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THE MATHEMATICAL ASSOCIATION OF VICTORIA

Meet the Assessors

Math Methods

2019

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Background and Contact details

THE MATHEMATICAL ASSOCIATION OF VICTORIA

- Trinity Grammar School, Kew
  - Senior Mathematics Teacher
  - Teaching Math Methods more than 20 years
  - mcnamaraa@trinity.vic.edu.au
- Chief Assessor Mathematical Methods Exam 2
  - 2006 onwards

MAV Meet The Assessors 2019 MM

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- The MAV has made the 2019 MAV Solutions to 2018 VCAA MM exams materials available as downloadable files.
- The files have been zipped so they can be downloaded easily. Files can be downloaded from:

http://www.mav.vic.edu.au/download/solutions_2018_VCAA_exams/Maths-Methods/

Username: methodical

Password: routine

Message from MAV

- *“A copy of the presenters PowerPoint will be made available to all workshop session attendees. At the conclusion of the workshop series (towards the end of Term 1) the PowerPoints will be uploaded to the MAV website downloads area. All participants will then receive an email with a link (similar to that for Exam resources) with a username and password allowing them to download the PowerPoint.”*

Assessment Report for Exam 2

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- ..\..\Math Methods 2018\MM2 examination report_forauthor_Nov18_30012019 (002) 31 January.docx

Scaling 2018

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- Mean 33.9 (34.0) and SD 8.4 (8.4)
- 20 (21)
- 25 (28)
- 30 (35)
- 35 (40)
- 40 (45)
- 45 (49)
- 50 (51)

Multiple Choice 2018

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- 95% answered Question 1 and 88% answered Question 2 correctly.

Question	Answer	Question	Answer
1	C	11	C
2	A	12	E
3	D	13	E
4	C	14	B
5	A	15	E
6	D	16	B
7	B	17	C
8	B	18	E
9	C	19	C
10	C	20	A

Question 1   C   95%

Question 1

Let $f : R \to R,\ f(x) = 4\cos\left(\frac{2\pi x}{3}\right) + 1.$

The period of this function is

A. 1

B. 2

C. 3

D. 4

E. 5

Question 2 A 88%

Question 2

The maximal domain of the function f is $R\backslash\{1\}$.

A possible rule for f is

A. $f(x)=\dfrac{x^2-5}{x-1}$

B. $f(x)=\dfrac{x+4}{x-5}$

C. $f(x)=\dfrac{x^2+x+4}{x^2+1}$

D. $f(x)=\dfrac{5-x^2}{1+x}$

E. $f(x)=\sqrt{x-1}$

E Question 13 (C 15) 59%

Question 13

In a particular scoring game, there are two boxes of marbles and a player must randomly select one marble from each box. The first box contains four white marbles and two red marbles. The second box contains two white marbles and three red marbles. Each white marble scores -2 points and each red marble scores $+3$ points. The points obtained from the two marbles randomly selected by a player are added together to obtain a final score.

What is the probability that the final score will equal $+1$?

A. $\frac{2}{3}$

B. $\frac{1}{5}$

C. $\frac{2}{5}$

D. $\frac{2}{15}$

E. $\frac{8}{15}$

Let W represent a white marble and R a red marble.

The only options are WW, WR, RW or RR.

To score $+1$ would require WR or RW.

$$\Pr(WR) + \Pr(RW) = \frac{4}{6}\times\frac{3}{5}+\frac{2}{6}\times\frac{2}{5}=\frac{8}{15}$$

D Question 6 (E 20) 58%

Question 6

Let f and g be two functions such that $f(x) = 2x$ and $g(x + 2) = 3x + 1$.

The function $f(g(x))$ is

A. $6x - 5$

B. $6x + 1$

C. $6x^2 + 1$

D. $6x - 10$

E. $6x + 2$

$$f(x) = 2x,\ g(x+2) = 3x+1$$

$$g(x) = 3(x-2)+1 = 3x-5$$

$$f\left(g(x)\right) = 2(3x-5) = 6x-10$$

E Question 12 (A/C 13) 58%

Question 12

The discrete random variable X has the following probability distribution.

x	0	1	2	3	6
$\Pr(X = x)$	$\frac{1}{4}$	$\frac{9}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{3}{20}$

Let μ be the mean of X.

$\Pr(X < \mu)$ is

A. $\frac{1}{2}$

B. $\frac{1}{4}$

C. $\frac{17}{20}$

D. $\frac{4}{5}$

E. $\frac{7}{10}$

$$\mu = \mathrm{E}(X) = \frac{9}{20} + \frac{1}{5} + \frac{3}{20} + \frac{9}{10} = \frac{17}{10}$$

$$\Pr(X < \mu) = \Pr\left(X < \frac{17}{10}\right) = \frac{1}{4} + \frac{9}{20} = \frac{7}{10}$$

C Question 9 (A 15) 57%

Question 9

A tangent to the graph of $y = \log_e(2x)$ has a gradient of 2.

This tangent will cross the y-axis at

A. 0

B. –0.5

C. –1

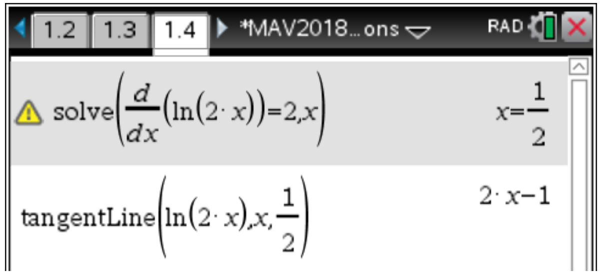
D. $-1 - \log_e(2)$

E. $-2\log_e(2)$

$$y = \log_e(2x)$$

$$\frac{dy}{dx} = \frac{1}{x} = 2,\ x = \frac{1}{2},\ y = 0$$

Tangent: $y = 2x - 1$, y-intercept is -1



E Question 15 (D 19) 49%

Question 15

A probability density function, f, is given by

$$f(x)=\begin{cases}\frac{1}{12}(8x-x^3) & 0\le x\le 2\\ 0 & \text{elsewhere}\end{cases}$$

The median, m, of this function satisfies the equation

A. $-m^4+16m^2-6=0$

B. $-m^4+4m^2-6=0$

C. $m^4-16m^2=0$

D. $m^4-16m^2+24=0.5$

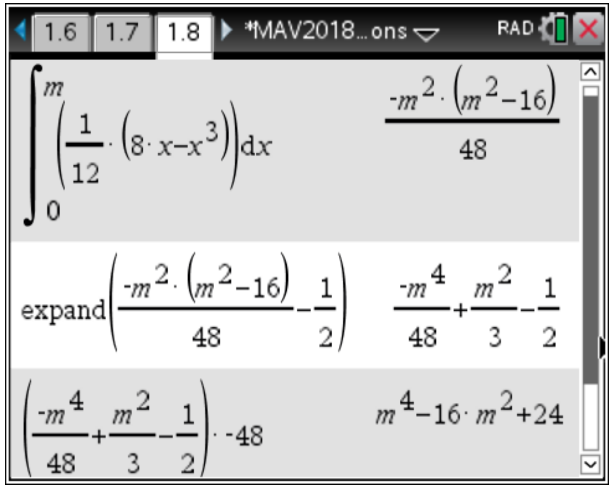
E. $m^4-16m^2+24=0$

$$f(x)=\begin{cases}\frac{1}{12}\left(8x-x^3\right) & 0\le x\le 2\\ 0 & \text{elsewhere}\end{cases}$$

$$\int_0^m f(x)dx=\frac{1}{2}$$

$$-\frac{m^4}{48}+\frac{m^2}{3}-\frac{1}{2}=0$$

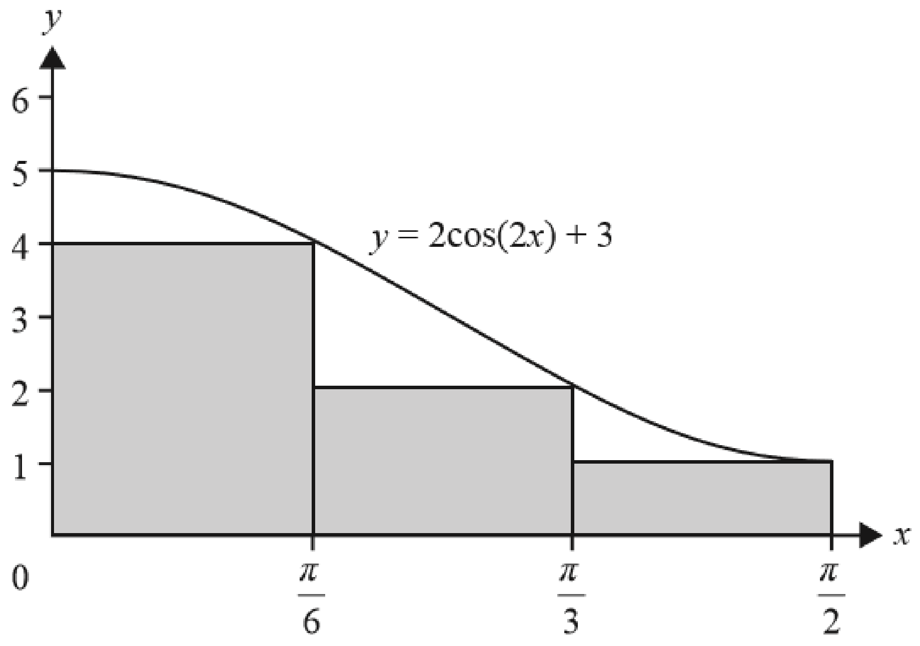
$$m^4-16m^2+24=0$$



B Question 16 (C 18) 49%

Question 16

Jamie approximates the area between the x-axis and the graph of $y = 2\cos(2x) + 3$, over the interval $\left[0, \frac{\pi}{2}\right]$, using the three rectangles shown below.



B Question 16 (C 18) 49%

Jamie's approximation as a fraction of the exact area is

A. $\frac{5}{9}$

B. $\frac{7}{9}$

C. $\frac{9}{11}$

D. $\frac{11}{18}$

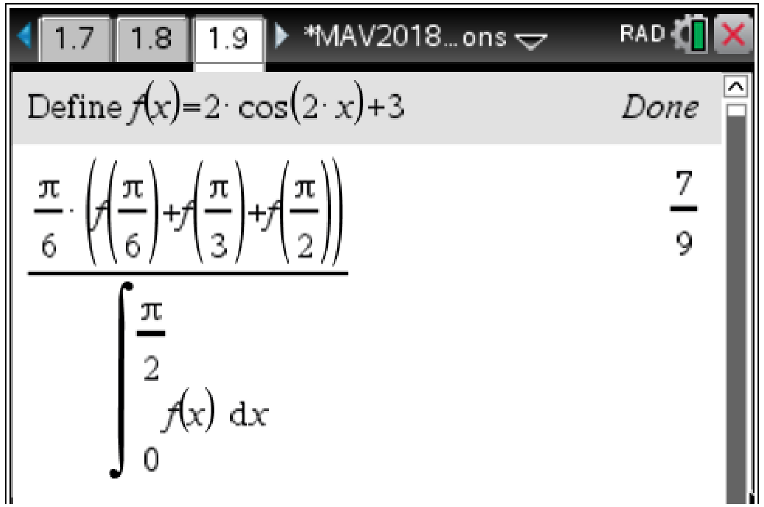
E. $\frac{7}{3}$

Let $f(x) = y = 2\cos(2x) + 3$

$$\text{Area of the rectangles} = \frac{\pi}{6}\left(f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{2}\right)\right) = \frac{7\pi}{6}$$

$$\text{Actual area} = \int_0^{\frac{\pi}{2}} f(x)dx = \frac{3\pi}{2}$$

$$\frac{\frac{7\pi}{6}}{\frac{3\pi}{2}} = \frac{7}{9}$$



D Question 3 (A 34) 48%

Question 3

Consider the function $f:[a, b) \to R, f(x) = \dfrac{1}{x}$, where a and b are positive real numbers.

