Maximum fun with Calculus

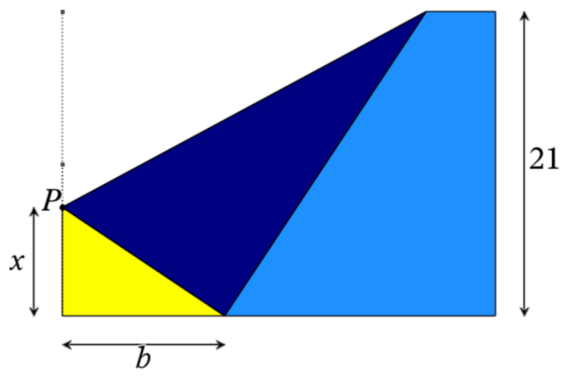
Introduction

This document contains a collection of interesting optimisation problems and investigations that include a significant calculus component. Most of the problems include opportunities to generate physical models or provide dynamic visuals to help students interact with the mathematics. The investigations can easily be turned into an Analysis Task suitable for Mathematical Methods or Specialist Mathematics. Most of the problems included here can be downloaded from the Texas Instruments Australia website complete with Student Question sheets, Teacher Notes, Answers and TI-Nspire files where applicable.

Origami

[Optimisation Problem]

An A4 piece of paper oriented in *landscape mode* contains a single fold such that the top left corner just touches the base of the paper. (Shown opposite)



The distance from the base of the page to point P is assigned to the variable 'x'.

Problem to solve:

Determine the maximum area of the triangle formed between the base, height and fold.

Comments:

A great way to start this problem is for students to fold the paper as shown and collectively record the height (x) and area (y). The points generated by the class should be plotted and a function determined to model the resultant curve. The plotted points and function definition provide visual feedback with regards to the validity of the function as it models the situation. A dynamic version of this is available on the Texas Instruments Australia website whereby points are automatically generated as students manipulate point P.

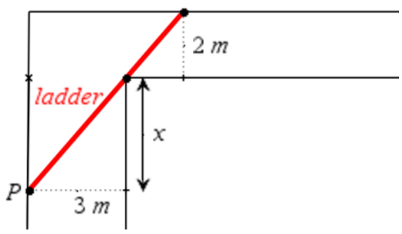
Calculus is used to determine the optimal value for the height (x) and corresponding area of the triangle. Of particular interest is the optimal height. How does this relate to the overall dimensions of the original paper? Is this always true? Why?

A lovely extension to this problem is to determine the maximum area of the dark-blue triangle, effectively the reverse side of the paper. There are many ways this problem can be explored.

Ladders

[Optimisation Problem]

There are many interesting 'ladder' problems including the "Falling Ladder" and the "Painter's Ladder". This problem considers the Painter's Ladder. The painter needs to navigate a relatively narrow hallway with his ladder. The painter would like to know: "What is the longest ladder that can get passed the corner at the end of the hallway?" This problem is summarised in the image shown opposite.



Problem to solve:

Determine the longest ladder length that will pass around the corner shown in the diagram. Assume that the ladder remains horizontal as it passes around the corner.

Comments:

A dynamic visual is available from the Texas Instruments website to illustrate the problem. One of the things that make this problem interesting is that calculus is used to determine the minimum value for a function corresponding to the maximum length of the ladder.

Eye – Spy an abnormal Calculus Problem

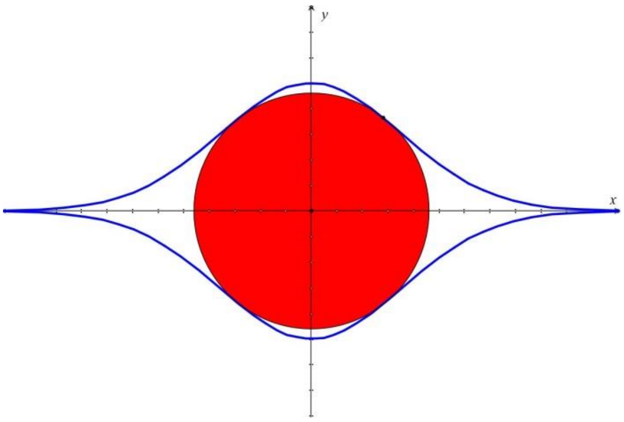
[Optimisation Problem]

This problem has several twists and turns that make it a deceptively simple optimisation problem.

