack of motivation and passive approach to learning. How can we recapture students' in-built curiosity and use this to explore a range of mathematical topics? How can we help students look beyond what is familiar and begin, once again, to question the things that are around them? This article looks at research surrounding the curiosity of young children and what happens to this once they enter more formalised schooling. It questions the effectiveness of common classroom practices and suggests a rethink of ways that problems are introduced to students in order to encourage a more active, engaging and rewarding approach to learning.

INTRODUCTION

An experienced colleague of mine once claimed that they could "do forty minutes on any maths topic by asking the students what they know about it." Initially, I thought this was a bold claim. Then I started thinking about my own teaching experience, and my role as an outreach officer for the Australian Mathematical Sciences Institute (AMSI). As part of this, I have the opportunity to meet teachers and visit classrooms across the country. Despite scheduling visits, I am, at times, asked to model lessons on topics that I have not even thought about. Faced with a lack of preparation, I have started lessons by asking questions to find out what students know about the topic. Using this strategy, I have managed to run rewarding lessons, where students were engaged in the learning, made new discoveries and began their own investigations. How is it that unplanned lessons can often become interesting and informative experiences? What is it about asking questions and using students' own curiosity to drive an investigation that can lead to unexpected outcomes? What can teachers learn about these unscripted lessons that can positively influence the way all lessons are conducted?

WHAT IS CURIOSITY?

Beginning lessons with questions about students' knowledge of a topic appears to help switch on students' curiosity and natural desire to discover information (Akesson, 2015; Musallam, 2013; von Renesse & Ecke, 2015; Zager, 2017) Research has suggested that children are born curious (Clements & Sarama, 2014; Day, 1982; deGrasse Tyson, 2017; Lowenstein, 1994; Robinson, 2006; Zager, 2017). They have an in-built desire to learn what is unknown and are "exhilarated by their own ideas and the ideas of others" (Clements & Sarama, 2014, p. 2).

Curiosity, itself, can be thought of then as a "motivated desire for information" (Lowenstein, 1994, p. 76). In fact, Lowenstein (1994) goes further to describe curiosity as the "critical motive that influences human behaviour" (p. 75). The challenge then for teachers is to identify and continue to engage the students who are curious and to find a way to motivate the others.

In an early study conducted by Day (1982), he highlighted three behaviours of curious students:

1. Shows an interest in new or complicated objects or events;
2. Explores new or complicated objects or events;
3. Persists in examining and exploring new or complicated objects or events in order to know more about them. (p. 21)

Day (1982) suggests that placing students in conditions that contain a level of uncertainty will in fact drive students to explore these conditions in order to reduce this uncertainty. If the blueprint for a curious classroom is this straightforward, why then are so many mathematics classrooms full of passive, unmotivated students?

WHAT HAPPENS TO CURIOSITY?

Historically, the education system has tended to emphasise the role of the teacher as "the leader, the source of information and the reinforcer of learning" (Day, 1982, p. 20). In typical mathematics classrooms, students are presented with teacher- or text book generated problems that they are required to solve, often using set procedures (Knuth, 2002; von Renesse & Ecke, 2015; Zager, 2017). These "solution-driven" (Knuth, 2002, p. 129) mathematical experiences, rarely provide students with the opportunity to pose their own problems or apply their discoveries to new situations (Akesson, 2015; Cunningham, 2004; Knuth, 2002; Zager, 2017). Over exposure to this kind of teaching has the effect of turning naturally curious students into "passive, apathetic teenagers" (Zager, 2017, p. 137).

Sir Ken Robinson, in his widely recognised TED talk on the problems with schools, builds on this idea. Robinson suggests that "we don't grow into creativity, we grow out of it. Or rather, we get educated out of it" (Robinson, 2006). The same notion can be applied to curiosity. Many common classroom practices are turning students off mathematics. In reality, students will only learn how to solve problems "by actively doing mathematics, rather than just listening to a teacher's explanations" (von Renesse & Ecke, 2015, p. 229). To halt this worrying trend, a shift in thinking, and a different approach to student engagement is required.

CREATING A CURIOUS CLASSROOM

> Here then is the recipe of learning. Take a student, place him in a situation of moderate uncertainty about some topic and get out of his way while he gets excited and attentive and directs his exploration to the source of his uncertainty. Moreover, research has demonstrated that he will enjoy his exploration and the accumulation of knowledge.
> (Day, 1982, p. 20).

This *recipe* for creating a curious classroom, suggested by Day (1982), clearly has some merit. Interestingly, more recently, Neil deGrasse Tyson has supported this idea, suggesting that the best advice he can offer, to a parent or an educator, to promote curiosity in children, is to "get out of their way" (deGrasse Tyson, 2017). As teachers, although such a notion may be appealing, a more structured approach to this method is needed.

Laura Akesson (2015) suggests that teachers need to "leave space for curiosity" in their lesson plans and encourage students to "ask the questions and come up with their own problems" (Akesson, 2015). Such sentiment is echoed by other researchers, including Cunningham (2004), who found that providing students with the opportunity to pose or extend problems can help students to develop "inquiring and curious disposition[s] towards mathematics" (Cunningham, 2004, p. 83). Cunningham goes on to suggest that such a process "can enhance the connections students make between mathematics and the real world, as well as promoting reasoning and reflection" (p. 83). Knuth (2002) also supports this idea, suggesting that "posing problems can also help students better understand the implications of a problem's solution" (p. 126).

This idea of allowing students to pose their own problems is not new. In fact, the National Council of Teachers of Mathematics (2000) in their *Principles and Standards for School Mathematics* suggest that "one of the critical requirements for successful problem solving is a productive disposition to problem pose" (NCTM, 2000, as cited in Cunningham, 2004, p. 89).

To develop and sustain a curious classroom is up to teachers to see themselves as creators of the environment in which curiosity and creativity can thrive. Liljedahl (2016) supports this belief, adding that teachers should aim to have a classroom that is "that is inhabited by thinking individuals as well as individuals thinking collectively, learning together and constructing knowledge and understanding through activity and discussion" (p. 364). To overcome possible obstacles to such an approach, Seely (2016) suggests that is equally important that teachers consider their own mindset. If teachers believe that a student's ability is limited, then they are "likely to set low expectations and potentially fail to adequately challenge students (Seely, 2016, p. 5).

Ramsey Musallam, in his 2013 TED Talk, builds on this idea further. He challenges educators to "have the guts to confuse our students, perplex them, and evoke real questions…" (Musallam, 2013). To achieve this, he suggests that teachers consider three rules when planning lessons:

1. Begin with questions;
2. Embrace the mess;
3. Practice reflection. (Musallam, 2013)

The types of questions and the language used by teachers to promote positive discussion and build on students' ideas must also be considered. Research undertaken by von Renesse and Ecke (2015) suggest that it important to use language that reaffirms students' attempts at learning. They suggest that phrases such as "I am wondering…" or "Tell me what you have done so far…" can be used in different contexts, regardless of whether the student's current thinking is correct or incorrect (von Renesse & Ecke, 2015). These types of statements can help to promote students' ideas, rather than shutting down potential thought processes.

As shown in Table 1, von Renesse and Ecke (2015) suggest a number of ways that teachers can promote thinking and encourage students to continue to explore problems.

Type	Example Prompt
Clarifying thinking	I am wondering about… Can you help me understand? Can you tell me what you are thinking? Tell me what you have done so far…
Recognising contradictions	What do you think? What is happening there? How is that different? What conditions have changed?
Activating prior knowledge	What do you notice? What does this remind you of? What is another example of? How is … related to … that we studied earlier? Remember when we looked at…
Revoicing	What I hear you say is… Can you repeat what was just said in your own words? Do you agree or disagree and why? Would someone like to add on?
Wait Time	Do you want more time to think? You can do this, I'll check in later I'm eager to see what you discover when I return
Modelling	If I use your strategy… How about trying…

Table 1. Prompts to Promote Student Thinking (adapted from von Renesse and Ecke, 2015, p. 231)

These prompts fit well with the "recipe for learning" suggested by Day (1982, p. 20). They also allow time for students to adapt, pose and explore their own problems. The following vignette provides an example of when I had the opportunity to employ this approach in a classroom.

HOW LONG IS A 30-CENTIMETRE RULER?

As part of my role as an outreach officer for AMSI, I arranged a visit to a school in the Upper Hunter region of NSW. Here, I was asked to model a lesson with some Year 6 students that involved converting decimal numbers and recording measurements. Having never met the students before, I wanted to check to see what the students already new about different measurement tools. To begin the lesson, I held up a 30-centimetre ruler and asked the class:

"How long is this?"

A number of students quickly responded, "Thirty centimetres."

"Is it?" I exclaimed.

This unplanned, slightly ironic response, would steer the lesson in an unexpected way. Almost immediately, the mood in the room was different. Possibly wanting to prove themselves right, or prove the teacher wrong, students soon found a tape measure that could be used to check the actual length of the ruler I had shown the class. Students gathered around as the 'original' ruler was measured, and the results shared.

"31.3 centimetres."

Immediately, general groans and questions erupted.

"What?"

"Really?"

"Are you sure?"

"That can't be right."

"Wait, I have a different ruler, let's check it."

The whole class pauses in anticipation.

"31.1 centimetres."

More groans and questions erupt. Without further prompting, a whole class resolve appears to form as all rulers in the room are checked and measurements confirmed. With many students having the same, school issued, ruler a decision is made to see if other classes have different rulers that can be checked. As a scavenger hunt ensues, the remainder of the students record, sort and compare the results. One student suggests taking photos of each ruler, to ensure that "we don't double up."

Ruler	Length (centimetres)
Wooden Micador College 30	31.1
White Plastic	32.7
Blue Plastic (with handle)	30.5
Wooden Rulex	31.3
COS Rule1060	31.2
Marbig Clear Plastic (975317P)	31.5
Marbig Plastic (975317B)	31.1
White Plastic	31.7
Texas Instruments Plastic	31.4
Junior (Department of Education)	30
Average	*31.25*

Table 2. Length of Rulers Measured in Year 6 class

Table 2 shows the results of the investigation. Interestingly, the students who searched the school for different rulers, were pleased to find one that was exactly 30 centimetres long. Further discussion, also took place about why the average length of the rulers measured that lesson was 31.25 centimetres. A search of the internet by students, revealed that a standard 30-centimetre ruler is sometimes called a 12-inch ruler ("Rulers," 2018). Twelve inches is equal to 30.48

centimetres. This fact, led some students to theorise that the length of a standard 30-centimetre ruler was designed to accommodate both units of measurement. As I left the classroom on that day, this theory was yet to be confirmed.

CONCLUSION

As this classroom vignette shows, the right question can turn a lesson. As teachers, we want students who are able to reason, justify their thinking, gather evidence, interpret data, collaborate with others, share their thinking, communicate ideas, and demonstrate a willingness to persevere with problems, and apply this understand to generate and solve their own problems (Akesson, 2015; Cunningham, 2004; Seely, 2016; Zager, 2017).

As teachers, we need to aim to meet students where they are at (Clements & Sarama, 2014), while at the same time encouraging them venture into regions unknown. We want students to build on what we have shown them, to explore new ideas and concepts and to continue to make new discoveries. To do this, we need to create the conditions that help promote student thinking, we need to ask questions that engage students' curiosity, and we need to have the courage to "get out of their way" (deGrasse Tyson, 2017) when the light switches on.

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Excel-ent maths

Robert Money, retired secondary teacher and
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Digital skills and enterprise skills are of central importance in 21st century workplaces. The ubiquity of the Excel spreadsheet in these workplaces provides a challenge for teachers to embed this technology in their mathematics curriculum. Some examples show how this can be implemented at junior secondary level.

HISTORY OF EXCEL

The electronic spreadsheet was essentially invented in 1979 by software pioneer Dan Bricklin, who started up Software Arts with Bob Frankston and created *VisiCalc*.

Microsoft released the first version of *Excel* for the Macintosh on September 30, 1985, and the first Windows version in November 1987. Excel can be "attached" to external software applications and is used as "front end" for analysis of larger data sets.[7]

EXCEL IN THE WORKPLACE

The *Quantitative Skills in 21st Century Workplaces* project (2014) [3] undertook research to identify and analyse the gaps between young peoples' quantitative skills in the senior years of schooling and the expectations of 21st century workplaces.

Mathematics teachers were supported to shadow young workers to identify the actual mathematical and problem solving skills in a range of diverse workplaces. The intention was to identify specific matches and mismatches between what is happening in schools and what is needed in workplaces. The teachers undertook some work-shadowing and interviews to develop case studies of quantitative thinking in action in local workplaces.

Some excerpts from the Report:-

> Many people in the workplace are engaged with technology, particularly in using spreadsheets and graphical outputs. There is an inter-dependency of mathematical skills and the use of technology in the workplace in ways that are not commonly reflected in current teaching practice.
>
> There is a need to identify and take opportunities to embed work-related technologies—particularly spreadsheets and computer generated graphics—in the mathematics curriculum and teaching in schools.

Identifying and Supporting Quantitative Skills of 21st Century Workers Final Report
(Commonwealth of Australia, 2014)

EXCEL IN THE MATHS CLASSROOM

Mathematics teachers can embed systematic development of learning about spreadsheets within the content areas of their Year 7 to 10 programs and the analysis tasks involved in STEM investigations. The authors believe that STEM skills are multi-disciplinary and will involve students in the design, constructions and use of data logging devices.

BENEFITS FOR STUDENTS DIGITAL SKILLS

Students will progress the development of their digital skills through the following four levels described by the UK Digital Skills Taskforce [6]:

1. **Digital muggle:** no digital skills required - projected 8% of the workforce
2. **Digital citizen:** use technology to communicate, find information and transact - projected 38% of the workforce

3. **Digital worker:** configure and use digital systems - projected 46% of the workforce

4. **Digital maker:** build digital technology - projected 8% of the workforce

BENEFITS FOR LEARNING MATHEMATICS

Consolidation of skills with calculations and use of formulae

Problem solving – finding patterns, generalizing, asking What-if

BENEFITS FOR LEARNING ENTERPRISE SKILLS

The New Basics Report (2017)[4] defines enterprise skills as

> **transferable skills** that enable young people to engage with a complex world and navigate the challenges they will inherit. ... they are skills that are required in many jobs. They have been found to be a powerful predictor of long-term job success. (p. 5)

Skills classified as enterprise skills include not only digital literacy but also problem solving, communication skills, teamwork, presentation skills, critical thinking, creativity, and financial literacy.

Excel can be used for the data collection and analysis involved in investigative tasks that involve the development of all of these enterprise skills.

EXCEL AND REAL DATA

With spreadsheets students can analyse and interpret real and quite large data sets, as distinct from the smaller, context-free "text book" examples. A real context allows students to make meaningful distinctions between mean, median, and mode, and to justify their decisions about the exclusion of any outliers. STEM projects typically involve quite large data sets imported from inexpensive data loggers directly into a spreadsheet. Other large data sets, such as from the Australian Bureau of Statistics [10] or sporting associations, are available in a spreadsheet format.

SOME EXAMPLES

1. **A STEM Investigation**

Students use their design, technology, and science skills to build a model house for the purpose of simple investigations of insulation capacity. More sophisticated analysis uses data logger data from temperature probes in an actual school building. The data logger typically stores the data in comma-delimited form. Students will become familiar with the use of programmable electronic devices to collect their own data. [8] They will use proprietary data loggers or may use extended STEM skills to build their own. Students can build data loggers for less than $20. [9] See Figure 1 for a summary, and Table 1 for sample data.

<table>
<tr><th>Excel learning</th><th>Maths content</th></tr>
<tr><td>Why use a spreadsheet?

Understanding that data need not "look like" a spreadsheet.

Retrieve data stored by a data logger.

Creating meaningful graphs of temperature-time data</td><td>Another form of technological calculation.

Interpreting the temperature - time graphs with appropriately chosen statistics on differences of temperature and rates of change.</td></tr>
<tr><td colspan="2">Enterprise skills – the critical and creative skills involved in making the links between the science, design and mathematical elements of a STEM investigation.</td></tr>
</table>

Figure 1. Characteristics of STEM investigation task

Date	Time	Temp Probe 1	Temp Probe 2	Temp Probe 3
28/11/2017	21:56:07	85.00[a]	26.31	26.19
28/11/2017	22:01:10	26.56	26.75	26.87
28/11/2017	22:06:13	26.69	26.75	27.06
28/11/2017	22:11:17	26.69	26.69	27.12
28/11/2017	22:16:21	26.62	26.62	27.12
28/11/2017	22:21:24	26.62	26.56	27.06
28/11/2017	22:26:28	26.62	26.5	27.06
28/11/2017	22:31:32	26.50	26.44	27.00
28/11/2017	22:36:35	26.50	26.37	27.00
28/11/2017	22:41:39	26.37	26.25	26.94
28/11/2017	22:46:43	26.19	26.19	26.94

Table 1. Sample Data for STEM Investigation Task

a. Actual reading.

Figures 2-9 describe characteristics of further activities and Tables 2 & 3 offer sample results

2. Pay calculations

Excel learning	Maths content
Copying a table into a spreadsheet Calculating a product Using CTRL-R to Fill Right Using the =sum function	Students develop a spreadsheet to obtain total weekly and fortnightly pay from hours worked at different pay rates. Level 8: Carry out the four operations with rational numbers and integers, using appropriate digital technologies (ACMNA183)
Enterprise Skills – teamwork and financial literacy in developing and reviewing the spreadsheet. Critical thinking, communication and presentation skills in relating this scenario to their own experience.	

Figure 2. Characteristics of pay calculations task

3. Finding a linear relationship

Excel learning	Maths content
Entering formulae, filling down (Ctrl-D) and inserting a graph.	Students find the rules to fit a table of values Level 8: Solve linear equations using algebraic and graphical techniques. Verify solutions by substitution (ACMNA194)
Enterprise skills – creativity and problem solving in setting and solving these tasks	

Figure 3. Characteristics of linear relationship task

4. Heights and Weights Lesson – using sports team data

Excel learning	Maths content
Understanding that a spreadsheet is able to complete this task. Downloading data into a spreadsheet Using indices in a formula Sorting data Using Excel assistance to insert a statistical formula Using Ctrl-R to Fill Right Insert a graph Change the number of decimal places in a value	Students calculate body mass index for a sports team and obtain statistical summary data and graphs. Level 7: Calculate mean, median, mode and range for sets of data. Interpret these statistics in the context of data (ACMSP171) Level 7: Construct and compare a range of data displays including dot plots (ACMSP170) Level 10: Construct and interpret box plots and use them to compare data sets (ACMSP249)
Enterprise skills – presentation skills and critical thinking in relating body size to sports skills	

Figure 4. Characteristics of heights and weights task

5. Two Heads Out of Three

Excel learning	Maths content
Formula for a random number Integer Part Of If statements Using Ctrl-D to Fill Down and Ctrl-R to Fill Right	Students simulate tossing three coins to find the proportion of 'two heads out of three' results. Level 6: Conduct chance experiments with both small and large numbers of trials using appropriate digital technologies (ACMSP145) Level 7: Assign probabilities to the outcomes of events and determine probabilities for events (ACMSP168)
Enterprise skills – problem solving for obtaining theoretical explanations; creativity in developing related problems and simulations; communication and presentation skills	

Figure 5. Characteristics of coin tossing task

6. Circles and Sectors

Excel learning	Maths content
Formulae involving π, squares and square roots Inserting shapes into a spreadsheet Identifying sources of error in the calculation	Students develop a spreadsheet to solve problems involving perimeters and areas of sectors and circles. Level 8: Investigate the relationship between features of circles such as circumference, area, radius and diameter. Use formulas to solve problems involving circumference and area (ACMMG197) Mortgage: Make it $250,000 at 8% over 30 years.
Enterprise skills – creativity and problem solving in making up and solving 'What if' questions.	