The range of f is

A. $\left[\dfrac{1}{a}, \dfrac{1}{b}\right)$

B. $\left(\dfrac{1}{a}, \dfrac{1}{b}\right]$

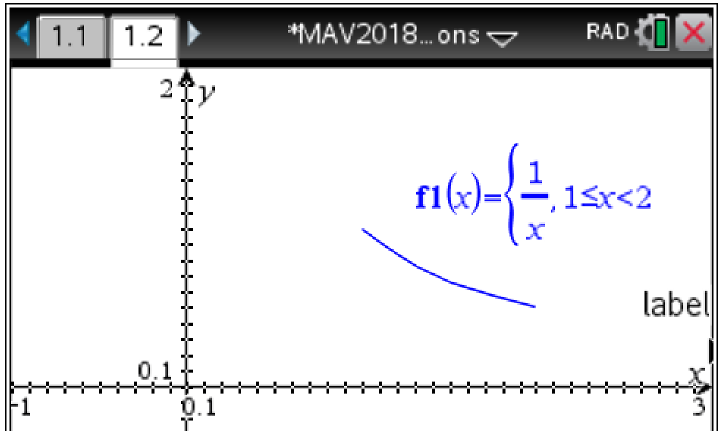
C. $\left[\dfrac{1}{b}, \dfrac{1}{a}\right)$

D. $\left(\dfrac{1}{b}, \dfrac{1}{a}\right]$

E. $[a, b)$

$$f:[a,\ b) \to R, f(x) = \frac{1}{x},\ f(a) = \frac{1}{a}, f(b) = \frac{1}{b}, f(a) > f(b)$$

Range $\left(\dfrac{1}{b}, \dfrac{1}{a}\right]$



C Question 4 (A 18) 48%

Question 4

The point $A(3, 2)$ lies on the graph of the function f. A transformation maps the graph of f to the graph of g, where $g(x) = \frac{1}{2} f(x-1)$. The same transformation maps the point A to the point P.

The coordinates of the point P are

A. $(2, 1)$

B. $(2, 4)$

C. $(4, 1)$

D. $(4, 2)$

E. $(4, 4)$

$A\ (3, 2)$, $g(x) = \frac{1}{2} f(x-1)$,

Dilate by a factor of a $\frac{1}{2}$ from the x-axis: $(3,\ 1)$

Translate 1 unit to the right: $(4,\ 1)$

C Question 17 (A 17) 45%

Question 17

The turning point of the parabola $y = x^2 - 2bx + 1$ is closest to the origin when

A. $b = 0$

B. $b = -1$ or $b = 1$

C. $b = -\frac{1}{\sqrt{2}}$ or $b = \frac{1}{\sqrt{2}}$

D. $b = \frac{1}{2}$ or $b = -\frac{1}{2}$

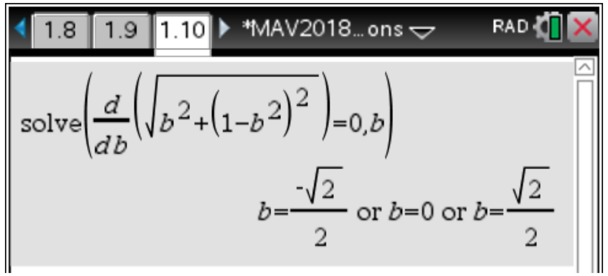
E. $b = \frac{1}{4}$ or $b = -\frac{1}{4}$

$y = x^2 - 2bx + 1$

Turning point $\left(b,\ 1 - b^2\right)$

Distance from the origin $d = \sqrt{b^2 + (1 - b^2)^2}$

$d'(b) = 0$, $b = \pm\frac{1}{\sqrt{2}}$



When $b = 0$ the distance from the turning point to the origin is 1 whereas when $b = \pm\frac{1}{\sqrt{2}}$ the distance from the turning point to the origin is $\frac{\sqrt{3}}{2}$ and so the answer is $b = \pm\frac{1}{\sqrt{2}}$.

B Question 8 (E 36) 41%

Question 8

If $\int_1^{12} g(x)\,dx = 5$ and $\int_{12}^{5} g(x)\,dx = -6$, then $\int_1^{5} g(x)\,dx$ is equal to

A. -11

B. -1

C. 1

D. 3

E. 11

$$\int_1^{12} g(x)dx = 5,\ \int_{12}^{5} g(x)dx = -6$$

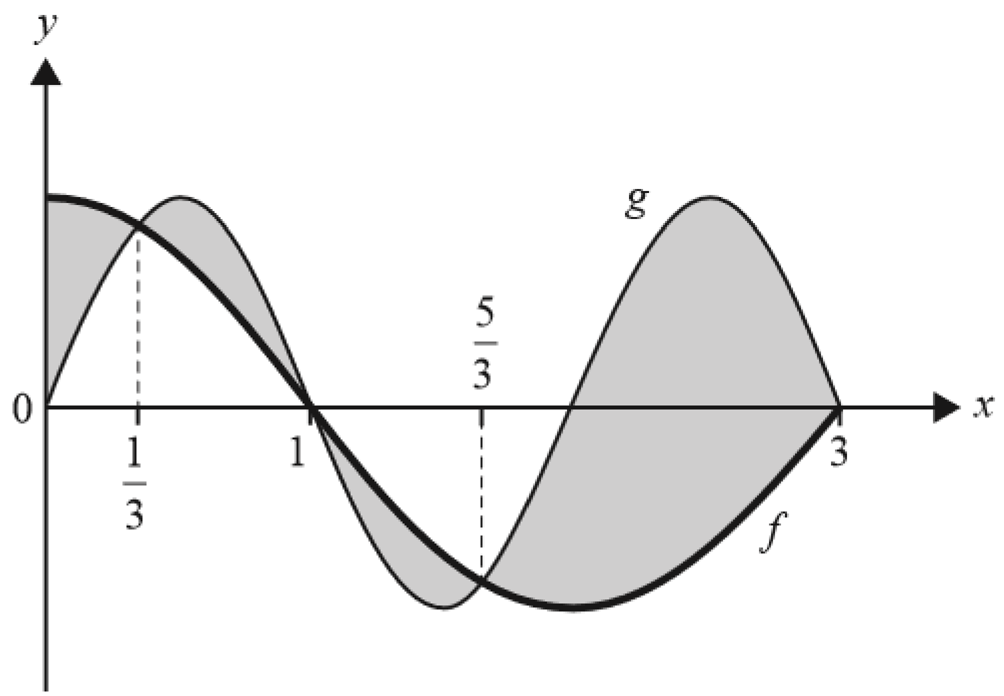
$$\int_1^{12} g(x)dx = \int_1^{5} g(x)dx + \int_5^{12} g(x)dx \ \text{ so } 5 = \int_1^{5} g(x)dx - (-6)$$

$$\int_1^{5} g(x)dx = -1$$

C Question 19 (E 42) 41%

Question 19

The graphs $f : R \to R,\ f(x) = \cos\left(\frac{\pi x}{2}\right)$ and $g : R \to R,\ g(x) = \sin(\pi x)$ are shown in the diagram below.



C Question 19 (E 42) 41%

An integral expression that gives the total area of the shaded regions is

$$\text{Area} = \int_0^{\frac{1}{3}}\left(f(x)-g(x)dx\right) - 2\int_{\frac{1}{3}}^{1}\left(f(x)-g(x)\right)dx - \int_{\frac{5}{3}}^{3}\left(f(x)-g(x)\right)dx$$

A. $\displaystyle\int_0^3\left(\sin(\pi x)-\cos\left(\frac{\pi x}{2}\right)\right)dx$

B. $\displaystyle 2\int_{\frac{5}{3}}^{3}\left(\sin(\pi x)-\cos\left(\frac{\pi x}{2}\right)\right)dx$

C. $\displaystyle\int_0^{\frac{1}{3}}\left(\cos\left(\frac{\pi x}{2}\right)-\sin(\pi x)\right)dx - 2\int_{\frac{1}{3}}^{1}\left(\cos\left(\frac{\pi x}{2}\right)-\sin(\pi x)\right)dx - \int_{\frac{5}{3}}^{3}\left(\cos\left(\frac{\pi x}{2}\right)-\sin(\pi x)\right)dx$

D. $\displaystyle 2\int_1^{\frac{5}{3}}\left(\cos\left(\frac{\pi x}{2}\right)-\sin(\pi x)\right)dx - 2\int_{\frac{5}{3}}^{3}\left(\cos\left(\frac{\pi x}{2}\right)-\sin(\pi x)\right)dx$

E. $\displaystyle\int_0^{\frac{1}{3}}\left(\cos\left(\frac{\pi x}{2}\right)-\sin(\pi x)\right)dx + 2\int_{\frac{1}{3}}^{1}\left(\sin(\pi x)-\cos\left(\frac{\pi x}{2}\right)\right)dx + \int_{\frac{5}{3}}^{3}\left(\cos\left(\frac{\pi x}{2}\right)-\sin(\pi x)\right)dx$

C Question 11 (A 38) 26%

Question 11

The graph of $y = \tan(ax)$, where $a \in R^+$, has a vertical asymptote $x = 3\pi$ and has exactly one x-intercept in the region $(0, 3\pi)$.

The value of a is

A. $\frac{1}{6}$

B. $\frac{1}{3}$

C. $\frac{1}{2}$

D. 1

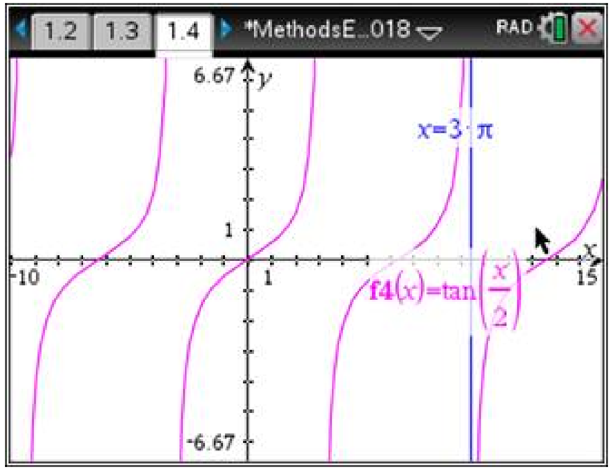
E. 2

$y = \tan(ax)$

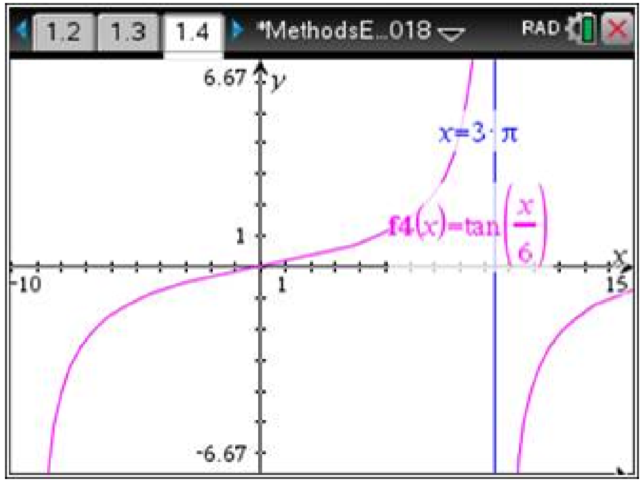
$y = \tan\left(\frac{x}{2}\right)$, Period $= 2\pi$

Asymptotes are at $x = \pi$, $x = 3\pi$

x-intercept is 2π



Option A The intercept is not in the required region $(0,\ 3\pi)$



A Question 20 (B 26) 20%

Question 20

The differentiable function $f: R \to R$ is a probability density function. It is known that the median of the probability density function f is at $x = 0$ and $f'(0) = 4$.

