THE MATHEMATICAL ASSOCIATION OF VICTORIA

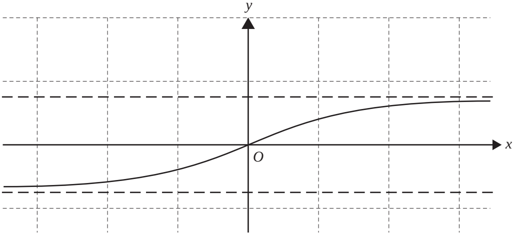
MAV Conference 2019

2018 Specialist Mathematics Exam 2

December 5 2019

Question 1

Part of the graph of $y = 12\tan^{-1}(x)$ is shown below.



The equations of its asymptotes are

A $y = \pm\dfrac{1}{2}$

B $y = \pm\dfrac{3}{4}$

C $y = \pm 1$

D $y = \pm\dfrac{\pi}{2}$

E $y = \pm\dfrac{\pi}{4}$

Question 1

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Solution

The horizontal asymptotes of $y = \tan^{-1} x$ are $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$. Therefore the asymptotes of $y = \frac{1}{2} \tan^{-1} x$ are $y = \frac{\pi}{4}$ and $y = -\frac{\pi}{4}$.

ANS: E (85%)

Question 2

Consider the function f with rule $f(x) = \dfrac{1}{\sqrt{\sin^{-1}(cx+d)}}$, where $c, d \in R$ and $c > 0$.

The domain of f is

A $x > -\dfrac{d}{c}$

B $-\dfrac{d}{c} < x \leq \dfrac{1-d}{c}$

C $\dfrac{-1-d}{c} \leq x \leq \dfrac{1-d}{c}$

D $x \in R \setminus \left\{-\dfrac{d}{c}\right\}$

E $x \in R$

Solution

We require that $\sin^{-1}(cx+d) > 0$. Therefore $0 < cx + d \leq 1$ and so $-\dfrac{d}{c} < x \leq \dfrac{1-d}{c}$ (since $c > 0$).

ANS: B (60%)

Question 3

Which one of the following, where A, B, C and D are non-zero real numbers, is the partial fraction form for the expression $\dfrac{2x^2+3x+1}{(2x+1)^3\,(x^2-1)}$?

A $\dfrac{A}{2x+1}+\dfrac{B}{x-1}+\dfrac{C}{x+1}$

B $\dfrac{A}{2x+1}+\dfrac{B}{(2x+1)^2}+\dfrac{C}{(2x+1)^3}+\dfrac{Dx}{x^2-1}$

C $\dfrac{A}{2x+1}+\dfrac{Bx+C}{x^2-1}$

D $\dfrac{A}{2x+1}+\dfrac{B}{(2x+1)^2}+\dfrac{C}{x-1}$

E $\dfrac{A}{2x+1}+\dfrac{Bx+C}{(2x+1)^2}+\dfrac{D}{x-1}$

Question 3

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Solution

The algebraic fraction $\dfrac{2x^2+3x+1}{(2x+1)^3\,(x^2-1)}$ simplifies to

$$\frac{1}{(2x+1)^2(x-1)}.$$

Therefore $\dfrac{2x^2+3x+1}{(2x+1)^3\,(x^2-1)}$ can be expressed in the form

$$\frac{A}{(2x+1)}+\frac{B}{(2x+1)^2}+\frac{C}{x-1}.$$

Note that option **E** is not valid as the question states that the real constants A, B, C and D are non-zero.

ANS: D (46%)

Question 4

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If $\cos(x) = -a$ and $\cot(x) = b$, where $a, b > 0$, then $\operatorname{cosec}(-x)$ is equal to

A $\dfrac{b}{a}$

B $-\dfrac{b}{a}$

C $-\dfrac{a}{b}$

D $\dfrac{a}{b}$

E $-ab$

Question 4

Solution

We have that

$$\operatorname{cosec}(-x) = \frac{1}{\sin(-x)} = \frac{-1}{\sin x}.$$

Now since

$$\cot x = \frac{\cos x}{\sin x} = b$$

and $\cos x = -a$ we have that

$$\frac{-1}{\sin x} = \frac{b}{a}.$$

ANS: A (49%)

Question 5

Let $z = a + bi$, where $a,\ b \in R \setminus \{0\}$.

If $z + \dfrac{1}{z} \in R$, which one of the following must be **true**?

A $\operatorname{Arg}(z) = \dfrac{\pi}{4}$

B $a = -b$

C $a = b$

D $|z| = 1$

E $z^2 = 1$

Question 5

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Solution

If $z = a + bi$ then

$$z + \frac{1}{z} = a + \frac{a}{a^2 + b^2} + \left(b - \frac{b}{a^2 + b^2}\right).$$

If $z + \dfrac{1}{z} \in R$ and $a, b \in R \setminus \{0\}$ then $b - \dfrac{b}{a^2 + b^2} = 0$ and so

$$\begin{aligned} b &= \frac{b}{a^2 + b^2} \\ \Rightarrow 1 &= \frac{1}{a^2 + b^2} \\ \Rightarrow 1 &= a^2 + b^2 \end{aligned}$$

ANS: D (41%)

Question 6

The complex numbers z, iz and $z + iz$, where $z \in C \setminus \{0\}$, are plotted in the Argand plane, forming the vertices of a triangle.
The area of this triangle is given by

A $|z|$

B $|z| + |z|^2$

C $\dfrac{|z|^2}{2}$

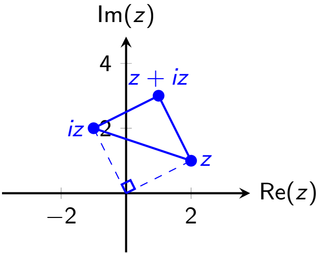
D $|z|^2$

E $\dfrac{\sqrt{3}|z|^2}{2}$

Question 6

Solution

Draw a diagram:



The area is $\frac{1}{2}|z|^2$.

