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THE MATHEMATICAL ASSOCIATION OF VICTORIA

Specialist Mathematics 2019 and Beyond

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Exam Instructions

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Instructions

Answer **all** questions in the spaces provided.

Unless otherwise specified, an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

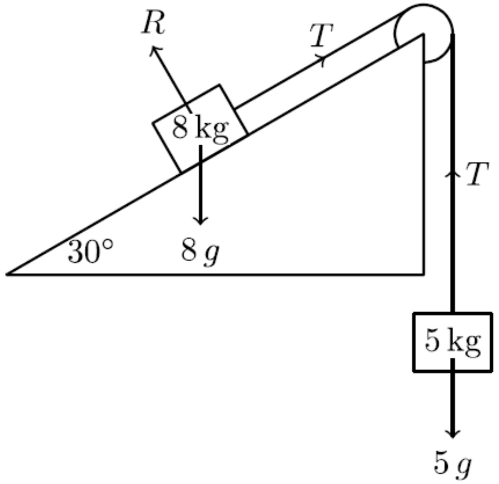
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

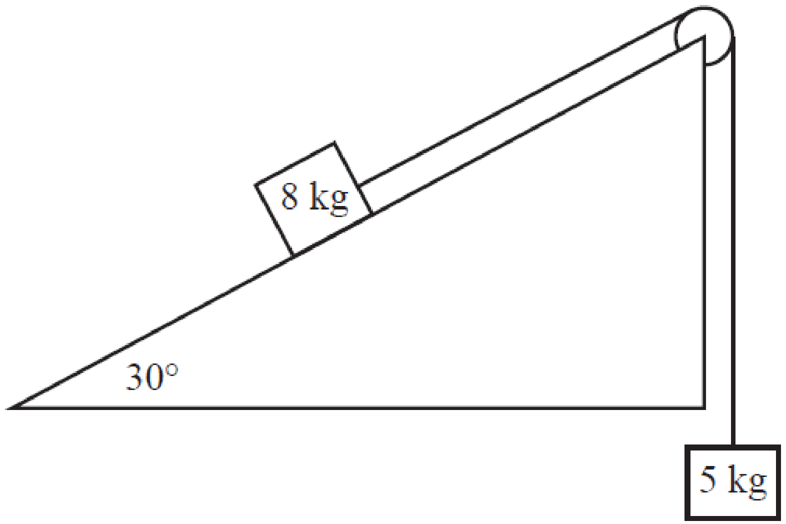
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1

Question 1 (4 marks)

Two objects of masses 5 kg and 8 kg are attached by a light inextensible string that passes over a smooth pulley. The 8 kg mass is on a smooth plane inclined at 30° to the horizontal. The 5 kg mass is hanging vertically, as shown in the diagram below.





a. On the diagram above, show all forces acting on both masses. 1 mark

Marks	0	1	Average
%	40	60	0.6

Question 1

This question was answered reasonably well. Common errors included:

- omitting the normal reaction force on the 8 kg mass
- introducing extra forces, for example, a friction force
- failing to indicate that the tension was constant throughout the string
- failing to label the forces. In particular, some students used labels that did not distinguish between the two masses.

Question 1

b. Find the magnitude, in ms^{-2}, and state the direction of the acceleration of the 8 kg mass. 3 marks

$$T - 8g\sin 30^0 = 8a$$

$$5g - T = 5a \quad R - 8g\cos 30^0 = 0$$

Adding parallel to vertical gives $5g - 8g\sin 30^0 = 13a$ so $a = \dfrac{g}{13}$

The 8 kg mass is accelerating at $\dfrac{g}{13}$ m/s^2 up the plane.

Marks	0	1	2	3	Average
%	18	24	17	41	1.8

Question 1

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Students could resolve forces in order to find an equation of motion. Some students correctly determined the force $5g - 4g = g$ but did not use the combined mass of 13 kg in order to find the acceleration. In particular, $\frac{g}{8}$ was a common incorrect response. Many students neglected to give the magnitude and/or the direction of motion of the 8 kg mass.

Question 2

Question 2 (4 marks)

a. Show that $1+i=\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$. 1 mark

$$r=\sqrt{1+1}=\sqrt{2}, \quad \theta=\arctan(1)$$

$$1+i=\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$$

Students were required to show that $1+i=\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$.

