THE MATHEMATICAL ASSOCIATION OF VICTORIA

MAKING +
CONNECTIONS
MATHEMATICS NUMERACY

INDUSTRY
PEDAGOGY
CURRICULUM
PROFICIENCIES
STEM
STUDENTS INDUSTRY PEDAGOGY
CURRICULUM PROFICIENCIES STEM
MAKING + CONNECTIONS
STUDENTS INDUSTRY PEDAGOGY
CURRICULUM PROFICIENCIES STEM
INDUSTRY
PEDAGOGY
CURRICULUM
PROFICIENCIES
STEM

STUDENTS INDUSTRY PEDAGOGY CURRICULUM PROFICIENCIES STEM MAKING + CONNECTIONS STUDENTS INDUSTRY PEDAGOGY CURRICULUM PROFICIENCIES STEM INDUSTRY PEDAGOGY CURRICULUM PROFICIENCIES STEM INDUSTRY PEDAGOGY CURRICULUM PROFICIENCIES STEM

CONFERENCE PROCEEDINGS

56th Annual Conference
mav.vic.edu.au/conference

2019 Mathematical Association of Victoria Annual conference proceedings

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Foreword

The theme of the 56th Annual Conference of the Mathematical Association of Victoria is *Making+Connections*. This theme is significant since making connections will lead to improvements in mathematics education for teachers and students. Connections that are made between levels (e.g., primary and secondary) can help to ease the transition for students and expand teachers' pedagogical repertoires. Furthermore, connections between practitioners and academics can deepen both groups' understandings of the teaching and learning process for mathematics.

Within the classroom setting, teachers need to help students develop an understanding of different representations of mathematical concepts. Students' comprehension of multiple representations can be supported through the use of a variety of materials, tools, and technologies. Beyond developing rich understandings of each mathematical concept or topic, it is important to establish strong cross-strand and cross-curricular connections. Additionally, linking curriculum, pedagogy, and assessment provides a coherent learning experience for students and thus supports their mathematical development. Finally, due to the social nature of working mathematically, student-student and student-teacher relationships need to be nurtured in order to provide an ideal setting in which to interact mathematically.

We were pleased to see that the authors who submitted conference papers were attentive to the significant ideas that underpin the conference theme of *Making+Connections*. In addition to the authors' careful alignment with the conference theme, it was encouraging to receive papers on a wealth of mathematical topics (e.g., multiplicative thinking, equality) and pedagogies (e.g., flipped learning, inquiry-based learning) that spanned across educational levels from early years to secondary.

We are extremely grateful to the volunteers for their willingness to review the papers within the short time frame given. We appreciate their commitment and dedication to supporting the authors, the MAV, and the conference. In particular, we thank Ann Downton and the MAV organising committee, together with the MAV staff members Jacqui Diamond and Louise Gray. They have worked tirelessly to ensure that this conference and the proceedings contribute to the improvement of teaching and learning of mathematics in Victoria. We are sure that the conference will be a success!

Jennifer Hall and Hazel Tan
Monash University
Editors

THE REVIEW PROCESS

The editors received 20 papers for blind and peer review. Specifically, eight papers were submitted for the double blind review process, in which the identities of author(s) and reviewer were concealed from each other. Two reviewers reviewed each of the papers. If they had a differing outcome, the editors reviewed the papers and made a decision regarding the outcome. In total, six blind papers were accepted for publication in the proceedings. A further 13 papers were submitted for the peer review process, in which non-blinded papers were reviewed by one reviewer each. This process resulted in eight papers being accepted for publication in the proceedings.