The transformation $T: R^2 \to R^2$ maps the graph of f to the graph of g, where $g: R \to R$ is a probability density function with a median at $x = 0$ and $g'(0) = -1$

The transformation T could be given by

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{-1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Gradient $= m = 4$

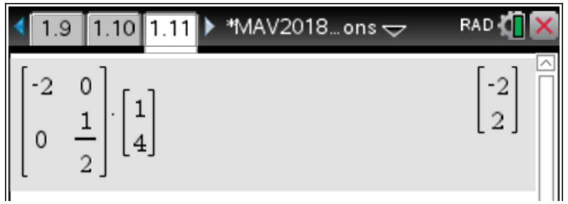
reflect in the y-axis $m = -4$

$$m = -\frac{4}{1}$$

Dilate by a factor of 2 from the y-axis $m = -\frac{4}{2} = -2$

Dilate by a factor of $\frac{1}{2}$ from the x-axis $m = -1$

The matrix $\begin{bmatrix} -2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ represents this transformation



A Question 20 (B 26) 20%

Question 20

The differentiable function $f: R \to R$ is a probability density function. It is known that the median of the probability density function f is at $x = 0$ and $f'(0) = 4$.

The transformation $T: R^2 \to R^2$ maps the graph of f to the graph of g, where $g: R \to R$ is a probability density function with a median at $x = 0$ and $g'(0) = -1$.

The transformation T could be given by

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{-1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Options B and E are reflections in the x-axis. Thus the functions are not PDFs.

Option C has no reflection so the gradient will not be negative.

Option D gives a gradient of -16.

E Question 18 (C 24) 14%

Question 18

Consider the functions $f : R^+ \to R,\ f(x) = x^{\frac{p}{q}}$ and $g : R^+ \to R,\ g(x) = x^{\frac{m}{n}}$, where p, q, m and n are positive integers, and $\frac{p}{q}$ and $\frac{m}{n}$ are fractions in simplest form.

If $\{x : f(x) > g(x)\} = (0, 1)$ and $\{x : g(x) > f(x)\} = (1, \infty)$, which of the following must be **false**?

A. $q > n$ and $p = m$

B. $m > p$ and $q = n$

C. $pn < qm$

D. $f'(c) = g'(c)$ for some $c \in (0, 1)$

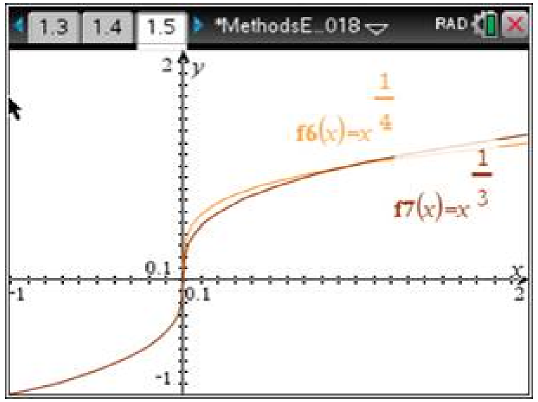
E. $f'(d) = g'(d)$ for some $d \in (1, \infty)$

$f'(d) = g'(d)$ for some $d \in (1, \infty)$ is false.

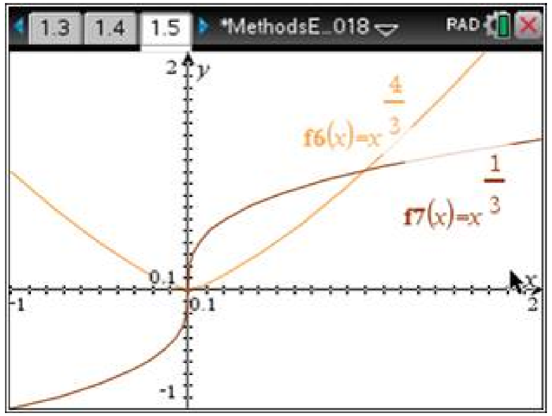
It is easy to see Options A to D can be true by substituting in values.

E Question 18 (C 24) 14%

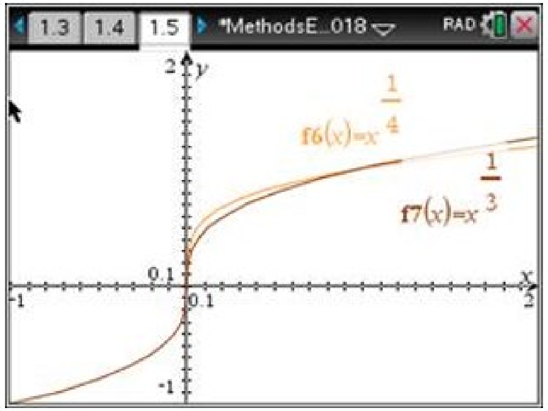
Option A True



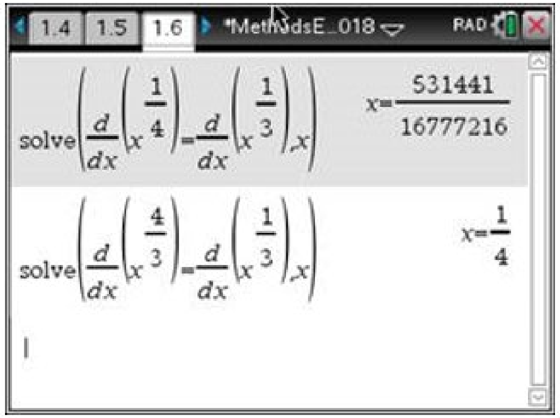
Option B True



Option C True



Option D True



Extended Answer

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- Give an exact answer unless otherwise stated.

$$\cancel{\frac{1}{3} = 0.3} \qquad \cancel{\frac{1}{3} \text{ OR } 0.3}$$

- No calculator syntax.
- Show working for questions worth more than one mark. (Rule and answer)
- Work to more decimal places than the required answer.
- Use the variables that are given within the question. (SAC questions)

Extended Answer

- Reread questions.
- Take time when drawing graphs – scale axes, check if coordinates are required, one sharp line…
- Don’t assume steps in *Show That* questions.
- Use brackets correctly.
- Transcribe formulas correctly (reread the calculator).
- Put units in the final answer.
- Check that the final answer makes sense.
- Use the calculator….check the entry

Question 1a

Marks	0	1	Average
%	5	95	1.0

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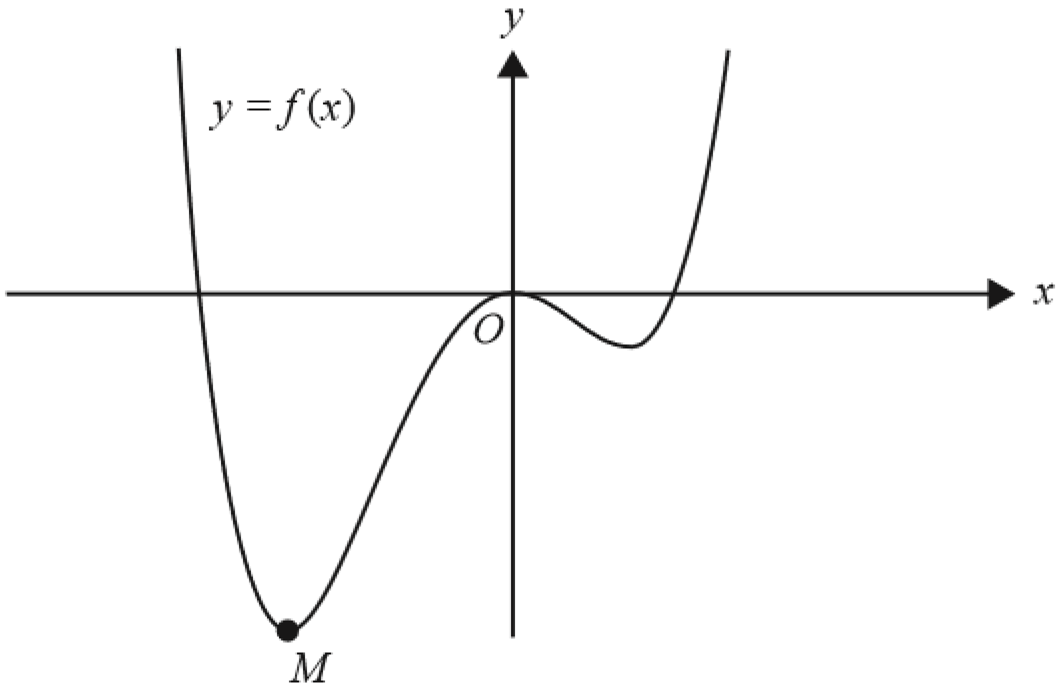
Question 1 (13 marks)

Consider the quartic $f: R \to R,\ f(x) = 3x^4 + 4x^3 - 12x^2$ and part of the graph of $y = f(x)$ below.



Question 1a

Marks	0	1	Average
%	5	95	1.0

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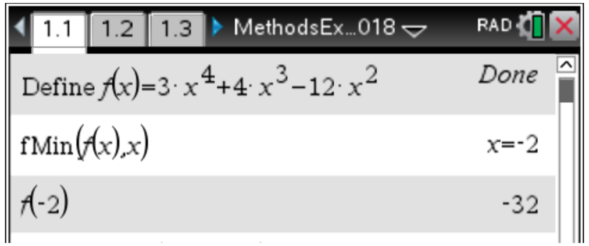
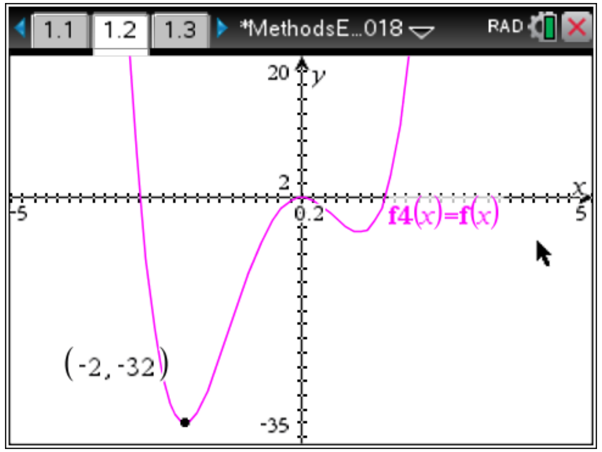
a. Find the coordinates of the point M, at which the minimum value of the function f occurs.

$$(-2, \ -32)$$





Question 1b

Marks	0	1	Average
%	35	65	0.7

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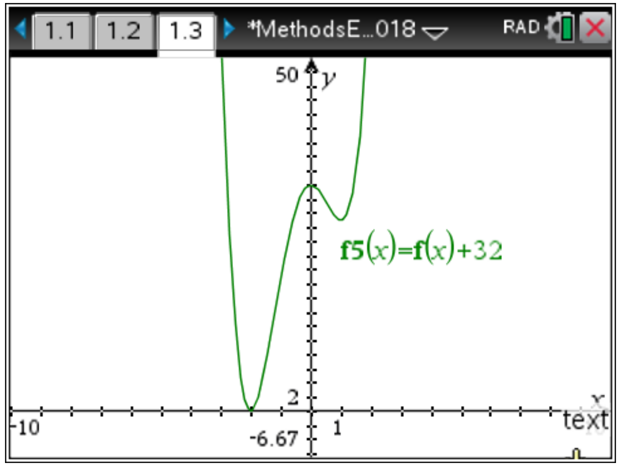
b. State the values of $b \in R$ for which the graph of $y = f(x) + b$ has no x-intercepts.