This question was answered well by most students. A common incorrect response was to write $\tan\left(\frac{1}{1}\right)=\frac{\pi}{4}$ rather than $\arctan\left(\frac{1}{1}\right)=\frac{\pi}{4}$.

Marks	0	1	Average
%	17	83	0.9

Question 2

b. Evaluate $\frac{\left(\sqrt{3}-i\right)^{10}}{(1+i)^{12}}$, giving your answer in the form $a+bi$, where $a, b \in R$. 3 marks

$$\sqrt{3}-i=2\,\text{cis}\left(-\frac{\pi}{6}\right)$$

$$\frac{\left(\sqrt{3}-i\right)^{10}}{(1+i)^{12}}=\frac{\left(2\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^{10}}{\left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)^{12}}$$

$$=\frac{2^{10}\,\text{cis}\left(-\frac{5\pi}{3}\right)}{2^{6}\,\text{cis}(3\pi)}$$

$$=16\text{cis}\left(-\frac{14\pi}{3}\right)=16\text{cis}\left(-\frac{2\pi}{3}\right)$$

$$=-8-8\sqrt{3}i$$

Marks	0	1	2	3	Average
%	12	3	48	37	2.1

Question 2

Most students realised that they needed to use polar form and de Moirvre's theorem. Quite a few students were not able to write $\sqrt{3}-i$ in polar form correctly with arguments of $\frac{\pi}{6}$, $\frac{5\pi}{6}$ and $\frac{\pi}{3}$ being given frequently. Students are reminded that a diagram placing the complex number in the correct quadrant can be helpful in avoiding errors. Of those students who obtained the result $16\text{cis}\left(-\frac{2\pi}{3}\right)$, some neglected to write the final answer in the required form or made errors in their attempt.

A small number of students attempted to expand brackets. This approach was rarely successful.

Question 3

Question 3 (4 marks)

Find the gradient of the curve with equation $2x^2\sin(y)+xy=\frac{\pi^2}{18}$ at the point $\left(\frac{\pi}{6},\frac{\pi}{6}\right)$. Give your answer in the form $\frac{a}{\pi\sqrt{b}+c}$, where a, b and c are integers.

$$4x\sin(y)+2x^2\cos(y)\frac{dy}{dx}+y+x\frac{dy}{dx}=0$$

$$4\times\frac{\pi}{6}\times\frac{1}{2}+2\times\frac{\pi^2}{36}\times\frac{\sqrt{3}}{2}\frac{dy}{dx}+\frac{\pi}{6}+\frac{\pi}{6}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\left(\frac{\sqrt{3}\pi^2}{36}+\frac{\pi}{6}\right)=-\frac{\pi}{2}$$

$$\frac{dy}{dx}=-\frac{18}{\sqrt{3}\pi+6}$$

Marks	0	1	2	3	4	Average
%	8	4	9	33	46	3.1

Question 3

Most students knew to use implicit differentiation in this problem and were successful in their application of the chain and product rules.

Many students attempted to find an expression for $\frac{dy}{dx}$ in terms of x and y. This was not necessary, with a more effective approach being to substitute $x = \frac{\pi}{6}$ and $y = \frac{\pi}{6}$ immediately following the implicit differentiation. Some students had difficulty with arithmetic.

Question 4

Question 4 (4 marks)

X and Y are independent random variables. The mean and the variance of X are both 2, while the mean and the variance of Y are 2 and 4 respectively.

Given that a and b are integers, find the values of a and b if the mean and the variance of $aX + bY$ are 10 and 44 respectively.

$E(aX+bY)=aE(X)+bE(Y)$ and $Var(aX+bY)=a^2Var(X)+b^2Var(Y)$

Mean: $2a+2b=10$ and so $b=5-a$

Variance: $2a^2+4b^2=44$ so $a^2+2b^2=22$

$a^2+2(5-a)^2=22$

$a^2+2(25-10a+a^2)=22$ and so $3a^2-20a+28=0$

$a=2$ or $\dfrac{14}{3}$

But a is an integer and so $a=2$. Hence $b=3$.

