THE UNIVERSITY OF
MELBOURNE

Developing fractional and algebraic thinking in middle years

Catherine Pearn & Max Stephens

The University of Melbourne

Introduction

Many researchers argue that a deep understanding of fractions is important for a successful transition to algebra.

The links between fractional knowledge and readiness for algebra have been highlighted by researchers such as Jacobs, Franke, Carpenter, Levi, and Battey, (2007) and Empson, Levi, and Carpenter, (2011).

Researchers such Kieren (1980) and Lamon (1999) believe that the basis for algebraic thought rests on a clear understanding of rational number concepts and the ability to manipulate common fractions. According to Wu (2001) the ability to efficiently manipulate fractions is: "vital to a dynamic understanding of algebra" (p. 17).

Siegler and colleagues (2012) used longitudinal data from both the USA and UK to show that, when other factors were controlled, competence with fractions and division in fifth or sixth grade is a uniquely accurate predictor of students' attainment in algebra and overall mathematics performance five or six years later.

Questions for today's session

- How does middle-years students' fractional competence and reasoning show evidence of emerging algebraic reasoning?
- Does the Structured Interview provide clear evidence of students' ability to generalise their solutions based on variations of a set of particular instances?
- Are similar strategies used by Australian and Chinese students in their written responses to the three reverse fraction tasks?

Algebraic Reasoning

Jacobs, Franke, Carpenter, Levi, and Battey (2007) emphasise the need to:

> … facilitate students' transition to the formal study of algebra in the later grades (of the elementary school) so that no distinct boundary exists between arithmetic and algebra" (p.261).

Three distinct aspects of algebraic reasoning identified by Jacobs et al. (2007) and by Stephens and Ribeiro (2012) are important for this study. They are students' understanding of:

- equivalence
- transformation using equivalence
- the use of generalisable methods

Algebraic Reasoning

Algebraic reasoning is a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways."

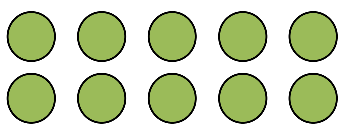
(Kaput & Blanton, 2005, p. 99)

In this session, algebraic reasoning is seen in terms of students' capacity to identify an equivalence relationship between a given collection of objects and the fraction this collection represents of an unknown whole, and then operate multiplicatively on both to find the whole.

Reverse Fraction Tasks (FST*)

Reverse Fraction Task 1

This collection of 10 counters is $^{2}/_{3}$ of the number of counters I started with.



How many counters did I start with? Explain how you decided your answer is correct.

Reverse Fraction Task 2

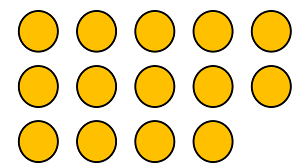
Susie's CD collection is $^{4}/_{7}$ of her friend Kay's. Susie has 12 CDs.

How many CDs does Kay have? ____

Show all your working.

Reverse Fraction Task 3

This collection of 14 counters is $^{7}/_{6}$ of the number of counters I started with.



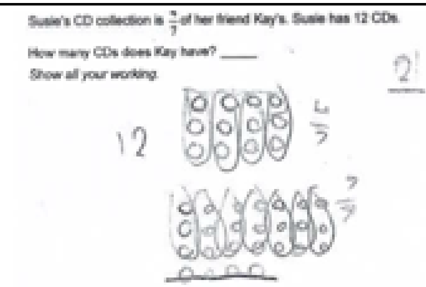
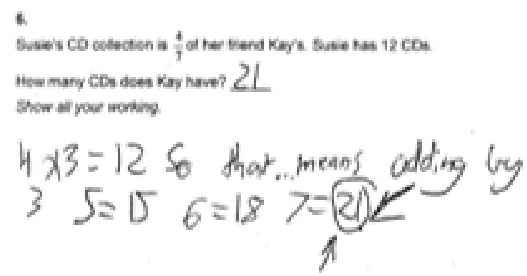
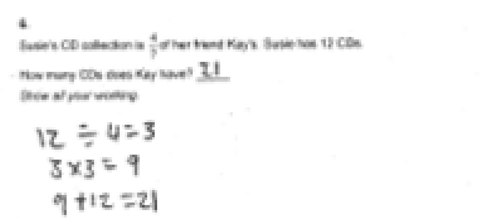
How many counters did I start with? Explain how you decided that your answer is correct.

* **Fraction Screening Test** (See for example, Pearn, Pierce & Stephens, 2017)

Reverse Fraction Tasks

1. Solve the three reverse fraction tasks using methods familiar to Years 5 – 8 students.
2. Write down your thinking. Include diagrams as needed.
3. Rank the questions in order of difficulty.
4. Identify possible misconceptions/difficulties students might encounter.
5. Classify your responses (in groups of 3) according to the framework.
6. Which kinds of responses would you expect from ‘your’ students in Years 5 – 8?