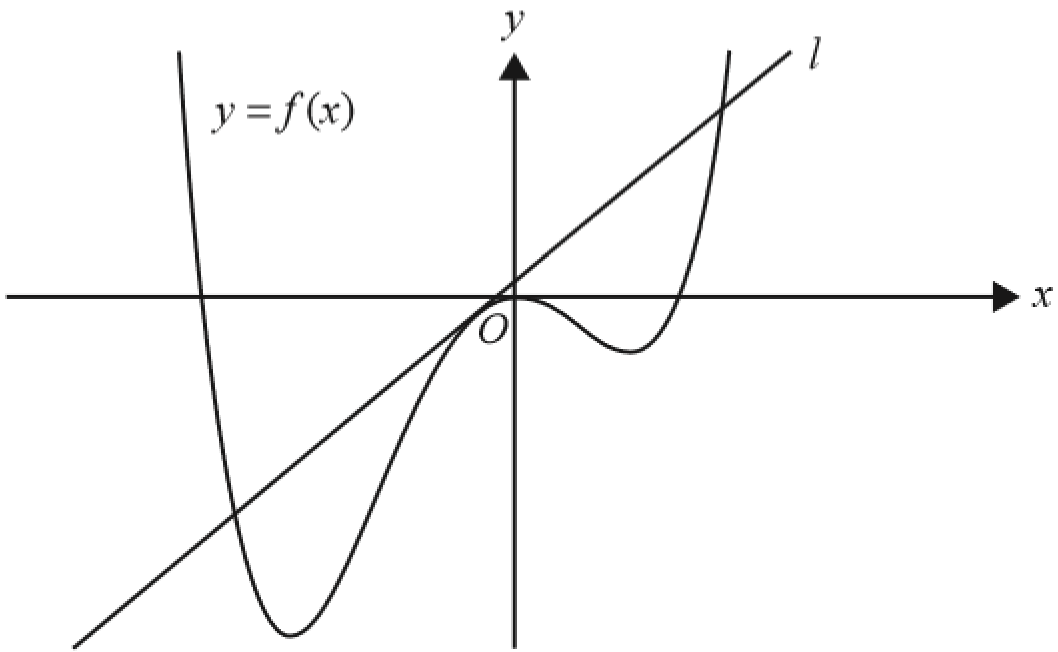
$$b > 32$$



Question 1c

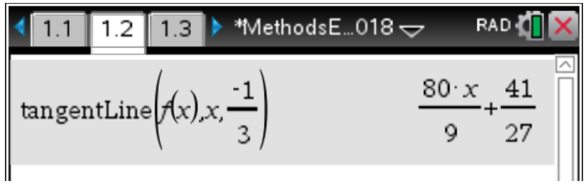
Marks	0	1	Average
%	27	73	0.7

Part of the tangent, l, to $y = f(x)$ at $x = -\frac{1}{3}$ is shown below.



c. Find the equation of the tangent l.

$$l(x) = \frac{80}{9}x + \frac{41}{27}$$



Question 1d

Marks	0	1	2	Average
%	20	13	67	1.5

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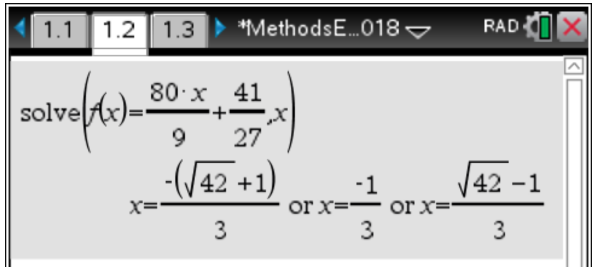
d. The tangent l intersects $y = f(x)$ at $x = -\frac{1}{3}$ and at two other points.

State the x-values of the two other points of intersection. Express your answers in the form $\frac{a \pm \sqrt{b}}{c}$, where a, b and c are integers.

Solve $l(x) = f(x)$ for x.

$$x = \frac{-1 \pm \sqrt{42}}{3}$$



Question 1e

Marks	0	1	2	Average
%	38	13	49	1.1

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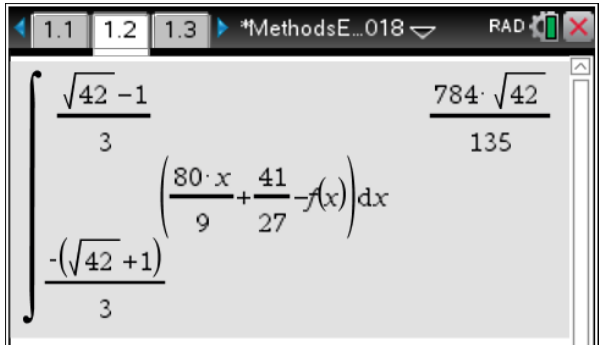
e. Find the total area of the regions bounded by the tangent l and $y = f(x)$. Express your answer in the form $\dfrac{a\sqrt{b}}{c}$, where a, b and c are positive integers.

To find the area between the two graphs find $\int(\text{upper curve} - \text{lower curve})dx$.

The point where the two curves touch is irrelevant for this calculation.

$$\text{Area} = \int_{\frac{-1-\sqrt{42}}{3}}^{\frac{-1+\sqrt{42}}{3}} \left(l(x) - g(x)\right) dx$$

$$= \frac{784\sqrt{42}}{135}$$



Question 1f

Marks	0	1	Average
%	51	49	0.5

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Let $p : R \to R,\ p(x) = 3x^4 + 4x^3 + 6(a-2)x^2 - 12ax + a^2,\ a \in R$.

f. State the value of a for which $f(x) = p(x)$ for all x.

Notice that there is no x or constant term in $f(x)$ so $a = 0$.

Question 1g

Marks	0	1	Average
%	43	57	0.6

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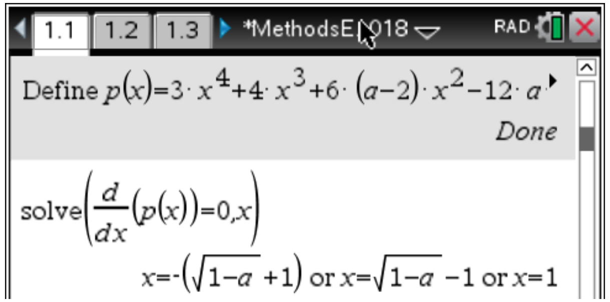
g. Find all solutions to $p'(x) = 0$, in terms of a where appropriate.

$$x = 1, x = -1 \pm \sqrt{1-a}$$



Question 1h.i.

Marks	0	1	Average
%	82	18	0.2

h. **i.** Find the values of a for which p has only one stationary point.

As there is already a stationary point at $x = 1$, we want the other two values of x to not exist.
This will occur when $1 - a < 0$, and so $a > 1$.

Question 1h.ii.

Marks	0	1	Average
%	47	53	0.5

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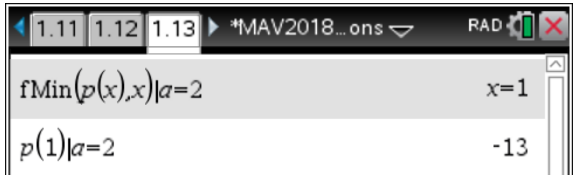
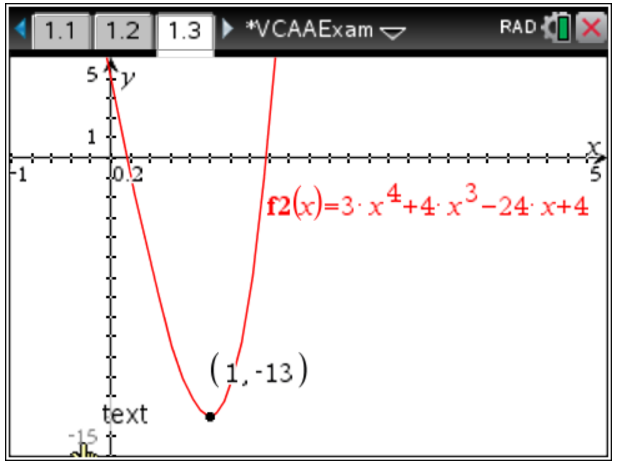
ii. Find the minimum value of p when $a = 2$.

$$a = 2,\ p(x) = 3x^4 + 4x^3 - 24x + 4$$

The minimum occurs when $x = 1$.

$p(1) = -13$ and so the minimum value is -13.





Question 1h.iii.

Marks	0	1	2	Average
%	92	4	4	0.1

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iii. If p has only one stationary point, find the values of a for which $p(x) = 0$ has no solutions.

Solve $p(x) > 0$ for a when $x = 1$.

$a > \sqrt{14} + 3$ as $a > 1$

1.12 1.13 1.14 *MAV2018...ons RAD

$\text{solve}(p(x)>0,a)|x=1$

$a<-(\sqrt{14}-3)$ or $a>\sqrt{14}+3$

Question 2a

Marks	0	1	2	Average
%	19	8	73	1.5

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Question 2 (10 marks)

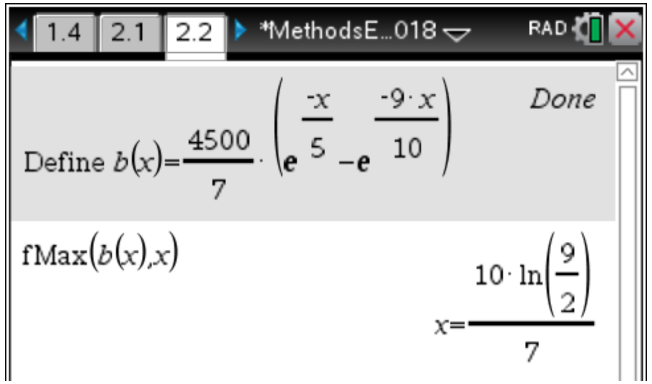
A drug, X, comes in 500 milligram (mg) tablets.