Figure 6. Characteristics of circles and sectors task

7. Double Your Money

Excel learning	Maths content
Dollar and % format of numbers. Filling down Use of $ sign to create absolute reference	Students develop a spreadsheet to solve problems in investment and annuities. Level 10: Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies (ACMNA229)
Enterprise skills – financial literacy; critical thinking about realistic situations; communication skills	

Figure 7. Characteristics of investment and annuities task

	A	B	C	D	E	F
1	**Initial investment**	$1,000.00	**Year**	**Start**	**Interest**	**Finish**
2	**Interest rate**	4%	1	$ 1,000.00	$ 40.00	$ 1,040.00
3	**Annual Extra**	$10.00	2	$ 1,050.00	$ 42.00	$ 1,092.00
4			3	$ 1,102.00	$ 44.08	$ 1,146.08
5			4	$ 1,156.08	$ 46.24	$ 1,202.32
6			5	$ 1,212.32	$ 48.49	$ 1,260.82

Matching Formulae

	A	B	C	D	E	F
1	**Initial investment**	1000	**Year**	**Start**	**Interest**	**Finish**
2	**Interest rate**	0.04	1	=B1	=D2*B$2	=D2+E2
3	**Annual Extra**	10	=C2+1	=F2+B$3	=D3*B$2	=D3+E3
4			=C3+1	=F3+B$3	=D4*B$2	=D4+E4
5			=C4+1	=F4+B$3	=D5*B$2	=D5+E5
6			=C5+1	=F5+B$3	=D6*B$2	=D6+E6

Table 2. Sample Results for Investment and Annuities Task

8. Investment Options

Excel learning	Maths content
Students use and extend a prepared spreadsheet. They report on the problems they have solved and the spread-sheeting techniques they have learnt.	Students make up and solve a range of financial investment problems by modifying the pages of a given spreadsheet. Future and Present Value Factors: Make it $1200, 7 years at 4% Amortization: Make it $1200, 7 years at 4% Present Value Calculation: Make it $1200, 7 years at 4% Monthly calculations: Make it $100, 84 months, 0.33% interest Car loan: Make it $25000, 5 years, 8% Mortgage: Make it $250,000 at 8% over 30 years.
Enterprise skills – financial literacy; problem posing and solving; communication skills	

Figure 8. Characteristics of investment options task

9. Double Your Money

Excel learning	Maths content
Students develop their own spreadsheet for use in data collection and analysis and in representation of results of an investigation. An example of this approach is "Population Data – What Questions Can We Ask? [10]	Students work in groups to undertake a statistical investigation of a topic of interest. Assessment criteria include depth of statistical analysis and all of the enterprise skills as listed above.
Enterprise skills – creativity; communication and presentation skills	

Figure 9. Characteristics of statistical investigation task

Statistical Area (SA1)	2136820	2136825	2136816
People	357	656	458
Male	164	330	224
Female	193	326	234
Median Age	40	31	47
Families	101	184	93
Average number of children	1.9	1.9	1.6
All private dwellings	144	240	193
Average number of people	2.5	2.8	2.2
Median income	$1,187	$1,489	$923
Median monthly repayment	$1,408	$1,798	$1,128
Median weekly rent	$250	$300	$217
Average number of vehicles	1.7	1.9	1.4

Table 3. Sample Results for Statistical Investigation Task [10]

THE WAY FORWARD

An internet search for sample Excel files [5] will give some view of the extent to which Excel spreadsheets are used in financial and other applications. Discussions with working parents could lead to a greater appreciation of the sorts of spreadsheet tasks undertaken in their workplaces. Workplace visits and work experience could also be helpful.

Schools can review the extent to which spreadsheets are encountered across the curriculum. Mathematics departments can use the key documents referred to here as a basis for review and extension of the spreadsheet activities included in their programs. The monitoring, assessment, and reporting of spread-sheeting skills can become a mathematics teaching responsibility.

Individual teachers can update their spreadsheet skills by working together and by using readily available tutorials [1, 2]. In classrooms students can work in teams, mentoring each other on the skills required for any particular spreadsheet-based task.

Networks of teachers may support one another to develop appropriate "numeracy" spreadsheet skills for students who do not attempt University entrance. [11]

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Acquiring the habit of digital innovation

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This paper is based on the findings of a PhD study of mathematics teachers' use of digital technology in lessons. Applying recent results of neuropsychology research, the study has found that the introduction of a digital innovation to a regular mathematics lesson gave rise to a cognitive conflict for some teachers of "habit versus innovation." The paper will address the importance of understanding both habits and innovation to learning, and how these may be mediated to produce new learning outcomes for a digital world. It will also offer practical suggestions for developing teaching and learning strategies that promote the habit of innovation for teachers and students alike.

INTRODUCTION

"Teachers' own beliefs and attitudes about the relevance of technology to students' learning were perceived as having the biggest impact on their success" (Ertmer, Ottenbreit-Leftwich, Sadik, Sendurur, & Sendurur, 2012).

When I began my study in 2012, this finding encapsulated research about using digital technology in lessons, and set me on a path of investigating mathematics teacher pedagogical beliefs and the use of digital technology.

LITERATURE

Rokeach (1968) described belief systems as consisting of beliefs and belief-substructures: attitudes and values. All beliefs have a cognitive component representing knowledge, an affective component representing emotional connotations, and a behavioural component representing the precursor to action. Beliefs link the information received with context and emotional impact. As such, they are personal, rather than universal.

In the study's framework the pressure for change in actions was represented by the theory of risk-taking (Priest, 1993). Change requires facing uncertainty. An action of uncertainty is one with several possible outcomes, perhaps some positive and some negative. Risk-taking was defined as acting in an uncertain situation with a known probability of a negative outcome amongst the positive outcomes. I changed my language to "facing uncertainty," but did not set aside the possibility of a negative outcome. When facing uncertainty, action was made possible when the personal perception of a negative outcome was offset by positive beliefs found in self-efficacy, control, and gains-to-be-made. The action might be repeated if perceived to be successful and supported (Priest, 1993).

The focus of my study was on mathematics teacher practice and the use of digital technology in the secondary school classroom. The research questions were formed from the literature review and hovered around questions on how, why, and which beliefs facilitated or obstructed the use of digital technology in the mathematics classroom — all of which were relatively straightforward. I also asked the question:

> What mathematics pedagogical practices no longer make sense in a contemporary learning environment?

METHODOLOGY

The methodology required a qualitative study of the field, cultural capital, and habitus (Bourdieu, 1977) associated with the teaching and learning of mathematics assisted, or not, by the use of digital technology, in a particular site of education, the classroom. The field was defined in terms of the classroom environment, the digital technology available to teacher and students in the classroom, and the student cohorts. The curriculum rules and schoolteacher responsibilities were examples of external influences on the field that acted on the inhabitants, and perhaps affected the loss or gain of cultural capital. Teacher habitus could not be observed directly, but instead beliefs were inferred from intentions, actions, statements, and experiences of the inhabitants (Reay, 2004).

Based on constructivist grounded theory (Chamaz, 2000), analysis of incidents of interest had two objectives: deriving belief meanings, and searching for patterns between teacher beliefs and digital technology uses. Incidents of interest revealed teacher beliefs about mathematics, how mathematics should be taught and learnt, and how digital technology was used or not used in the lesson. Beliefs about mathematics and mathematics education were compared to Levin and Wadmany's (2006) categories for interpreting teachers' pedagogical views and teaching models. Beliefs about digital technology use were categorised by an analogy of digital roles as master, servant, partner, and extension of self (Goos, Galbraith, Renshaw, & Geiger, 2003), and in terms of teacher experience.

PARTICIPANTS

Six mathematics teachers from one Catholic all-girls secondary school agreed to participate in the study. Data was collected from audio-recorded lesson observations, post-observation interviews, and stories about the participants' experiences in learning mathematics and becoming mathematics teachers. A final interview and field notes also contributed data. Participants were observed in four or five mathematics lessons of 60-75 minutes each, in 2015.

Three teachers taught VCE Mathematical Methods Unit 1-2 (CAS). A requirement of the VCE Study Design was to use a computer algebraic system to develop mathematical ideas, produce results, and carry out analyses in situations requiring problem solving, modelling, or investigative techniques and approaches. The CAS application *Wolfram Mathematica* featured in data collected for the VCE teachers.

Alan was the leader of the VCE team and set the pace for the curriculum delivery. In his mid-forties, he was an experienced teacher of Mathematical Methods, and a fully qualified mathematics teacher with an honours degree in Mathematics.

Three teachers taught Year 7 Mathematics. A new school leadership initiative of combined classes and team-teaching was in place for Year 7 lessons in 2015. The Year 7 teacher participants were trialling a mathematics teaching/learning online application, *Maths Pathway*, to support the initiative. *Maths Pathway* featured in the Year 7 teachers' data. The team acted in complete agreement about the curriculum, its delivery, and the how, which, and why of digital technology use.

Helen was the leader of the team and integral developer of the established Year 7 curriculum. In her early fifties, with a Masters Degree in Neuropsychology, Helen came to teaching mathematics via a qualification in special needs education. She had not majored in Mathematics at university, the more usual qualification for a mathematics teacher.

FINDINGS

As Ertmer et al. (2012) had found, the participant teachers had difficulty in relating digital technology use to purpose and to pedagogical advantage. This was one of many claims from literature that the study supported. However, in general, all participant teachers demonstrated difficulty in relating their lesson intentions to learning outcomes other than content. They taught mathematics as a disconnected "bubble" of content according to the curriculum. The completeness of lessons had no direct need for digital or any other sort of innovation with the exception of the requirement by the VCE Study Design to facilitate the use of a CAS application.

ALAN

Alan's teaching model was based on discovery learning, an inquiry-based teaching model, in which learning was knowledge construction, influenced by the context, and relative to the accomplishment of a goal (Levin & Wadmany, 2004). Alan continually challenged his students with small experimental, discovery or inquiry activities with specific purposes. He spoke about achieving "your goal." His entire class of 12 students comprised a cohesive learning group with the teacher in control of the classroom, but mathematics learning as the centre of attention for teacher and students.

In this environment, Alan and his students used *Mathematica* seamlessly and without comment, as "an extension of self" (Goos, et al., 2004). In four observed lessons Alan modelled all curriculum requirements for digital technology use with a fluidity that demonstrated that this was normalised behaviour in his classrooms.

On several occasions Alan displayed a degree of uncertainty about general digital technologies, despite his lengthy exposure to them. The need to email a file to students in the lesson elicited a flurry of mutterings from Alan. His uncertainty has been attributed to a lack of internal control over the technologies, and to his reliance on others if something went wrong. He was also self-conscious of my scrutiny of his digital technology uses, and he was aware of the pressure to use more technology in order to be considered "modern."

I also observed his launch of the *I Will Derive* YouTube clip. Alan back-referenced the clip in subsequent observed lessons, dancing an orange pen along the curve of a displayed function, "as in the video." At the final interview, Alan said that students had related to the two You Tube clips that he had shown, and he found that the clips had animated his class. He said he would use them again and that he would search out YouTube clips in future if he had time. He still thought the *I Will Derive* clip had no benefit, and had been unable to recognise the visual advantage that the clip had offered and which he had subsequently copied.

HELEN

Helen's use of digital technology for mathematical advantage was limited to a suite of flipcharts that provided the basis for efficient and effective communication. This one digital application aligned with the Year 7 dominant mathematics teaching strategy of transmitting information and knowledge.

Helen had rejected the leadership initiative to team-teach and had moved her class to a separate classroom. She used *Maths Pathway* for five weeks in lessons, but basically as an individual worksheet generator for her students. She then rejected the worksheets because they did not keep up with the established curriculum. She thought that they were weak and had zero tolerance for mistakes. At the final interview, Helen expressed doubts about other initiatives that the school had promoted, and these included *SMART* pre- and post- testing.

However, Helen was willing to try to change, as evidenced by her attempts to use *Maths Pathway* and *SMART*, and she had been successful in creating flipcharts for the established curriculum. Her reasoning about rejecting change was not always rational, and many of her ideas seemed disconnected. At her final interview she said about the leadership initiatives: "We needed to jump on board and I thought we were doing really well. Why do we have to change every year?"

MAKING SENSE

An underlying assumption I had made was that beliefs about student learning outcomes drove teacher actions in the classroom. Was it possible that the actions of teachers in the classroom had become disconnected from learning outcomes? How could that happen and, if so, what was driving teacher actions in the classroom?

The Power of Habit (Duhigg, 2012) described how new habits were created just like normal learning "… by putting together a cue, a routine and a reward" (p. 49). Repetition cultivated a craving that drove the loop. If this was like normal learning, then was it possible those teachers' well-practised classroom strategies were habits driven by a craving, and not by beliefs about learning? If so, did this make a difference to the adoption of digital technology for pedagogical advantage?

LITERATURE REVISITED

The literature on teacher habits and beliefs about learning revealed that there were two behavioural processes of relevance to this study, namely goal-directed cognitive behaviour and habitual automatic behaviour (Shah, 2013). The processes were controlled in essentially different components of the brain, and each had advantages and disadvantages.

Goal-directed behaviour used prediction of the outcome of an action in order to select and initiate the action. The process was flexible and didn't require much experience, but did require cognitive effort, memory, and computations (Shah, 2013). Goal-directed behaviour initiates learning.

Habitual processes were trained by repeated experiences gained from goal-directed behaviour in a recurring context (Neal, Wood, Labrecque, & Lally, 2011). These processes could also be built from random behaviour or by copying behaviour (Daw, Niv, & Dayan, 2005). The advantage of habitual behaviour was speedy, effortless, and often unconscious decision-making.

Habit was the psychological disposition to repeat past behavior that, when successful, initiated an intrinsic reward. The behaviour became associated with a cue, which was an aspect of the context of repetition. The cue for the routine could be almost anything in the context, even an idea. The reward was personal and related to the senses or emotions. Strong habits were not triggered by goals (Neal et al., 2011).

These findings raise the idea that many teacher actions in the classroom may be initiated by craving and not by beliefs. Given the automatic and unconscious nature of habits, the teacher may not be aware of these actions.

A conflict in the brain arises when a habit meets goal-directed behaviour. The conflict is mediated by a sense of certainty. Applying these ideas to the choice a teacher might make between the use of a digital technology innovation in a lesson, loaded with uncertainty about its purpose and outcomes, against a teacher's long-term habits, loaded with certainty and speed, the technology choice comes off second-best.

TEACHER HABITS

The application of these results to data collected gave a possible explanation for why some teachers were able, and others not able, to use digital technology in the mathematics classroom for pedagogical advantage.

Alan's pedagogical approaches were goal-directed for both teaching and learning activities. He set small, thought provoking activities for students. He modelled reasoning and making connections when he unpacked concepts in lessons. With the goal of fulfilling curriculum requirements and sufficient experience, *Mathematica* had been seamlessly woven into this environment.

Helen believed in traditional methods and seemed reluctant to change. However, she was willing to try, and she had been successful in creating flipcharts. I came to the conclusion that Helen was bound by strong habit when it came to the teaching and learning of mathematics, and an underlying lack of self-belief about mathematics and about using digital technology fed into her uncertainty about change. Her flipcharts were made outside the classroom.

PRACTICAL IMPLICATIONS

This study casts new light on an elusive problem of change that lies in the power of habit to sustain the status quo. An advantage of habit is automatic decision-making. A disadvantage is that habitual actions disrupt consideration of new ideas and associated action choices.

The findings of this study provide mathematics teachers with the possibility of change through an understanding of habits. To confront habits is a personal journey for each teacher, and this is conflicted by a lack of awareness. Raising self-awareness of teaching practices may require input from others. Surrounding yourself with a group of like-minded colleagues to share ideas and perspectives, and to provide encouragement and support, benefits the process of confronting habits.

Changing habits is a challenge that can be overcome. In particular, there is an interim stage of habit formation where either the original goal or the cue can initiate the action. In this stage, goals can readily be changed and a new action initiated.

To change a strong habit is more challenging. One way is to keep the cue and the reward, and to change the routine in the middle. I have experienced personally a version of habit routine change. In the course of this study, I recognised that I sustained an habitual idea "I cannot write" that filled me with anxiety (the reward) every time I sat down to do so (the cue). I have used this habitual idea to cue a second idea "I am learning to write." Now, when I sit down at the computer, I am able to continue the learning task, instead of fretting about the writing task. This is the sequential cueing that

provides the basis for developing habitual action processes.

The final suggestion is to create new habits that make more sense than the old ones. There are a number of actions related to the concept of facing uncertainty that facilitate the development of new digital approaches. An important initial step is to know the digital purpose, both how the technology is to be used and the benefit to be gained. Prepare for change outside the classroom in order to avoid cognitive conflict with classroom habits. Preparation includes gaining competency, confidence, self-belief, control, and considering the back-up plan in the event of a technology hiccup. Finally, introduce the innovation at the beginning of the lesson before habits kick in, and then reflect on the outcome. Have my purposes been achieved? If so, repeat.

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Sustaining and scaling up research-based professional learning for mathematics teachers

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Education research journals regularly report on small-scale studies that have been successful in changing mathematics teachers' classroom practices. But it is rare to find professional development programs for teachers of mathematics that have been scaled up and sustained over time. Research findings are often specific to the context and conditions in which they were conducted, strengthening calls for greater attention to scalability and sustainability of professional learning at both the school and system level.

The aim of the session is to communicate the findings from the aforementioned study which addressed the following research questions to the Mathematical Capability Clusters:

- What practices are effective in establishing a coherent instructional system supporting mathematics teachers' development of ambitious teaching practices?
- To what extent do teacher networks and mathematics coaching of teachers contribute to the effectiveness of the cluster model in changing mathematics teaching practice?
- What features of schools and district instructional leadership have contributed to the sustainability and scaling up of the Mathematics Capability Cluster model?

The Mathematics Capability Cluster (MCC) model is a cost neutral State Schooling project unique to the North Coast Region of Queensland (including Moreton Bay, Sunshine Coast, Hervey Bay, and Bundaberg). It brings together primary and secondary school teachers and principals to analyse student performance data, create diagnostic tasks that reveal students' mathematical understanding, and promote teaching practices that both address students' learning difficulties and ways to embed critical and creative thinking into all classrooms. The effectiveness of this school-based professional learning community approach is evidenced by reported improvements in teacher confidence and knowledge, changes to mathematics teaching and assessment practices, noticeable gains in student outcomes, and reported increased enjoyment and willingness to tackle challenging problems. The MCC was built on what our principals and teachers identified in regional seminars carried out in 2015. Principals requested access to in-school, subject based, ongoing professional learning conducted by highly regarded experts. These experts would need to win teachers' hearts and minds by having the capacity to model effective practices with 5 to 18 year olds. They would be able to contextualise research–informed practices to align with school, Queensland Department of Education, and Australian Curriculum priorities. They would need to have demonstrated credibility from other school leaders for impact and being a agent of sustainable change. Since 2015 schools have invested over $1.3 million of their internal professional development budgets to empower their teachers through the MCC. The MCC is providing the necessary conditions for research to become translated into classroom practice.

The development of these conditions has been an evolutionary process, supported by regional unity and urgency to perpetuate change through subject expertise. The initial barriers to this urgency for change were overcome through the strategic roles and responsibilities of the Regional Expert, namely the assessment calibre, data integrity, and the validity and consistency for how schools enact curriculum intent. What began as a single school initiative in 2014 was able to develop into one of Queensland's largest mathematical cluster initiatives, covering a diverse range of geographical and socio-economic backgrounds. Professional recognition for this work, coming from the Queensland School of the Year 2015, created the necessary urgency and commitment to create scalability with fidelity and flexibility across multiple cluster settings. A second School of the Year award in 2017 for Excellence in Primary Years supported school leaders in the continuation to invest in and promote this professional development model.

Drawing on the literature on teacher professional development, the session will analyse the MCC in terms of (1) the extent to which the content of the program represented a *coherent instructional system* supporting teacher development of ambitious teaching practices; (2) support for *collective action*; and (3) instructional *leadership*.

1. Factors contributing to sustainability and scaling up: An instructional system

1.1. Explicit goals are established for students' mathematics learning.

- Development of students' reasoning and problem solving, as well as fluency and understanding, by reducing teacher tensions in representing the interdependence between the proficiency strands.
- Improvement of students' dispositions towards mathematics(engagement, relevance, and resilience)
- Celebration and communication of short term wins (student attitude and aptitudes, teacher enthusiasm, and reported self efficacy)

1.2. A vision of high quality instruction is communicated, specifying practices leading to attainment of student learning goals.

- A series of whole day workshops, three times a term for a minimum of 18 months
- A strong focus on contextualising tools and practices to fit existing programs

1.3. Instructional materials and tools are designed to support teachers' development of the target practices, and to build their Australian Curriculum and assessment knowledge.

- Teachers are supported to develop instructional tools themselves, in collaboration with colleagues
- The approach has resulted in teachers reclaiming their professional knowledge

1.4. Regional professional development is focused on the target practices and materials sustained over time, and builds school-based professional learning communities.

- Demonstration lessons by the Regional Expert were a powerful impetus for teachers to transform the way they plan, teach, and assess mathematics

1.5. Assessment is aligned with student learning goals.

- Teachers developed diagnostic tasks (Show Mes) that focused on key concepts across Early years to Junior Secondary.
- Teachers modified existing summative assessment tasks (Prep to Year 10) through our criteria of validity, cognition, complexity, relevance, and inclusivity

1.6. There are additional supports for struggling students.

- Lower and higher achieving students benefit equally from the crafted and endorsed Cognitive Activation Tasks which press students to explain their thinking

2. Factors contributing to sustainability and scaling up: Collective action

- Teacher networks within and between cluster schools were central to the sustainability and spread of the initiative; teachers felt safe and supported to take risks.
- For secondary school teachers, networking with all 41 High Schools in the region increased everyone's knowledge of numeracy across the curriculum

3. Factors contributing to sustainability and scaling up: Instructional Leadership

- A balance of regional support, commitment, and friendly accountability for all school principals through visible attendance, termly reports, hosting workshops.
- Strategies for selecting delegates, roles at school, community engagement workshops.