Problem to solve:

Two 'bell shaped' shaped curves neatly encase a circle, what is the largest area of the circle if the bell shaped curves have the equations:

$$f(x) = e^{-x^2} \quad \text{and} \quad g(x) = -e^{-x^2}$$



Comments:

If students assume that the gradient of the function will be the same as the gradient of the circle when the area is a maximum, they will arrive at more than one solution. The problem is that whilst this is a necessary condition, it is not sufficient. Consider the case where the circle passes through the point (0, 1). The gradient of the two curves and the gradient of the circle are the same at this point; the problem is that the circle does not fit entirely between the curves.

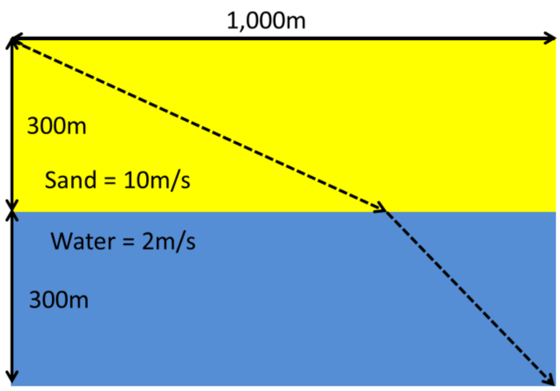
Students can over complicate the calculus and solution process if they do not consider some simple alternatives. By considering the symmetry of the problem and the fact that the radius of the circle will always be positive simplify the set up and corresponding calculus. When determining the area of the circle, students should remember that this involves r^2, therefore it is not entirely necessary to determine r.

Give this problem a try; you will be surprised how easy the answer falls out!

Cunning Running

[Optimisation Problem]

A common problem given to students to solve involves finding the quickest path across two different surfaces. In this problem Renee is a life guard, she is about to save a person in the water. She must first run across the sand, her maximum speed on the sand is 10m/s. She will then swim through the water at a speed of 2m/s. (Distances shown opposite). She could travel the shortest distance, a straight line, however as she can run faster than she swims, it would be advisable to travel less distance in the water.



Problem to solve:

Find the quickest time and the path that Renee should take in order to achieve this time.

Comments:

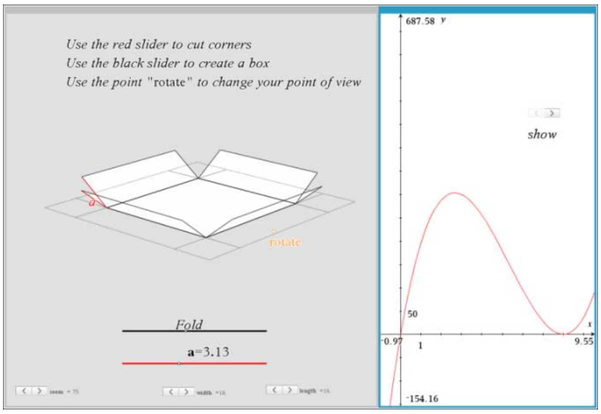
Is calculus the only way to solve this problem? Some of your students may find an easier way.

Open Box Problem

[Optimisation Problem]

A common problem given to students:

> *"Determine the maximum volume of an open box formed by cutting squares from the corner of a rectangular piece of card with dimensions a x b."*



It is a great problem that can be done practically with students collecting measurements to ensure their algebraic representation of the problem aligns with the practical results.

Problem to solve:

a) Why doesn't the maximum volume occur when box is the shape of a cube?

b) Does the point of inflexion on the volume graph have any meaning with regards to the problem?

Comments:

So question (a) attempts to draw a correlation with the maximum area of a rectangle with fixed perimeter forming a square. The problem is more closely aligned with the variation of this where one boundary of the rectangle is not required (typically a river).

The practical answer is that we are not covering the top of the box so all sides are not considered equal when it comes to forming the volume. Algebraically we can see that we would need to at least start with a square piece of card if the end result is to be a cube.