The amount, b, of drug X in the bloodstream, in milligrams, t hours after one tablet is consumed is given by the function

$$b(t) = \frac{4500}{7}\left(e^{\left(-\frac{t}{5}\right)} - e^{\left(-\frac{9t}{10}\right)}\right)$$

a. Find the time, in hours, it takes for drug X to reach a maximum amount in the bloodstream after one tablet is consumed. Express your answer in the form $a\log_e(c)$, where $a, c \in R$.

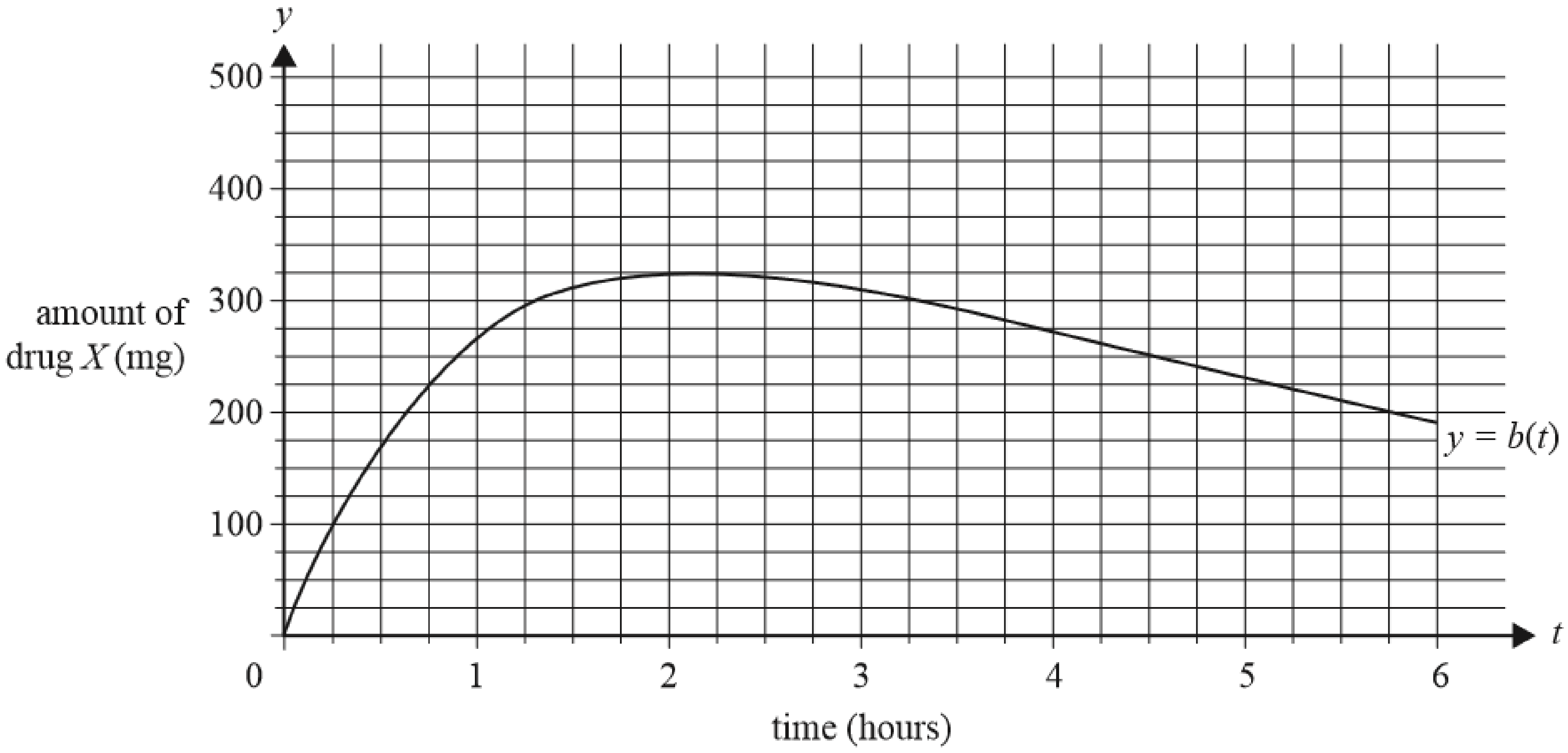
The maximum value of b occurs when $t = \dfrac{10\log_e(4.5)}{7}$ hours.



Question 2b

Marks	0	1	2	Average
%	16	15	70	1.5

The graph of $y = b(t)$ is shown below for $0 \le t \le 6$.



Question 2b

Marks	0	1	2	Average
%	16	15	70	1.5

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b. Find the average rate of change of the amount of drug X in the bloodstream, in milligrams per hour, over the interval [2, 6]. Give your answer correct to one decimal place.

$$\text{Average rate of change} = \frac{b(6)-b(2)}{6-2} = -33.5 \text{ mg/h correct to one decimal place}$$

Question 2c

Marks	0	1	2	Average
%	39	6	56	1.2

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c. Find the average amount of drug X in the bloodstream, in milligrams, during the first six hours after one tablet is consumed. Give your answer correct to the nearest milligram.

$$\text{Average amount of drug} = \frac{1}{6}\int_0^6 (b(t))\,dt = 256 \text{ mg to the nearest integer}$$

Question 2d.i.

Marks	0	1	2	Average
%	35	29	35	1.0

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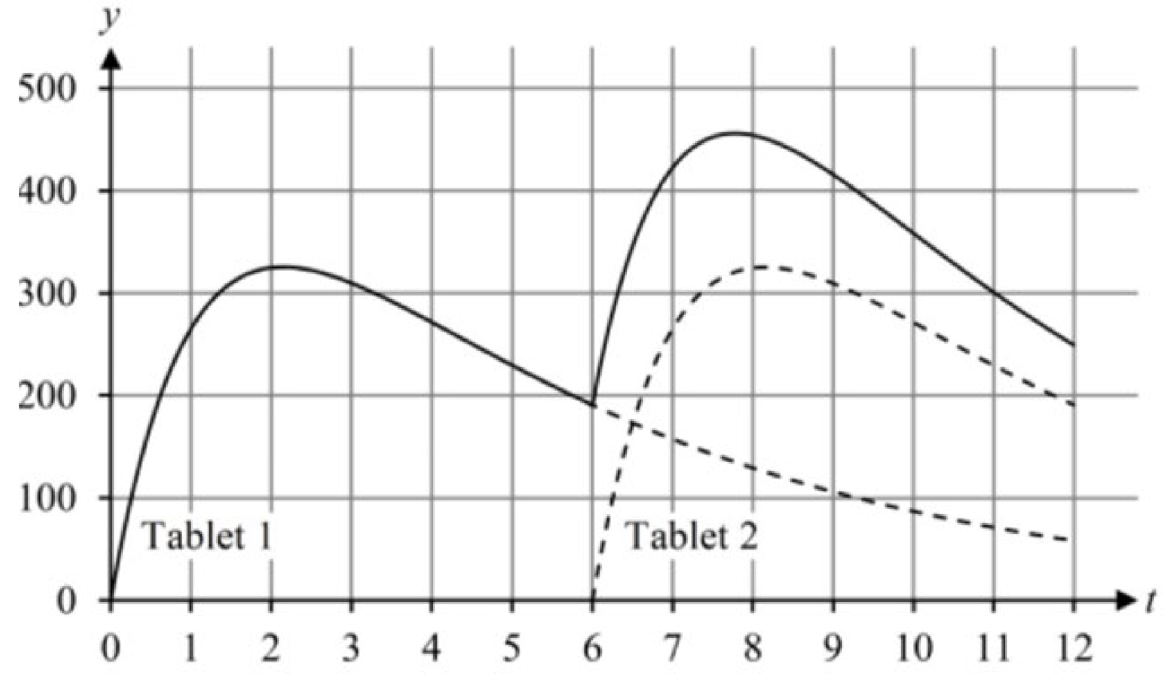
d. Six hours after one 500 milligram tablet of drug X is consumed (Tablet 1), a second identical tablet is consumed (Tablet 2). The amount of drug X in the bloodstream from each tablet consumed independently is shown in the graph below.

i. On the graph above, sketch the total amount of drug X in the bloodstream during the first 12 hours after Tablet 1 is consumed.



Question 2d.ii.

Marks	0	1	2	Average
%	74	6	19	0.4

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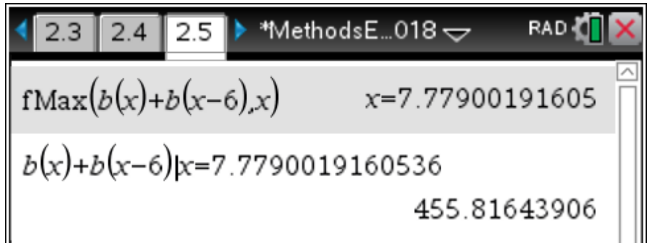
ii. Find the maximum amount of drug X in the bloodstream in the first 12 hours and the time at which this maximum occurs. Give your answers correct to two decimal places.

Total amount of drug $= b(t) + b(t-6)$

Maximum amount of drug is 455.82 mg, correct to two decimal places.
This occurs when $t = 7.78$ h, correct to two decimal places.

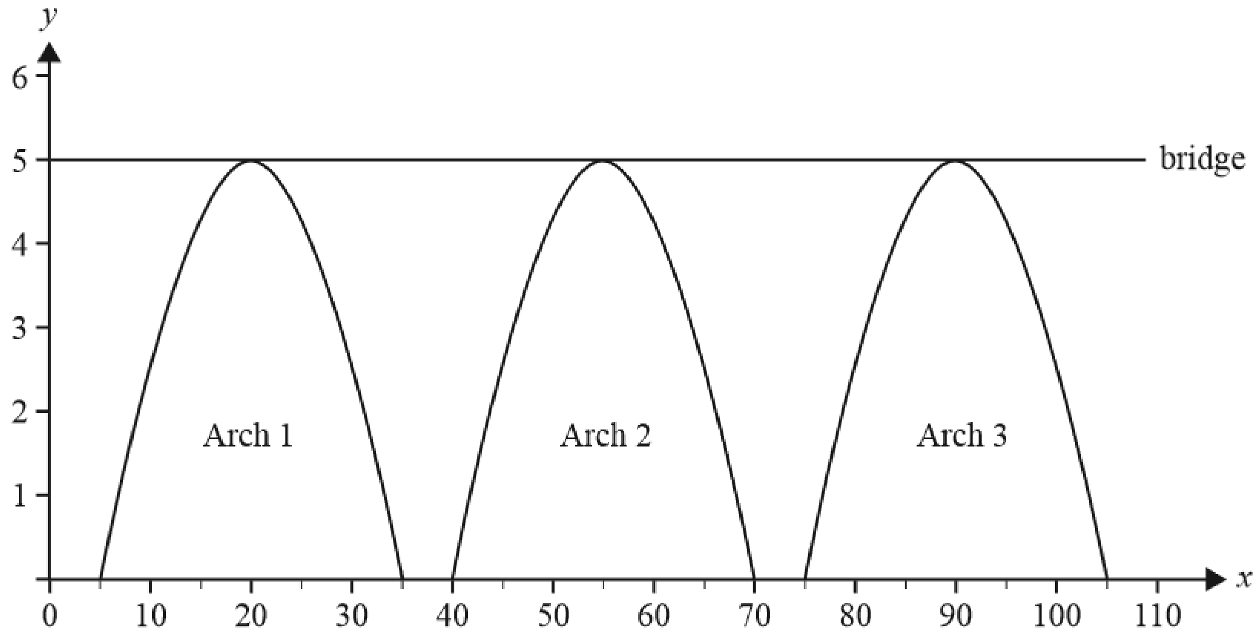


Question 3

Question 3 (11 marks)

A horizontal bridge positioned 5 m above level ground is 110 m in length. The bridge also touches the top of three arches. Each arch begins and ends at ground level. The arches are 5 m apart at the base, as shown in the diagram below.