Marks	0	1	2	3	4	Average
%	5	8	8	36	43	3

Question 4

From the information given, students needed to write down a pair of simultaneous equations $2a+2b=10$, $2a^2+4b^2=44$ and then solve for a and b. Common problems included failing to reject the non-integer solution and only stating the solution with minimal or no working. Students are reminded that in a question worth more than one mark, appropriate working must be shown.

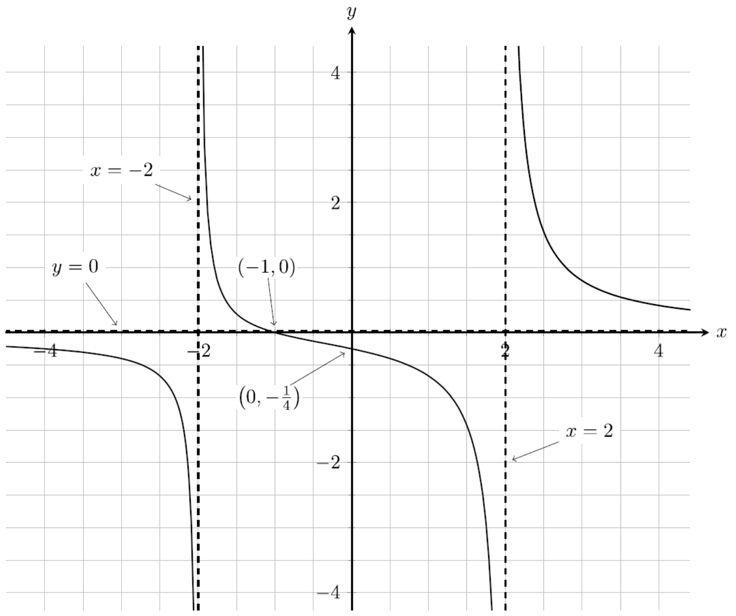
Algebraic errors were common, with some students having difficulty solving a quadratic equation.

Quite a few students 'squared' both sides of the first equation to obtain $4a^2+4b^2=100$.

Question 5

Question 5 (4 marks)

Sketch the graph of $f(x)=\dfrac{x+1}{x^2-4}$ on the axes provided below, labelling any asymptotes with their equations and any intercepts with their coordinates.



Marks	0	1	2	3	4	Average
%	9	21	35	20	15	2.1

Question 5

Most students realised that $x = -2$ and $x = 2$ were vertical asymptotes, although the horizontal asymptote $y = 0$ was often not stated. Students who found the axis intercepts were not always able to position them correctly on the axes. Some students showed a stationary point of inflection on their graph or were missing the outer branches.

Question 6

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Question 6 (3 marks)

A particle of mass 2 kg moves under a force $\underset{\sim}{F}$ so that its position vector $\underset{\sim}{r}$ at any time t is given by $\underset{\sim}{r} = \sin(t)\,\underset{\sim}{i} + \cos(t)\,\underset{\sim}{j} + t^2\,\underset{\sim}{k}$. Distances are measured in metres and time is measured in seconds.

Find the change in momentum, in kg ms^{-2}, from $t = \frac{\pi}{2}$ to $t = \pi$.

$$\dot{\underset{\sim}{r}}(t) = \cos t\underset{\sim}{i} - \sin t\underset{\sim}{j} + 2t\underset{\sim}{k}$$

$$\text{At } t = \frac{\pi}{2},\ \underset{\sim}{v} = \cos\left(\frac{\pi}{2}\right)\underset{\sim}{i} - \sin\left(\frac{\pi}{2}\right)\underset{\sim}{j} + \pi\underset{\sim}{k} = -\underset{\sim}{j} + \pi\underset{\sim}{k}$$

$$\text{At } t = \pi,\ \underset{\sim}{v} = \cos(\pi)\underset{\sim}{i} - \sin(\pi)\underset{\sim}{j} + 2\pi\underset{\sim}{k} = -\underset{\sim}{i} + 2\pi\underset{\sim}{k}$$

$$\text{Change in momentum} = 2\left(-\underset{\sim}{i} + \underset{\sim}{j} + \pi\underset{\sim}{k}\right)$$