ANS: C (58%)

Question 7

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A curve is described parametrically by $x = \sin(2t)$, $y = 2\cos(t)$ for $0 \le t \le 2\pi$. The length of the curve is closest to

A 9.2

B 9.5

C 12.2

D 12.5

E 38.3

Solution

Use length of the curve is

$$\int_0^{2\pi} \sqrt{(2\cos(2t))^2 + (-2\sin t)^2}\, dt = 12.1944.$$

ANS: C (78%)

Question 8

Using a suitable substitution, $\int_0^{\frac{\pi}{6}} \tan^2(x)\sec^2(x)\,dx$ can be expressed as

A $\int_0^{\frac{1}{\sqrt{3}}} \left(u^4 + u^2\right)\,du$

B $\int_1^{\frac{2}{\sqrt{3}}} \left(u^4 + u^2\right)\,du$

C $\int_0^{\frac{1}{\sqrt{3}}} u\,du$

D $\int_0^{\frac{\pi}{6}} u^2\,du$

E $\int_0^{\frac{1}{\sqrt{3}}} u^2\,du$

Question 8

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Solution

Let $u = \tan x$. Then $\dfrac{du}{dx} = \sec^2 x$ and so $du = \sec^2 x\, dx$.

When $x = 0$, $u = 0$ and when $x = \frac{\pi}{6}$, $u = \frac{1}{\sqrt{3}}$. Therefore the integral can be expressed as

$$\int_0^{\frac{1}{\sqrt{3}}} u^2\, du.$$

ANS: E (71%)

Question 9

A solution to the differential equation $\dfrac{dy}{dx} = \dfrac{2}{\sin(x+y) - \sin(x-y)}$ can be obtained from

A $\displaystyle\int 1\, dx = \int 2\sin(y)\, dy$

B $\displaystyle\int \cos(y)\, dy = \int \operatorname{cosec}(x)\, dx$

C $\displaystyle\int \cos(x)\, dx = \int \operatorname{cosec}(y)\, dy$

D $\displaystyle\int \sec(x)\, dx = \int \sin(y)\, dy$

E $\displaystyle\int \sec(x)\, dx = \int \operatorname{cosec}(y)\, dy$

Question 9

Solution

$\sin(x+y) - \sin(x-y) = 2\cos x \sin y$ and so the differential equation is

$$\frac{dy}{dx} = \frac{1}{\cos x \sin y}.$$

Separating gives

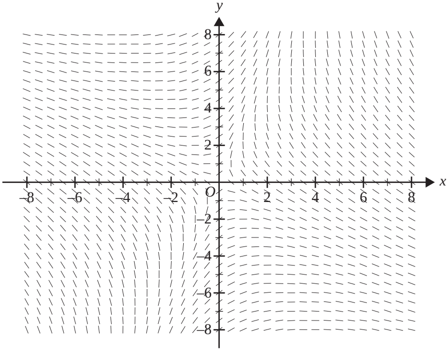
$$\frac{dx}{\cos x} = \sin y \, dy$$

and therefore

$$\int \sec x \, dx = \int \sin y \, dy.$$

ANS: D (69%)

Question 10



The differential equation that best represents the direction field above is

A $\dfrac{dy}{dx} = \dfrac{2x + y}{y - 2x}$

B $\dfrac{dy}{dx} = \dfrac{x + 2y}{2x - y}$

C $\dfrac{dy}{dx} = \dfrac{2x - y}{x + 2y}$

D $\dfrac{dy}{dx} = \dfrac{x - 2y}{y - 2x}$

E $\dfrac{dy}{dx} = \dfrac{2x + y}{2y - x}$

Question 10

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Solution

We can eliminate options until we arrive at the correct answer.
Note that (for instance) when $x = 3$ and $y = 6$ that $\frac{dy}{dx}$ is undefined. This eliminates options **C** and **E**.
If $y = 0$ then $\frac{dy}{dx} < 0$. This eliminates option **B**.
If $x = 0$ then $\frac{dy}{dx} > 0$. This eliminates option **D**.
The only remaining option is **A**.

ANS: A (65%)

Question 11

Consider the vectors given by $\underset{\sim}{a} = m\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j}$, where $m \in R$.
If the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 30°, then m equals

A $\sqrt{2} \pm 1$

B $2 \pm \sqrt{3}$

C $\sqrt{3}, \ \dfrac{1}{\sqrt{3}}$

D $\dfrac{\sqrt{3}}{4 - \sqrt{3}}$

E $\dfrac{\sqrt{39}}{13}$

Question 11

Solution

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 2m$ and $|\underset{\sim}{a}| = |\underset{\sim}{b}| = \sqrt{m^2+1}$. The acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 30° so

$$\cos 30^\circ = \frac{\sqrt{3}}{2} = \frac{2m}{m^2+1}.$$

Use CAS to solve this equation and find that $m = \sqrt{3}$ or $m = \dfrac{1}{\sqrt{3}}$.

ANS: C (80%)

Question 12

If $|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{a}| + |\underset{\sim}{b}|$ and $\underset{\sim}{a}, \underset{\sim}{b} \neq \underset{\sim}{0}$, which one of the following is **necessarily true**?

A $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$

B $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

C $\underset{\sim}{a} = \underset{\sim}{b}$

D $\underset{\sim}{a} = -\underset{\sim}{b}$

E $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$

Question 12

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Solution

It is necessarily true that if $|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{a}| + |\underset{\sim}{b}|$ then $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$.

ANS: A (36%)

Question 13

The position vector of a particle that is moving along a curve at time t is given by $\underset{\sim}{r}(t) = 3\cos(t)\underset{\sim}{i} + 4\sin(t)\underset{\sim}{j},\ t \geq 0$.
The **first** time when the speed of the particle is a minimum is

A 3

B $\dfrac{\pi}{2}$

C $\tan^{-1}\left(\dfrac{4}{3}\right)$

D $\dfrac{3\pi}{2}$

E 9

Question 13

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Solution

We have $\dot{\underset{\sim}{r}}(t) = -3\sin t\underset{\sim}{i} + 4\cos t\underset{\sim}{j}$ and so

$$|\dot{\underset{\sim}{r}}(t)|^2 = 9 + 7\cos^2 t.$$

This is first a minimum when $\cos t = 0$. That is, when $t = \dfrac{\pi}{2}$.