Original dimensions: x , $a-2x$ and $b-2x$

Volume: $v(x)=x(a-2x)(b-2x)$.

So if $a=b$ then the volume would have two edges of the same length. This leaves us to compute and solve the derivative:

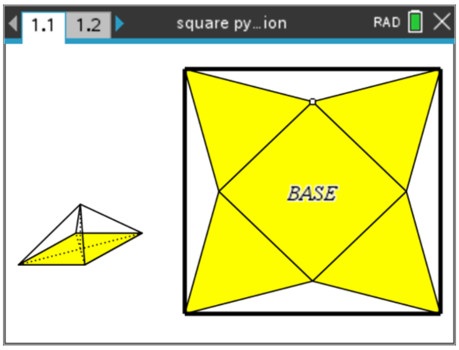
$solve\left(\dfrac{d(x(a-2x)^2)}{dx}=0,x\right)$ produces a feasible solution at $x=\dfrac{a}{6}$ which is clearly not a cube.

What about question (b)? This is a much more challenging problem. The visual representation provides a clue. This investigation is left to the reader with only the following clues:

i) Consider starting with a square card.

ii) The solution relates to question (a).

Related Problems to solve:



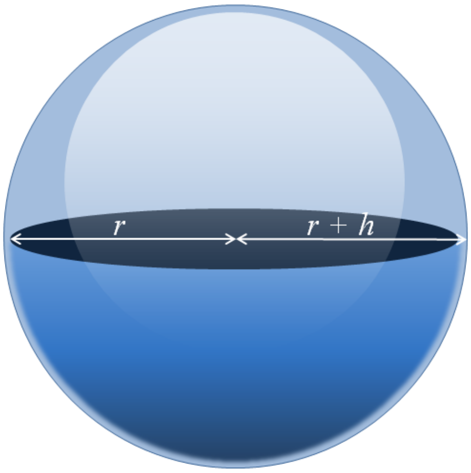
- Maximum volume of a closed cylinder
  Start by asking which provides a greater volume (open cylinder) rolling an A4 piece of paper in portrait or landscape?

- Maximum volume of a square based pyramid.
  This one has been written by Shane Dempsey and requires students to cut the net from a square piece of paper.

Spheres and Circles

[Investigation]

While this problem doesn't involve optimisation, it is has been included in this collection as an illustration of the wide variety of uses of calculus. Teachers and students often comment that it is interesting that the circumference of a circle is the derivative of the area, and similarly that the surface area of a sphere is the derivative of the volume.



Volume of a sphere: $v=\dfrac{4\pi r^3}{3}$

Surface area of a sphere: $\dfrac{dv}{dr}=4\pi r^2$

Is this coincidental? Differentiation involving limits is often left to first principles, but this problem lends itself to a similar approach.

Imagine we have a sphere of radius r and we coat it with a very thin layer of *stuff* to make a shell. If that layer of *stuff* is really, really thin then it effectively models the surface area of the original sphere. The diagram shown here has the thickness of the *stuff* as *h*.

Let the volume of the sphere be: $v(r)=\dfrac{4\pi r^3}{3}$

The volume of the sphere whilst covered in the *stuff* will be: $v(r+h)=\dfrac{4\pi(r+h)^3}{3}$

If we look at the limit as $h\to 0$ then $\dfrac{dv}{dr}=\lim\limits_{h\to 0}\dfrac{v(r+h)-v(r)}{h}$.

Explore this limit on your calculator and see what happens.

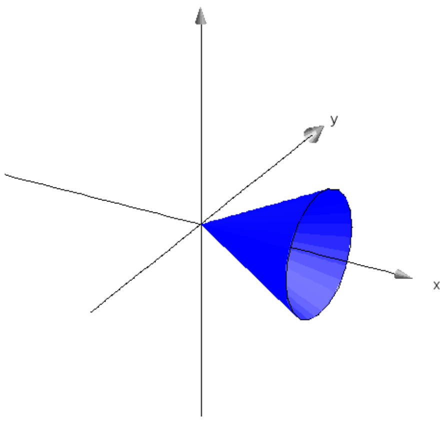
Problem to solve:

Use the above approach to explore the area of a circle and a slightly larger circle that just fits around the outside.