Marks	0	1	2	3	Average
%	12	14	42	31	1.9

Question 6

The majority of students correctly differentiated $\underset{\sim}{r}(t)$ to find $\dot{\underset{\sim}{r}}(t) = \cos t\underset{\sim}{i} - \sin t\underset{\sim}{j} + 2t\underset{\sim}{k}$ and substituted $t = \frac{\pi}{2}$ and $t = \pi$. Some students made errors in their arithmetic when attempting to evaluate $2\left(\dot{\underset{\sim}{r}}(\pi) - \dot{\underset{\sim}{r}}\left(\frac{\pi}{2}\right)\right)$. A number of students were unable to evaluate the trigonometric expressions correctly. Some students thought that a scalar result was required.

Students who interpreted this question as asking for the average rate of change of momentum to be dimensionally consistent with the units and did this correctly were awarded marks accordingly.

Question 7

Question 7 (3 marks)

Given that $\cot(2x)+\frac{1}{2}\tan(x)=a\cot(x)$, use a suitable double angle formula to find the value of a, $a \in R$.

$$\cot(2x)+\frac{1}{2}\tan(x)=\frac{1}{\tan(2x)}+\frac{\tan(x)}{2}$$

$$=\frac{1-\tan^2(x)}{2\tan(x)}+\frac{\tan(x)}{2}$$

$$=\frac{1-\tan^2(x)+\tan^2(x)}{2\tan(x)}$$

$$=\frac{1}{2\tan(x)}$$

$$a=\frac{1}{2}$$

Marks	0	1	2	3	Average
%	10	6	19	66	2.4

Question 7

This question was answered well, with most students converting $\cot(2x)$ into $\frac{1}{\tan(2x)}$ and using a double angle formula. A number of students used $\sin$ and $\cos$ but were less successful than those who used the more direct approach. A number of students thought that $\cot(2x)$ was equal to $\frac{1}{\cos(2x)}$. Some students gave their final answer as $a = 2$.

Question 8

Question 8 (4 marks)

A tank initially holds 16 L of water in which 0.5 kg of salt has been dissolved. Pure water then flows into the tank at a rate of 5 L per minute. The mixture is stirred continuously and flows out of the tank at a rate of 3 L per minute.

a. Show that the differential equation for Q, the number of kilograms of salt in the tank after t minutes, is given by 1 mark

$$\frac{dQ}{dt} = -\frac{3Q}{16+2t}$$

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$= 0 - \frac{3Q}{\text{Volume}}$$

$$= -\frac{3Q}{16+2t}$$

Marks	0	1	Average
%	56	44	0.5

Question 8

This problem required students to recognise a difference of rates. The most common error was a failure to explicitly note that the rate in was zero.

Question 8

b. Solve the differential equation given in **part a.** to find Q as a function of t. Express your answer in the form $Q = \dfrac{a}{(16+2t)^{\frac{b}{c}}}$, where a, b and c are positive integers. 3 marks

$$\int \frac{dQ}{Q} = \int -\frac{3\,dt}{16+2t}$$

$$\log_e(Q) = -\frac{3}{2}\log_e(16+2t) + c$$

$$Q = A(16+2t)^{-3/2}$$

$$t = 0,\ Q = \frac{1}{2}$$

$$\frac{1}{2} = \frac{A}{64}$$

$$Q = \frac{32}{(16+2t)^{\frac{3}{2}}}$$

Marks	0	1	2	3	Average
%	27	12	38	23	1.6

Question 8

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The majority of students realised that this was a separable differential equation, but many made errors in the subsequent integration with the arbitrary constant of integration frequently missing. Some students made transcription errors that fundamentally changed the problem. Others encountered arithmetic or algebraic issues. Many students took the common factor of 2 from the $16+2t$ expression and evaluated $\frac{1}{2}\int \frac{1}{8+t}\,dt$. This unnecessary manipulation made subsequent calculations more difficult for these students.