Let x be the horizontal distance, in metres, from the left side of the bridge and let y be the height, in metres, above ground level.



Question 3a

Marks	0	1	Average
%	5	95	0.9

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Arch 1 can be modelled by the function $h_1 : [5, 35] \to R, h_1(x) = 5\sin\left(\frac{(x-5)\pi}{30}\right)$.

Arch 2 can be modelled by the function $h_2 : [40, 70] \to R, h_2(x) = 5\sin\left(\frac{(x-40)\pi}{30}\right)$.

Arch 3 can be modelled by the function $h_3 : [a, 105] \to R, h_3(x) = 5\sin\left(\frac{(x-a)\pi}{30}\right)$.

a. State the value of a, where $a \in R$.

There is a translation of the graph of h with rule $h(x) = 5\sin\left(\frac{x\pi}{30}\right)$, 75 m to the right and so $a = 75$.

Another approach is to look at the domain of $h_3(x)$. It is [75, 105].

Question 3b

Marks	0	1	Average
%	22	78	0.8

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b. Describe the transformation that maps the graph of $y = h_2(x)$ to $y = h_3(x)$.

The graph of $h_2(x) = 5\sin\left(\frac{(x-40)\pi}{30}\right)$ has been translated 35 m to the right to get to the graph of $h_3(x) = 5\sin\left(\frac{(x-75)\pi}{30}\right)$.

Question 3c

Marks	0	1	2	3	Average
%	8	8	18	66	2.4

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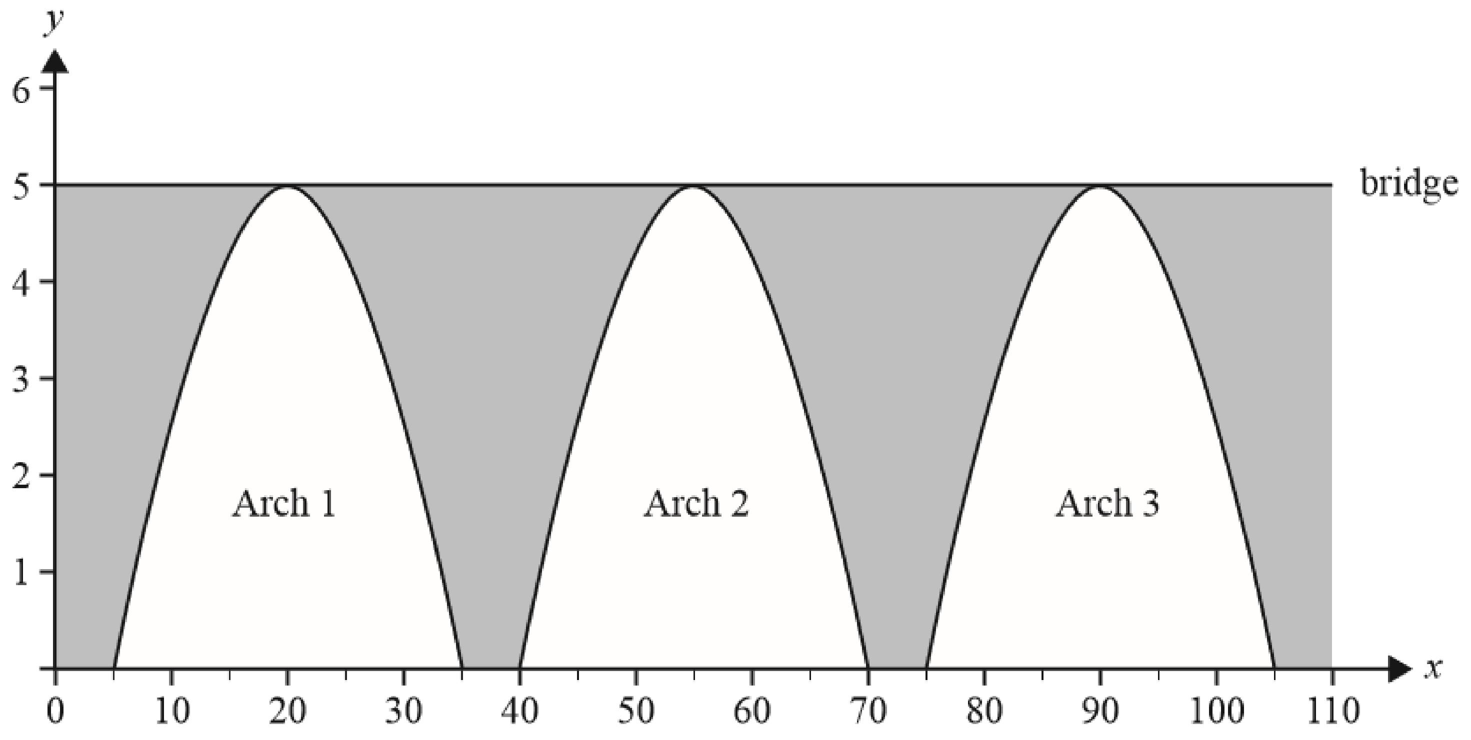
The area above ground level between the arches and the bridge is filled with stone. The stone is represented by the shaded regions shown in the diagram below.



Question 3c

Marks	0	1	2	3	Average
%	8	8	18	66	2.4

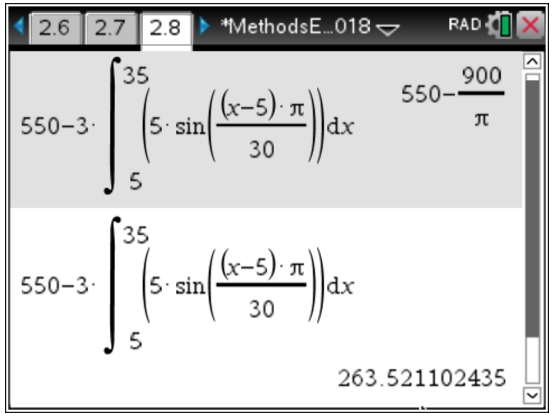
THE MATHEMATICAL ASSOCIATION OF VICTORIA

c. Find the total area of the shaded regions, correct to the nearest square metre.

Shaded area = area of the rectangle $-3 \times$ area of an arch

$$= 5 \times 110 - 3\int_{5}^{35} \left(5\sin\left(\frac{(x-5)\pi}{30} \right) \right) dx$$

$= 264$ m^2, correct to the nearest square metre

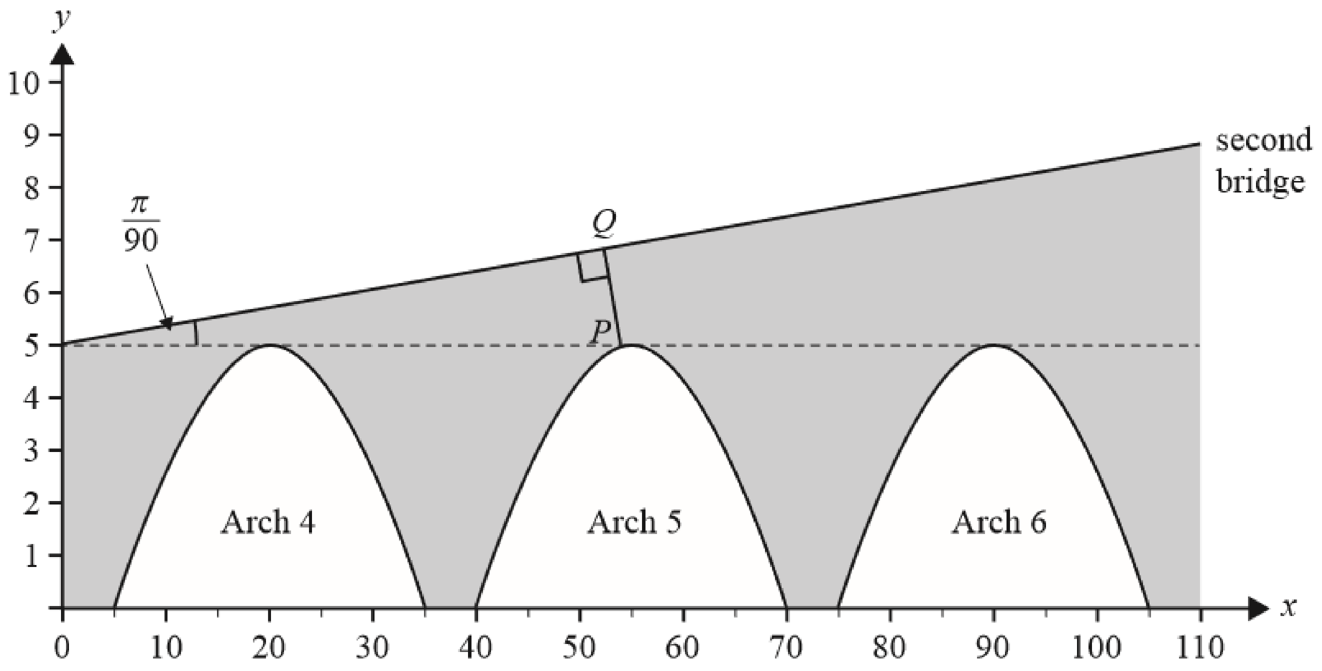


Question 3d

Marks	0	1	Average
%	39	61	0.6

A second bridge has a height of 5 m above the ground at its left-most point and is inclined at a constant angle of elevation of $\frac{\pi}{90}$ radians, as shown in the diagram below. The second bridge also has three arches below it, which are identical to the arches below the first bridge, and spans a horizontal distance of 110 m.

Let x be the horizontal distance, in metres, from the left side of the second bridge and let y be the height, in metres, above ground level.



Question 3d

Marks	0	1	Average
%	39	61	0.6

d. State the gradient of the second bridge, correct to three decimal places.

$$\tan\left(\frac{\pi}{90}\right) = 0.035\text{, correct to three decimal places}$$

Question 3e

Marks	0	1	2	Average
%	46	13	41	1.0

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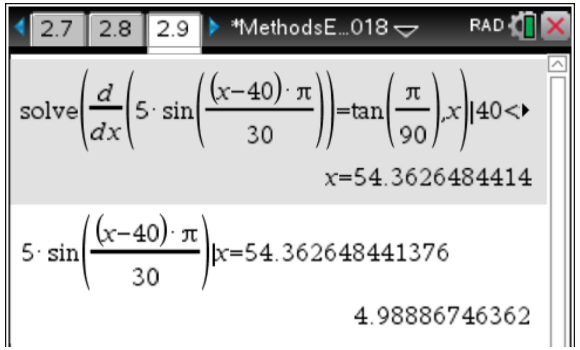
P is a point on Arch 5. The tangent to Arch 5 at point P has the same gradient as the second bridge.

e. Find the coordinates of P, correct to two decimal places.