- Regional leaders successfully balance the often competing goals of instructional management and instructional improvement

The Mathematical Capability Clusters go some way towards addressing the call made by Beswick, Anderson, and Hurst (2016) for researchers to give more attention to issues of scale and sustainability of professional learning initiatives. Implications for the next step of the MCC would focus on

1. What can be done to encourage this type of networking?
2. How can principals be persuaded that this is an important investment?
3. How can the role of these Regional Experts be sustained and scaled up to build leadership capacity in all regions and states.

This paper is based on a longer article by Goos, Bennison, and Proffitt-White (2018).

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Problem Solving: What did you learn?

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Problem solving has a special importance in the learning of mathematics. Research on mathematical problem solving focuses on analysing the extent to which problem solving activities play a crucial role in learners' understanding and use of prior mathematical knowledge. Mathematical problems are central in providing practice in applying mathematical knowledge to foster students learning. Hence, what students learn from the experience of solving mathematical problems should be given emphasis. In this study, recreational problems, better known as mathematical puzzles, were used to explore what pre-service teachers learnt from their experience of solving them. The study used a survey design with open response questions to collect data. The data obtained from this survey was then used to outline a series of questions for future instructions when using these puzzles that would guide the individual to explore the underlying rules to solve the puzzles rather than merely solving them.

INTRODUCTION

Problem solving has a special importance in the learning of mathematics. The framework used to think about the process in problem solving in the Malaysian Mathematics curriculum emphasizes the use of Polya's problem solving stages. The stages as prescribed in textbooks, depict problem solving as a linear process, involving a series of steps which gives a misleading picture of the demands of problem solving, especially non-routine problems. This depiction implies that solving mathematical problems is a procedure to be memorised (Schoenfeld, 1988, 1989)and rehearsed with one goal in mind, which is to obtain the correct answer. Exercises which provide ample opportunity to employ the same procedure to obtain the correct answer reinforce the memorised procedure. This process instils in students that every mathematical problem has a correct answer and obtaining the correct answer or answers is the only goal of solving mathematical problems. The linear framework, however, provides a consistent manner in which problem solvers present their solution after solving a problem (Wilson, Fernandez, & Hadaway, 1993).

Wilson, Fernandez, and Hadaway (1993) also claim that the framework for problem solving is a dynamic and cyclic interpretation of Polya's stages (1973). I agree with this claim that problem solving is a repeating cyclic process based on my personal experience of solving problems and observing others (my colleagues and students) solve problems. The process is largely related to the ways in which individuals orientate themselves with the information provided in the problem, their mathematical content knowledge of the problem and their previous experiences of the mathematics underlying the problem.

According to Garofalo and Lester (1985) students are largely unaware of the processes involved in problem solving and addressing this issue within problem solving instruction may be important. To become a good problem solver in mathematics, one must develop a base of mathematical knowledge. How effective one is in organizing that knowledge also contributes to successful problem solving. According to Wilson et al (1993) state that one of the activities essential to promote learning from problem solving are exploring problem contexts. They also claim that "it is what you learn after you have solved the problem that really counts".

In this study, recreational problems, better known as mathematical puzzles, were used to explore what pre-service teachers learnt from their experience of solving them. A mathematical puzzle is related to mathematical facts and objects and requires mathematics to solve it. The puzzles used in this study can be solved by trial and error with the application of basic addition and subtraction operations and logical reasoning. The data obtained from this survey was then used to outline a series of questions for future instruction when using these puzzles that would guide the individual to explore the underlying rules to solve the puzzles rather than merely solving them. This then would help students learn a lot more from their experience of solving the given puzzles.

METHODOLOGY

THE STUDY

In this study pre-service primary school mathematics teachers were required to solve a specific type of mathematical puzzle (see Figure 1) and explore the rules to create such puzzles. The aim of this study was to determine what the participants of this study (who will be referred to as students in this paper) learnt and what they failed to learn from the activity of solving the given puzzles. The study also aimed to improve instruction based on the findings so that the problem solving experience using these puzzles would have a greater impact on learning.

RESEARCH QUESTIONS

1. Were students able to recall basic mathematical knowledge to solve the given puzzles?
2. Were the students able to justify why a given puzzle did or did not have a solution?
3. Were the students able to provide rules that could be used to check whether a given puzzle could or could not be solved before they attempt to solve the puzzle?
4. Were the students able to create their own puzzles that had solutions?
5. Were the students able to create their own puzzles that did not have solutions?
6. What did the students learn from their experience from the activity with the given puzzles?

PARTICIPANTS

The participants of the study were nine second year Mathematics-major students who are trainee teachers been trained to become graduate primary school teachers. These trainees are high grade achievers in the public examinations for mathematics.

METHOD

The study used a survey design with open response questions to collect data. The survey form was similar to a test except that there was no time limit imposed to complete the survey. The form provided an example on how to solve the puzzle and then two puzzles for the students to solve. The survey form is as shown in Figure 1. Question 1 can be solved while question 2 can be solved if B = 0, that is, it assumed to have no stones, otherwise it has no solution. The aim of the questions was to provide a starting point to explore the difference between the two puzzles and develop a deeper understanding of the puzzles. The two questions required the recall of basic addition and subtraction operations and logical reasoning to solve them. As Schoenfeld (1982) stated, the problem solving process is ultimately a dialogue between the problem solver's prior knowledge, his attempts, and his thoughts along the way. Hence finding the solutions for the puzzles provided data to answer the first research question. For each correct answer the student was awarded a score of 1, and 0 for an incorrect answer.

Question 3 in the survey form is aimed to require the students to explore the two puzzles and to provide a justification for their responses to questions 1 and 2. This would prevent them from merely giving the responses based on their ability to solve or not solve the puzzles. The justifications would provide insight into the line of thinking of the students, and data to answer the second research question. A score of 1 was awarded for responses which gave correct justification in numeric or algebraic form and a 0 for incorrect justifications.

Question 4 in the survey form required the students to make generalisations – to formulate the rules and to justify them. This provides data concerning what the students had learnt which they could then apply to any given puzzle of the same type; and also data for the third research question. For each correct generalisation or rule a score of 1 was awarded and 0 if no generalisations were made.

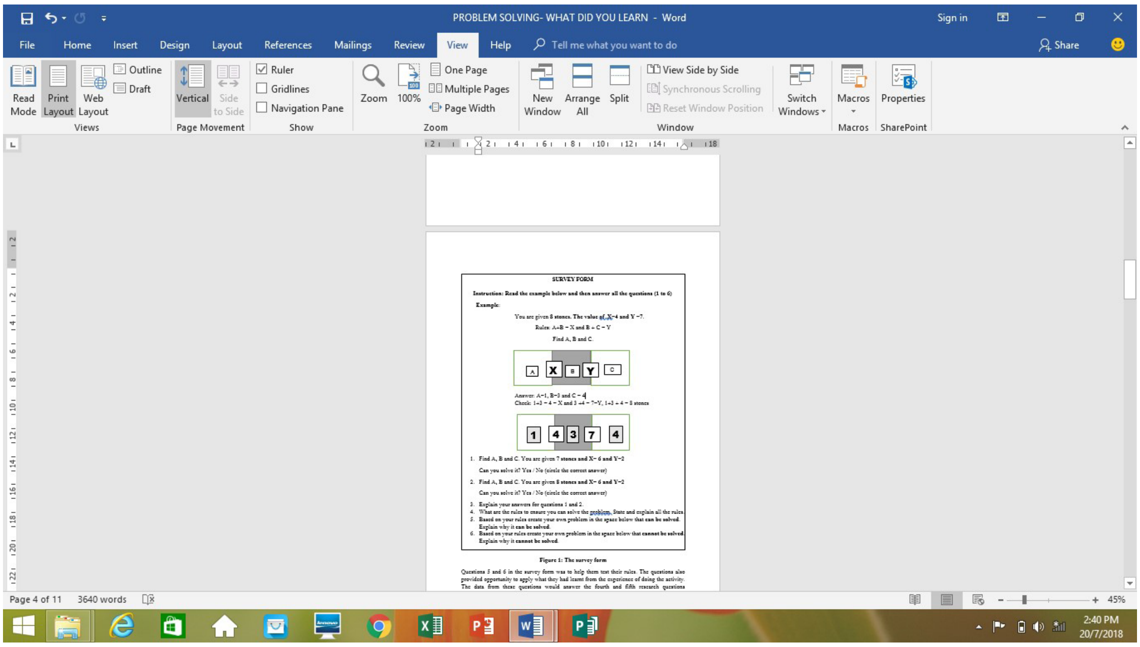

Figure 1. The survey form.

Questions 5 and 6 in the survey form were intended to help them test their rules. The questions also provided an opportunity to apply what they had learnt from the experience of doing the activity. The data from these questions would answer the fourth and fifth research questions, respectively. If students created a puzzle that could be solved and explained, based on their rules in question 4, they would be awarded a score of 1. If they created a puzzle that could not be solved, and/or provided an incorrect explanation, a score of zero was awarded. Likewise for question 6, if a puzzle that had no solution was created, and explained based on their rules in question 4, they would be awarded a score of 1. If they created a puzzle that could be solved and/or provided an incorrect explanation a score of 0 was awarded for question 6.

The questions in the survey form were guiding the students on how to explore the given puzzles, draw on salient points, express generalisations and finally apply what they had learnt to create puzzles according to the conditions stated in the questions. The process demands application of higher order thinking skills. The data for the first five research questions would collectively provide data for the last (sixth) research question.

The scores were used to provide the quantitative data. Based on the quantitative data areas of strength (percentage of students giving the correct response is more than 80%) and weaknesses of the line of questioning were determined. The findings in the quantitative and qualitative data were then used to make a list of amendments to improve instruction to provide a rich and fruitful experience, and hence have a greater impact on the students' learning.

DATA ANALYSIS AND DISCUSSION

Table 1 shows the result of the analysis of the nine students' responses for questions 1 to 6, including the percentage of correct responses from all nine students. The data shows that all students could solve the puzzle in question 1 and stated that it has a solution. One student gave the answer question for 1 also for question 2, and stated it could be solved. He had ignored the change in the total number of stones for question 2. However, the overall performance shows that all 9 students could solve these puzzles.

Data for question 3 shows that this is an area of weakness. Only one student recognised that the total number of stones determines whether the puzzle has a solution or not (see Figure 2). This may be due to the fact that both puzzles in questions 1 and 2 had a solution. To help students realise that the total number of stones played a role in determining whether the puzzle had a solution, a third puzzle with more than 8 stones should have been given. This extra puzzle

would also have helped students identify Rule 1 for question 4.

Question number	Correct response	Percentage of correct response = (Total score awarded/9) x 100
1	Yes with correct answer given for A, B, and C	100% (Strength)
2	Yes with correct answer given for A, B, and C No with no answer given or incorrect answer filled.	89% (Strength)
3: Q1	Yes because number of stones given is 7 which is less than $6 + 2 = 8$	11% (Weakness)
3: Q2	Yes because number of stones given is 8 which is equal to $6 + 2 = 8$ (if $B = 0$ or $C = 0$) OR No because number of stones given is 8 which is equal to $6 + 2 = 8$. Number of stones equals $6 + 2 = 8$ (if $B \neq 0$ and $C \neq 0$)	11% (Weakness)
4 (Rule 1)	Number of stones less than or equal to $X + Y$ or maximum number of stones $= X + Y$	22% (Weakness)
4 (Rule 2)	Minimum number of stones $= X$ (given that $X > Y$)	0% (Weakness)
4 (Rule 3; e = even number, o = odd number)	If total number of stones is an odd number, then the solution is o + o + o (X and Y even) or e + e + o (X even and Y odd or vice versa) or e + o + e (X and Y odd) If the total number of stones is an even number, then e + e + e (X and Y even) or o + o + e (X even and Y odd or vice versa) or o + e + o (X and Y odd). (But Rules 1 and 2 must be satisfied first)	11% (Weakness)
5	Create puzzle with solution and explanation based on rules 1 to 3	56% (Weakness)
6	Create puzzle without solution and explanation based on rules 1 to 3	33% (Weakness)

Table 1. Percentage of Correct Responses

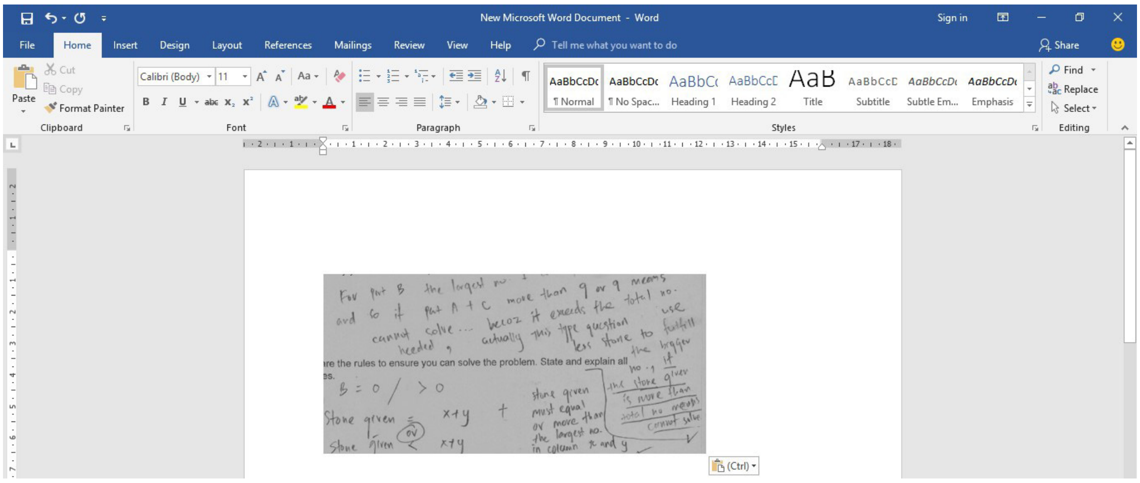

Figure 2. Response of student with the correct response for question 3 and for question 4 based on rule 1.

Another puzzle with fewer than six stones, for example 5 stones, may have guided the students to realise that a lower limit existed for the total number of stones in a given puzzle. This may have then helped the students to state Rule 2 for question 4; which none of the nine students realised as a factor that determines whether the puzzle has a solution or not.

As for stating rule 3 in question 4, only one student managed to explain it partially, based on the puzzle in question 1 (see Figure 3).

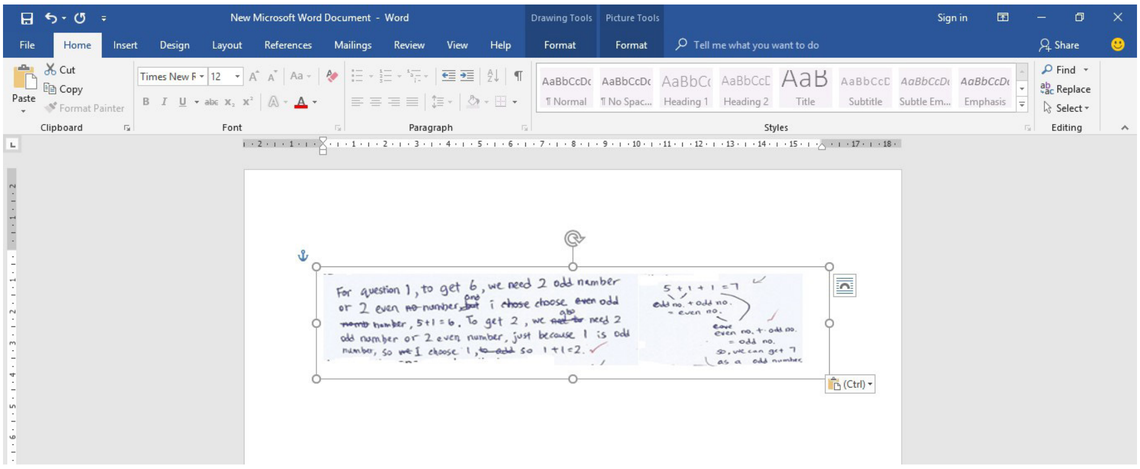

Figure 3. Response of student with the correct response for 4 based on part of rule 3.

Figure 4 shows the response of a student who attempted to use rule 3 in question 4, but became confused and hence stated incorrect rules. She claimed that if two odd numbers appear in the question (she is referring to the X and Y values), the stones provided must be an even number because the combination of two odd numbers is an even number. Likewise, she claimed that if two even numbers appear in the question (she is referring to the X and Y values), the stones provided must be an odd number because two even numbers can be made up of two odd numbers. Finally she claimed that if one odd and one even number appears in the question (she is referring to the X and Y values), the stones provided must be an even number. She has confused the X and Y values with the A, B, and C values.

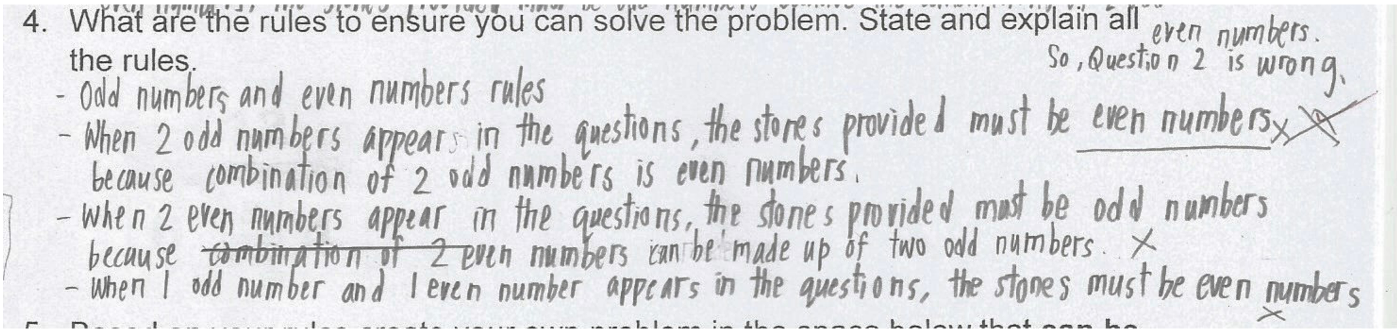

Figure 4. Response of student with the incorrect response for question 4 based on rule 3.

For students to realise all the parts of rule 3, they need to explore systematically for an even number of given stones and the various combinations of X and Y (X and Y both even numbers; X and Y both odd numbers, and finally X an even number and Y and odd number or vice versa), but the total number of stones must abide by rules 1 and 2. Similarly they need to explore systematically an odd number of given stones. Without a systematic evaluation, confusion would result and incorrect rules would be stated. Hence, instruction must demand that these various combinations be explored, and the respective rules stated from their findings.

Question 5 has 56% correct response rate, which is five of the nine students being able to create and provide an explanation based on Rules 1, 2, and/or 3. However, of the five, three used rule 1 only (see Figure 5) and two used part of rule 3 only (see Figure 6). The two who used Rule 3, however, had been unaware that they had abided by Rule 1: that is, the number of stones was less than X + Y.

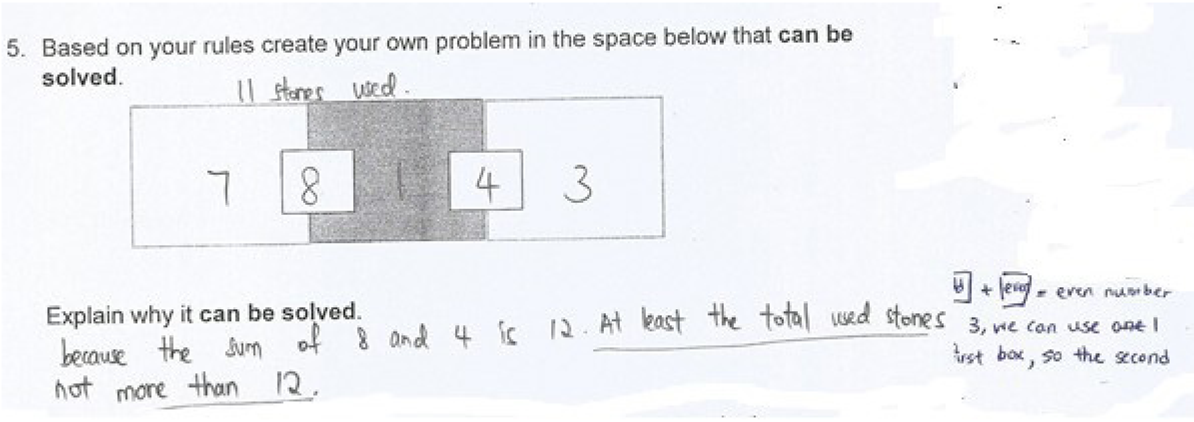

Figure 5. Puzzle created based on rule 1.