ANS: B (65%)

Question 14

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The scalar resolute of $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{k}$ in the direction of $\underset{\sim}{b} = -\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ is

A $-\dfrac{9\sqrt{13}}{13}$

B $-\dfrac{9}{14}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$

C $-\dfrac{9\sqrt{14}}{14}$

D $-\dfrac{9}{13}\left(3\underset{\sim}{i} - 2\underset{\sim}{k}\right)$

E $-\dfrac{\sqrt{14}}{2}$

Question 14

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Solution

$\underset{\sim}{a} \cdot \underset{\sim}{b} = -3 - 6 = -9$ and $|\underset{\sim}{b}| = \sqrt{1+4+9} = \sqrt{14}$. Therefore $\underset{\sim}{a} \cdot \widehat{\underset{\sim}{b}} = -\dfrac{9}{\sqrt{14}}$.

ANS: C (75%)

Question 15

A constant force of magnitude P newtons accelerates a particle of mass $8\,\text{kg}$ in a straight line from a speed of $4\,\text{m}\,\text{s}^{-1}$ to a speed of $20\,\text{m}\,\text{s}^{-1}$ over a distance of $15\,\text{m}$.
The magnitude of P is

A 9.8

B 12.5

C 12.8

D 100

E 102.4

Question 15

Solution

Use the formulas $v^2 = u^2 + 2as$ and $F = ma$. First we find the acceleration:

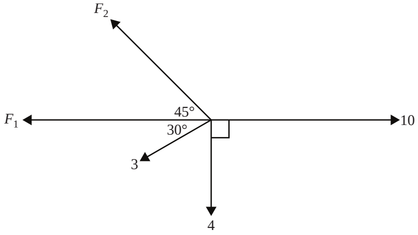
$$20^2 = 4^2 + 30a \Rightarrow a = \frac{64}{5}.$$

Then $F = 8 \times \dfrac{64}{5} = 102.4.$

ANS: E (69%)

Question 16

The diagram below shows a mass being acted on by a number of forces whose magnitudes are labelled. All forces are measured in newtons and the system is in equilibrium.



The value of F_2 is

A $\frac{\sqrt{2}}{2}\left(8+3\sqrt{3}\right)$

B $\frac{11\sqrt{2}}{2}$

C $\frac{3\sqrt{2}}{2}$

D 7.78

E 7.0

Question 16

Solution

Consider the forces acting vertically:

$$\frac{1}{\sqrt{2}}F_2 = \frac{1}{2} \times 3 + 4 = \frac{11}{2}.$$

Therefore $F_2 = \dfrac{11\sqrt{2}}{2}$.

ANS: B (64%)

Question 17

A tourist standing in the basket of a hot air balloon is ascending at $2\,\mathrm{m\,s^{-1}}$. The tourist drops a camera over the side when the balloon is 50 m above the ground.

Neglecting air resistance, the time in seconds, correct to the nearest tenth of a second, taken for the camera to hit the ground is

A 2.3

B 2.4

C 3.0

D 3.2

E 3.4

Question 17

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Solution

Use the formula $s = ut + \frac{1}{2}at^2$. Setting the positive direction as down we have

$$50 = -2t + \frac{1}{2} \times 9.8t^2$$

and so $t \approx 3.4$ seconds.

ANS: E (48%)

Question 18

A 95% confidence interval for the mean height μ, in centimetres, of a random sample of 36 Irish setter dogs is $58.42 < \mu < 67.31$.
The standard deviation of the height of the population of Irish setter dogs, in centimetres, correct to two decimal places, is

A 2.26

B 2.27

C 13.60

D 13.61

E 62.87

Question 18

Solution

The mean $\overline{x} = \dfrac{58.42 + 67.31}{2} = 62.865$. The standard deviation s is found by solving

$$62.865 + 1.96 \times \frac{s}{\sqrt{36}} = 67.31.$$

We find that s is closest to 13.61.

ANS: D (62%)

Question 19

The gestation period of cats is normally distributed with mean $\mu = 66$ days and variance $\sigma^2 = \dfrac{16}{9}$.

The probability that a sample of five cats chosen at random has an average gestation period greater than 65 days is closest to

A 0.5000

B 0.7131

C 0.7734

D 0.8958

E 0.9532

Question 19

Solution

$E\left(\overline{X}\right) = 66$ and $\text{Var}\left(\overline{X}\right) = \frac{\frac{16}{9}}{5} = \frac{16}{45}$.

Then $\Pr\left(\overline{X} > 65\right) = 0.9532$.

ANS: E (57%)

Question 20

The scores on the Mathematics and Statistics tests, expressed as percentages, in a particular year were both normally distributed. The mean and the standard deviation of the Mathematics test scores were 71 and 10 respectively, while the mean and the standard deviation of the Statistics test scores were 75 and 7 respectively.
Assuming the sets of test scores were independent of each other, the probability, correct to four decimal places, that a randomly chosen Mathematics score is higher than a randomly chosen Statistics score is

A 0.2877

B 0.3716

C 0.4070

D 0.7123

E 0.9088

Question 20

Solution

We have that $M \sim N\left(71, 10^2\right)$ and $S \sim N\left(75, 7^2\right)$. Let $X = M - S$. Then

$$\begin{aligned} E(X) = E(M - S) &= E(M) - E(S) \\ &= 71 - 75 \\ &= -4 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(X) = \mathrm{Var}(M - S) &= \mathrm{Var}(M) + \mathrm{Var}(S) \\ &= 10^2 + 7^2 \\ &= 149. \end{aligned}$$

So, $Pr(X > 0) = 0.3716$.

ANS: B (56%)

Consider the function $f : D \to R$, where $f(x) = 2\arcsin\left(x^2 - 1\right)$.

a. Determine the maximal domain D and the range of f. (2 marks)

Solution

dom $f = D = \left[-\sqrt{2}, \sqrt{2}\right]$ and ran $f = [-\pi, \pi]$.

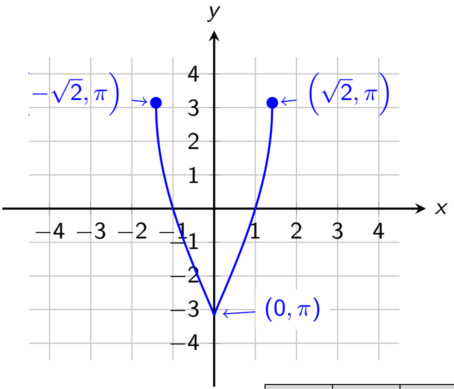
Marks	0	1	2	Average
%	16	16	68	1.5

Issues:
- open endpoints
- decimal approximations
- failing to state the range

Question 1 II

b. Sketch the graph of $y = f(x)$ on the axes below, labelling any endpoints and the y-intercept with their coordinates. (3 marks)

Solution



Issues:
- no labels as required
- precision
- y-intercept shown as a turning point

Advice: Set CAS window to match the exam.