In total, 22 papers are published in the Mathematical Association of Victoria's 56th Annual Conference Proceedings (four keynote papers, 14 full papers, and four summary papers). There were 18 reviewers involved in the process, all of whom provided thoughtful feedback and were outstanding in responding quickly to requests.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Contents

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Keynotes

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Reasoning as a mathematical habit of mind

Mike Askew, University of the Witwatersrand

Of the four proficiencies – understanding, fluency, problem solving and reasoning – reasoning is often seen as the 'gold standard' of mathematics: harder to learn than the other fluencies and only accessible to a minority of learners. An alternative view is that mathematical reasoning is an extension of the sort of everyday reasoning that we all engage in and so engaging in mathematical reasoning should be accessible to all learners, not just a select few. Research evidence shows that, contrary to another popular belief, reasoning is not necessarily dependent on fluency needing to be learnt first. In this paper, I examine some of the research into what it means to reason mathematically and suggest some practical examples for bringing reasoning into the centre of mathematics lessons in ways that it can happen 'little and often' in mathematics lessons, and encourage all students to come to develop mathematical reasoning as a 'habit of mind'.

INTRODUCTION

Some years back, when I was teaching at Monash University, I was involved with some action research in a Melbourne primary school. We gave the 4th, 5th, and 6th grade students the following three equations to complete.

$$8 + 4 = [\ ] + 5$$

$$32 + 19 = [\ ] + 20$$

$$68 - 39 = [\ ] - 40$$

The overall success rates on the first two items were 58% and 53% respectively, with many of the incorrect answers being the result of students giving answers of 12 and 51, interpreting the open box as simply needing to be filled with the answer to the calculation on the left-hand side of the equal side. On the third item, the success rate dropped to 26%.

In devising these questions, we had hoped that the students would take a reasoning approach to think about answers: that 32 + 19 had to be equivalent to 31 + 20 because increasing 19 by 1 to 20 would mean decreasing 32 by 1, to preserve the equality, or that the difference between 68 and 39 would be the same as the difference between 69 and 40 because increasing the 39 by 1 would mean needing to increase the 69 by 1, to preserve the difference. We thought that engaging in such reasoning was not overly challenging and so were surprised by these facility levels, particularly on the third item.

To try to understand why this subtraction item was proving to be so much more difficult we talked to several of the students involved. What becomes clear, through those conversations, was that the vast majority had set about answering the questions by carrying out a series of calculations, for example, working out that 32 + 19 was 51 and then figuring out what needed to be added to 20 to make 51. The fact that calculating 68 – 39 was harder than calculating either of the addition calculations was the main contributing factor to the much lower facility on that item. It seemed that the default mode of thinking for these students was 'calculate first', an approach that my experience with students elsewhere suggests is common.

Since then, I've become increasingly interested in whether, and how, we can help students move from this 'calculate first' habit of mind, to pausing before calculating, to thinking about the underlying mathematical structures of which such questions are particular examples. In other words, developing a 'reasoning first' habit of mind'.

CALCULATE OR REASON?

Nunes, Bryant, Sylva, and Barros (2009) make the distinction between arithmetic ("learning how to do sums and using this knowledge to solve problems", p. 3) and mathematical reasoning ("learning to reason about the underlying relations in mathematical problems they have to solve", p. 3). In a series of research studies, Nunes and colleagues have looked at the distinction between these and the impact each might have on mathematical attainment and progression.

Working with 6-year-olds, near the beginning of their schooling, Nunes and colleagues (Nunes et al., 2007)

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

demonstrated that the children's reasoning abilities were good predictors of their performance in mathematics 16 months later. As part of that study, the researchers also assessed their working memories and general cognitive abilities – controlling for each of these showed that the predictive power of the ability to reason mathematically was independent of either working memory or general cognitive ability. While the results showed some overlap of the effect of reasoning, general cognitive ability and working memory, the authors conclude that it is worth attending to mathematical reasoning in its own right.

In later work, Nunes, Bryant, Barros, and Sylva (2012) further explored the question of whether the two different abilities - mathematical reasoning and arithmetical skills - make distinct contributions to students' attainment in mathematics, doing so by tracking learning in a five-year longitudinal study. Their overall conclusion from that research is that while reasoning and arithmetical abilities do independently contribute to predicting progress in learning mathematics, of the two "mathematical reasoning was by far the stronger predictor" (p. 136) and that teaching must address improving reasoning skills as well as, and separately to, calculating skills, particularly given that "this study provides a clear empirical basis for distinguishing mathematical reasoning as a form of conceptual knowledge and knowledge of arithmetic as separate constructs" (p. 153).