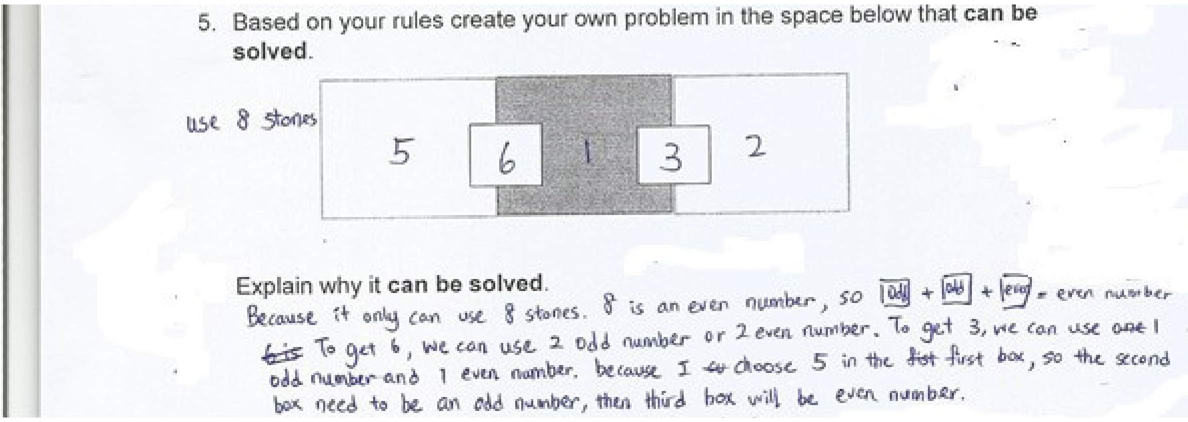

Figure 6. Puzzle created based on rule 3.

There were, however, two students who did not use any of the rules to create a puzzle that could be solved. They used the "working backwards" method by first finding the values of A, B, and C; then adding A + B to obtain X, and B + C to obtain Y; and finally calculating the number of stones by adding A, B, and C. These two students were not awarded a score of 1 because they had not used any of the rules from a generalisation (see Figure 7).

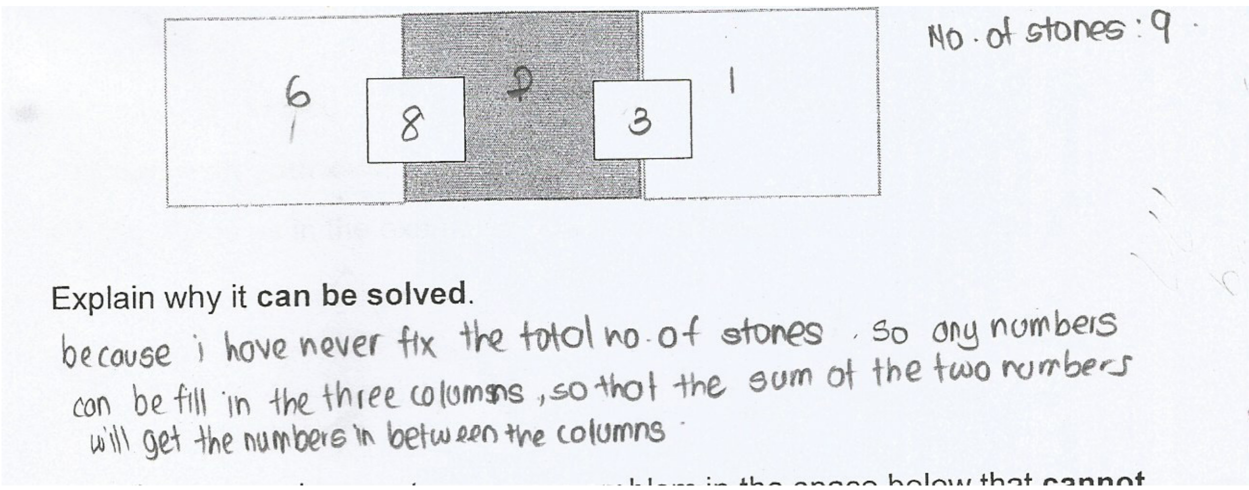

Figure 7. Puzzle created by working backwards.

The data shows that only 33% of the students, three out of nine, were able to create a puzzle that could not be solved. All three had used rule 1. Although one did not state rule 1 as a response for question 4, for questions 5 and 6 this student applied rule 1 to justify why her puzzle could be solved (question 5) and could not be solved (question 6).

The two students who used rule 3, and were not aware of rule 1, were unable to create a puzzle that could not be solved. One of them was totally incapable of creating a puzzle that had no solution, while the other created a puzzle that had a solution but thought that it did not have a solution (see Figure 8). The justification given was also incorrect (see Figure 8) because in question 4 she did not analyse the puzzle systematically, and hence failed to obtain a complete set of conditions for rule 3.

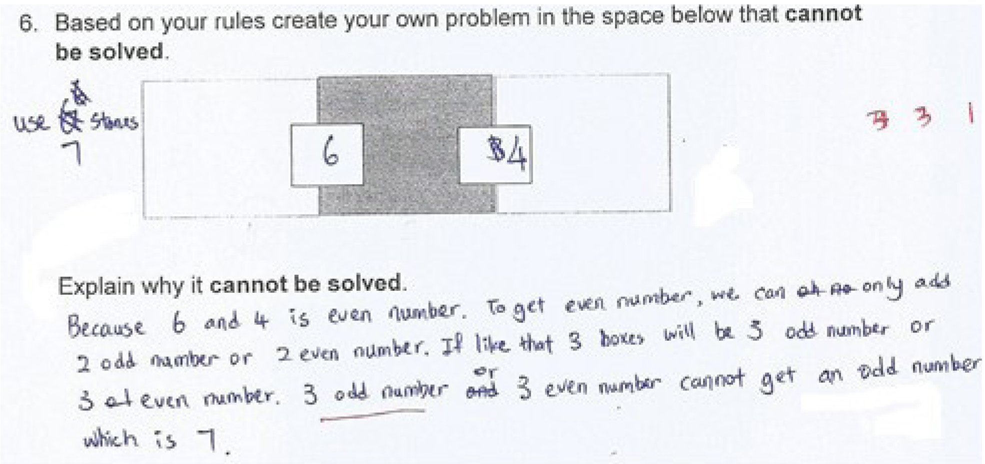

Figure 8. Puzzle created assumed to have no solution but does have a solution for 7 stones.

Working backwards did not make it easy to create a puzzle that has no solution. Both students who used the working backwards method for question 5, failed to create a correct response for question 6. From the data of this study, creating a puzzle without a solution demands knowledge of some of the rules and has proved to be more challenging than creating a puzzle of this type with a solution. However, it is a challenge only if the condition that A or B or C can be zero, so that the maximum number of stones is X + Y. If this condition is not imposed, then creating a problem with no solution is extremely easy as was shown by one student. This student, however, was not awarded a score for her answer to question 6 because it contradicted her answer to question 2, which she answered as having a solution although the number of stones is equal to X + Y. For question 6 she stated that it had no solution because the number of stones was the same as X + Y (see Figure 9).

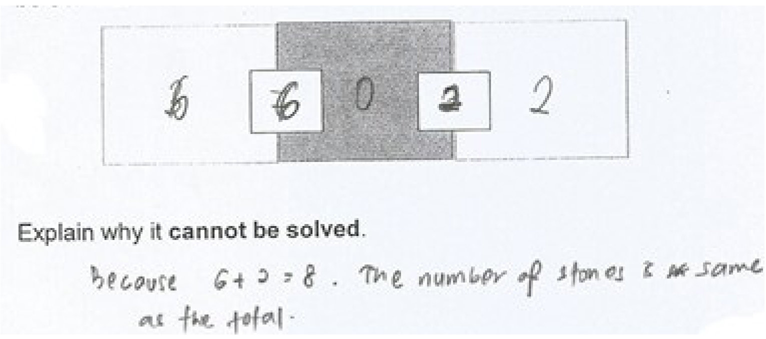

Figure 9. Puzzle without solution created but contradicts the students' response in question 2.

Based on the data of this study, the following list outlines the amendments recommended to improve instruction when using this type of puzzle so as to provide a rich and fruitful experience, and hence have a greater impact on the students' learning.

1. The puzzles must have a minimum of five different values for the number of stones given (to develop Rules 1 and 2 as follows :

(a) number of stones = X + Y (to obtain maximum number of stones)

(b) number of stones = X (if X > Y) (to obtain minimum number of stones)

(c) number of stones > X + Y (no solution)

(d) number of stones < X (if X > Y) (no solution)

(e) number of stones greater than X (if X > Y) and less than X + Y

2. For all cases in part 1 set the condition that A or B or C can be equal to zero, that is only one of these values can be equal to zero. (This is required to prevent the creation of a puzzle without a solution to be non-challenging).

3. Require students to explore the change in values of A, B and C with the changing values of X and Y for odd and even number of stones (To develop Rule 3)

4. Require students to state rules that identify a puzzle with a solution.

5. Require students to create puzzles with and without solutions and to provide justification for the puzzle created based on their rules.

CONCLUSION

The solving of these type of puzzles can be easily achieved by trial and error, and the experience does not have a greater impact on the students' learning. Using guiding questions to analyse the relationship between the total number of stones given and the value of X and Y provides a rich experience giving the opportunity for the development of systematically analysing the puzzles and for developing new knowledge. To ensure that these puzzles provide a rich experience, the recommendations made above should be taken into consideration.

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The challenges of teaching with challenging tasks: developing prompts

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Teaching with challenging tasks in the early and middle years of primary school can support the development of student reasoning and unleash critical and creative mathematical thinking; however, teaching with challenging tasks can be challenging. Some issues that might arise for teachers when considering teaching with such tasks are: How do you develop (and use) appropriate enabling and extending prompts to support and extend all learners? How should you structure lessons involving challenging tasks? How do you introduce challenging tasks without creating classroom management issues? Although all of these questions are important and warrant examination, the focus of the current paper is on unpacking enabling and extending prompts. The author draws on his firsthand experience of teaching challenging tasks to students in Foundation to Year 4 to explore this issue.

WHAT ARE CHALLENGING TASKS?

Challenging tasks are complex and absorbing problems with multiple solution pathways, which include enabling and extending prompts to support differentiated learning (Sullivan & Mornane, 2014).

Challenging tasks should:

- engage students;
- be solvable through multiple means (i.e., have multiple solution pathways) and may have multiple solutions;
- involve multiple mathematical steps;
- have at least one enabling prompt and one extending prompt developed prior to delivery of the lesson;
- be initially perceived as challenging by the majority of students and involve students spending considerable time working on the task (e.g., at least 10 minutes),

Generally, lessons with challenging tasks are structured such that work on a challenging task (Explore phase) precedes a whole-group mathematical discussion (Discuss/Summary phase) (Stein, Engle, Smith, & Hughes, 2008). How can the teacher ensure that all (or almost all) students are in a position to potentially contribute to this mathematical discussion? Facilitating an inclusive discussion is dependent on all students engaging meaningfully with the primary learning focus, which is in turn contingent on the teacher employing carefully developed enabling and extending prompts.

DEVELOPING APPROPRIATE ENABLING AND EXTENDING PROMPTS TO SUPPORT ALL LEARNERS

Enabling prompts are designed to reduce the level of challenge through: (a) simplifying the problem, (b) changing how the problem is represented, (c) helping the student connect the problem to prior learning and/ or (d) removing a step in the problem (Sullivan, Mousley, & Zevenbergen, 2006). Extending prompts are designed for students who finish the main challenge and expose students to an additional task that is more challenging; however this requires them to use similar mathematical reasoning, conceptualisations, and representations to the main task (Sullivan et al., 2006).

There are a number of things to keep in mind when developing (and using) enabling and extending prompts, some of which are particularly relevant when working with early primary students (Foundation, Year 1, Year 2). These considerations can be separated into how prompts should be used *during* the lesson, and how prompts should be developed *prior* to the lesson.

HOW SHOULD PROMPTS BE USED IN THE CLASSROOM?

When teaching with challenging tasks, it may be helpful to refer to the enabling prompts using the more student friendly term "hint sheets." Children should be encouraged to access the enabling prompt of their own volition, after spending at least some time grappling with the problem in what Sullivan and colleagues have termed the "zone of confusion" (Sullivan et al., 2014, p. 11). Initially, there may be a tendency for enabling prompts to be underutilised by students even when they are unproductively stuck on a task for long periods, either because they are not used to taking it on themselves to decide when they need help, or because there is some perceived stigma around using the "hint sheet." If this seems to be the case in your classroom, it might be worthwhile to praise students for being proactive in accessing the enabling prompt, in part to develop the norm that taking initiative to help oneself with one's own learning is a sign of maturity and independence, rather than implying "failure" with the task. The goal is to build a culture where students take responsibility for their own learning, and to foster the belief amongst students that they can become better mathematicians through effort, calculated risk-taking and resourcefulness.

To support this culture, enabling prompts should be in the same place for every session where challenging tasks are being used (e.g., on the teacher's chair at the front). As a more general point when teaching with challenging tasks, developing well established, predictable routines seems particularly important, both for communicating clear (and high) expectations and for building student autonomy (Russo & Hopkins, 2018).

By contrast, you may wish to consider referring to the extending prompt as "the super-challenge," and including it on the flip-side of the challenging task handed out to students. It is suggested that the teacher might have a role to play in subtly "eyeballing" a student's work to make sure that they have made genuine progress with the main task before moving on to the extending prompt. Although the intention is to encourage students to take responsibility for determining whether they are ready to be extended, initially this may require some monitoring as students are often keen to "keep up" with their peers. This is obviously likely to be less of an issue if a culture of personal responsibility is well established within the classroom.

HOW SHOULD PROMPTS BE DEVELOPED?

Enabling prompts can be presented in many different forms. However, as much as possible, enabling prompts should use visual cues that support the relevant mathematical thinking (e.g., images, tables, diagrams, concept cartoons), rather than relying on lots of text. This increases the likelihood that students will be able to successfully use such prompts independently of teacher support.

When developing prompts to support work on a challenging task, there might be a tendency to treat anything that makes the task easier as an appropriate enabling prompt, and anything that makes the task harder as an appropriate extending prompt. Given that the goal is to engage all students in meaningful mathematical work, this general rule of thumb might be good enough to achieve this ends. However, ideally, decisions around what are the most appropriate enabling and extending prompts for a given challenging task should be influenced by the primary learning objective of a given lesson. The advantage of this more considered approach to designing prompts is that getting all students to focus on the same primary learning objective lays the foundation for a meaningful and inclusive discussion around the relevant mathematics at the end of the lesson. Even when using the same challenging task, changing the focus of the learning may well lead to a change in one or both of the prompts. Therefore, the ideal role of the prompts is not just to make the task easier or harder, but to deepen engagement with the primary learning focus.

LEARNING OBJECTIVES SHOULD SHAPE PROMPTS

The idea that learning objectives should shape the nature of the associated prompts is best illustrated through examples. The remainder of this article elaborates on three challenging tasks, and explores how modifying the learning objectives influences the best choice of prompt to include.

HOW MANY FINGERS TASK

Consider the challenging task, targeted at Year 1 and 2 students (How Many Fingers task): *Without leaving your seat, or talking to anyone, can you work out how many fingers are in the room right now? Show how you worked it out* (see Russo, 2015). This task appears to have at least two learning objectives embedded within it:

- Problem representation: "To find a quasi-abstract or abstract means of representing a concrete counting-based mathematical problem" (Russo, 2015, p. 10).
- Skip-counting patterns: "Counting by 5's or 10's can be a more efficient way of working out how many things there are in a collection" (Russo, 2015, p. 10).

Although both problem representation and skip-counting patterns might be viewed as important learning objectives, prioritising these objectives supports the development of prompts. This would seem to be particularly important when developing the enabling prompt, because focusing on one specific learning objective generally means de-emphasising (or even removing) the other objective(s). The suggestion is that this should be an active and pre-mediated choice by the teacher, rather than something left to chance.

For instance, with the current example task, if the teacher determined that problem representation was the primary learning objective, an appropriate enabling prompt may be: "Without leaving your seat, or talking to anyone, can you work out how many people are in the room right now? Show how you worked it out" (Russo, 2015, p. 11). This maintains the emphasis on finding a means of appropriately representing the worded problem, whilst removing the focus on skip-counting. We might expect the child to model all the people in the room at their tables using *Unifix*, or perhaps draw a picture conveying this information.

Conversely, if exploring skip-counting patterns was the primary learning objective, a more appropriate enabling prompt may be to show students an image and ask them to determine the number of fingers in the image (see Figure 1). This removes the emphasis on problem representation so that students can engage with skip-counting patterns. Notice that the image chosen in Figure 1 does not allow students to accurately count all the fingers by 1s, as several fingers are obscured. Therefore, to accurately ascertain how many fingers are in the image, students would likely need to skip-count (or use multiplication), or at the very least conceptualise the problem using this structure (even if they resorted to attempting to count by 1s).

Figure 1. Enabling prompt: How many fingers are in this picture? How did you work it out?

Similarly, if the primary emphasis was on problem representation, an appropriate extending prompt may be something like: "How many fingers are there in all of the Year 1 and Year 2 classes combined?". Assuming there are multiple Year 1 and 2 classes, this problem is considerably more challenging to represent than the original task. Specifically, it is more abstract, likely requiring a "top-down" conceptualization (e.g., knowledge of approximately how many students are in

each class, and skip-counting "this many times"), rather than a "bottom-up" conceptualization (e.g., drawing pictures of students sitting at tables, and using this pictorial representation to count the fingers).

By contrast, an appropriate extending prompt for the Fingers task if the emphasis was on skip-counting patterns might be: "I forgot to tell you; thumbs don't count as fingers!". This invites students to skip-count by more challenging patterns (e.g., counting by 4s or 8s, rather than 5s or 10s) to solve this extension and/ or engage with multiplicative ideas, in particular a preliminary version of the distributive property (e.g., students may use their existing problem representation to count back by 2s, leveraging off the idea that a group of 8 is 2 less than a group of 10). However, it is worth noting that ensuring that the extending prompt builds on the primary learning objective is probably less important than is the case for the enabling prompt, as these students will still be able to contribute meaningfully to the whole-group mathematical discussion.

BLOCK OF CHOCOLATE TASK

The next challenging task was sourced from the Australian Curriculum work sample portfolio available through the ACARA website. It is targeted at students in Year 2 and can be referred to as the Block of Chocolate task. It is presented as follows: *I have a 30 piece block of chocolate (in the shape of a rectangle). What might my chocolate look like? Record as many possibilities as you can* (ACARA, 2014).

Two learning objectives inherent in the Block of Chocolate task might be:

- Drawing an array: For students to be able to visualise a discrete set of objects, and represent and record the set of objects as an array.
- Many factors: For students to realise that some numbers can be represented by multiple arrays; that is, they have many factors (e.g., 24 can be represented as 8 × 3 or 6 × 4 or 12 × 2).

An appropriate enabling prompt if the primary learning focus was on drawing an array might be to reduce the number of pieces of chocolate being considered; for example, 10 pieces, rather than 30 pieces. Using a smaller set of objects will likely support students' capacity to appropriately visualise the problem, and allow them to use their pre-existing computational knowledge to mentally organise the pieces into an array (e.g., 10 is the same as 2 rows of 5).

Alternatively, if the primary focus was on students discovering that some numbers have many factors and can be represented by multiple arrays, it might be more appropriate to provide students with concrete materials to manipulate; in this case, square counters to represent the pieces of chocolate (see Figure 2). Again, although both learning objectives seem important, prioritising your objective helps to determine which prompt might be more suitable in a given learning context.

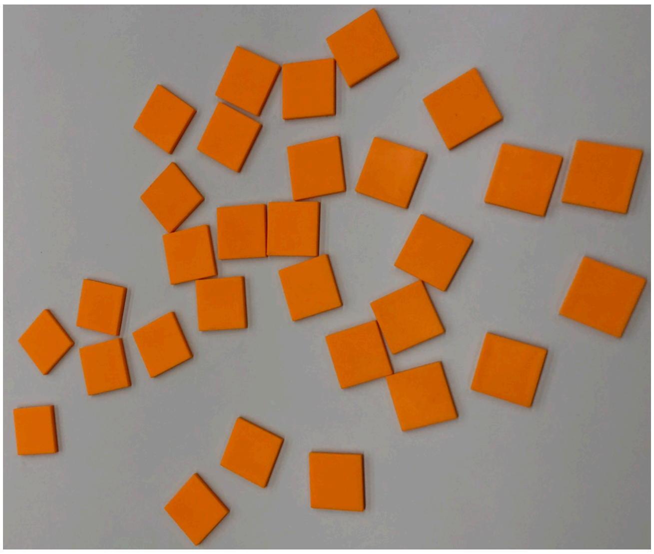

Figure 2. Enabling prompt: Can you organise these 30 pieces of chocolate into a rectangular block? How many different rectangular blocks of chocolate can you create using all 30 of these pieces?