Marks	0	1	2	3	Average
%	6	13	32	50	2.3

c. Find $f'(x)$ for $x > 0$, expressing your answer in the form $f'(x) = \dfrac{A}{\sqrt{2-x^2}}$, $A \in R$. (1 mark)

Solution

$$f'(x) = \frac{4}{\sqrt{2-x^2}}, \; x > 0.$$

Marks	0	1	Average
%	20	80	0.8

d. Write down $f'(x)$ for $x < 0$, expressing your answer in the form $f'(x) = \dfrac{B}{\sqrt{2-x^2}}$, $B \in R$. (1 mark)

Solution

$$f'(x) = \frac{-4}{\sqrt{2-x^2}}, \; x < 0.$$

Marks	0	1	Average
%	19	81	0.8

Question 1 IV

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e. The derivative $f'(x)$ can be expressed in the form $f'(x) = \dfrac{g(x)}{\sqrt{2-x^2}}$ over its maximal domain.

i. Find the maximal domain of f'. (1 mark)

Not well done.
- endpoints included
- 0 not excluded

Solution

dom $f' = \left(-\sqrt{2}, 0\right) \cup \left(0, \sqrt{2}\right) = \left(-\sqrt{2}, \sqrt{2}\right) \setminus \{0\}$.

Marks	0	1	Average
%	79	21	0.2

ii. Find $g(x)$, expressing your answer as a piecewise (hybrid) function. (1 mark)

Solution

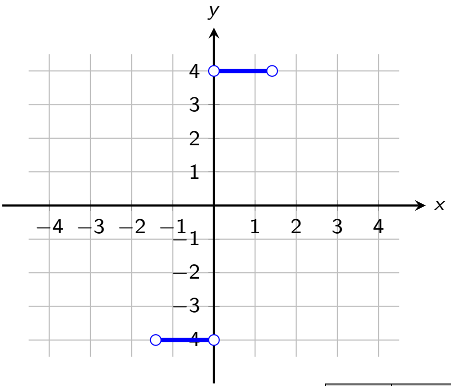
$$g(x) = \begin{cases} 4 & 0 < x < \sqrt{2} \\ -4 & -\sqrt{2} < x < 0 \end{cases}$$

Marks	0	1	Average
%	51	49	0.5

Some answers were not functions.

iii. Sketch the graph of g on the axes below. (2 marks)

Solution



Success followed from part ii.

Many attempts to graph f'(x) were seen.

Marks	0	1	2	Average
%	43	18	39	1

a. State the centre in the form (x, y), where $x, y \in R$, and state the radius of the circle given by $|z - (1 + 2i)| = 2$, where $z \in C$. (1 mark)

Solution

The centre is $(1, 2)$ and the radius is 2.

Marks	0	1	Average
%	30	71	0.7

Incomplete answers were seen.

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b. By expressing the circle given by $|z+1| = \sqrt{2}\,|z-i|$ in cartesian form, show that this circle has the same centre and radius as the circle given by $|z-(1+2i)| = 2$. (2 marks)

Solution

Let $z = x + iy$. Then

$$\begin{aligned}
|z+1|^2 = 2|z-i|^2 &\Rightarrow (x+1)^2 + y^2 = 2\left(x^2 + (y-1)^2\right) \\
&\Rightarrow x^2 + 2x + 1 + y^2 = 2x^2 + 2y^2 - 4y + 2 \\
&\Rightarrow x^2 + y^2 - 4y - 2x + 1 = 0 \\
&\Rightarrow (x-1)^2 - 1 + (y-2)^2 - 4 + 1 = 0 \\
&\Rightarrow (x-1)^2 + (y-2)^2 = 4
\end{aligned}$$

This is a circle with centre $(1, 2)$, radius 2. "show that" - requires an explicit answer

Marks	0	1	2	Average
%	16	31	54	1.4

c. Graph the circle $|z+1| = \sqrt{2}\,|z-i|$ on the Argand diagram below, labelling the intercepts with the vertical axis. (2 marks)

Solution

The circle given by $|z+1| = \sqrt{2}|z-i|$ has Cartesian equation $(z-1)^2 + (y-2)^2 = 4$. The points of intersection of the circle with the vertical axis have coordinates $\left(0, 2-\sqrt{3}\right)$ and $\left(0, 2+\sqrt{3}\right)$.

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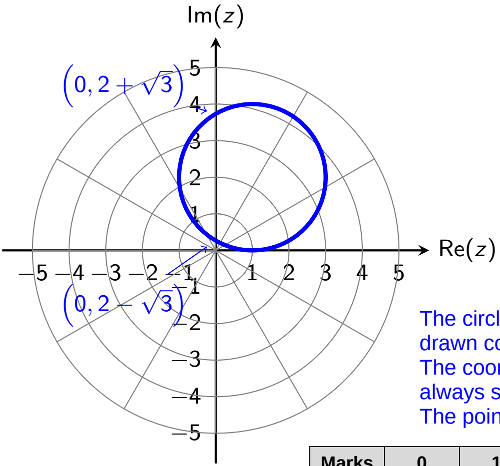
Solution - continued



The circle was generally drawn correctly.
The coordinates were not always shown.
The points are real numbers.

Marks	0	1	2	Average
%	23	19	58	1.4

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The line given by $|z-1| = |z-3|$ intersects the circle given by $|z+1| = \sqrt{2}\,|z-i|$ in two places.