Such research showing that students' ability to reason about mathematical relations is the stronger predictor of later mathematical achievement raises the question as to whether differences in attainment in mathematical reasoning reveal 'natural' differences in mathematical ability, whether they are the result of different experiences, and whether reasoning can be taught. To explore this, Nunes et al. (2007) carried out an intervention with a group of six-year-olds to see if it were possible to develop their mathematical reasoning. The instructional approach taken included small groups working on practical problems that embodied the inverse relationship between addition and subtraction, and pushing for reasoning by covering up manipulables, so that children could not count them. The researcher working with the group explicitly directed the students to the correct reasoning if they were initially reasoning erroneously. The intervention also worked on additive composition, using coins that encouraged reasoning about counting on, and solving one-to-many correspondence problems involving two variables (e.g., three lorries are each carrying four tables, how many tables are delivered?).

The students taught in this way were found to progress further in mathematics than a control group, an effect that was still in evidence in delayed post-test assessments carried out 13 months later. The authors thus conclude that the learning of number facts is not a sufficient basis for early mathematics, and that young children can engage with reasoning about number relations (without calculating) and that teaching should address this.

While further research is needed into the impact of explicitly teaching for reasoning, rather than arithmetic, if we believe that the majority of students can and should be able to learn a substantial amount of mathematics, then we have to assume, until proven wrong, that everyone can improve their ability to reason mathematically and be as successful at that as at calculating. We have to engage with how we can help everyone reason mathematically and consider whether our teaching is getting the balance right between reasoning and calculating.

REASONING AS A FIRST RESORT

It is popularly believed that students' thinking has to have reached a certain 'stage' before they can reason mathematically, such a view supported by Piaget (1964) and his 'stages of thinking' theory in which he argued that 'concrete operational' thinking preceded 'formal operational' a stage of thinking that might only be attained by the early teen years and that "the ordering of these stages is constant and has been found in all the societies studied" (p. 178). Although such 'stage theory' is may not be explicitly talked about as much, it does seem still to shape discussions of teaching and learning.

Researchers now think that Piaget got this part of his theory wrong (Claxton, 2015) and that the structures for thinking that we use throughout our lives are much in place from an early age, certainly from when we have learned to talk. What changes as we get older is the amount we can think about – the quantity – rather than how we actually think about things – the quality. Young children can reason about abstract relationships in appropriate contexts. For example, they may not be able to answer 3 ÷ 4 presented 'out of context' but can find practical ways to share three chocolate bars between four

children. A flip side of this critique of 'stages of thinking' is the evidence that the need for 'concrete' representations does not diminish as we grow older, although our awareness of the 'concreteness' of our thinking may be hidden in the extensive use of linguistic metaphors (Lakoff & Johnson, 1980).

Let us look at an example. Are these statements true or false?

$$3 \times 6 = 3 \times 5 + 3$$

$$27 \times 40 = 27 \times 39 + 27$$

$$326 \times 18 = 327 \times 17 + 327$$

A calculating approach would be to say that 3×6 is 18 and $3 \times 5 + 3$ is also 18, so the two expressions are equal.

A reasoning approach is to consider if there is a way of 'reading' the equation in a way that reveals the underlying structure of each expression and so lays bare whether or not those structures are equivalent. That raises the question of how best to 'read' 3×6 - as three multiplied by six (three taken six times) or as three times six (six taken three times)? A moment's reflection might suggest that the reading of three multiplied by six helps reveal the connection between the two sides of the equation: on the right-hand side, we have six groups of three and on the left-hand side, five groups of three, plus another three, so the equality is true. As an aside, such reasoning helps students realise that different 'readings' of mathematical expressions may be helpful in different contexts. That, say, 3.75×5 might be better read as five lots of 3.75 than as 3.75 lots of 5, but that 3.75×12 might be thought of as 3 lots of 12 plus three-quarters of 12, and in each case, reasoning leading to an answer more quickly than setting out an algorithm or reaching for a calculator.