Comments:

You can follow this same idea with the volume of a cube and the corresponding total surface area of the cube. You will however you need to be very careful. Our circle example would not work if we were using the diameter. The same applies to the cube. See what happens if x represents 'half' the side length.

Can we use these same ideas for working out the total surface area of less uniform shapes using solids of revolution? Try working out the total surface area of a cone. Note that the formula generated will be based on the vertical height of the cone rather than the slant height which is the most common form provided to students.



Wine Glasses

[Investigation]

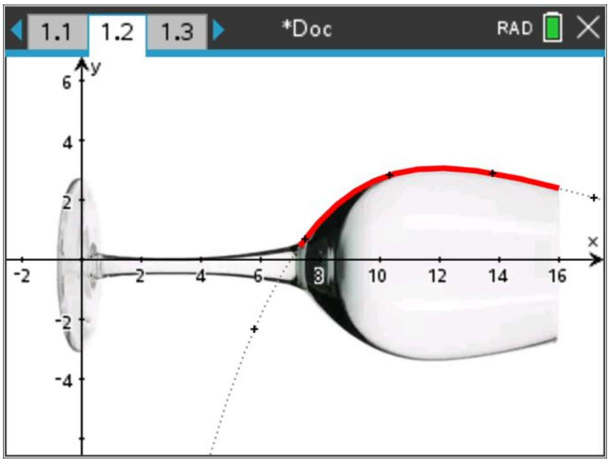
Wine glasses come in a range of shapes and sizes depending on their purpose. The tall stem is so that you don't have to place your hands on the body of the glass, this is to avoid heating the wine with your hands. Red wine glasses tend to have a wider body to help the wine breathe and a relatively narrow rim to allow the concentrated aromatic flavours to rise and tantalise the pallet. The glass must also be designed to hold a standard drink (175ml for wine) and preferably at the widest point of the body. The rim to body area ratio should be around 1 : 1.5. Finally, stability plays a role in the design of the glass, if the stem is too long or the base too small the glass will be unstable and tip over too easily. There is a lot more to designing a good wine glass than first meets the eye. Considering some bottles of wine sell for $100.00's and more, it makes sense to design the perfect glass from which to serve the wine.

Students can design the wine glass to meet all the specifications, starting with a standard wine glass and using functions to model the curvature. To ensure the curves used to model the glass profile are functions, tip the glass on its side.

Mathematics:

- Standard drink = 175ml (wine) – This should occur at the widest section of the glass (turning point)
- Ratio between y ordinates at the turning points and top of the glass can be used to help match the rim to body ratio.
- What is the ideal stem length? (Now it's a STEM activity!) A piecewise function could be used to ensure the stem and vessel curves join smoothly.
- To increase the complexity of the task the thickness of the glass could also be included in the modelling. Is it okay to simply translate the function modelling the outside of the glass or does this pose a problem with regards to glass thickness?

A range of calculus concepts can be dealt with in this task. If you are really adventurous, try using a 3D printer to produce a real model of your perfect wine glass.



Students often struggle with an open investigation, starting with a similar problem can help students continue independently. Keeping with the same context, the class might start by designing a 'white wine' glass with a different set of parameters and then be required to design the remaining glasses in the 'set'.

Roller Coasters

[Investigation]

Theme park rides provide a wonderful context for mathematical modelling. The construction of a roller coaster is a marvellous example of engineering, mathematics and physics. From a technology aspect students can get visual feedback by placing an image in the background of the graphing screen. Students can build an understanding of how the degree of a polynomial determines features such as the quantity of turning points, the nature of odd and even degree polynomials and curvature.

In the example shown here a quartic function has been used to represent a section of the roller coaster. The x axis was aligned approximately with the horizon and the image scaled using known information. The turning points on the function were used to model the peaks and troughs of the track which provided no freedom to adjust the curvature to align the remaining sections, even those within the domain of the model. Students are invited to consider other functions, including piecewise functions that may produce a more appropriate model with greater flexibility.

If an appropriate scale is set, students can also include calculations of:

- Ride length
- Gravitational potential energy
- Kinetic energy
- Speed
- Time

Some or all of these calculations can be done automatically using a program that relies on the function stored in $f_1(x)$ which leaves students to analyse and interpret. The calculations are approximate and based on the quantity of intervals used. From a visual perspective it is easy to follow when speed is graphed against position.