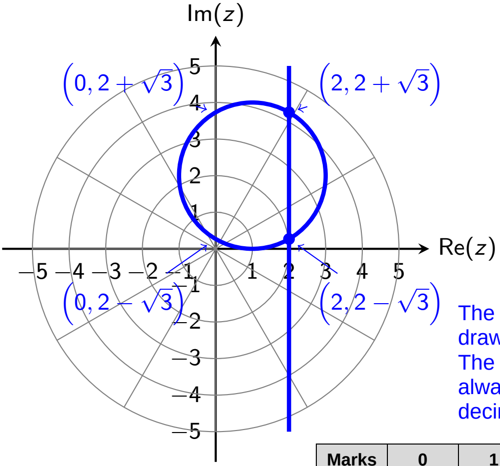
d. Draw the line given by $|z-1| = |z-3|$ on the Argand diagram in **part c.** Label the points of intersection with their coordinates. (2 marks)

Solution

The line given by $|z-1| = |z-3|$ has the Cartesian equation $x = 2$. The coordinates of the points of intersection of the circle with the line are $\left(2, 2+\sqrt{3}\right)$ and $\left(2, 2-\sqrt{3}\right)$.

Question 2 VI

Solution - continued



The line was generally drawn correctly.
The coordinates were not always shown or given as decimal approximations.

Marks	0	1	2	Average
%	27	18	55	1.3

e. Find the area of the minor segment enclosed by an arc of the circle given by $|z+1| = \sqrt{2}\,|z-i|$ and part of the line given by $|z-1| = |z-3|$.

(3 marks)

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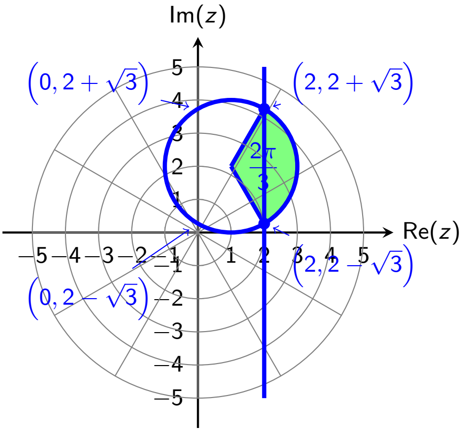
Solution

It is helpful to consider the diagram below:



Question 2 IX

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Solution - continued

The angle at the centre of the circle is $\frac{2\pi}{3}$ and so the area of the minor segment is

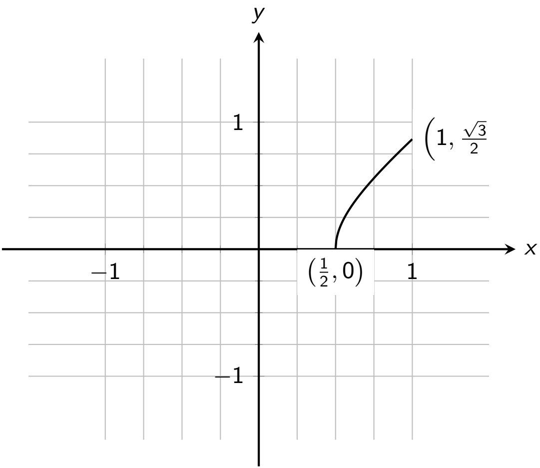
$$\frac{1}{2} \cdot 2^2 \left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right) \right) = 2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) = \frac{4\pi}{3} - \sqrt{3}.$$

Marks	0	1	2	3	Average
%	50	15	4	31	1.2

Standard formulas were more successful than integration approaches.
Many did not start with a correct sector angle.

Question 3 I

Part of the graph of $y = \frac{1}{2}\sqrt{4x^2 - 1}$ is shown below.



The curve shown is rotated about the y-axis to form a volume of revolution that is to model a fountain, where length units are in metres.

a. Show that the volume, V cubic metres, of water in the fountain when it is filled to a depth of h metres is given by $V = \frac{\pi}{4}\left(\frac{4}{3}h^3 + h\right)$. (2 marks)

Solution

Since $4y^2 = 4x^2 - 1$ we have $x^2 = \frac{4y^2+1}{4}$. So

$$
\begin{aligned}
V &= \pi \int_0^h x^2 \, dy \\
&= \frac{\pi}{4} \int_0^h \left(4y^2 + 1\right) \, dy \\
&= \frac{\pi}{4} \left[\frac{4}{3}y^3 + y\right]_0^h \\
&= \frac{\pi}{4} \left(\frac{4}{3}h^3 + h\right).
\end{aligned}
$$

About half the students set up a correct integral. Many of these did not explicitly show that this leads to the required volume.

Marks	0	1	2	Average
%	40	27	32	0.9

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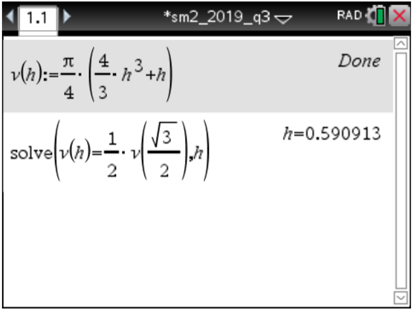
b. Find the depth h when the fountain is filled to half its volume. Give your answer in metres, correct to two decimal places. (2 marks)

Solution

The fountain is full when $h = \dfrac{\sqrt{3}}{2}$. So solve $V(h) = \dfrac{1}{2}V\left(\dfrac{\sqrt{3}}{2}\right)$. This gives $h = 0.59$ to two decimal places.



A variety of approaches was seen.

Failing to halve the volume was a common error.

Marks	0	1	2	Average
%	28	17	56	1.3

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The fountain is initially empty. A vertical jet of water in the centre fills the fountain at a rate of 0.04 cubic metres per second and, at the same time, water flows out from the bottom of the fountain at a rate of $0.05h$ cubic metres per second when the depth is h metres.

c. i. Show that $\dfrac{dh}{dt} = \dfrac{4 - 5\sqrt{h}}{25\pi\left(4h^2 + 1\right)}$. (2 marks)

Solution

We have that $\dfrac{dV}{dt} = 0.04 - 0.05\sqrt{h} = \dfrac{1}{100}\left(4 - 5\sqrt{h}\right)$ and $\dfrac{dV}{dh} = \dfrac{\pi}{4}\left(4h^2 + 1\right)$.