A similar reading of $27 \times 40 = 27 \times 39 + 27$ reveals the same underlying structure: 40 lots of 27 is the same as 39 lots of 27 plus a further 'lot' of 27. This structure is preserved for $326 \times 18 = 327 \times 17 + 327$.

At the time of writing, students in England have to take three national assessments at the end of primary school (when they are 11 years old). The first paper comprises a series of pure arithmetical calculations, and the second and third papers are deemed to be assessing reasoning (although I would say many of the questions on these two papers are still arithmetical questions, now just put into a context). On the 2016 papers, the final question on the third paper started with providing the statement that $5542 \div 17 = 326$. Students were asked to explain how they could use this result to find the answer to 326×18.

Given the position of this question being the final item, the test setters presumably assumed that it was the hardest question across the three tests. Judging by the success rate on this question, it was the most difficult: Only 26% of pupils were able to answer the question correctly, the lowest score on any item across the three tests. However, if one can engage in the sort of 'reading' and reasoning outlined above, then the only challenge this problem presents is realising that $5542 \div 17 = 326$ tells you that $326 \times 17 = 5542$: it is then a small step to say that 326×18 is going to be $5542 + 326$.

We can speculate as to why this question (and others like it) are so poorly answered, and I expect a most likely one is that many students tried to carry out some calculations in order to arrive at the answer. Some years ago, Gray and Tall (1994) noted differences between students who relied on using a set procedure for calculating and students who could use flexible approaches. Not only were the mathematical achievements of the pupils focused on procedures lower than those of the flexible thinkers, but also the authors argued that the procedural thinkers actually engaged in a more difficult form of mathematics, which contributed to the difference in attainment. In the case of questions like the one discussed above, reverting to calculating also makes the mathematics more difficult than it need be.

Looking at the sort of reasoning that supports being able to answer such questions without calculating reveals important insights: the identical underlying structure of the three examples above illustrates how, in many instances, the ability to reason is independent of the size of the numbers involved and, importantly, independent of being able to actually carry out the calculations. The thinking involved in getting an answer does not have to wait until the final year of primary. Talking about why $3 \times 6 = 3 \times 5 + 3$ can be done with much younger students, who may well be able to go on to explore the similarity of the underlying structure to $27 \times 40 = 27 \times 39 + 27$. Having the mathematics habit of mind of not simply reading numerical expressions as 'instructions to calculate' but asking 'what is the general structure of which this

is a particular instance?' could mean students being able to talk and reason about why $326 \times 18 = 327 \times 17 + 327$ is necessarily, logically, true, long before they had the knowledge and skills to actually carry out the calculation of 326×18.

REASONING CHAINS

Reasoning chains provide a way of thinking about and strategically working on calculations and discouraging students from defaulting to a 'calculate first' mindset. Reasoning chains take only around ten minutes to work through, meaning that they can be done little and often and so encourage students to develop a 'reasoning' habit of mind.

For maximum impact, reasoning chains have to be designed around three or four number sentences that have been chosen to build and connect in some way, through careful variation within and between examples. Many resources providing examples for students to work on present a variety of examples to work on, but not all resources pay close attention to the variation between examples: although sounding similar, variety and variation promote different learning experiences. For example, a page of calculations providing practice in, say, division may be compiled in such a way that, although the numbers might get larger or more awkward to work with in later examples, in essence, the examples could be re-ordered without the experience of working through them being dramatically changed. Repeated exposure to such exercises can result in students coming to think that, having completed a question, it can then be mentally 'set aside' in turning to the next question.

Variation is different both in the design of exercises and in what students' attention may be focused on when working through the examples. Variation is based on a theory of learning positing that we learn not only from awareness of the similarities between things but also the difference, the variation, between examples (Marton & Booth, 1997). Designing exercises based in variation means carefully considering the changes from one example to the next and so the order of the examples matters. That is done with the intention that the students' attention is not simply on each individual example, but is drawn to the connections, similarities, and differences between the examples: The experience of thinking about what is going on across a sequence of examples is greater than just the sum of attending to each example on its own. At heart, variation aims to raise student's awareness that mathematics is a network of interconnected ideas, not a random collection of tasks.

A reasoning chain example illustrates how such variation might play out in practice.