FOUR SEASONS TASK

The last challenging task is most appropriate for students in Years 3 and 4 (although it has been used with younger students), and is called the Four Seasons task. Without any preliminary discussion, students are asked: *In Australia, what is the longest season?* (see Russo, 2016). There are likely to be several learning objectives embedded within this task, including:

- Digging deeper: For students to appreciate that mathematical problems can involve counter-intuitive findings that will only be revealed after a student works through a task methodically.
- Applying knowledge: For students to use their knowledge of how the seasons and months are constructed to solve a mathematical problem.
- Computation with addition: For students to use appropriate computational strategies for adding a sequence of two-digit numbers.

Any one of these learning objectives might serve as the primary focus for the lesson. Let us take perhaps the most obvious scenario first, and imagine that the objective of digging deeper is the primary focus. Although ideally we would still like students to bring to mind and apply their knowledge of the seasons and months to solve the problem, and to employ appropriate computational strategies, we would be willing to sacrifice these aims in order for students to engage with the idea that mathematical tasks can involve counter-intuitive findings that only reveal themselves through methodical application. Consequently, we might give students access to tools that allow them to hone in on the primary learning focus through removing the emphasis on knowledge application and computation. To this end, our enabling prompt might in fact be a calendar and a calculator!

If, instead, our primary focus was on computation with addition, we might provide a table for students which removes the applying knowledge and the digging deeper objectives and turns the task into one which emphasises multi-digit addition. Such a prompt is presented in Figure 3.

Summer	Autumn	Winter	Spring
December 31 days	March 31 days	June 30 days	September 30 days
January 31 days	April 30 days	July 31 days	October 31 days
February 28 days (29 in leap years)	May 31 days	August 31 days	November 30 days

Figure 3. Enabling prompt: Can you work out exactly how many days there are in each season through completing the table?

Although most teachers would likely concur that the calendar and the calculator intuitively seems a more appropriate enabling prompt than the table included in Figure 3, prioritising the various learning objectives embedded in the task emphasises why this is in fact the case. The task seems more in the realm of a rich problem solving inquiry than an elaborate context for addition practice. Hopefully this contrast provides further evidence for the contention that not all prompts are created equal, and that identifying and focusing on the primary learning objective is a fruitful means through which to develop and refine prompts.

CONCLUDING THOUGHTS

Teaching with challenging tasks is inherently challenging for teachers for many reasons, one of which is the expectation that they will create and apply appropriate enabling and extending prompts to support student learning. The purpose of this paper has been to provide some initial guidance to teachers both around how to develop prompts prior to the lesson, and how to use prompts in the lesson. In particular, there has been emphasis on how the process of identifying

and clarifying the primary learning objective can help direct the development of prompts. Carefully developed prompts can do much more than simply make a task easier or harder; they have the potential to deepen engagement with the primary learning focus and enrich the subsequent mathematical discussion. Teachers may wish to consider adapting existing problem-solving tasks that are available through professional journals (e.g., *Australian Primary Mathematics Classroom, Prime Number*) and websites (e.g., *Nrich, reSolve*) by developing specific enabling and extending prompts that suit their specific learning focus.

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STEM resources from ReSolve: maths by inquiry

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This article introduces several new STEM resources that have special relevance to teaching mathematics. All resources are freely available from <www.resolve.edu.au>. Units can be taught separately or put together to make a coherent multi-year program. Year 7 and 8 topics include the mathematics of motion and of music (including conducting experiments to gather, plot, and interpret data). There is a series of five units that develop skills in modelling the real world, using contexts such as wait times for theme park rides, designing packaging, analysing statistics on risk and accidents, and vehicle cornering. Three units use coding to explore data visualisation, patterns in nature, and simulating games. These highlight algorithmic thinking in mathematics.

INTRODUCTION

THE STEM AGENDA

Over the past decade, governments in many countries have come to recognise the importance of technological innovation to the health of their economies, and for citizens to have access to high-wage jobs in an increasingly competitive and globalised world. For example, the Victorian Minister for Education, in his introduction to “VicSTEM: STEM in the Education State” (HREF1), noted:

> Our employers are increasingly looking for workers who are creative problem solvers, innovative and critical thinkers, and able to use new technologies. STEM skills are also integral to Victoria’s priority sectors. These have the potential for remarkable growth, driving up economic output and creating over 400,000 jobs for Victorians by 2025. (HREF 1, introduction).

Teachers come to the STEM agenda through their concern to maximise the life chances of each of the individual students in their care. The STEM skills required are built on a strong foundation in “technological” subjects in the Victorian Curriculum, namely Science, Design and Technologies (including Engineering principles and systems), Digital Technologies, and Mathematics. However, two important components of the STEM agenda are to recognise how solutions to real world problems often bring together knowledge from various disciplines, and also to recognise that the disciplines themselves need to be taught with an orientation towards fostering practical problem solving, critical thinking, creativity, and communication.

There are many ways in which mathematics teaching contributes to the STEM agenda. Appropriate contributions vary markedly with the age and interests of students, from children beginning to learn mathematics to senior students in VCE and VCAL. This paper provides some examples of teaching with an emphasis on STEM education within mathematics in school education in Years 7 to 10. In these years of school, many areas of application of mathematics open up, but at the same time there needs to be a strong focus on developing basic mathematical skills for students with varying abilities and varying interests.

RESOLVE: MATHEMATICS BY INQUIRY

The reSolve: Mathematics by Inquiry project (HREF2) is an initiative of, and funded by, the Australian Government Department of Education and Training, through the Australian Academy of Science and the Australian Association of Mathematics Teachers. Both authors of this paper are employed on the project through the Academy of Science, especially for developing the “special topics” in conjunction with teams from Australia and overseas. By the end of 2018, the reSolve website will provide open access for teachers to

- approximately 100 individual lessons, covering all strands of the Australian and Victorian Curriculum and all year levels F to 10;
- 8 special topics, which offer innovative units of work, including using new technologies in real world contexts (several of these are described below);
- 8 professional learning modules to assist teachers in promoting a spirit of inquiry in all mathematics teaching.

All these components illustrate three basic tenets of good mathematics, described in the reSolve Protocol (HREF2) and summarised as:

- reSolve mathematics is purposeful
- reSolve tasks are challenging and inclusive
- reSolve classrooms have a knowledge-building culture.

The by-line of the reSolve resources is “promoting a spirit of inquiry” (HREF2). This does not imply that mathematics should all be taught through any particular style of “inquiry learning.” Instead it means that in every lesson, whatever pedagogy is being followed, there should be an expectation of mathematical reasoning and problem solving. and capacities to think critically and creatively should feature. It means that students should ask questions and seek answers, with the expectation that mathematical knowledge can help them find good answers.

The reSolve resources demonstrate three approaches to STEM, all of which have a place in students’ overall mathematical diet:

(i) teaching about the process of mathematical modelling

(ii) using real world contexts to teach about target content

(iii) employing integrated units to demonstrate mathematics in use in other subjects.

Table 1 presents a summary of the STEM-related units. The following sections describe in more detail two reSolve resources one on the teaching of mathematical modelling, fitting (i) above, and one showing a unit integrating mathematics and science, fitting (iii) above.

Name (year levels)	Summary	Approximate class time
Teaching about the modelling process		
Mathematical Modelling (Years 9 - 10)	A set of 5 independent units that focus on the process of mathematical modelling, including making assumptions, defining relationships, and evaluating outcomes. Varied real world contexts.	5 lessons per unit
Using real world contexts to teach about target content		
Bringing the Real World into Algebra (Years 9 - 10)	These stand-alone lessons use digital images of real world objects to develop understanding of the effects of parameters in linear, quadratic, and exponential functions.	1 – 3 lessons per topic
Mechanical Linkages and Deductive Geometry (Years 7 - 10)	Students make physical models of the linkages that enable the movement of everyday objects, such as ironing boards, tool boxes, and cherry pickers. Students then use dynamic geometry models to highlight the underlying geometry and then prove that the motion is as required using geometry theorems.	1 – 5 lessons per year level
Mathematical Inquiry into Authentic Problems (Years F - 6)	Ten inquiry units, each centred on solving a real world problem, such as "what fraction of a bottle should be filled with water to make it good for bottle flipping?" The need for providing mathematical evidence is highlighted, and regular checkpoints provide attention to key mathematical concepts and skills.	4 lessons per year level
Integrated units to demonstrate mathematics in use in other subjects		
Modelling Motion (Years 5 - 7)	Developing concepts and the mathematics of motion, integrated with Science. Students follow Galileo's experiments to measure changing and constant speed, and graph the influence of gravity and friction.	4 - 7 lessons
CD Cars (Year 7)	Students make and improve a car with CD wheels, integrating mathematics with Design and Technologies. They use video to measure and calculate speeds and graph data.	5 lessons
Battle of the Sounds (Year 8)	Students make "musical" instruments and tune them, integrating maths with Science and Music. Students use inverse proportion functions and solve equations.	5 lessons
Mathematics and Algorithmic Thinking (Years 9 - 10)	Three units with "behind the scenes" components integrated with Digital Technologies. They address how computation is changing maths, with examples from data visualisation, simulation, and in identifying simple algorithmic rules behind complex patterns in nature.	From 5 to 15 lessons per unit, with multiple stopping points.

Table 1. Summary of Main reSolve STEM Resources (accessible from HREF1).

TEACHING ABOUT MATHEMATICAL MODELLING PROCESS

Mathematical modelling is the process of using mathematics to solve real world problems. Mathematical models are used for prediction and for understanding how the various factors influencing a phenomenon balance out. For example, there are economic models that predict the effect of a change in tax policy, and climate models that predict the effect of increased greenhouse gases in the atmosphere. At home, we all use mathematical models such as: (a) growth charts that indicate whether a baby is putting on sufficient weight, (b) a list of recommended doses of analgesic for children of different weights or ages, and (c) apps that predict the time taken to drive from A to B. These are all mathematical models: (a) assumptions are made based on data where possible (e.g., the average speed likely on a freeway and a side road, the weight of a 12-year-old), (b) mathematical relationships are described (e.g., time = distance divided by time), and (c) calculations produce a useable prediction.

The special topic on Mathematical Modelling examines the processes involved in creating and using mathematical models, using examples from a variety of real world contexts, STEM and otherwise. The models employ mathematics that is reasonably well known to the students in order to reduce the cognitive demand of the mathematical techniques so that learners can focus on the modelling process. The process of mathematical modelling is described by the "modelling cycle" shown in Figure 1, and special attention is given to the actions of moving between the real world and the mathematical world. These actions are formulating the model by identifying assumptions and relationships, and later interpreting the mathematical results to see if they give sensible real world guidance for decision making.

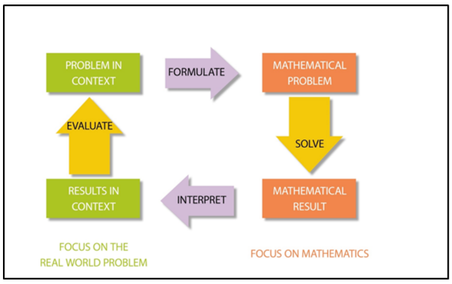


Figure 1. The modelling cycle.

The first unit[1] looks at queuing problems. Students are put in the role of a theme park worker who must decide where to place markers so that customers waiting in line for a popular ride will know when they can expect to wait 15, 30, 45, and 60 minutes in the queue. Relevant and irrelevant information about the ride, including the height, length, maximum speed, g-force, arrangement of riders in the cars, etc. is provided. A major theme of this unit is that mathematical modelling starts with simple models that are improved. The main mathematics involved is proportional reasoning.

After the introductory unit, the other units can be selected independently. A full implementation could use the five units, one per term, across Years 9 and 10. The second unit focuses on formulating a model, especially identifying relevant variables and generating the relationships between them. The processes are illustrated by modelling the effects of pricing decisions and input costs on profit. The other units are based around problems involving the design of packaging, the space needed for vehicles to turn corners (with clear road safety messages), and drawing conclusions about "how risky is life" from data from the Australian Bureau of Statistics (see Figures 2 & 3). For example, students have to comment on the naïve explanation of the graph shown in Figure 2.

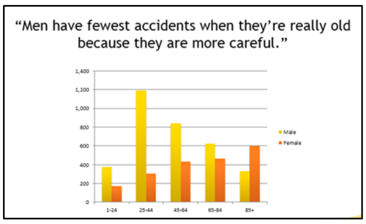


Figure 2. Graph of total accidents by age in Australia in 2015

1. *The units were produced by Professor Geoff Wake and colleagues of the Shell Centre at the University of Nottingham in conjunction with the reSolve team.*

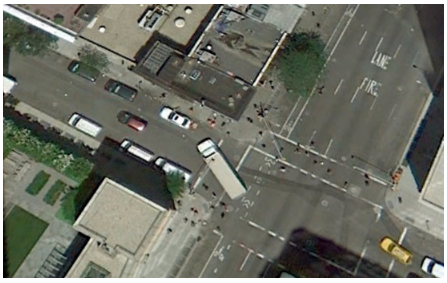


Figure 3. Large vehicles need a lot of space to turn a corner.

INTEGRATED SCIENCE AND MATHEMATICS UNIT

Finding activities that use mathematics authentically to integrate the STEM subjects has often proved challenging. *Modelling Motion* provides students with an authentic STEM experience where mathematics is central to understanding the science. Our modern understanding of motion began with Galileo's measurement and modelling of simple motions of a ball on a slope. The *Modelling Motion* lessons draw on Galileo's experiments to systematically develop students' ability to understand how objects move and to represent motion, using graphs derived directly from the real situation, with all the uncertainties and experimental errors that entails. Many general investigative skills for good data collection and data analysis and the requirements of a fair test of a hypothesis arise in all lessons. There are many opportunities for students constructing, sharing, interpreting, and comparing their models in small groups and in whole-class discussions. Activities and projects are included to extend students' understanding.

The overarching goal is to give students a strong foundation for understanding speed, and to understand acceleration and deceleration as changes of speed. In terms of mathematical development, this unit strengthens proportional reasoning and prepares the way for further learning about rate of change, of which speed is a primary example. Students also measure, make graphs of data, and interpret the graphs to learn more about the behaviour of the moving objects. It is well known by science educators that many students have misconceptions about speed, so the unit carefully leads from students' initial understanding of speed as just a name for a physical experience towards a quantified attribute linking distance and time. The later lessons explore Newton's first law, that unbalanced forces cause speed to change.

The focus of the first lesson is to understand speed as distance travelled per unit time. Students make "streamer graphs" of the distance travelled in a second by a student aiming to walk at a constant speed – first slow and then faster. A streamer graph from this activity is shown in Figure 4. Students will transition from their intuitive understanding of speed just as *faster* and *slower* towards distance travelled in one second. Two important goals are for students to know that the strips in the graph represent both the actual distance travelled in one second, and the speed of the walker, and that longer strips represent faster speeds.

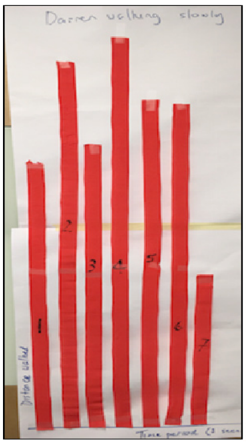
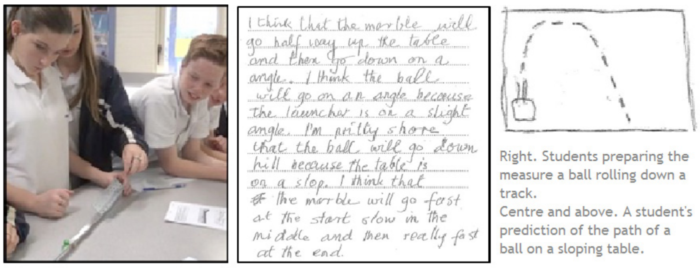


Figure 4. A streamer graph made when Darren attempts to walk at a constant speed.

In the next lessons, students make streamer graphs of balls rolling down (and later up) a track on a slope, as shown in Figure 5. Students discuss how the graphs show acceleration and they can confront misconceptions about the way speed changes due to gravity. They look at how the distances travelled per second grow steadily, but observe that the total distance travelled since the start increases very quickly with time. They predict the height from which a ball needs to be dropped to fall for 1 second and test the prediction. Understanding of gravity as a force is consolidated. Students learn that experimental data represented graphically can be used as a model to predict the results of further experiments. Later, students experiment with the complex motion of a ball rolling up, down, and across a sloping table, and predict how to hit a target at the edge of the table by varying the angle of the launcher. A student's initial prediction of the motion is shown in Figure 5.

In later lessons students construct their own forcemeters and make streamer graphs to show the motion of a trolley car pulled by unbalanced forces from the front and behind. Figure 6 shows five predictions of what the streamer graphs (i.e., distance travelled in one second) will look like. These reveal that students have little intuition about what will happen. Individual students predict that the car gets slower, or faster, and that speed changes uniformly or suddenly. Students also compare the motion of the ball rolling on surfaces with different friction, and come to see friction as a force.



Right. Students preparing the measure a ball rolling down a track.
Centre and above. A student's prediction of the path of a ball on a sloping table.

Figure 5. Students experiment with rolling balls along sloping tracks and surfaces.

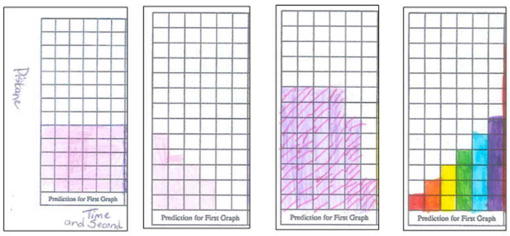


Figure 6. Five students' predictions the speed of a car pulled by unbalanced forces.

REFLECTIONS ON MATHEMATICS AND STEM

THE STEM AGENDA IS GOOD FOR MATHEMATICS TEACHING

Mathematics is a key skill set for the aspirations of a STEM agenda. The agenda of making mathematics a subject that students can really use for their own purposes, at home, at play, or at work is also an important one for all teachers of mathematics to address. Whilst many mathematics teachers are primarily attracted to mathematics because of its structural beauty, intrigue, and the pleasure of solving mathematics problems, most students and most people in the community value mathematics for its usefulness. Mathematics teaching must deliver on that.

MATHEMATICS IS NOT ONLY FOR STEM

Mathematics applies to nearly all areas of human endeavour, not only those clearly under the STEM label, and so the range of real world contexts which are important to include in teaching mathematics goes well beyond STEM. Going beyond STEM is especially helpful to motivate students whose interests and passions are outside typical STEM topics. Because of the enormous spread of digitisation of "information" (Brynjolfsson & McAfee, 2014) – digitisation of music, colour, faces, purchasing preferences, networks of friends – mathematics and logical thinking have wider applications than ever before.

MATHEMATICS CANNOT BE TAUGHT MOSTLY THROUGH INTEGRATION

The essence of a STEM unit is that the real world context drives the learning. This has advantages of seeing mathematics in action. However, the fact that it is inherently bound to a context means that such an approach does not do justice to the mathematics. In history, it has generally been the case that mathematics has arisen from the need to understand one or a few real world situations. Patterns and relationships are observed within these situations, and abstract mathematics is developed to describe and explain them, and to make predictions. This theoretical mathematics is not tied to one context but is abstract. The surprising power of the abstraction, though, is that for all the central topics of school mathematics, it has been found that the mathematics developed originally from one situation applies to many. This is illustrated in Figure 7. A STEM unit, bound to a context, can motivate mathematical knowledge and give practice in mathematical skills. However, thoroughly learning a topic requires seeing how it works with many variations, arising from different contexts, or different types of numbers, different magnitudes, different data types, etc. Good mathematics teaching must have space for students to learn how to get power from the abstraction of mathematics.

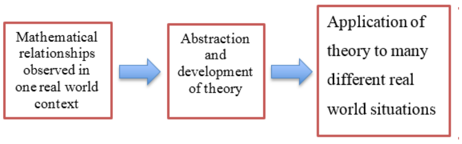


Figure 7. A new mathematics topic is often inspired by one context but applies to many.