So

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV} \cdot \frac{dV}{dt} \\ &= \frac{4 - 5\sqrt{h}}{100 \times \frac{\pi}{4}\left(4h^2 + 1\right)} \\ &= \frac{4 - 5\sqrt{h}}{25\pi\left(4h^2 + 1\right)}\end{aligned}$$

"Show that..."
- an explicit solution is required, steps must be clear.
- correct use of brackets

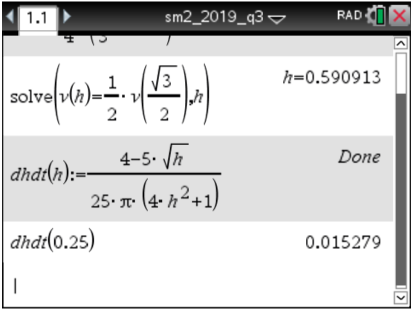
Marks	0	1	2	Average
%	34	30	36	1

Question 3 V

ii. Find the rate, in metres per second, correct to four decimal places, at which the depth is increasing when the depth is 0.25 m. (1 mark)

Solution

When $h = 0.25$, $\dfrac{dh}{dt} = 0.0153$ to four decimal places.



Most students were able to use the supplied derivative.

Some answers were not in the required decimal form.

Marks	0	1	Average
%	26	74	0.8

d. Express the time taken for the depth to reach 0.25 m as a definite integral and evaluate this integral correct to the nearest tenth of a second. (2 marks)

Solution

The time taken for the depth to reach 0.25 m is given by

$$\int_0^{0.25} \frac{dt}{dh}\,dh = \int_0^{0.25} \frac{25\pi\left(4h^2+1\right)}{4-5\sqrt{h}}\,dh$$
$$= 9.8.$$

Transcription errors in the integrand were occasionally present.

Marks	0	1	2	Average
%	34	10	56	1.2

So the time taken is 9.8 s.

e. After 25 seconds the depth has risen to 0.4 m. Using Euler's method with a step size of five seconds, find an estimate of the depth 30 seconds after the fountain began to fill. Give your answer in metres, correct to two decimal places. (2 marks)

Solution

Using Euler's method and noting that when $t = 25$ the depth is 0.4 m and $\dfrac{dh}{dt}$ when $t = 25$ is equal to $\left.\dfrac{dh}{dt}\right|_{h=0.4}$ we find

$$\begin{aligned} h(30) &\approx h(25) + 5 \times h'(25) \\ &= 0.4 + 5 \times \left.\frac{dh}{dt}\right|_{h=0.4} \\ &= 0.43. \end{aligned}$$

Many students did not explicitly demonstrate use of Euler's method as required.

Marks	0	1	2	Average
%	64	10	25	0.6

So the depth of the water in the fountain 30 seconds after it begins to fill is approximately 0.43 m.

f. How far from the top of the fountain does the water level ultimately stabilise? Give your answer in metres, correct to two decimal places. (2 marks)

Solution

$\dfrac{dh}{dt} = 0$ when $h = \dfrac{16}{25}$. So the water stabilises at $\dfrac{\sqrt{3}}{2} - \dfrac{16}{25} = 0.23$ m below the top of the fountain.

Most set h' = 0. Many did not subtract from the height.

Marks	0	1	2	Average
%	61	15	24	0.7

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Two yachts, A and B, are competing in a race and their position vectors on a certain section of the race after time t hours are given by

$$\underset{\sim}{r}_A(t) = (t+1)\underset{\sim}{i} + \left(t^2 + 2t\right)\underset{\sim}{j} \text{ and } \underset{\sim}{r}_B = t^2\underset{\sim}{i} + \left(t^2 + 3\right)\underset{\sim}{j}, \ \ t \geq 0$$

where displacement components are measured in kilometres from a given reference buoy at origin O.

a. Find the cartesian equation of the path for each yacht. (2 marks)

Solution

Consider yacht A: $x = t + 1 \Rightarrow t = x - 1$ and since $y = t^2 + 2t$ the Cartesian equation of the path of yacht A is

$$\begin{aligned} y &= (x-1)^2 + 2(x-1) \\ &= x^2 - 2x + 1 + 2x - 2 \\ &= x^2 - 1. \end{aligned}$$

For yacht B, note that $x = t^2$ and $y = t^2 + 3$ so the Cartesian equation of the path of yacht B is $y = x + 3$.

Marks	0	1	2	Average
%	11	8	81	1.7

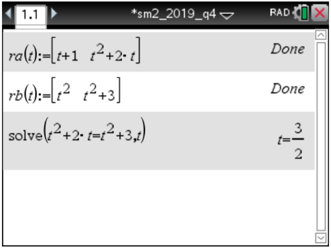
b. Show that the two yachts will not collide if they follow these paths.

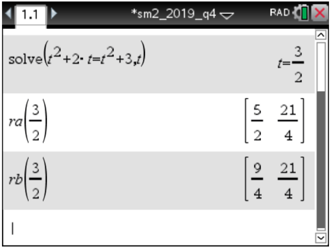
(2 marks)

Solution

The two yachts have the same y-coordinate when $t^2 + 2t = t^2 + 3 \Rightarrow t = \dfrac{3}{2}$.

Since $\underset{\sim}{r}_A\left(\dfrac{3}{2}\right) = \dfrac{5}{2}\underset{\sim}{i} + \dfrac{21}{4}\underset{\sim}{j}$ and $\underset{\sim}{r}_B\left(\dfrac{3}{2}\right) = \dfrac{9}{4}\underset{\sim}{i} + \dfrac{21}{4}\underset{\sim}{j}$ the two yachts do not collide.





A variety of correct approaches were seen.

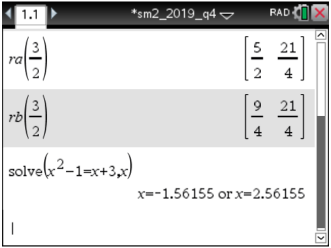
Marks	0	1	2	Average
%	19	8	72	1.6

Question 4 III

c. Find the coordinates of the point where the paths of the two yachts cross. Give your coordinates correct to three decimal places. (2 marks)

Solution

The paths of the yachts cross when $x^2 - 1 = x + 3$. The coordinates of the point where the paths cross are $(2.562, 5.562)$.



Many otherwise correct responses were not expressed in the required form.