9×12

18×6

3×36

The first calculation presented, would, I expect, easily be answered by students, and not a lot of time would need to be given over to establishing how the answer was arrived at. A diagram drawn up alongside would be of a 9 by 12 array (on the assumption that the students are familiar with arrays and that an open array showing the 9, 12 and 108 is sufficient - that they can picture the 96 individual squares that would all be drawn in).

Presenting the second calculation ($18 \times 6 =$) should, again, not be too challenging. The key pedagogic move is in drawing students' attention to the fact that the answer is the same as the first one and having a conversation is about why this might be. Working with the array image, the array sketched out for the first problem can be divided into two 8 by 6 arrays, and rearranged to create a 16 by 6 array.

With the third problem ($3 \times 36 =$), reasoning comes to the fore by asking the students, once it has been established that the answer is, once again, 108, to sketch for themselves, what needs to be done to the original array to show why the answer is constant. The 'chain' can then be brought together by making explicit how each of these is an example of the associative law:

$9 \times 12 = 9 \times (2 \times 6)$ $9 \times 12 = (3 \times 3) \times 12$

$9 \times (2 \times 6) = (9 \times 2) \times 6$ $(3 \times 3) \times 12 = 3 \times (3 \times 12)$

$(9 \times 2) \times 6 = 18 \times 6$ $3 \times (3 \times 12) = 3 \times 36$

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

As described above, for each of these equivalences it is important to build up images alongside the symbols. This embodiment of the underlying mathematical ideas through visual and diagrammatic representations leads to deep understanding, together with extended notation. Symbolic representations on their own are often too 'compact', simply recording the outcome of the underlying mathematical structure, not revealing and making explicit that structure.

SAMPLE REASONING CHAINS

Addition and subtraction

$54 + 10$

$54 + 30$

$54 + 29$

The big idea: the associative rule $(54 + 29 = 54 + (20 + 9) = (54 + 20) + 9$ or $54 + 29 = 54 + (30 - 1) = (54 + 30) - 1$

$54 - 10$

$54 - 30$

$54 - 29$

The big idea: the associative rule $(54 - 29 = 54 - (20 + 9) = (54 - 20) - 9$ or $54 - 29 = 54 - (30 - 1) = (54 - 30) + 1$

Multiplication

18×10

18×2

18×12

The big idea: the distributive rule $(18 \times 12 = 18 \times (10 + 2) = (18 \times 10) + (18 \times 2)$

6×8

12×8

12×16

The big idea: doubling and the associative rule $(12 \times 8 = (2 \times 6) \times 8 = 2 \times (6 \times 8)$ and

$12 \times 16 = 12 \times (8 \times 2) = (12 \times 8) \times 2)$

CONCLUSION

In the school I talked about at the start of this paper, after 4 weeks of a daily, ten-minute reasoning chain, we re-assessed the Grade 4, 5 and 6 students on similar items with the following success rates:

$9 + 3 = [\ ] + 4$	96%
$44 + 19 = [\ ] + 20$	92%
$56 - 19 = [\ ] - 20$	55%

These are a considerable improvement on the pre-test results, although obviously reasoning about subtraction needed further work. However, they show that even in a short time, students can be nudged away from calculating as the default towards reasoning about mathematical structure, an important move if we accept Nunes et al.'s (2009) finding that "mathematical reasoning, even more so than children's knowledge of arithmetic, is important for children's later achievement in mathematics" (p. 1).

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

REFERENCES

Claxton, G. (2015). *Intelligence in the flesh: Why your mind needs your body much more than it thinks*. New Haven, CT: Yale University Press.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: A 'proceptual' view of simple arithmetic. *Journal for Research in Mathematics Education*, *25*(2), 116-140.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. Chicago, IL: University of Chicago Press.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, NJ: Lawrence Erlbaum Associates.