Now P is a point on the arch given by the equation: $h_2(x) = 5\sin\left(\frac{(x-40)\pi}{30}\right)$. It should be noted that this point is **not** on the line given by the equation $y = 5$. The tangent at P is parallel to the second bridge and so $h_2'(x) = \tan\left(\frac{\pi}{90}\right)$.



Solving $h_2'(x) = \tan\left(\frac{\pi}{90}\right)$ for x gives $x = 54.36$, correct to two decimal places.

$h_2(54.3626...) = 4.99$, correct to two decimal places and so the coordinates of P are (54.36, 4.99).

Question 3f

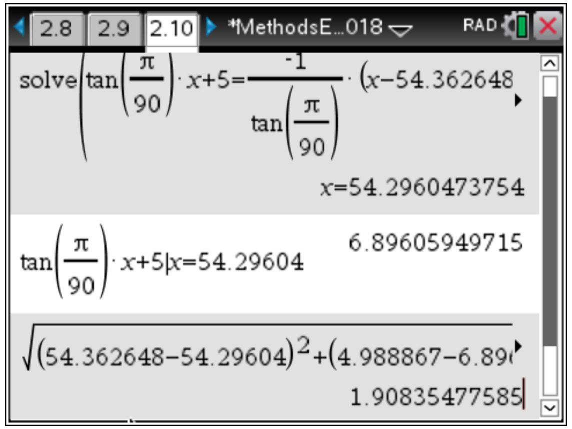
Marks	0	1	2	3	Average
%	56	17	8	19	0.9

THE MATHEMATICAL ASSOCIATION OF VICTORIA

f. A supporting rod connects a point Q on the second bridge to point P on Arch 5. The rod follows a straight line and runs perpendicular to the second bridge, as shown in the diagram on page 18.

Find the distance PQ, in metres, correct to two decimal places.

The equation of the line for the second bridge is $y_1 = \tan\left(\frac{\pi}{90}\right)x + 5$.



The equation of the perpendicular line passing through P and Q is

$$y_2 - 4.988... = -\frac{1}{\tan\left(\frac{\pi}{90}\right)}(x - 54.363...).$$

The point Q is the intersection of these two lines. Solving the equations simultaneously gives

Q (54.29…, 6.896…).

Distance $PQ = \sqrt{(54.36... - 54.29..)^2 + (4.98... - 6.896...)^2} = 1.91$ m, correct to two decimal places.

Question 3f

Marks	0	1	2	3	Average
%	56	17	8	19	0.9

OR

The equation of the line for the second bridge is $y_1 = \tan\left(\frac{\pi}{90}\right)x + 5$.

$Q(x,\ 0.0349..x + 5)$

Distance from P to any point on the bridge is

$$D = \sqrt{(54.36... - x)^2 + (4.98... - (0.0349..x + 5))^2}$$

$\frac{dD}{dx} = 0$ when $x = 54.29...$

$D(54.29...) = 1.91$ m, correct to two decimal places

Question 4a

Marks	0	1	Average
%	13	87	0.9

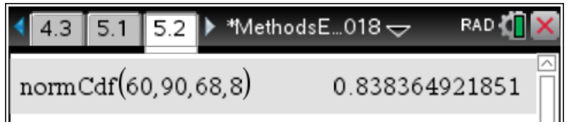
Question 4 (16 marks)

Doctors are studying the resting heart rate of adults in two neighbouring towns: Mathsland and Statsville. Resting heart rate is measured in beats per minute (bpm).

The resting heart rate of adults in Mathsland is known to be normally distributed with a mean of 68 bpm and a standard deviation of 8 bpm.

a. Find the probability that a randomly selected Mathsland adult has a resting heart rate between 60 bpm and 90 bpm. Give your answer correct to three decimal places.

Let $M \sim N(68, 64)$.



$\Pr(60 < M < 90) = 0.838$, correct to three decimal places

Question 4b.i.

Marks	0	1	Average
%	43	57	0.6

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b. **i.** Find $\Pr(H|S)$, correct to three decimal places.

$$\Pr(H \mid S) = \frac{\Pr(H \cap S)}{\Pr(S)} = \frac{0.09}{0.29} = 0.310 \text{, correct to three decimal places.}$$

Question 4b.ii.

Marks	0	1	Average
%	56	44	0.4

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ii. Are the events H and S independent? Justify your answer.

No, the events are not independent.

If they were independent then $\Pr(H \mid S) = \dfrac{\Pr(H \cap S)}{\Pr(S)} = \dfrac{\Pr(H) \times \Pr(S)}{\Pr(S)} = \Pr(H)$, but

$0.310... \neq 0.1587$.

OR

If the events are independent then $\Pr(H \cap S) = \Pr(H) \times \Pr(S)$. But $0.09 \neq 0.1587 \times 0.29 = 0.046...$ and so they are not independent.

Question 4c.i.

Marks	0	1	2	Average
%	25	9	66	1.4

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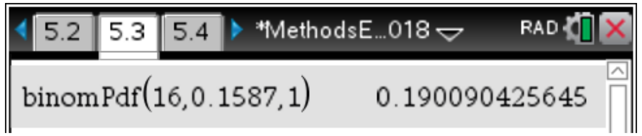
c. i. Find the probability that a random sample of 16 Mathsland adults will contain exactly one person with a slow heart rate. Give your answer correct to three decimal places.

Let $X \sim \text{Bi}(16, 0.1587)$.

$\Pr(X = 1) = 0.190$, correct to three decimal places



Question 4c.ii.

Marks	0	1	2	Average
%	54	9	36	0.8

THE MATHEMATICAL ASSOCIATION OF VICTORIA

ii. For random samples of 16 Mathsland adults, $\hat{P}$ is the random variable that represents the proportion of people who have a slow heart rate.

Find the probability that $\hat{P}$ is greater than 10%, correct to three decimal places.

$\Pr\left(\hat{P} > 0.1\right)$

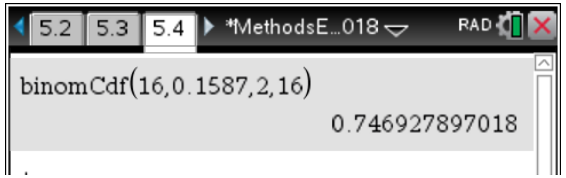
$\hat{P} > 0.1$ means that $X > \dfrac{1}{10} \times 16$. So $X > 1.6$. Since X is binomial and therefore discrete, $\Pr\left(X \geq 2\right)$ is evaluated.

$\Pr\left(\hat{P} > 0.1\right)$

$= \Pr\left(X > 1.6\right)$

$= \Pr\left(X \geq 2\right)$

$= 0.747$ correct to three decimal places



Question 4c.iii.

Marks	0	1	2	Average
%	86	6	8	0.2

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iii. For random samples of n Mathsland adults, $\hat{P}_n$ is the random variable that represents the proportion of people who have a slow heart rate.

Find the least value of n for which $\Pr\left(\hat{P}_n > \frac{1}{n}\right) > 0.99$

$$\Pr\left(\hat{P}_n > \frac{1}{n}\right) > 0.99$$

Remember that $\hat{P}$ stands for proportion and lies in the interval [0, 1]. n is an integer (number of people). So $\Pr\left(\hat{P} > \frac{1}{n}\right) = \Pr\left(X > \frac{1}{n} \times n\right) = \Pr(X > 1) = \Pr(X \geq 2) > 0.99$ needs to be solved.

$$\Pr(X > 1) = \Pr(X \geq 2) > 0.99$$

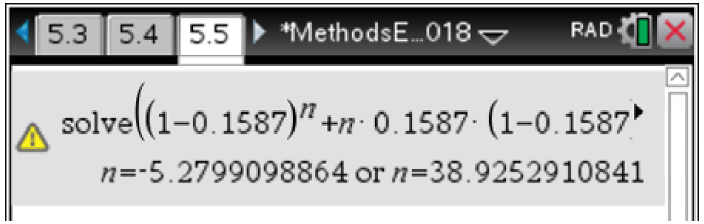
$$1 - \left(\Pr(X = 0) + \Pr(X = 1)\right) > 0.99$$

$$\Pr(X = 0) + \Pr(X = 1) < 0.01$$

$$(1 - 0.1587)^n + \binom{n}{1} 0.1587(1 - 0.1587)^{n-1} < 0.01$$

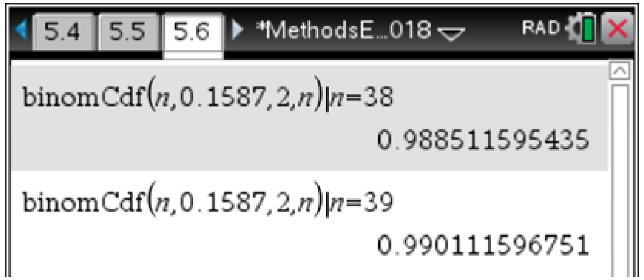
$n = 38.925\ldots$

$n = 39$ is the least value.



OR

Trial and error



Question 4d.i.

Marks	0	1	Average
%	55	45	0.4

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The doctors took a large random sample of adults from the population of Statsville and calculated an approximate 95% confidence interval for the proportion of Statsville adults who have a slow heart rate. The confidence interval they obtained was (0.102, 0.145).

d. **i.** Determine the sample proportion used in the calculation of this confidence interval.

If the confidence interval is (0.102, 0.145) then $\hat{p} = \dfrac{0.102 + 0.145}{2} = 0.1235$.

Question 4d.ii.

Marks	0	1	Average
%	89	11	0.1

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ii. Explain why this confidence interval suggests that the proportion of adults with a slow heart rate in Statsville could be different from the proportion in Mathsland.

The 95% confidence interval for Statsville, (0.102, 0.145), does not contain the Mathsland proportion which is 0.1587.

Question 4e

Marks	0	1	2	Average
%	38	9	53	1.2

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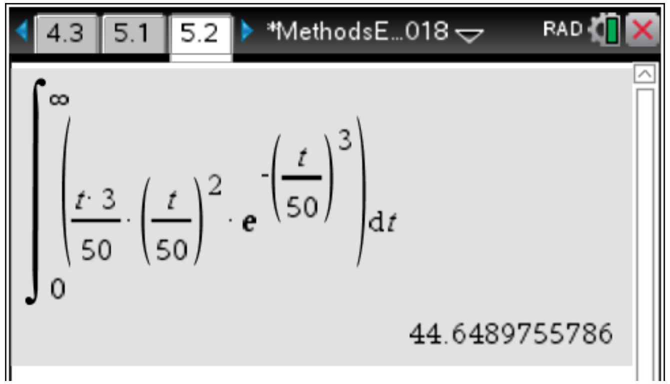
Every year at Mathsland Secondary College, students hike to the top of a hill that rises behind the school.