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Evaluating the impact of evidence based practices on the teaching and assessing mathematical proficiency

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One can argue that Mathematics in much more complex than three Strands and that it involves more of a mental disposition and persistence. In order to develop this disposition and persistence, as "mathematics is what mathematicians do" (Milgram, 2007, p. 2), one must be emerged and challenged within a task. Mathematical Proficiency is one way of explaining this mathematical disposition and persistence, and through Evidence Based Practices one can begin to see what is meant by Mathematical Proficiency. Relating the learning and teaching of mathematics to the three Content Strands of AusVELS in isolation encourages our practices to have very little connection with actual mathematics. So what cognitive changes do we want to promote in children so that they can be successful in learning mathematics? Recognising that no term captures completely all aspects of expertise, competence, knowledge, and facility in mathematics, The US National Research Council has chosen the term "Mathematical Proficiency" to capture what is necessary for anyone to learn Mathematics successfully. However, questions still remain; how do Evidence Based Practices support the successful learning and teaching of Mathematical Proficiency? How can teachers accurately assess Mathematical Proficiency using Evidence Based Practices? And in what ways can students' responses be used to assess their Mathematical Proficiency?

RATIONALE

This study was conducted in a Catholic primary school in the inner suburbs of Melbourne. At the time there were 325 enrolled students with 20 classroom teaching staff. The students come from mainly European backgrounds with parents of second or third generation. The purpose of this project was to identify which evidence based practices could assist students in developing mathematical proficiency, and the most effective practices teachers could use to assess their students' mathematical proficiency. The long term objective was that teachers would use more problem based teaching and challenging tasks in their regular learning and teaching of Mathematics. NAPLAN and PAT M data had identified that the students had a surface level understanding of concepts and topics but struggled with deeper level questions. It was also discovered that students struggled to articulate the learning dispositions they used and needed, not only in mathematics but in all learning areas. The project began with the introduction of Problem Solving sessions from all years, Foundation to Six. When solving problems, students work through a Problem Solving Plan that reflects the four Mathematical Proficiency strands outlined in AusVELS (now replaced by the Victorian Curriculum F-10, http://ausvels.vcaa.vic.edu.au/). Their solutions are then assessed on a rubric which also reflects the four Mathematical Proficiency strands.

WHAT IS MATHEMATICS?

Teaching and learning of Mathematics needs to be well understood. Therefore, a shift, or change, needs to take place so that Mathematical Proficiency is developed in conjunction with the skills and strategies embedded throughout the Content Strands. This involves a culture in which Mathematics is taught, learnt, and assessed through problem based teaching and learning, in a classroom where tasks are open ended, so that there are multiple entry and exit points and where a low floor / high ceiling can be easily identified. It is where students feel challenged in their learning, not too challenged so that they are disengaged, but also challenging enough so that they don't find it boring. This challenge also needs to be supported by enabling and extending prompts, to encourage students to persist and experience success. Finally, the classroom needs to be a safe place, one where students can make mistakes and learn from them. It needs to be a classroom where their learning is not a secret, so that students are aware of where they are going, how they are going, and what their next learning steps are going to be. This self-regulated understanding of learning should also be supported with learning tools such as rubrics, goal setting, and strategy building organisers.

HOW DO WE CHANGE THE MINDSET?

Leading and embedding a change provides its own challenges. A clear vision is needed but also keeping in mind that "there is no step-by-step shortcut to transformation" (Fullan, 2002, p. 75). The goal needs to be that teachers see themselves as evaluators of their effects on students and "teachers need to use evidence-based methods to inform, change, and sustain these evaluation beliefs about their effects" (Hattie, 2012, p. 18). This type of change takes time, it cannot be rushed, and requires a lot of groundwork before delivery. The delivery of the data also requires strategic planning, as its use for formative interpretations is important for future direction.

Multiple models of change are available to school leaders, an effective one being Nicholls's (2002) Process of Management: "1 Establish a purpose, 2 Build a shared vision, 3 Develop shared plans, 4 Lead the action and 5 Evaluate the results" (p. 18). However, consideration also needs to be made in reference to individual school Annual Action Plans and School Improvement Plans. Therefore, schools needed to determine:

- What knowledge and skills do the students need?
- What knowledge and skills do the teachers need?
- How can professional knowledge be deepened and skills refined?
- How can students be engaged in new learning experiences?
- What has been the impact of the change process?

When leading change, the leader must move back and forth from transformational leader to instructional leader, and identify the appropriate times for each. Furthermore, goals need to come from necessities rather than desires, so that they can be deeply embedded into the history and culture of the school (Zaleznik, 2004). Overall, it is a learning process that has to be communicated consistently and redundantly (de Pree, 1993), and a journey in which staff need to take ownership because "if the staff members of a school are going to work together to implement change, they must be able to discuss issues and make sound decisions" (Ridden, 1991, p. 19). Once a shift in the learning and teaching of Mathematics has undergone change and it is embedded in the school culture, data can be collected to determine whether Evidence Based Practices assist in the teaching and learning of Mathematical Proficiency. So what does this look like in practice?

CONTEXTS AND INSTRUMENTS

The data was collected from Problem Solving sessions at each level from Foundation to Year 6. George Polya (1945) summarised the core steps in problem solving:

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Look back

The school's Problem Solving Plan was developed from Polya's work, but altered to suit the needs of the students and to reflect the Mathematical Proficiency Strands. It included:

1. Understand the Problem
2. Seek Relevant/Irrelevant Information
3. Select/Choose a Strategy
4. Take Action
5. Look Back/Check
6. Explain/Justify/Share

The school had developed a Problem Solving Strategy Scope and Sequence in order to ensure all strategies had been taught and learnt at an accurate developmental age. However, one could not assume that it was being followed.

Another task was to identify whether the Problem Solving tasks used were challenging tasks. One needs to keep in mind that there are verbal and non-verbal aspects to problem solving and both need to be considered. According to Van de Walle, Karp, and Bay-Williams (2013), a problem is defined as "any task or activity for which the students have no prescribed or memorized rules or methods, nor is there a perception by students that there is a specific 'correct' solution method" (p. 34). A problem is where the engaging aspect of it must be due to the mathematics, and it must require justifications and explanations. If a task or problem contains all of these features, then it can be considered a problem or challenging task, and the learner can be said to be "immersed in Problem Solving."After completing a Problem Solving Moderation task, students were asked two questions:

1. Think about the problem you just completed.

I prefer problem solving tasks to be:

- much harder
- about the same
- much easier

2. Think about the problem you just completed.

I prefer to do problem solving tasks:

- by myself
- working with other students
- by listening to the teacher's explanations first

Responses were collected and collated and percentages of each of the responses were recorded for Year 1/2, Year 3/4, and Year 5/6.

Another data source was a walk-through during Problem Solving sessions. Not only would this data identify student engagement and disposition towards mathematics but it would also provide an opportunity to observe the mathematical proficiency of students. Four classes were observed, from Foundation, Year 1/2, Year 3/4 and Year 5/6, and the data collected and recorded as percentages.

In order to assess the Mathematical Proficiency of their students, the school had developed a rubric to reflect the Proficiency Strands. Through this rubric, students' Understanding, Fluency, Reasoning, and Problem Solving could each be assessed from limited to advanced. Samples of this rubric along with corresponding work samples were collected and analysed. This process was conducted through a whole school moderation session, so that teachers could moderate within and across levels using challenging tasks which included enabling and extending prompts. Throughout the sessions discussion took place on the benefits and difficulties in using the rubric as an assessment tool, and what evidence the work samples provided.

Finally, a review of problem solving was conducted by the staff. Teachers were asked to fill out a table identifying what was working well, what could be better, how well they and their students had been using the problem solving plan, how well they had been able to assess their students against the rubric, and some recommendations that could improve these sessions. This data served multiple purposes in the research:

1. to identify teacher understanding of the proficiency strands, as they will be required to acknowledge the importance of each problem solving steps and its corresponding proficiency strand
2. to determine students' mathematical proficiency, as teachers needed to identify aspects of the plan that students found difficult

3. to assist in the development of the students' mathematical proficiency, as teachers were required to analyse ways they could improve problem solving sessions in order to support their students
4. to find out if Challenging Tasks support the successful learning and teaching of Mathematical Proficiency, as teachers needed to identify ways they could further support their students.

Data was collected and analysed from students, teachers, and tasks. This was to ensure the data reflected the three research questions that guided the project:

1. How do Evidence Based Practices support successful Learning and Teaching of Mathematical Proficiency?
2. How can teachers accurately assess Mathematical Proficiency using Evidence Based Practices?
3. In what ways can students' responses be used to assess their Mathematical Proficiency?

HOW DO EVIDENCE BASED PRACTICES SUPPORT SUCCESSFUL LEARNING AND TEACHING OF MATHEMATICAL PROFICIENCY?

EVIDENCE BASED PRACTICE: PROBLEM SOLVING STRATEGIES
EVIDENCE BASED PRACTICE: PROBLEM SOLVING PLAN

A problem solving task is only as challenging as the students' experience and, as such, students need to be immersed in a range of experiences so that they can be consistently challenged. One way of doing this is through the learning and teaching of various problem solving strategies.

Strategy	Make a list or table	Look for a pattern	Guess, check and improve	Experiment (idea and test)	Simplify the problem	Make a model	Draw a picture	Think logically	Work backwards	Act it out
Scope and sequence	F - Year 6	F - Year 6	F - Year 6	Years 5/ 6	Years 4-6	Years 5/6	F - Year 6	Years 3-6	Years 1-6	F - Year 4
Percentage of each year level using the strategy	50%	75%	75%	25%	75%	0%	100%	50%	25%	50%
Year levels	3/4, 5/6	1/2, 3/4, 5/6	F, 3/4, 5/6	1/2	1/2, 3/4, 5/6	-	F, 1/2, 3/4, 5/6	3/4, 5/6	5/6	F, 1/2

Figure 1. Problem solving strategies audit.

Figure 1 shows that teachers had not placed a large focus on the strategies being taught and learnt. It was also evident that "Draw a picture" was the most frequently used strategy. All year levels had recalled using the strategy, and it also appeared to be the one most teachers felt comfortable teaching and referring to.

Foundation	Guess, check and improve	Draw/use a picture	Act it out				
Year 1/2	Look for/use a pattern	Experiment (Idea and test)	Simplify the problem	Draw/use a picture	Act it out		
Year 3/4	Make a list or table	Look for/use a pattern	Guess, check and improve	Simplify the problem	Draw/use a picture	Think logically	
Year 5/6	Make a list or table	Look for/use a pattern	Guess, check and improve	Simplify the problem	Draw/use a picture	Think logically	Work backwards

Figure 2. Strategies used in each level.

Figure 2 displays a clearer picture of what was taking place from Foundation to Year 6. Although there was a scope and sequence of strategies, the data showed that it had not been followed, meaning that their students were not exposed to a range of challenging learning experiences. However, the data also displayed that all year levels were using a variety of strategies, and not simply focusing on a particular one or two.

HOW CAN TEACHERS ACCURATELY ASSESS MATHEMATICAL PROFICIENCY USING EVIDENCE BASED PRACTICES?

Evidence Based Practice: Rubrics

Evidence Based Practice: Problem Solving Plan

Evidence Based Practice: Challenging Tasks

The rubric (Figure 3) was created as a tool to assess the Mathematical Proficiency of students. As it evolved, it also became a tool for students to access, as it described what teachers were looking for and what success looked like. As the rubric developed so did the teachers' understanding of the Mathematical Proficiency Strands.

Content Strand:	NUM & ALG MEA & GEO STA & PROB	Name:		Year Level:	
Task:	ENABLING MAIN EXTENSION	Limited	Emerging	Developing	Advanced
Understanding	Generic	Did not use the key mathematical words and did not make connections to what they already knew.	Used some mathematical ideas, words and symbols correctly.	Made connections to some ideas and used most mathematical words and symbols correctly.	Chose, used and showed relevant ideas and connected them together. Used mathematical words and symbols correctly.
	Task Specific	*Needed help with drawing a picture of what the problem looked like.*	*Drew a picture of what the problem looked like.*	*Identified a pattern while drawing a picture of the problem.*	*Identified a pattern in the seats and rows and applied it to the problem.*
Fluency	Generic	Very few calculations were correct or did not choose the correct strategy, and therefore did not find any solutions.	Some calculations were correct, some methods were appropriate but did not select the most efficient strategy.	Used appropriate calculations, with efficient methods to find a solution with few errors and the strategy selected was efficient.	Working out was complete with no errors, calculations were presented clearly and the strategy selected was the most efficient.
	Task Specific	*Needed assistance counting the number of seats in each row.*	*Started counting each seat in each row to solve the problem.*	*Began counting each seat but soon identified a pattern and applied it.*	*Used the pattern they identified within the seats and rows to find the ticket number.*
Reasoning	Generic	Did not explain thinking and the steps taken and the representations were not made clear.	Some thinking and some of the steps taken were shown.	Most of the steps taken were shown and an explanation or justification was clear for most of their thinking and selected problem solving strategy.	The steps in the solution were shown, they used examples to explain and justify their thinking and they justified why their problem solving strategy was the most efficient.
	Task Specific	*Could not explain how they solved the problem.*	*Needed help explaining the steps they took and/or the strategy they used.*	*Most of the steps they took were clear and reference to their problem solving strategy was made.*	*All steps were clearly explained, verbally or in working, with clear reference to a problem solving strategy and justified why their working was the most efficient.*
Problem Solving	Generic	Did not show how they used the problem solving plan.	Showed some planning using the problem solving plan and used some of their own thinking but needed assistance.	Showed how they used the problem solving plan along with some of their own creative thinking when attempting to solve the problem.	Clearly showed how they planned and solved the problem and correctly used the problem solving plan to make sure that they solved the problem correctly.
	Task Specific	*Didn't use the problem solving plan.*	*Showed some evidence of using the problem solving plan and their own thinking, but needed help making links to the problem.*	*Used the problem solving plan along with their own thinking and applied it when attempting to solve the problem. Looked back to reflect on the reasonableness of their solution.*	*Displayed how they planned and used the problem solving steps and reflected on how they solved the problem how how effective their working was.*

Figure 3. Original rubric.

The original rubric (Figure 3) was broken up into the four Proficiency Strands: Understanding, Fluency, Reasoning, and Problem Solving, and the students were assessed for each of these strands according to *Limited, Emerging, Developing*, and *Advanced*. Along with a generic description of each of the areas, a Task Specific description was also included. After using the rubric, the teachers developed misconceptions on the Mathematical Proficiency Strands as confusion arose as to what was being assessed — the generic statement or the task specific statement — and, as such, teachers found themselves assessing student work rather than their Mathematical Proficiency.

	Number & Algebra Measurement & Geometry Statistics & Probability	Enabling Task Main Task Extension Task	Name:	Year Level:
Proficiency Rubric	**Limited**	**Emerging**	**Developing**	**Advanced**
Understanding	Highlight none, some or irrelevant information in the question. Unable to choose strategy or chose an inefficient one.	Highlight some relevant information in the question and with support choose a strategy.	Highlight the most relevant information in the question and choose an appropriate strategy (might not be the most efficient).	Highlight all relevant information in the question, choose an appropriate and efficient strategy.
Fluency	Unable to apply the strategy. Working is unorganised or non-existent.	With support apply a strategy and adjust it according to their knowledge. Show some working.	Apply a strategy and show some evidence of adjusting it according to their mathematical knowledge. Show working that is reasonably clear and easy to follow.	Evaluate the strategy as they go to see what is working and what isn't, adjusting it according to their mathematical knowledge. Show working clearly and in an order.
Reasoning	Identify the mathematics used.	Name the strategy and identify the mathematics used.	Name the strategy used, why they chose it and how it helped answer their question. Explain the mathematics they used.	Name and explain their strategy with detail and justify and evaluate why they chose it. Explain and justify the mathematics they used.
Problem Solving	Unable to make connections from their working to the Problem Solving plan.	Use their own thinking but needs assistance identifying how they used the Problem Solving plan.	Follow the Problem Solving plan, along with their own thinking but may have neglected to look back to ensure they solved the problem.	Clearly plan their work, follow the Problem Solving plan and show evidence of successfully completing each step.

Figure 4. Edited rubric.

From this, the rubric was changed to include only generic statements (Figure 4).Teachers could use it with any challenging task, as well as use it to report back to students and parents about the development of a student's Mathematical Proficiency. Teachers found this more beneficial than the original rubric, however some misconceptions about the Proficiency Strands still remained. It was also determined that reasoning was difficult to assess without a task specific focus. Furthermore, the modified rubric became decomposed into to a Problem Solving Plan to be used by the students. Without teachers being able to make links between the plan and the rubric, Mathematical Proficiency was not accurately assessed.

Number & Algebra		Measurement & Geometry	Statistics & Probability	Level of Task:	Name:	Year Level:
Proficiency Rubric	Proficiency Strands	**Problem Solving Steps**	**Limited**	**Emerging**	**Developing**	**Advanced**
Problem Solving	***Understanding***	The Question	Highlight none, some or irrelevant (not key) information in the question.	Highlight some relevant/key information in the question	Highlight most relevant/key information in the question	Highlight all relevant/key information in the question
		Choosing a Strategy	Unable to choose strategy or chose an inefficient one.	With support choose a strategy.	Choose an appropriate strategy (might not be the most efficient).	Choose an appropriate and efficient strategy.
	Fluency	Take Action-Strategy	Unable to apply the strategy.	With support apply a strategy and adjust it according to their knowledge.	Apply a strategy and show some evidence of adjusting it according to their mathematical knowledge.	Evaluate the strategy as they go to see what is working and what isn't, adjusting it according to their mathematical knowledge.
		Take Action-Working Out	Working is unorganised or non-existent.	Show some working.	Show working that is reasonably clear and easy to follow.	Show working clearly and in an order.
	Reasoning	Look Back / Explain and Justify	Unable to explain their thinking or strategy	Name the strategy or explain their thinking	Name the strategy, explain in steps how they used it in the task and, mathematical steps involved and their thinking	Name the strategy, explain in steps how they used the strategy, write how the strategy worked and what mathematics they used. Explain what they would do differently, if anything (strategy, mathematics, working out)

Figure 5. Final rubric.

In order to meet these needs, the rubric was modified once again (Figure 5). The structure remained the same, however one adjustment was made. The previous rubric enabled teachers to assess their students' Problem Solving as its own proficiency strand. This rubric was altered to show that assessing Problem Solving was incorporated into assessment of the other three strands. It was also changed to reflect the links between the Problem Solving Plan and the Mathematical

Proficiency Strands. This had two purposes, one to develop teacher understanding, and the other to assist when using the rubric with the students. As such, each of the three remaining Proficiency Strands was broken up according to their link to the Problem Solving Plan, and generic statements were created to identify what teachers were assessing in each of these areas.

IN WHAT WAYS CAN STUDENTS' RESPONSES BE USED TO ASSESS THEIR MATHEMATICAL PROFICIENCY?

Evidence Based Practice: Challenging Tasks

Evidence Based Practice: Enabling and Extending Prompts

Evidence Based Practice: Student Voice

Evidence Based Practice: Exit Slips

The analysis of work samples against a rubric provided an insight into how students' responses to challenging tasks assisted teachers in assessing their Mathematical Proficiency. During the school's moderation process, discussion took place around the challenge of tasks and their impact on accurately assessing students. Furthermore, it was also noted here that enabling and extending prompts assisted in ensuring that students were actively engaged with a challenging task, as teachers were able to assess their students' mathematic proficiency against the rubric more accurately. The difficulty in assessing a student's reasoning from a work sample was highlighted and further explored.

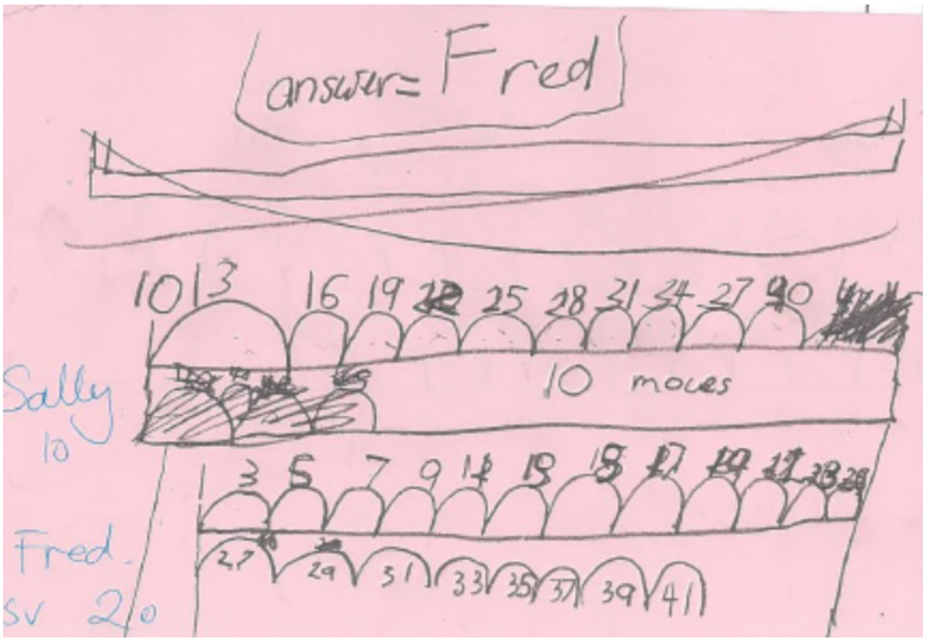

Figure 6. Work sample (Year Two).