Marks	0	1	2	Average
%	25	41	34	1.1

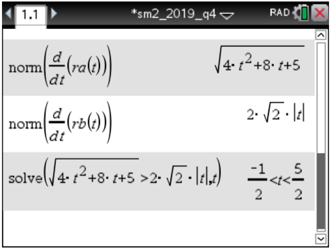
Question 4 IV

One of the rules for the race is that the yachts are not allowed to be within 0.2 km of each other. If this occurs there is a time penalty for the yacht that is travelling faster.

d. For what values of t is yacht A travelling faster than yacht B? (2 marks)

Solution

Using CAS we find that $|\dot{\underset{\sim}{r}}_A| = \sqrt{4t^2 + 8t + 5}$ and $|\dot{\underset{\sim}{r}}_B| = \sqrt{8t^2}$. So $|\dot{\underset{\sim}{r}}_A| > |\dot{\underset{\sim}{r}}_B|$ when $0 \le t < \frac{5}{2}$.



Common errors:

- comparing magnitudes of position vectors
- using velocities instead of speeds

Marks	0	1	2	Average
%	43	32	25	0.8

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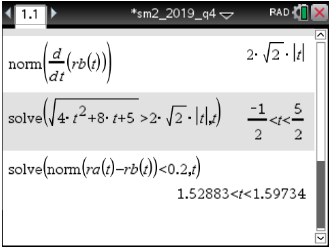
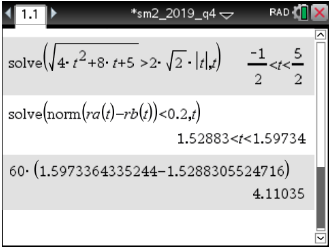
e. If yacht A does not alter its course, for what period of time will yacht A be within 0.2 km of yacht B? Give your answer in **minutes**, correct to one decimal place. (2 marks)

Solution

To determine the time when the two yachts are within 0.2 km of each other we must solve $|\underset{\sim}{r}_B(t) - \underset{\sim}{r}_A(t)| < 0.2$.

It is found that $1.53 < t < 1.60$ (two decimal places). So the time in minutes is 4.1 min.





A serious (but common) misconception:
- using a difference of magnitudes
Correct units: minutes were required.

Marks	0	1	2	Average
%	54	18	28	0.8

Question 5 I

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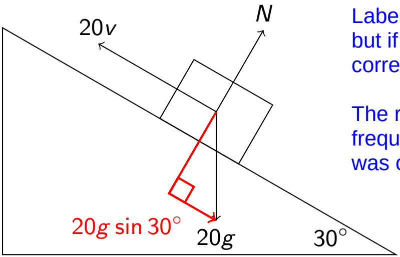
Luggage at an airport is delivered to its owners via a 15 m ramp that is inclined at 30° to the horizontal. A 20 kg suitcase, initially at rest at the top of the ramp, slides down the ramp against a resistance of v newtons per kilogram, where $v\ \mathrm{m\,s^{-1}}$ is the speed of the suitcase.

a. On the diagram below, show all forces acting on the suitcase during its motion down the ramp. (1 mark)

Solution



Labels were not required here but if given they should be correct (or general).

The resistance force was frequently labelled 'v'. (this was often corrected in part b.)

Marks	0	1	Average
%	56	44	0.5

b. i. By resolving forces parallel to the ramp, write down an equation of motion for the 20 kg suitcase. (1 mark)

Solution

Equation of motion, not just net force.

Considering the forces acting parallel to the plane only, with $F = ma$ we have

$$20a = 20g\sin 30^\circ - 20v$$
$$\Rightarrow 20a = 10g - 20v.$$

Marks	0	1	Average
%	41	59	0.6

ii. Hence, show that the magnitude of the acceleration, $a\,\mathrm{m\,s}^2$, of the suitcase down the ramp is given by $a = \dfrac{g-2v}{2}$. (1 mark)

Solution

Solve the equation from b. i. for a to find

$$a = \frac{1}{20}(10g - 20v)$$
$$= \frac{g-2v}{2}.$$

Marks	0	1	Average
%	34	66	0.7

c. By expressing a in an appropriate form, find the distance x metres that the suitcase has slid as a function of v. Give your answer in the form $x = bv + c\log_e\left(\dfrac{c}{c-v}\right)$, where $b, c \in R$. (2 marks)

Solution

Use the acceleration equivalent $v\dfrac{dv}{dx}$. So

$$v\frac{dv}{dx} = \frac{g-2v}{2}$$
$$\Rightarrow \int \frac{2v}{g-2v}\,dv = \int dx$$

Most students set up a DE using the appropriate form of acceleration.

and so

$$\begin{aligned} x &= \int \frac{2v-g+g}{g-2v}\,dv \\ &= \int \left(-1+\frac{g}{g-2v}\right) dv \\ &= -v - \frac{g}{2}\log_e(g-2v) + c \end{aligned}$$

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Solution - continued

Since $x(0) = 0$ we have

$$0 = -\frac{g}{2}\log_e g + c$$
$$\Rightarrow c = \frac{g}{2}\log_e g.$$

So

$$\begin{aligned} x &= -v - \frac{g}{2}\log_e(g - 2v) + \frac{g}{2}\log_e g \\ &= -v + \frac{g}{2}\log_e\left(\frac{g}{g - 2v}\right) \\ &= -v + \frac{g}{2}\log_e\left(\frac{\frac{g}{2}}{\frac{g}{2} - v}\right) \\ &= -v + 4.9\log_e\left(\frac{4.9}{4.9 - v}\right). \end{aligned}$$

Many students who solved the DE successfully did not give the solution in the required form.

Marks	0	1	2	Average
%	28	49	23	1

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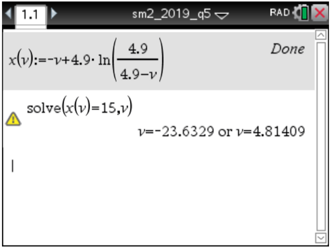
d. Find the velocity of the suitcase just before it reaches the end of the ramp. Give your answer in m s^{-1}, correct to two decimal places. (1 mark)

Solution

Use CAS to find that when $x = 15$, $v = 4.81$.



Difficulties with 5c. led to problems here.

Marks	0	1	Average
%	74	26	0.3

Question 5 VI

e. i. Write down a definite integral that gives the time taken for the suitcase to reach a speed of $4.5\,\mathrm{m\,s^{-1}}$. (1 mark)

Solution

Incorrect integrands - typically the reciprocal of the correct one.