Framework for reverse fraction task strategies: Task 2

Classification	Explanation	Example
Diagram dependent	Students use explicit partitioning of diagrams before using additive or subtractive strategies	 Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? ____ Show all your working. 21 12
Additive/ subtractive	Students use additive or subtractive methods without explicit partitioning a diagram. Students find the number of objects needed to represent the unit fraction and then use counting or repeated addition to find the number of objects needed to find the whole.	 6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? 21 Show all your working. 4 x3 = 12 So that means adding by 3 5= 15 6=18 7=21
Partially multiplicative	Students use both multiplicative and additive methods. In the example, they calculate the missing fractional part (3/7) and then add it onto the original quantity.	 6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? 21 Show all your working. 12 ÷ 4 = 3 3 x 3 = 9 9 + 12 = 21

Framework for reverse fraction task strategies: Task 2

Fully multiplicative	Students use fully multiplicative methods. Students find the quantity represented by the unit fraction using division and then multiply the quantity of the unit fraction to find the whole.	6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? 21 Show all your working. $12 \div 4 = 3$ $3 \times 7 = 21$ (CDs)
Advanced multiplicative	Students use more advanced multiplicative methods to solve the reverse fraction questions. These include the correct use of appropriate algebraic notation to find the whole, or a one-step method to find the whole by dividing the given quantity by the known fraction.	Suzie = 12 Kay = x $12 = \frac{4}{7}x$ $12 \div \frac{4}{7}$ $= 3 \times 7 = 21$ $x = 21$

Our sample

Year Level	Australian students	Chinese students
Year 5	9	6
Year 6	8	6
Total	17	12

These students:

- completed three reverse fraction tasks and successfully solved and explained their solutions to at least two of the three tasks
- were interviewed using the Structured Interview

Classification of students' responses

Response strategy	Fraction Task 1		Fraction Task 2		Fraction Task 3	
	AU	CH	AU	CH	AU	CH
Incomplete				1		1
Diagram dependent			1		1	
Additive/subtractive			1			
Partially multiplicative	10		5		9	
Fully multiplicative	6	5	9	4	6	4
Advanced multiplicative	1	7	1	7	1	7

- Australian students used a range of strategies
- Chinese students used either fully or advanced multiplicative strategies

Examples of ***Advanced multiplicative*** strategy

Reverse Fraction Task 1	Reverse Fraction Task 2	Reverse Fraction Task 3
$\frac{2}{3}x = 10$ $\frac{2}{3}x \div \frac{2}{3} = 10 \div \frac{2}{3}$ $x = 15$	$\frac{4}{7}x = 12$ $x = 12 \times \frac{7}{4}$ $x = 21$	$\frac{7}{6}x = 14$ $\frac{7}{6}x \div \frac{7}{6} = 14 \div \frac{7}{6}$ $x = 12$
$10 \div \frac{2}{3} = \cancelto{5}{10} \times \frac{3}{\cancelto{1}{2}} = 15$ (个)	$12 \div \frac{4}{7} = \cancelto{3}{12} \times \frac{7}{\cancelto{1}{4}} = 21$ (张)	$14 \div \frac{7}{6} = \cancelto{2}{14} \times \frac{6}{\cancelto{1}{7}} = 12$ (个)
$10 \div 2 \times 3 = 15$ (个)	$12 \div 4 \times 7 = 21$	$14 \div 7 \times 6 = 12$

Results for reverse fraction tasks

The three reverse fraction tasks were in a format that would have been unfamiliar to both Australian and Chinese students.

Successful Chinese students used either fully or advanced multiplicative strategies whereas Australian students tended to use a range of strategies including partially multiplicative strategies.

The Structured Interview responses demonstrated that students' strategies varied from strictly arithmetical (computational fluency), beginning to generalise (additive, multiplicative) to the fully generalised demonstration of algebraic reasoning (verbal, symbolic).

The Structured Interview

The Structured Interview was designed to provide stronger evidence of generalised thinking (Pearn & Stephens, 2017).

The Structured Interview includes reverse fraction tasks similar to three reverse fraction tasks but with progressive levels of abstraction, moving from particular examples and becoming more generalised.

Students were only interviewed if they had successfully solved at least two of the three reverse fraction tasks.

Did students use same strategies for the responses to the interview tasks as they had for the written tasks?

The Structured Interview

First variation (Questions 1, 2, 3).