Question 9

Question 9 (5 marks)

A curve is specified parametrically by $\underset{\sim}{r}(t) = \sec(t)\,\underset{\sim}{i} + \frac{\sqrt{2}}{2}\tan(t)\,\underset{\sim}{j},\ t \in R$.

a. Show that the cartesian equation of the curve is $x^2 - 2y^2 = 1$. 2 marks

$$x = \sec(t),\ y = \frac{\sqrt{2}}{2}\tan(t)$$

$$x^2 = \sec^2(t),\ y^2 = \frac{1}{2}\tan^2(t)$$

$$\sec^2(t) - \tan^2(t) = 1$$

$$x^2 - 2y^2 = 1$$

Marks	0	1	2	Average
%	13	14	73	1.6

Question 9

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This question was answered well, with most students realising that a substitution into a trigonometric identity was required.

Question 9

b. Find the x-coordinates of the points of intersection of the curve $x^2 - 2y^2 = 1$ and the line $y = x - 1$.

1 mark

$$x^2 - 2(x-1)^2 = 1$$

$$x^2 - 4x + 3 = 0$$

$$x = 1 \text{ or } x = 3$$

Marks	0	1	Average
%	30	70	0.7

Question 9

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Students needed to substitute $y = x - 1$ into the equation $x^2 - 2y^2 = 1$ and solve the resulting quadratic equation for x. A number of students gave the coordinates of the points of intersection and in some cases did not do this correctly.

Question 9

c. Find the volume of the solid of revolution formed when the region bounded by the curve and the line is rotated about the x-axis. 2 marks

$$V = \pi\int_1^3 \left(\frac{x^2-1}{2} - (x-1)^2\right)dx$$

$$= \pi\int_1^3 \left(-\frac{x^2}{2} + 2x - \frac{3}{2}\right)dx$$

$$= \pi\left[-\frac{x^3}{6} + x^2 - \frac{3}{2}x\right]_1^3$$

$$= \frac{2\pi}{3}$$

Marks	0	1	2	Average
%	60	21	19	0.6

Question 9

Students found this question challenging. Some students did not apply the formula for the volume of a solid of revolution correctly. Many students made algebraic or arithmetic errors. A small number of students realised that the volume required could be found by finding the volume obtained by rotating the region bounded by the hyperbola, the x-axis and the lines $x=1$ and $x=3$ about the x-axis and then subtracting the volume of an appropriate cone.

Question 10

Question 10 (5 marks)

The position vector of a particle moving along a curve at time t seconds is given by

$\underset{\sim}{r}(t) = \frac{t^3}{3}\underset{\sim}{i} + \left(\arcsin(t) + t\sqrt{1-t^2}\right)\underset{\sim}{j}$, $0 \le t \le 1$, where distances are measured in metres.

The distance d metres that the particle travels along the curve in three-quarters of a second is given by

$$d = \int_0^{\frac{3}{4}} \left(at^2 + bt + c\right) dt$$

Find a, b and c, where $a, b, c \in Z$.

Marks	0	1	2	3	4	5	Average
%	35	22	24	17	0	2	1.3

Question 10

$$\frac{dx}{dt} = t^2, \ \frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}} + \sqrt{1-t^2} - \frac{t^2}{\sqrt{1-t^2}} = \frac{2-2t^2}{\sqrt{1-t^2}}$$

$$d = \int_0^{3/4} \left(\sqrt{t^4 + \frac{4\left(1-t^2\right)^2}{1-t^2}} \right) dt$$

$$= \int_0^{3/4} \left(\sqrt{t^4 + 4(1-t^2)} \right) dt$$

$$= \int_0^{3/4} \left(\sqrt{(t^2-2)^2} \right) dt$$

$$= \int_0^{3/4} \left(\left| t^2 - 2 \right| \right) dt$$

$$= \int_0^{3/4} \left(2 - t^2 \right) dt \qquad \text{as } t^2 - 2 < 0$$

$$a = -2, b = 0, c = 2$$

Question 10

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Only a few students obtained full marks for this question. Most students recognised that the arc length formula needed to be applied, but some had difficultly differentiating $\arcsin(t)+t\sqrt{1-t^2}$. A number of students applied the product and chain rule correctly to the $t\sqrt{1-t^2}$ term and ignored the $\arcsin(t)$ term. Many students had difficulty simplifying $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$ and were unable to proceed further. Of those students who were able to find that $d=\int_0^{\frac{3}{4}}\sqrt{\left(t^2-2\right)^2}\,dt$, only a small number recognised the significance of the domain $0 \le t \le 1$.