As shown in Figure 6, work samples did show how Mathematical Proficiency could be assessed, however the use of the rubric made it possible. It must also be noted that work samples of students in Years Three to Six were easier to assess, as student thinking was more evident. Work samples from students in Foundation to Year Two still provided information, however, without observations, an accurate assessment of Mathematical Proficiency was difficult to determine. This was more evident in Foundation as some samples provided very little information.

Students need to feel challenged when completing tasks, and collecting student voice is one method of knowing how students are feeling. Through an exit slip, students were asked two questions and responses were collated (see Tables 1 & 2).

I prefer problem solving tasks to be.....			
	Year 1/2	Year 3/4	Year 5/6
much harder	17/80 = 21%	10/43 = 23%	11/54 = 20%
about the same	51/80 = 64%	26/43 = 60.5%	35/54 = 65%
much easier	12/80 = 15%	7/43 = 16.5%	8/54 = 15%

Table 1. Student Voice Question 1

I prefer to do problem solving tasks.....			
	Year 1/2	Year 3/4	Year 5/6
by myself	38/80 = 47.5%	8/43 = 18.5%	14/54 = 26%
working with other students	26/80 = 32.5%	34/43 = 79%	36/54 = 66.5%
by listening to the teacher's explanation first	16/80 = 20%	1/43 = 2.5%	4/54 = 7.5%

Table 2. Student Voice Question 2

Table 1 shows some pleasing results, as an emphasis needs to be placed on the students feeling challenged. However, across the year levels, an average of 20% of students felt as though they could be further challenged. Therefore, more research needs to be done to ensure all students are engaged in a mathematics classroom that meets their needs.

FINDINGS AND CONCLUSION

Looking back at the data presented, one can begin to identify how Teaching and Assessing the Mathematical Proficiency of students using Evidence Based Practices can be achieved. First, the tasks presented to students need to be challenging. Furthermore, in order to assess Mathematical Proficiency against a rubric, each student needs feel challenged. If the task is too easy, students are assessed as "advanced" across the Proficiency Strands. If the task is too hard, then students' Mathematical Proficiency appears to be "limited." This is further supported through the implementation of enabling prompts for students experiencing difficulty and extending prompts for students needing further challenge. Teachers articulated that once students were given a challenging task, either from the beginning or through an enabling or extending prompt, then Mathematical Proficiency could be assessed through student work samples, against the rubric.

It is also evident that students need opportunities to develop Mathematical Proficiency through a range of problem solving strategies. Through the problem solving strategies audit it was discovered that a variety of strategies had been taught and used. This enabled the students to build a knowledge bank of strategies that could be recalled when engaging in a challenging task. This toolkit of strategies was apparent in student work samples, as evidence of different problem solving strategies appeared across multiple work samples, thus making Mathematical Proficiency easier to assess as well as showing a transfer of learning.

One of the most evident outcomes of this research was the importance of a rubric. This was not only so that teachers could assess Mathematical Proficiency, but as a tool to use with students to develop a shared language. The rubric also assisted with consistency across the year levels, and supported teacher judgements throughout moderation processes.

The need for peer interactions was another integral aspect in the teaching and assessing of Mathematical Proficiency. It was evident that the change in teaching practice had promoted peer interactions, as a high percentage of students suggested that they enjoy working with others when solving problems. However, as Hattie (2012) noted:

> receiving feedback from peers can lead to a positive effect relating to reputation as a good learner, success, and reduction of uncertainty, but it can also lead to a negative effect in terms of reputation as a poor learner, shame, dependence, and devaluation of worth. (p. 147)

Therefore, a supportive classroom culture needs to be developed in order to promote positive peer interactions.

Finally, the research in this project also highlighted the importance of collecting student voice, in order to identify if students are being challenged and what can be done to ensure that they are actively engaged in Mathematics. This collection of student voice should be an ongoing practice to ensure that teaching practices reflect the needs of the students as "it is only those teachers who have the mind frame that students' perceptions are important who make the sustained efforts needed to engage students more in learning" (Hattie, 2012, p. 38).

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A matter of time

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Time can be a challenging concept. There is more to the teaching and learning of time than the reading of clocks and calendars. For deep learning to take place, four major Components of time and the relationships between these Components, need to be understood. These components are presented in a Framework for the Learning and Teaching of time that forms the basis of an assessment tool for students in the middle years of primary school.

INTRODUCTION

Time has been measured for centuries based on natural phenomena such as the movement of the sun and the stars, but now the measurement of time is more precise. Common time measuring tools are the calendar, which is used to measure and record the time the Earth takes to revolve around the Sun, and the clock, which is used to measure the time the Earth takes to rotate once on its axis. To understand the use of clocks and calendars, it is necessary to appreciate that these tools are not only used to read the current time, but also to measure the passing of time.

As the Earth revolves around the Sun we experience seasons, and, as the Earth rotates, day and night occur. Understanding these cyclic processes enables children to be actively engaged in the present, build notions of the past, and anticipate and plan for the future (Friedman, 2000; Hudson & Mayhew, 2011).

Studies into the mathematics used by Australian adults in their everyday, non-occupational lives by Northcote and McIntosh (1999) and Northcote and Marshall (2016) have shown that calculations of time were the most common context for calculation; time calculations comprised 25% of all calculations in the 1999 study, rising to 30% in the 2016 study. Our reliance on time makes it an important topic in the mathematics curriculum. While this deeper conceptual understanding of time is clearly part of a developmental process which occurs gradually from infancy to adolescence (Friedman, 2011; Piaget, 1969; Trosborg, 1982), it is important to note the distinction between telling the time and the concept of time. Children may be trained to read the dials on a timepiece, but still have difficulty in understanding the concept of time (Dickson, Brown, & Gibson, 1984).

A FRAMEWORK FOR THE LEARNING AND TEACHING OF TIME

Thomas (2018) identified four major Components[1] of time: an awareness of time, succession, duration, and the measurement of time   all considered essential for a deep understanding of time. These major Components were incorporated into a *Framework for the Learning and Teaching of Time*. Double headed arrows link the four major Components to indicate that each component is of equal importance and inter-dependent. Key ideas are listed under each major component for further clarification (See Figure 1). There is not a sequential hierarchy embedded in the Framework, as each component needs to be learned alongside every other component.

1. After a long discussion with my supervisors, the important aspects of time were listed as major Components, with the capital C.

A Framework for the Learning and Teaching of Time.

Awareness of time.

- Any event on the time continuum can be used as a reference (e.g., an occurrence, a period of time).
- The language of time includes specific terms (e.g., yesterday, tomorrow) and informal words and phrases (e.g., in a jiffy, soon).
- Temporal patterns occur with regularity (e.g., daily and weekly routines, months of the year).
- Psychological time is an individual's perception of time.

Succession.

- Two or more different events are organized sequentially.
- An understanding of succession and seriation is needed to iterate units of time.
- Events can occur simultaneously (at the same time).
- The relationships between units of time need to be understood to solve problems of succession.
- The names of days and months follow a recurring pattern while years are named in numerical order.
- Succession involves the present, the past and the future.

Duration.

- Duration is an unbroken interval of time between two successive events.
- To add, subtract, multiply and divide units of time requires an understanding of the duration of the units.
- Events can be performed in equal times (isochronal).
- The relationships between units of time need to be understood to solve problems of duration involving more than one unit.
- A unit of time is constant, being equal in length of time to any other unit of time bearing the same name.
- The duration of an event can be measured in units of time from the very small to the very large.

Measurement of time.

- The passage of time is measured in specific units which are based on natural phenomena reliant on the movement of the Earth in space (e.g., days, years).
- Units formulated to measure time more precisely have become entrenched in our culture (e.g., second, minute, hour).
- A point in time is meaningful when its position is located on the time continuum.
- To understand the measurement of time, the structure and operation of time measuring devices need to be understood.
- Scientific developments have made the measurement of time increasingly accurate (e.g., an atomic clock).
- To measure time accurately, the relationships between units of time need to be understood.

Figure 1. A framework for the teaching and learning of time.

ASSESSMENT OF STUDENTS

A one-to-one task based interview consisting of 60 items to assess students' understanding of three of the four major Components of time and key associated ideas was developed by Thomas (Thomas, 2018; Thomas, Clarke, McDonough, & Clarkson, 2017; Thomas, McDonough, Clarkson, & Clarke, 2016). The students in Years 3 and 4 were deemed to have an awareness of time. The interview had a range of anticipated responses. A scoring method based on the work of Clements and Ellerton (1995) whereby two points were given if the student's response indicated a full understanding of an item, one point for a response that indicated a partial understanding, and zero points if the student's response indicated no understanding of the item. Each item was linked to a key idea on the Framework.

The results from 27 interviews were compiled to indicate the most challenging items from the interview, and the students who were experiencing the greatest difficulty (see Table 1). By listing the students in the top row and the interview items in the first column, each student's score for each item could be displayed. Each student's score was tallied vertically (e.g., M05G3 scored a total of 91/138). Scores for each item were tallied horizontally (e.g., item 3 scored 19/54).

Interview 1	M01B4	M02G3	M03G3	M04G4	M05G3	M06B3	M07G4	M08B3	M09G3	M10B4	M11B3	M12G4	M13B3	M14G4	M15B4	M16G4	M17B3	M18B4	M19B4	M20B3	M21B4	M22B3	M23B4	M24B3	M25G4	M26B3	M27G3	Total score for class
1	0	0	1	1	1	1	1	1	0	1	1	0	1	1	0	0	1	1	0	1	0	1	0	1	0	1	0	16
2	2	1	0	2	1	2	2	1	2	2	1	2	0	1	1	1	2	1	1	0	2	0	1	2	1	0	0	31
3	1	1	1	1	0	2	2	1	0	1	1	0	0	1	1	0	0	2	0	0	1	0	1	1	0	1	0	19
4	0	1	0	1	1	2	1	1	1	2	1	1	0	1	1	1	0	0	1	0	1	1	0	0	1	1	0	20
5	1	0	0	1	0	1	1	1	1	1	0	1	0	2	1	2	0	1	1	1	1	0	0	0	1	1	1	20
6	2	1	1	2	1	2	2	2	2	2	0	2	2	2	2	2	2	1	2	2	1	2	0	2	2	2	1	44
7	2	0	0	1	0	0	1	2	0	1	0	2	0	2	1	0	0	2	2	0	0	0	0	0	2	0	2	20
8	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	54
9	0	0	0	1	0	0	2	0	0	1	0	0	2	1	0	0	2	0	0	0	0	0	0	0	0	2	0	11
10	2	2	0	2	0	2	0	0	1	2	0	2	0	1	2	2	2	0	2	1	0	0	2	2	2	0	0	29
11	2	0	0	2	0	2	0	0	2	2	0	0	2	0	0	2	0	2	0	2	0	2	2	2	2	0	0	26
Scores for Items 12 to 50																												
51a	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	0	0	2	2	0	2	2	2	2	48
51b	2	2	0	2	2	2	2	2	0	2	2	2	0	0	2	2	2	2	0	0	2	0	1	2	2	2	0	37
51c	2	0	0	2	2	2	2	2	0	2	1	1	0	1	1	2	1	2	0	0	2	0	0	1	2	2	0	30
52	2	2	0	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	0	50
53a	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	54
53b	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	0	2	2	2	2	2	2	2	52
53c	2	2	0	2	2	2	2	2	1	2	2	1	2	2	2	2	2	2	0	0	1	2	0	2	1	1	0	39
54c	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	54
54d	0	1	0	1	2	1	2	2	1	2	1	1	0	0	0	1	1	1	0	0	2	2	0	2	1	1	0	25
55	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0	1	0	6
Total	**95**	**92**	**54**	**114**	**91**	**108**	**122**	**109**	**81**	**124**	**99**	**87**	**79**	**96**	**106**	**112**	**89**	**99**	**77**	**71**	**99**	**87**	**48**	**109**	**111**	**101**	**61**	

Table 1. Student Scores for Items 1 -10 and 51a - 55

THE MATHEMATICAL ASSOCIATION OF VICTORIA

The scores for each item were ranked from lowest to highest, with the lowest scores deemed to be the most challenging. Key ideas behind items that scored less than 40 (75%) were incorporated into an eight lesson intervention.

The interview items which scored 40 or less were spread across all key ideas from the three major Components being assessed. The more challenging Succession items related to ordering events sequentially and iterating units of time. Challenging Duration items related to duration of time between successive events, and the measuring of duration using the hands of a clock. The concept that time is measured challenged many students. Other challenging Measurement items related to the relationship between the movement of Earth in space and the Measurement of time, converting between units of time, and the use of clocks and calendars to measure time. An inspection of Figure 1 will link these key ideas to the four time Components.

Lesson ideas

An eight lesson intervention covered a range of time concepts and was designed to flow from one lesson to the next as students' learning progressed. The following lessons and activities proved effective. Each lesson began with a story selected to bring the students together, introduce the focus of the lesson, introduce new vocabulary, and instigate discussion.

Round and round we go
Focus: Measurement

The rotation of the Earth on its axis and its revolution around the Sun are important concepts to understand when measuring time. The length of a day, the seasons, and the duration of a year are reliant on the Earth's movement in space.

This lesson commenced with the book *On Earth* (Karas, 2005) which described the movement of the Earth in space. To demonstrate both the rotation and revolution of the Earth, the children modelled the movement of the Earth and the Sun in our solar system using a large fit-ball covered in yellow plastic and an inflatable beach ball printed with the world's continents. A bright light represented the Sun's light.

While the Sun remained in the centre of the room and illuminated the Earth, the Earth was carried around in a large elliptical shape by a student. During the re-enactment, the remaining students were prompted to talk about their observations of the Earth's movements, such as rotation and revolution, the tilt of the Earth, and the seasons. The students were encouraged to consider what time could be measured by knowing the duration of the Earth's revolution and rotation (the duration of a day is measured using a clock, and the duration of the year is measured using a calendar).

At the completion of a revolution, the student holding the model of the Earth was directed to complete the circuit with a quarter turn demonstrating that one revolution is 365¼ days. As the calendar year includes only complete days, the students could see that after one year the calendar was a quarter of a day out of alignment with the revolution. The students counted and tallied each remaining quarter of a day from each revolution until four revolutions had passed and a whole day had been accumulated.

Tick, tock: Measuring seconds and minutes
Foci: Duration, Measurement

Many of the students considered the purpose of a clock was to "tell the time" at any given moment, and did not appreciate that the movement of the hands on the clock measures the time that is passing as the day progresses.

The term *Duration* was defined as "the interval between two successive events, that is, the time taken for an event or action to happen." The story *Clocks and More Clocks* (Hutchins, 1970) introduced the concept of duration. This book tells the story of Mr Hutchins who, as he moves through his house from room to room, finds that the time on his clocks is not the same, and he is at a loss to explain why this occurs.

The students worked in pairs sharing a small battery operated clock in working order. The tick of the clock is an important cue for the passing of time. After checking that their clock was working and had all the necessary features for this exercise (12 numbers, minute lines, three hands being the second, minute, and hour hands), the students were

informed that they would not be “using” or reading the numbers as they would be concentrating on the minute lines. The students had to observe what happened on the clock face in one second and answer the question, “How is the clock informing (showing, telling) you that one second has passed?” (The second hand had moved from one line to the next, the clock had ticked, and the second hand had crossed the space from one minute line to the next.)

It was important for the students to understand that they were measuring the duration of a second, which was the length of time taken for the second hand to move from one line to the next. The students counted number of spaces between the lines that had been crossed for a duration of 10 seconds, and then for a duration of 20 seconds. Students were reminded that although these lines are called the *minute* marks, they also used to mark the passing of the seconds.

Students worked in groups of three to four to investigate the movement of the minute hand for 60 seconds duration, and to use the minute markers to count and record the minutes passing. Discussion between the students was encouraged.

Fifteen minutes of fame
Foci: Duration, Measurement

Students worked in pairs to develop a timetable allowing each student in the room to enjoy 15 minutes of fame. Many students developed their timetables beginning on the hour, half hour and quarter hour times, as seen in Table 2. Of interest is the students’ self-correction of 12:60 to 1:00.

Start	Name	Famous for	Finish
12:00	01B4	Astronaut	12:15
12:15	19B4	Shops	12:30
12:30	21B4	Teaching	12:45
12:45	04G4	Being pretty	~~12:60~~ 1:00
~~12:60~~ 1:00	26B3	Inventing	1:15
1:15	Student not in study	Singing	1:30
1:30	18B4	Robber	1:45
1:45	12G4	Swimming	2:00
2:00	07G4	Maths	2:15
2:15	15B4	Scary	2:30
2:30	Student not in study	Singing	2:45
2:45	11B3	Tunes	3:00

Table 2. A 15 Minutes of Fame Timetable

From time to time
Focus: Measurement

Eight one-minute sand timers were placed on the table. The teacher explained that the sand timers had to be checked as the manufacturer had made them cheaply without considering their accuracy. The task was to determine the duration of time for the sand to run through each sand timer, and to record the results. This activity encouraged the developing skills of reading seconds and minutes on a clock. The sand timers were found to produce both inaccurate and inconsistent results.

ANALOGUE V DIGITAL

This question has been asked recently: Do we really need analogue clocks, with the prevalence of digital clocks in our houses and work places? This question overlooks the prominence of analogue clocks that are still prevalent in public spaces. When students are learning about time, an analogue clock has a number of distinct advantages over a digital clock. Analogue clocks allow us to:

- ‘See’ the passing of time (Awareness of time).
- ‘Hear’ the passing of time (Awareness of time).
- Observe that durations of each hour (minute/second) are equal (Duration).
- Observe the sequencing of hours (Succession).
- Measure the duration of seconds/minutes/hours (Measurement of time).

CONCLUSION

Undue focus is given to the reading of clocks and calendars with little emphasis on the Components of time. Research by Thomas (2018) indicates a need for students and teachers to be given the opportunity to use the Framework for the Learning and Teaching of Time and to develop a deep understanding of the major Components and their interrelationship.

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Reviewed papers

"Take a chance on me!"

Lewis Gunn, Red Cliffs Secondary College

CONTEXT

The following is an overview of the implementation of an inquiry based learning approach in a year 8 mathematics class at Red Cliffs Secondary College in Victoria. The year 8 class has 22 students, one student with an Individual Education Plan (IEP) and a variety of mathematical abilities (On Demand score range of 5.4). In 2018, the implementation of inquiry based learning was incorporated into the school's year 7 to 10 maths study design. All students are required to complete a three tiered project or assignment each term as a component of their maths studies. This overview focuses on a year 8 probability project where students were asked to create a board game by combining their knowledge of probability and pre-existing board games. Whilst a key reason for implementing the inquiry based learning approach into the mathematics classroom in this manner was to maximise the engagement of all students, the approach was found to be particularly successful in targeting those students who were disengaged from the more traditional maths classroom practices.

BENEFITS OF IMPLEMENTATION

- Maximise engagement of all students
- Successfully target students who are disengaged from their learning
- Provide multiple points of entry and exit that align with the curriculum
- Develop enterprise skills
- Assess students' achievement against a range of different criteria
- Provide experiential learning with a focus on creating (higher order thinking)
- Make student learning visible
- Enable a student voice

HOW THE INQUIRY BASED LEARNING APPROACH WAS IMPLEMENTED

The three tiered project with multiple exit points was designed to be both accessible to all students in the class and to enable all students to succeed. Students worked on the project over a week. At the project's commencement, students were provided with clear instructions and criteria to follow. This provided a framework to scaffold and the support them in their work without encroaching on the goal that they be as creative as possible. Throughout the week, students were asked to assess their own work against the set criteria and evaluate their performance and progress. Additionally, students were asked to assess each other's work and offer feedback based on the set criteria. This resulted in students solving problems that arose for not only their projects but also for each other's. Furthermore, students were often observed having rich mathematical discussions surrounding the design and outcomes of each other's projects.

KEY POINTS OF IMPLEMENTATION:

- A 3 tiered assignment or project
- Learning intention and success criteria differentiated to reflect the 3 tiers
- Project implemented over one week

- Focused on the process of creating rather than the end product
- Focused on the reflection process
- Peer assessed
- Self-assessed
- Critical analysis of process

"Enterprise skills are transferable skills that enable young people to engage with a complex working world and which have been found to be a powerful predictor of long-term job success" (FYA, 2017, p. 8).