- number quantities change
- fractions remain the same as the reverse fraction tasks
- no diagrams

1. Imagine that I gave you 12 counters which is $^{2}/_{3}$ of the number of counters I started with.
How many counters did I start with?
Explain your thinking.

2. Susie has 8 CDs.
Her CD collection is $^{4}/_{7}$ of her friend Kay's.
How many CDs does Kay have? ______
Explain your thinking.

3. Imagine that I gave you 21 counters which is $^{7}/_{6}$ of the number of counters I started with
How many counters did I start with?
Explain your thinking.

The Structured Interview

Second variation (Questions 4, 5, 6)

- same three fractions
- different number quantities
- introduction of '*any number*' in part b

4a. If I gave you 18 counters, which is $^2/_3$ of the number of counters I started with, how would you find the number of counters I started with?	4b. If I gave you any number of counters, which is also $^2/_3$ of the number I started with, what would you need to do to find the number of counters I started with?

The Structured Interview

Third variation (Question 7)
- '*any fraction*' combined with '*any number*' building on known fraction and any number (generalisation)

7. What if I gave you any number of counters, and they represented any fraction of the number of counters I started with, how would you work out the number of counters I started with? Can you tell me what you would do? Please write your explanation in your own words.

The Structured Interview

1. Work in pairs. One person interviews, the other completes the tasks.
2. Record responses.
3. As pair what information did you get? Did the Structured Interview lead the interviewee to change the methods used in the original three tasks.

The Emerging Algebraic Reasoning Framework

Level	Description
Computational fluency Partial	Solved only some questions with method restricted to given fractions and quantities.
Computational fluency Complete	Solved all questions with given fractions and quantities but unable to answer more than one question with 'any quantity'.
Generalising - Additive	Solved all questions with given fractions and quantities. Used additive or mixed methods to solve questions with 'any quantity'. No appropriate generalized multiplicative response for 'any fraction' and 'any quantity'.
Generalising-Multiplicative	Solved all questions with given fraction and 'any quantity' using multiplicative methods. No appropriate generalised response to 'any fraction' and 'any quantity'.
Algebraic generalisation - Verbal	Solved all questions with known fractions and 'any quantity' using consistent multiplicative methods. Students verbalised but did not symbolise full generalisation to 'any fraction' and 'any quantity'.
Algebraic generalisation - Symbolic	Solved all questions with known fractions and 'any quantity' and generalised using consistent multiplicative methods. Appropriate algebraic notation used to solve 'any fraction' and 'any quantity' task.

The Emerging Algebraic Reasoning Framework

Level of developing algebraic reasoning	Australian		Chinese	
	Year 5 (n = 9)	Year 6 (n = 8)	Year 5 (n = 6)	Year 6 (n = 6)
1. Computational fluency - partial			1	2
2. Computational fluency - complete	1		1	
3. Generalising - additive	1	3	2	1
4. Generalising - multiplicative	1	2		1
5. Algebraic generalisation - verbal	5	1	1	
6. Algebraic generalisation - symbolic	1	2	1	2

Students' strategies varied from:

- strictly arithmetical (computational fluency)
- beginning to generalise (additive, multiplicative)
- fully generalised demonstration of algebraic reasoning (verbal, symbolic)

Results from Structured Interview

- Many students calculated the number of objects in the whole group when given a specific fraction and the number of objects representing that fractional part.
- When students were given ‘any number of counters’ some gave no response while one explained: “I can’t do this because I don’t know the number of circles.” There was no evidence of generalisation in this group.
- Some students demonstrated that they were *beginning* to generalise using strategies for the tasks with ‘any number of counters’ but were unable to complete the third variation with ‘any fraction’ and ‘any number’.
- Students who were successful with the third variation (‘any fraction’, ‘any number’ used a range of explanations, verbal and symbolic, to fully generalise their solutions.

What have we learned and what can we do?

Need to recognise that there are many successful approaches to solving the original three tasks.

Students may choose an additive or diagrammatic strategy because it is the easiest one available. While some are capable of multiplicative thinking you need to identify those who depend on additive and diagrammatic methods.

Variations in the fractions and the numbers is important for providing students an opportunity to see patterns and relationships.

Moving from additive/diagrammatic to multiplicative cannot be done in one step. Students need to use a mix of additive and multiplicative methods at first.

Giving students a rule such as ‘divide by the fraction’ will give answers but wont necessarily allow generalisation and promote algebraic thinking.

The bar model (Singapore) is one powerful representation that can be used in conjunction with other materials.

THE UNIVERSITY OF
MELBOURNE

Thank you

Cath Pearn:
cpearn@unimelb.edu.au

Max Stephens:
m.Stephens@unimelb.edu.au