The time taken by a randomly selected student to reach the top of the hill has the probability density function M with the rule

$$M(t)=\begin{cases}\frac{3}{50}\left(\frac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3} & t\geq 0\\ 0 & t<0\end{cases}$$



where t is given in minutes.

e. Find the expected time, in minutes, for a randomly selected student from Mathsland Secondary College to reach the top of the hill. Give your answer correct to one decimal place.

$$\int_0^{\infty}(t\times M(t))\,dt=44.6\text{, correct to one decimal place}$$

Question 4f

Marks	0	1	Average
%	44	56	0.6

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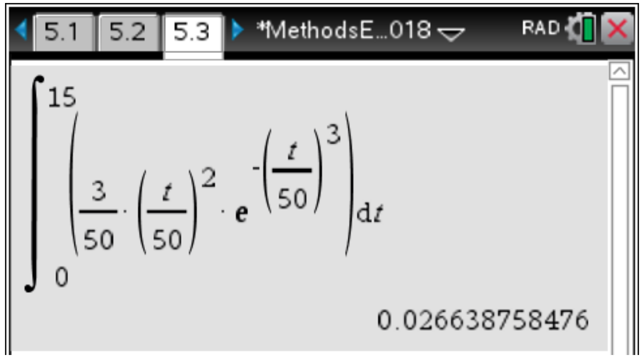
Students who take less than 15 minutes to get to the top of the hill are categorised as ‘elite’.

f. Find the probability that a randomly selected student from Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places.

$$\int_0^{15} (M(t))\,dt = 0.0266 \text{, correct to four decimal places}$$



Question 4g

Marks	0	1	2	Average
%	89	6	5	0.2

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g. The Year 12 students at Mathsland Secondary College make up $\frac{1}{7}$ of the total number of students at the school. Of the Year 12 students at Mathsland Secondary College, 5% are categorised as elite.

Find the probability that a randomly selected non-Year 12 student at Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places.

Solve $0.05 \times \frac{1}{7} + x \times \frac{6}{7} = 0.0266\ldots$ for x.

$x = 0.0227$, correct to four decimal places

Question 5a

Marks	0	1	2	Average
%	32	19	49	1.2

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Question 5 (10 marks)

Consider functions of the form

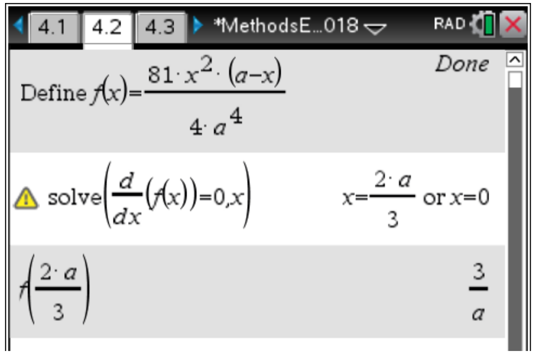
$$f : R \to R,\ f(x) = \frac{81x^2(a-x)}{4a^4}$$

and

$$h : R \to R,\ h(x) = \frac{9x}{2a^2}$$

where a is a positive real number.

a. Find the coordinates of the local maximum of f in terms of a.



$\left(\frac{2a}{3}, \frac{3}{a}\right)$ is the set of coordinates for the local maximum.

The other point, (0, 0) is a local minimum.

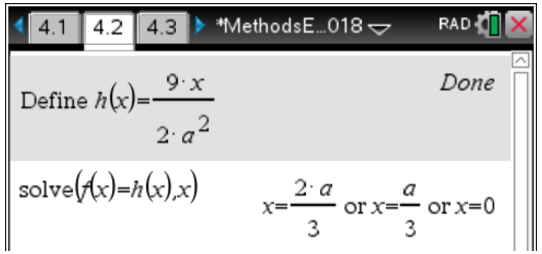
Question 5b

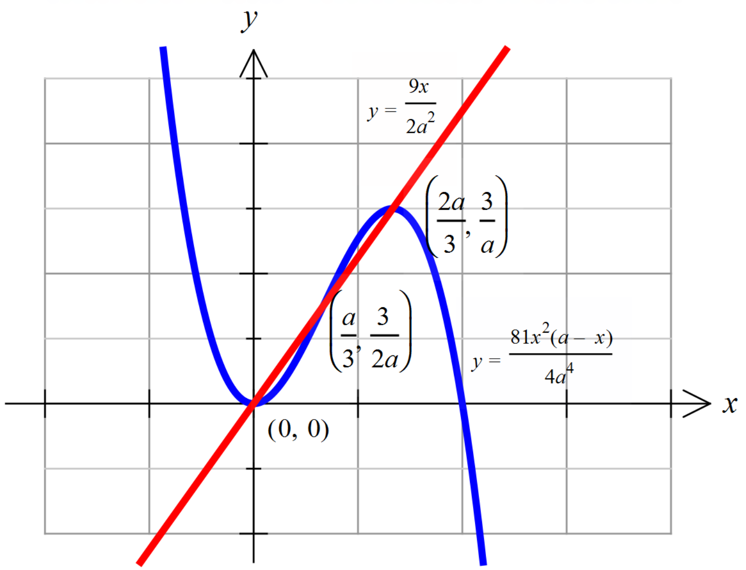
Marks	0	1	Average
%	38	62	0.6

b. Find the x-values of all of the points of intersection between the graphs of f and h, in terms of a where appropriate.

Solve $f(x) = h(x)$ for x.

$$x = \frac{2a}{3} \text{ or } x = \frac{a}{3} \text{ or } x = 0$$





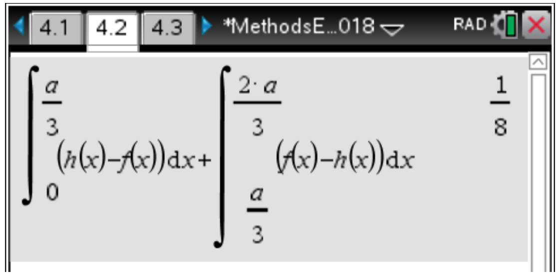
Question 5c

Marks	0	1	2	Average
%	60	6	34	0.8

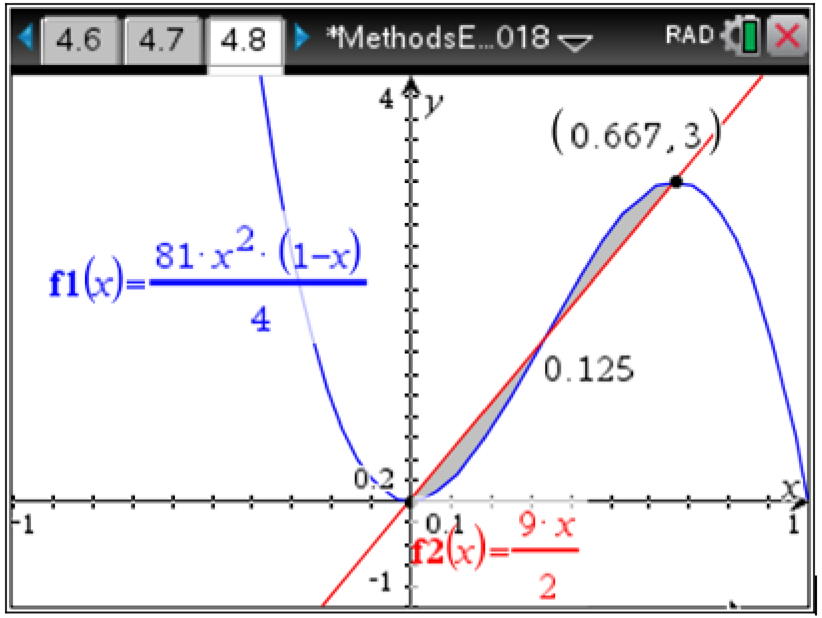
THE MATHEMATICAL ASSOCIATION OF VICTORIA

c. Determine the total area of the regions bounded by the graphs of $y = f(x)$ and $y = h(x)$.

$$\int_0^{\frac{a}{3}} (h(x) - f(x))\,dx + \int_{\frac{a}{3}}^{\frac{2a}{3}} (f(x) - h(x))\,dx = \frac{1}{8}$$



Graphs when $a = 1$



Question 5d

Marks	0	1	Average
%	36	64	0.6

THE MATHEMATICAL ASSOCIATION OF VICTORIA

Consider the function $g:\left[0, \frac{2a}{3}\right] \to R,\ g(x) = \frac{81x^2(a-x)}{4a^4}$, where a is a positive real number.

d. Evaluate $\frac{2a}{3} \times g\left(\frac{2a}{3}\right)$.

$$\frac{2a}{3} \times g\left(\frac{2a}{3}\right) = 2$$

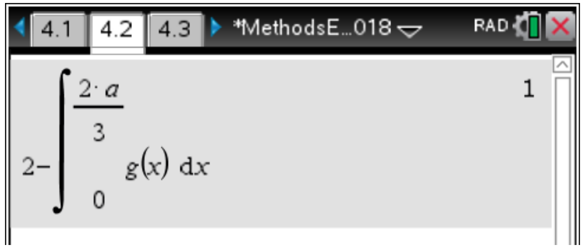
Question 5e

Marks	0	1	2	Average
%	87	6	7	0.2

THE MATHEMATICAL ASSOCIATION OF VICTORIA

e. Find the area bounded by the graph of g^{-1}, the x-axis and the line $x = g\left(\frac{2a}{3}\right)$.

$$\frac{2a}{3} \times \frac{3}{a} - \int_0^{\frac{2a}{3}} \left(g(x)\right)dx = 1$$



Question 5f

Marks	0	1	Average
%	92	8	0.1

THE MATHEMATICAL ASSOCIATION OF VICTORIA

f. Find the value of a for which the graphs of g and g^{-1} have the same endpoints.

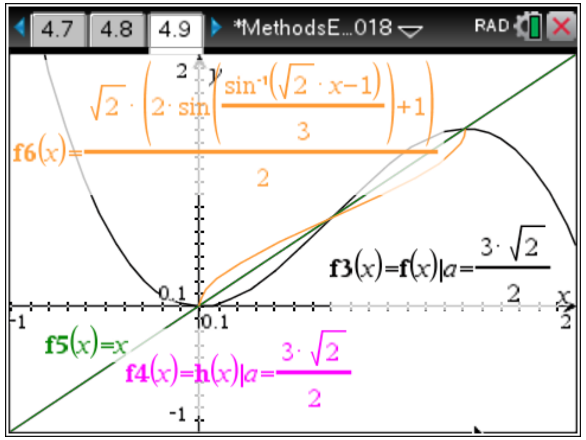
$$\frac{2a}{3}=\frac{3}{a},\ a=\frac{3\sqrt{2}}{2}$$

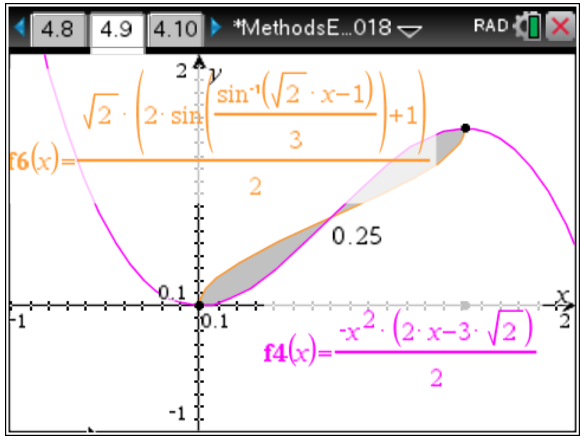
Question 5g

Marks	0	1	Average
%	97	3	0.0

g. Find the area enclosed by the graphs of g and g^{-1} when they have the same endpoints.

The area enclosed by g and g^{-1} is $2\times\frac{1}{8}=\frac{1}{4}$.





Thank you and Feedback

- FEEDBACK

- The MAV would appreciate your feedback hence we ask you to please take a couple of minutes to complete the following online survey:

https://www.surveymonkey.com/r/VCEPDMelb19