POST IMPLEMENTATION FINDINGS

It was evident that after the students had completed the project that they were able to articulate the problems they had faced and how they overcame them. They demonstrated a thorough understanding of the learning along with a hunger to understand more about probability. It was clear that throughout the process students were engaged. One teaching colleague who observed the class as they worked on their projects commented, "I have never seen Toby engage with a task so deeply." The On Demand data for the class over the term showed a substantial average growth of 0.64.

KEY FINDINGS:

- Engagement levels increased
- Students were able to articulate problems they faced and the ways in which they planned to solve them
- Students showed a greater understanding of probability and the implications this has in a real life context
- Meaningful discussions arose from the group work and the development of enterprise skills was evident
- All students either met or exceeded the achievement standard

While the project served as a productive assessment task for the class, the data of student learning and engagement informed the faculty on how to expand their pedagogical approach to planning innovative teaching and learning experiences. The project created "teachable moments" which were student driven. An example of this was students wanting to use multiple probability elements within the game: At once this resulted in students asking how to calculate the probability of two events happening. These "teachable moments" were valuable as it expanded student curiosity and enabled student voice.

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Algorithmic and computational thinking using R software

Nazim Khan, St Mary MacKillop College, Canberra

AN INTRODUCTION TO R

To install R on your computer go to the home website of R: http://www.r-project.org/ and do the following (assuming you work on a windows computer):

- click download CRAN in the left bar
- choose a download site
- choose Windows as target operation system
- click base
- choose Download R 3.0.3 for Windows and choose default answers for all questions

SOME EXAMPLES OF R COMMAND

R as a calculator:

```
> 10^2+36
```

R can be used to assign or reassign values to variable:

```
> a
> a*6
> a=a+7
> rm(a)
> rm(list=ls())
```

R can be used to perform operations with Scalars, Vectors and Matrices:

```
> v=c(5,7,9)
> mean(x=v)
> rnorm(20)
> rnorm(20, mean=4.5, sd=1.5)
> x=rnorm(100)
> plot(x)
> help(rnorm)
> example(rnorm)
```

More on vectors using R:

```
> v1=c(1,3,5,7,9)
> v1
> v1[5]
> v1[3]=8
> v1
> v2=seq(from=0,to=2,by=0.5)
> v2
> sum(v1)
> v1+v2
```

Matrices using R:

```
> m=matrix(data=c(2,4,6,8,10,12),ncol=3)
> m
> m[2,3]
> m[1,]
> mean(m)
```

Graphics using R:

```
> plot(rnorm(200), type= "l", col= "blue")
> hist(rnorm(200))
```

Programming using R:

```
> k=4
> if(k<6)
   {
   m=3
   }else{
   m=11
   }
> m

> p=c(1,2,3,4)
> q=c(5,6,7,8)
> h=p[q==5 | q==8]
> h
```

For-Loop using R:

```
> m=seq(from=1, to=10]
> n=c()
> for(i in 3:12)
      {
      n[i]=m[i]*14
      }
> n
```

Functions using R:

```
> f=function(x,y)
      {
      z=x^2
      return(y+z)
      }
>f(x=6,y=8)
```

Real-world data analysis using Google Sheets

Nazim Khan, St Mary MacKillop College, Canberra

UNIVARIATE AND BIVARIATE DATA ANALYSIS

The following investigation task will be used to demonstrate the implementation of Google Sheets to undertake Univariate and Bivariate Data Analysis: *Determinants of Economic Corruption: A Cross-Country Data Analysis*

The data for this project was obtained from the following web link:
http://www.sscnet.ucla.edu/polisci/faculty/treisman/Papers/what_have_we_learned_data.xls

A refined version of this data will be provided during the workshop.

1. Is there an association between economic corruption and political rights?

a) Construct an appropriate segmented bar chart to compare economic corruption levels (Low, Moderate, High) between countries with varying levels of political rights.

b) Summarise the information in the chart to answer the research question: **Is there an association between economic corruption and political rights? Explain your answer.**

2. What is the best predictor for economic corruption?

a) Calculate the summary statistics for each variable under study.

b) Construct a histogram for economic corruption and describe the distribution.

c) Calculate pairwise Pearson's Correlation Coefficient, r.

d) From the results in (c) determine:

I. Which explanatory variable is the best predictor for economic corruption? Explain. why? Provide a plausible reason why this explanatory variable could be the best predictor for economic corruption.

II. Identify three cases where there is a strong correlation ($|r| > 0.80$) between the explanatory variables and provide a plausible explanation for the result obtained.

e) Further investigation of the relationship between economic corruption and the explanatory variable identified in part (d) (I).

I. Construct a scatter plot with a superimposed trend line.

II. Fit a Least Squares Regression Model to the data. Interpret the results.

III. Construct a Residual Plot. Interpret the result.

I V. Calculate the Coefficient of Determination. Interpret the result.

f) Include other explanatory variables in the model fitted in part (e) (II) above to fit a Multiple Linear Regression Model. What is the effect on the coefficient of determination?

g) The intercept of the linear model in part (e) above can be made meaningful by centering the explanatory variable about its mean and then fitting a new Least Squares Regression Model. Perform this task, state the equation of the new model and interpret the intercept. What do you notice about the slope parameter?

3. Is the effect of GDP on economic corruption same in countries with varying levels of political rights?

a) Construct **separate** scatter plots (each with trend line, regression equation and coefficient of determination) to examine the association between economic corruption and GDP between countries with **greater** political rights and those with **fewer** political rights.

b) Based on your results in part (a), is the effect of GDP on economic corruption same in countries with varying levels of political rights? Provide a plausible explanation for the observed result in context of the data under study.

4. Is there an association between regime type and economic corruption?

a) Construct a scatter plot (with trend line, regression equation and coefficient of determination) to examine the association between economic corruption (numerical variable) and regime type (binary variable – use the variable 'RegimeType' where 0 = less democratic and 1 = more democratic).

b) State the regression equation. Interpret the results.

c) Based on the results in parts (a)-(c) is there an association between regime type and economic corruption? Provide a plausible explanation for this relationship/no-relationship.

TIME SERIES DATA ANALYSIS

A real-world time series data and Google Sheets will be used to demonstrate the construction of time-series plot, smooth time series data by using a simple moving average, calculate seasonal indices by using the average percentage method, deseasonalise a time series by using a seasonal index, fit a least-squares line to model long-term trends in time series data.

Challenge, persist, and share

Stacey Lamb, St. Bernard's Primary School, Wangaratta

With the right amount of challenge, a positive mindset and the platform to share their learning and engage in ideas and strategies, students can learn at a high level. This paper is about the success of challenging mathematics pedagogy, students' positive mindset in mathematics, and how students' sharing ideas and their learning has changed the mathematics classroom.

CHALLENGE IN MATHEMATICS

The term *challenging* applies to tasks that require students to process multiple pieces of mathematical information simultaneously and make connections between them, and for which there is more than one possible solution or solution method (Sullivan & Davidson, 2014). Challenging maths tasks are given at the start of the lesson without prior explicit teaching. This concept is built around the belief that students learn mathematics best when doing it for themselves and using strategies that work best for them. Sullivan et al. (2014) explain that when teachers give students an opportunity to explore a concept prior to an explanation, no matter whether the teacher or the students provide the explanation, the students think more deeply about the content.

Students are asked to work on a challenging task silently for a small amount of time (e.g., 5-10 minutes) before then looking and talking with peers. While this is happening, the teacher is roaming the room looking at the students' mathematical strategies and thinking. The Organisation for Economic Co-operation and Development (OECD, 2014) states that students learn mathematics best when they are engaged in building connections for themselves between mathematics and their world.

Traditional mathematics classrooms have an approach of taking students from knowing how to do the mathematics up to teachers making the questions more difficult until students don't know how to find the answer. In contrast, the pedagogy of challenging tasks allows students to go from not knowing to knowing the answer through persistence and building connections. The OECD (2014) states that giving a problem that requires students to think for an extended time, and presenting problems without obvious ways of arriving at a solution, helps students to learn from their mistakes, thereby increasing student drive. This learning and teaching approach builds confidence and allows students to think for themselves.

FOSTERING MINDSET AND ENCOURAGING PERSISTENCE

Challenge requires persistence and persistence is a skill for life. Sullivan et al. (2014) state in their research findings that students prefer maths to be challenging. Research undertaken by Carol Dweck and Jo Boaler has provided evidence linking positive growth mindset and mathematics achievement. In their work, both Boaler (2016) and Dweck (2006) state that valuing mistakes has changed the mathematics classroom. Teachers and students valuing mistakes and exhibiting a positive growth mindset can have incalculable benefits. Boaler states that when people change their mindsets and start to believe that they can learn at high levels, they change their learning pathways and achieve higher learning. Teachers can help develop positive mindsets by modelling behaviour and taking risks, as well as creating a positive classroom culture and allowing students to have opportunities to struggle and make mistakes.

For too long students have had strong and often negative feelings around mathematics. By changing the mathematics classroom and developing a positive culture these perceptions can change. Allowing students to use resources and any effective and efficient mathematical strategies to solve the problem can help. Teachers can teach and develop positive mindsets by changing the tasks from closed to open questions, and modelling and valuing risk taking behaviours. Jo Boaler (2016) explains that good teachers have told students that mistakes are useful because they show learning, however new brain science evidence says that the brain sparks and grows when a mistake is made.

MATHS TALK/ SHARING THE LEARNING

Communication is an essential component of learning mathematics, and student talk is a powerful tool for students to communicate and reason about their mathematical ideas (Way & Bobis, 2017). Sharing their ideas and mathematical strategies empowers the student who is sharing, and can provide new learning opportunities for the students listening. Students sharing their mathematical understandings can support their peers' learning, as it reveals understandings or misconceptions, and repeats and reinforces important ideas, deepens understanding, develops mathematical language, and allows for elaboration or clarification (Way & Bobis, 2017).

Reasoning is one of the proficiencies in the Victorian Curriculum, which refers to students developing an increased capacity for logical, statistical, and probabilistic thinking and actions. (VCAA, 2016). Students are reasoning if they are explaining their thinking, justifying strategies, transferring learning, proving they are right or wrong, making inferences and comparing. The reasoning needs to be their own. Students are more likely to reason if they have developed a strategy, connection, or justification for themselves (Sullivan et al., 2014).

In the mathematics classroom, teachers choose students to share their work and thinking. Selecting and determining which student and idea the teacher will focus on during the discussion is crucial. According to Smith and Stein (2011, p. 43): "Selecting can be thought of as the act of purposefully determining what mathematics the students will access and help build on their mathematical understanding." By selecting, the teacher has control over the discussion and ensures mathematics learning is the main focus of the discussion. The teacher always chooses the student, so that no surprises or 'off track' topics are brought up. The teacher is in full control and can reinforce the mathematical strategy and learning explained, and encourage other students to experiment and try what they have heard. Humphreys and Parker (2015) see this as a dance between supporting and stretching students mathematical understanding. Smith and Stein (2011) refer to this as the connection stage, where teachers focus on the mathematical meaning and relationship, and make links between mathematical ideas and representations.

CHANGE IN THE CLASSROOM

A strong positive classroom culture that values challenge and persistence, and welcomes student sharing has resulted in mathematical growth and a change in mindset from both teachers and students towards the curriculum area. In many of our classrooms, our class discussions now have a strong focus on how the students arrived at the answer and the feelings of struggle and challenge that were felt. Students will speak of the mathematical strategy they used to get an answer, and will listen carefully to other students to get ideas of another strategy that they could try. A major focus on a positive maths mindset for teachers and students created change in the school. Teachers learnt about mindset and created lessons and discussions about what mindset is and how it can be positive and help in the learning.

During a student voice survey on mathematics, a group of Grade 5 and 6 students expressed their enjoyment and success in mathematics, stating, "I feel like I am getting better at maths and I can learn from anyone in the class', another quoted saying: "I like the challenge and then the feeling I get once I have solved it and learnt something." Evidence of a positive growth mindset can be seen beyond our mathematics classroom. Recently, a pupil at our school who loves playing football injured his ankle coming into the finals. He stated to the class that he "was in the pit and struggling, but would be ok and will learn a lot from this experiences." In the classroom, persistence has now become normal behaviour. Students accept and look forward to the challenge, and embrace it by simply trying their best while using different mathematical ideas and strategies. Students share their thinking proudly and listen intently to others, looking for ideas and new strategies.

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Numerating the community: What is mathematics?

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When asking a general member of the community "What is Mathematics?," we receive various kinds of answers such as: Maths is about money and shopping; it is about measuring things; it is about problem solving; I don't know, Maths is not my favourite subject; some people may even say "I hate Maths."

When asking students, at any level, the same question "What is Mathematics?," we also get various answers such as: It is about equations, fractions, numbers and sums, money, shopping, problem solving, and so on.

In this paper, I would like to share with my readers: parents, educators, and students, some ideas from my own teaching experience.

INTRODUCTION

Firstly, all the above answers are correct, including the response: "I hate Maths." That's why we should say the word *Mathematics* in full and not *Maths*. Mathematics is a way of thinking where we use patterns and logical reasoning to find answers to questions, solutions to problems, and also to make predictions. Mathematics, I believe, is the science of problem solving since humankind first walked on the Earth.

Secondly, we need to make Mathematics more fun, more productive and more attractive to everyone, and we should not consider it as a subject for study at all educational levels only, but as a part of our community cultural activities. General community members don't need to know Pythagoras's theorem, Matrix theory, advanced algebra, nor the sequences and series theories to deal with daily problems that involve numbers, measurements, money and business, and so on.

However, as educators and community members, we need to build and develop the numeracy culture within the wider community. In that, I would like to share with you the following Mathematical situations or problems (some of it are real situations).

SOME STORIES

Story #1: My neighbour Bill was faster than me in calculating the total scores for his favourite AFL team and my team's scores (he was winning). While I was busy writing and multiplying matrices, Bill said for each team to: "Multiply the number of goals by 6 and then add it to the number of behinds."

Story #2: One day I gave my Year 8 students this home work:

Calculate the total number of "blades of grass" in your backyard. Hint: ask your parents for assistance.

Some parents became angry and said to their kids, "Tell your teacher to come and count it himself;" some other parents said, "What a funny teacher!". However, a few parents and students were very calm and reasonable by suggesting that they could consider a small square in the backyard (e.g., 10 cm by 10 cm), count the number of blades of grass in this square and then multiply the answer by the area of the backyard.

Story #3: One of my favourite Mathematical riddle given to students and parents is:

"Is seven an odd number or an even number"? Most of them would say that seven is an odd number. Then I ask: "How can you make seven **even**"? I normally receive so many different numerical solutions, but none of these solutions is correct because the answer is not numerical!

Story #4: My student-teacher Josh brought the following puzzle to class (one he learnt from his father):

Read this statement l0l0l0 = 950 and state if it is a true statement or a false statement.

Of course, it is false. You are allowed to add one single line to make it a true statement e.g., I, \, /, --.

Only a few students solve this puzzle, and only after a short discussion on the ways of reading numbers such as “nine fifty” for the number 950 (which could, for example, be the time). What do you think?

Story #5: Add all numbers from one to one hundred. Of course, you can use the arithmetic sequence sum formula to get that. But students and people in general may notice, when considering the set of numbers 1, 2, 3, ... to 10, that 1 + 10 = 11, 2 + 9 = 11 etc… So there are 5 sums of 11 and the sum of the numbers from 1 to 10 is 55. Therefore, they may generalise that to the big sum 1 to 100, giving a total of 5050.

Story #6: One day I was presenting at the Mathematics Conference in Canberra. I started my presentation with the following question (as a warm up activity): Can you fit 10 horses into 9 stables?

The surprise was that only one teacher got the correct answer after I drew the stables (see Figure 1):

Figure 1. Drawing the empty stables.

The teacher wrote (see Figure 2):

T	E	N	H	O	R	S	E	S

Figure 2. Solving the stables problem.

The teacher received a bar of chocolate as a present from me and I got some funny comments from the audience such as: “Do you have to ask this silly question?”

Story #7: I learnt the following story from a university colleague in Melbourne. He learnt the story himself from a Bedouin in Saudi Arabia when he was working with ARAMCO. The story goes: A man had three sons and he had left a will to them. The will stated that the older son will receive ½ of the heritage. The second son will receive ¼ and the youngest son will receive 1/5 of the heritage. However, after the death of the father, they found 19 camels for them to share according to that will. The three sons didn’t know how to divide the 19 camels between them according to the will.

Do you know how to solve this problem mathematically or do you want to hear how the Bedouin solved it?

Don’t laugh! You may need to go to Saudi or the Central Australia and borrow (only borrow) one camel. This may help you to solve the problem: Because the number 19 is not divisible by 2, 4, or 5, the number 20 is the closest number that you can divide by 2, 4, and 5. Finally, check the results and make a comment.

Story #8: You have attended a football match in your local area. You paid $16 dollars for two adult tickets and 3 kids’ tickets. Your friend paid $19 for 3 adult tickets and 2 kids’ tickets. You noticed that the old man, who sells the tickets, did not have a calculator but he was using a Chart (Grid) in front of him. Can you complete the grid (Figure 3) to show the prices for all possible combinations (up to 10 by 10) for adults’ tickets and kids’ tickets. By the way, what is price of each type of ticket?

ADULT TICKETS	9										
	8										
	7										
	6										
	5										
	4										
	3		$19								
	2			$16							
	1										
	0										
		0	1	2	3	4	5	6	7	8	9
	KIDS TICKETS										

Figure 3. Chart showing ticket prices

Story #9: Every student will shake hands with every other student in the class. What is the total possible number of handshakes that 25 students in class may have?

I believe that this is one of the most engaging Maths problems that you may present to your students at all levels. Give the students the opportunity to practise in class. Let them work in groups, shake hands, draw pictures, and record the results. They may come to you with some unexpected ideas, such as counting the number of diagonals in various polygons, or adding series of numbers, e.g. $4 + 3 + 2 + 1$; $7 + 6 + 5 + 4 + 3 + 2 + 1$, etc… Let students think and come up with many possible solutions, encourage them to present their results in a table of values, check patterns, and see if they can find a rule that connects the relation between the number of students and the number of handshakes. Encourage them also to use Microsoft *Excel*, which may help them to graph their gathered data and think of a possible and desired relation (*Excel* will lead you to that relation).

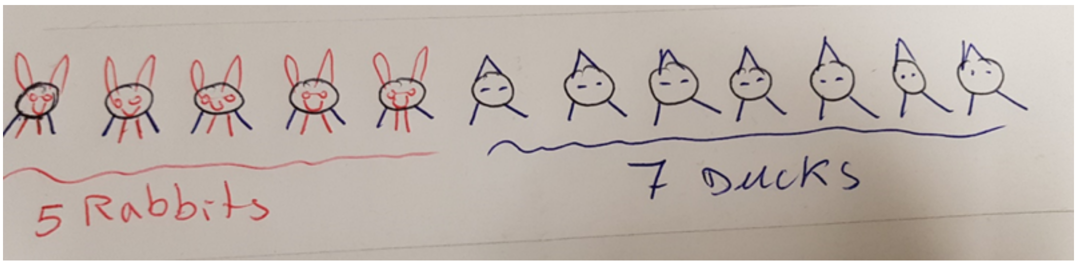
For students at year 10 and above, encourage them to use arithmetic series' properties to find the total number of all combinations of two for any group of people.

Story #10: In my backyard, I have rabbits and ducks. I counted 12 heads and 34 legs. How many rabbits and how many ducks were there?

Some students may use the guess and check method (trial and error method) to find the answer, or may write equations, and some other students may even draw pictures to visualise the problem.

What can you do to solve this problem? Give your students opportunities to present their ideas by using various strategies in solving such a numerical problem including the use of algebra and solving simultaneous equations.

Some students in grades 5, 6 and 7 used a drawing strategy to solve this problem.



As you may notice, all of these problems can be presented to students at all levels; you need to set up questions that suit their abilities and your academic expectations at that level, even when using the same problem. Students should be able to use their abilities and their mathematical skills to develop various strategies in solving the same problem; tell them that there is no "magic and unique strategy" that can be used to solve all mathematics problems even in the same area of mathematics.

However, mathematics teachers need to develop the culture of problem solving in their class. Start your class with a warm-up activity related to the topic that you are teaching, posing a relevant problem for students to solve. It is preferable that the problem is an open-ended problem or one that can be solved in different ways. From time to time give students homework that includes problem solving and encourage them to share this problem with parents and other family members. Furthermore, encourage students to come up with their own mathematics problems and share ideas and solutions with the class.

Finally, devote more time in your class to problem solving activities, and discuss the idea of using the problem-solving approach to solve mathematical problems. For your students, encourage them to think out of the box when dealing with any mathematical problem.

Remind them to use always this wisdom: Think, Do, and Link.