Nunes, T, Bryant, P., Barros, R., & Sylva, K. (2012). The relative importance of two different mathematical abilities to mathematical achievement. *British Journal of Educational Psychology*, *82*(1), 136-156. doi:10.1111/j.2044-8279.2011.02033.x

Nunes, T, Bryant, P., Evans, D., Bell, D., Gardner, S., Gardner, A., & Carraher, J. (2007). The contribution of logical reasoning to the learning of mathematics in primary school. *British Journal of Developmental Psychology*, *25*(1), 147-166.

Nunes, T., Bryant, P., Sylva, K., & Barros, R. (2009). *Development of maths capabilities and confidence in primary school (Research report DCSF RR-118)*. London, England: Department for Children, Schools and Families.

Piaget, J. (1964). Part I: Cognitive development in children: Piaget development and learning. *Journal of Research in Science Teaching*, *2*(3), 176-186.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Big ideas: Connecting across the mathematical curriculum

Mike Askew, University of the Witwatersrand

Teaching and learning mathematics is sometimes treated in a bottom-up approach, by selecting and ordering a number of small objectives that, we hope, build to something over time – a little like picking a handful of Lego bricks and seeing what you can build from them. Complementary to this is a top-down perspective – knowing in advance what you want to build and choosing the bricks appropriately. Thinking about the big ideas in teaching and learning mathematics provides this top-down perspective in the mathematics classroom. In this paper, I look at some big ideas in multiplicative reasoning as examples of how having the big picture in mind can support teaching that not only focuses on reasoning and problem solving, as well as fluency, but also helps develop learning over time by building on, connecting and refining ideas, rather than continually introducing new ones.

INTRODUCTION

We know that learning mathematics is more powerful, deep, and lasting when students make connections between different mathematical ideas. Yet, teaching often reverts to focusing on 'topics', on the content for a particular year group, with the assumption that either the ideas will continue to be useful in later years of learning, or that some ideas will need to be 'unlearned' later. For example, typically we know that, even if not explicitly taught, many students, through the choice of examples that they are given, construct erroneous generalisations such as 'division makes smaller' or 'to multiply by ten, you add a zero', both of which do not serve them well when working with rational numbers. Is it inevitable that there are such contradictions in learning that need to be overcome or can such 'bumps in the road' be smoothed out as we go?

I suggest that one way to not only make learning a smoother path but also reduce teaching loads (with the curriculum being overloaded with new topics) is to look at threads that run through mathematics as a discipline – threads that might inform teaching as a continuity rather than constant accretion of new ideas and that are big ideas.

WHAT IS A BIG IDEA?

There is no agreed definition of 'big ideas' but there are certain criteria that, for me, define a big idea in mathematics teaching and learning. First, the idea must be big enough to connect seemingly disparate aspects of mathematics, but not so big that it is unwieldy. For example, 'mathematics is all about problem solving' is a big idea of sorts, but too big to be helpful. An idea like 'fractions, decimals, and percentages are different ways to represent equal quantities' is a big idea that links separate aspects of the curriculum together, yet it is small enough to be thought about in practical terms.

Second, the idea must have currency across many years of schooling because it means that students get to revisit big ideas across the year groups. The ideas will grow and develop, but there will be a core of similarity on which learning can build. All students can be engaged in thinking about a big idea but perhaps at different developmental levels; working with big ideas is a means of dealing with classroom diversity and being inclusive. It is beyond the scope of this paper to look at several big ideas, so I illustrate how they might influence teaching by taking the exemplar case of multiplicative reasoning. First, I consider why it is important to distinguish between the big ideas of multiplicative reasoning and additive reasoning, and then consider some of the ways that we might work on multiplicative reasoning in class that help students realise its distinctiveness. Consider these two problems:

I have eight apples in a bag and I put six more apples into the bag. How many apples are there altogether?

I have eight bags and I put six apples into each bag. How many apples are there altogether?

The calculations that model each of these word problems are obvious: $8 + 6 = 14$ and $8 \times 6 = 48$. A little less obvious, perhaps, is the number of numbers, including the answer, that there are in each problem. At first sight, it looks as though there are three numbers in each: 8, 6, and 14 in the addition and 8, 6, and 48 in the multiplication. There is a subtle fourth number included in the multiplication, tucked away in the word 'each': the number one. Making this hidden 'one' explicit reveals that the second problem is a simple, direct ratio problem: 1 bag is to 6 apples as 8 bags is to 48 apples,

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

1:6 as 8:48. We assume that students are not ready to work with ratio and proportion until the upper years of primary, but most introductory multiplication problems that students work on in the early primary years are simple ratio problems.