Length of track sections: calculated using arc-length $\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ [Stored in 'distance']

Speed: Loss of potential energy transferred to kinetic. [Stored in 'speed']

Time: Based on speed and track length. [Stored in 'time']

X Position: Based on number of sections calculated. [Stored in 'xp']

Note: The initial position (A) entered into the program **must** be the highest point on the ride, otherwise the initial kinetic energy would be required, so too the mass of the roller coaster.

We can easily see why mass is not required if we take A to be the start of the journey and friction is ignored. If 100% of the potential energy is transferred to kinetic energy then:

$E_p = mgh$ and $E_k = \frac{1}{2}mv^2$

$\therefore v = \sqrt{2gh}$ where *h* is the overall change in height determined by the function.

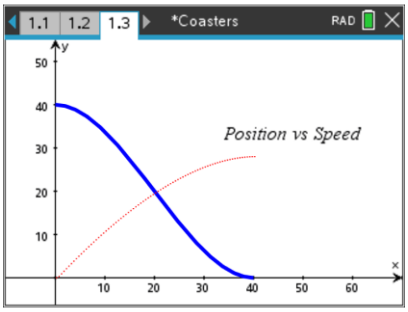
Students can also explore how concavity affects the ride experience.

Water Slides

[Investigation]

For an easier starting context, students can consider water slides that typically have downhill sections only. There are numerous water slides around the world worth studying. “Summit Plummet” at Disney’s Blizzard theme park in Florida is a relatively simple design. Consisting of a starting height of approximately 40 metres and an almost vertical drop, riders can achieve speeds of approximately 100km/h. The radar located at the base of the drop displays rider speeds and boasts the highest speed of the day.

The following curves are all modelled on a drop of 40 metres over a horizontal distance of 40 meters. The blue line on each graph represents the shape of the water slide; the red dotted line is the speed at each point on the slide, assuming zero resistance. Note that for water slides it is necessary to have a decreasing function!

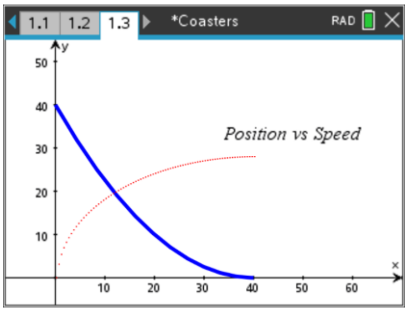


Equation: $y = \frac{x^3}{800} - \frac{3x^2}{40} + 40$

Time: 6.78s

Length: 58.8m

Average speed: 8.67m/s (31.18km/h)

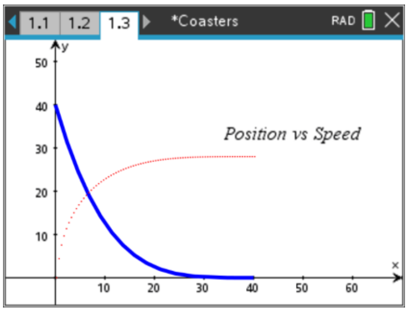


Equation: $y = \frac{(x-40)^2}{40}$

Time: 3.63s

Length: 59.66m

Average Speed: 16.44m/s (59.18km/h)

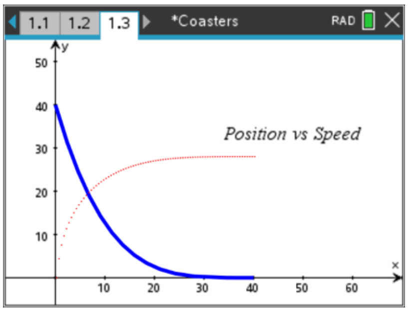


Equation: $y = \frac{(x-40)^4}{40^3}$

Time: 3.61s

Length: 64.5m

Average Speed: 17.87m/s (64.3km/h)



Equation: $y = 40e^{\frac{-x}{4}}$

Time: 3.64s

Length: 69.9m

Average Speed: 19.21m/s (69.2km/h)

Note: All calculations above are approximate based on 0.5m track intervals. Maximum speed is the same for all rides as the program assumes no energy loss.