Since $\dfrac{dt}{dv} = \dfrac{2}{g-2v}$, a definite integral that gives the time taken for the suitcase to reach a speed of $4.5\,\mathrm{m\,s^{-1}}$ is

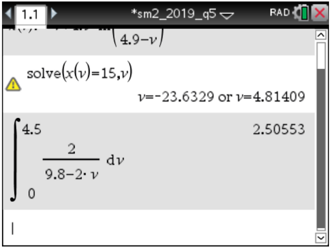
$$\int_0^{4.5} \frac{2}{g-2v}\,dv.$$

Marks	0	1	Average
%	59	41	0.4

ii. Find the time taken for the suitcase to reach a speed of $4.5\,\mathrm{m\,s^{-1}}$. Give your answer in seconds, correct to two decimal places. (1 mark)

Solution

The time in seconds taken for the suitcase to reach a speed of $4.5\,\mathrm{m\,s^{-1}}$ is 2.51.



Correct responses in part e.i. followed through to e.ii.

Marks	0	1	Average
%	59	41	0.4

The heights of mature water buffaloes in northern Australia are known to be normally distributed with a standard deviation of 15 cm. It is claimed that the mean height of the water buffaloes is 150 cm.
To decide whether the claim about the mean height is true, rangers selected a random sample of 50 mature water buffaloes. The mean height of this sample was found to be 145 cm.
A one-tailed statistical test is to be carried out to see if the sample mean height of 145 cm differs significantly from the claimed population mean of 150 cm.
Let $\overline{X}$ denote the mean height of a random sample of 50 mature water buffaloes.

a. State suitable hypotheses H_0 and H_1 for the statistical test. (1 mark)

Solution

$H_0 : \mu = 150$

$H_1 : \mu < 150$

Well done.
Common errors included poor notation or not showing a one-tailed test.

Marks	0	1	Average
%	30	70	0.7

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b. Find the standard deviation of $\overline{X}$. (1 mark)

Solution

Since $\text{Var}\left(\overline{X}\right) = \dfrac{15^2}{50} = \dfrac{9}{2}$, the standard deviation $\text{sd}\left(\overline{X}\right) = \dfrac{3}{\sqrt{2}}$.

Marks	0	1	Average
%	16	84	0.9

Well done.

A variety of correct forms were accepted.

Question 6 III

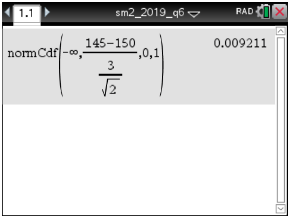
c. Write down an expression for the p-value of the statistical test and evaluate your answer correct to four decimal places. (2 marks)

Solution

The p-value is

$$\Pr\left(\overline{X} < 145 \mid \mu = 150\right) = \Pr\left(Z < \frac{145-150}{\frac{3}{\sqrt{2}}}\right)$$

$$= 0.0092.$$



Most students correctly found p. Some did not give the required 4 decimal places. Calculator syntax sometimes replaced correct working.

Marks	0	1	2	Average
%	22	18	60	1.4

d. State with a reason whether H_0 should be rejected at the 5% level of significance. (1 mark)

Solution

As the p-value is $0.0092 < 0.05$, there is strong evidence against the null hypothesis H_0 at the 5% level of significance.
H_0 should be rejected in favor of the alternative hypothesis.

Marks	0	1	Average
%	24	76	0.8

A reason is required.

Some transcription errors in otherwise correct answers.

Question 6 V

e. What is the smallest value of the sample mean height that could be observed for H_0 to be **not** rejected? Give your answer in centimetres, correct to two decimal places. (1 mark)

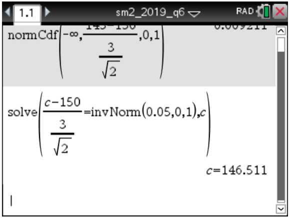
Solution

For H_0 not to be rejected (at the 5% level of significance) we require that

$$\Pr\left(Z \leq \frac{c-150}{\frac{3}{\sqrt{2}}}\right) = 0.05$$

$$\Rightarrow \frac{c-150}{\frac{3}{\sqrt{2}}} = -1.64485$$

$$\Rightarrow c = 146.51$$



146.52 was also accepted.

Marks	0	1	Average
%	52	48	0.5

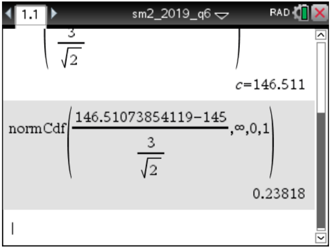
f. If the true mean height of all mature water buffaloes in northern Australia is in fact 145 cm, what is the probability that H_0 will be accepted at the 5% level of significance? Give your answer correct to two decimal places.

(1 mark)

Solution

If H_0 is accepted at the 5% level of significance then the smallest value that the sample mean be is 146.51 cm. So we find

$$\Pr(X > 146.51 \mid \mu = 145) = \Pr\left(Z > \frac{146.51 - 145}{\frac{3}{\sqrt{2}}}\right)$$
$$= 0.24.$$



Mostly left blank.

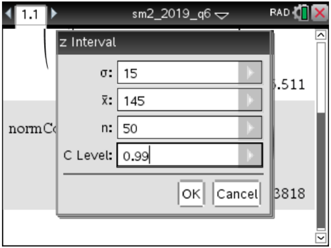
Marks	0	1	Average
%	89	11	0.1

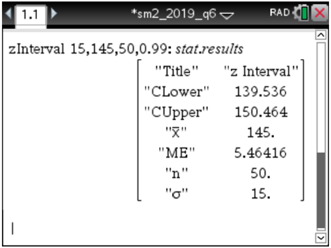
Question 6 VIII

g. Using the observed sample mean of 145 cm, find a **99% confidence interval** for the mean height of all mature water buffaloes in northern Australia. Express the values in your confidence interval in centimetres, correct to one decimal place. (1 mark)

Solution

A 99% confidence interval for the mean height of all mature water buffaloes in northern Australia (to one decimal place) is $(139.5, 150.5)$. This is found easily using CAS:





Many students used a 95% confidence interval.

Marks	0	1	Average
%	48	52	0.5