Rather than work explicitly on the big ideas behind ratio and proportion, we more often introduce multiplication as repeated addition, and therein lies a problem. For example, when the item in Figure 1 was given to a representative sample ($n = 309$) of 12- to 13-year-olds (England's Year 8 of schooling), only 14% could provide a correct answer (Hart, 1981). Replicating this research with a larger representative sample ($n = 754$), Küchemann, Hodgen, and Brown (2011) only found 12% of students were able to produce a correct answer, with many reasoning, incorrectly, that as $8 + 4 = 12$, then the answer would be obtained by adding 4 to 9 (i.e., defaulting to additive reasoning rather than multiplicative). Evidence from such studies demonstrating that students often resort to thinking additively when they need to be reasoning multiplicatively points to the importance of treat additive reasoning as a big idea that is a separate way of thinking about situations rather than the big idea of multiplicative reasoning.

These 2 letters are the same shape. One is larger than the other.
AC is 8 units. RT is 12 units.

The curve AB is 9 units. How long is the curve RS ?   . . . .
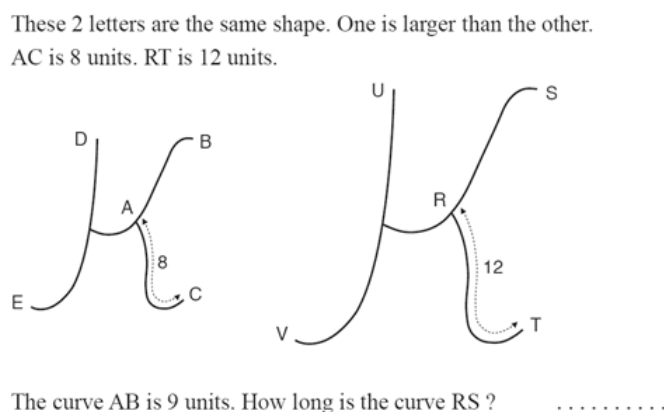
Figure 1. Curly Ks item (Hart, 1981).

One reason that students muddy the additive and multiplicative arises from the emphasis on first introducing multiplication as repeated addition. There is, of course, a link between multiplication and repeated addition, but it is a subtle one. First, there are genuine repeated addition problems. For example, suppose a conference room is set out with circular tables that seat 10 people. As people arrived, six sat at one of the tables, another six sat at the next table, and so on until eight tables had six people sitting at them. How many people were sitting at tables? Again, this can be modelled by 6×8. It seems very like the apples in bags problem, but the key difference is that there is no 'rule' about how many have to sit at each table, in the way that above there was the 'rule' that each bag had to contain exactly six apples. In the seating situation, it just so happened that the same number of people sat at each table, but it could as easily have been the case that five sat at a table, or seven, or whatever. Thinking of the way in which people arrived, the situation was $6 + 6 + 6 + 6 + 6 + 6 + 6 + 6$, not, say, $6 + 8 + 4 + 7 + 6 + 5 + 9 + 6$. It just so happened that the sequence of additions repeated itself and so lends itself to being more efficiently calculated by multiplication than repeated addition. When you look, however, at the problems students most often face when introduced to multiplicative situations, they rarely, if ever, are given 'genuine' repeated addition problems. They are much more likely to meet simple ratios: There are three plates on the table. Each plate has five biscuits on it. How many biscuits are there altogether? Alternatively, there are five cars in the carpark. Each car has four wheels. How many wheels are there altogether? As soon as there is an explicit or implicit 'each' or 'per' in a situation, then it is a simple ratio.

A second connection comes from the observation that simple ratio problems can, and will, be solved by students through repeated addition, especially if they are only just beginning to learn about multiplication. If they do not know the answer to 6×8, then students will find the total number of apples in eight bags by repeatedly adding six. We need to be clear and not confuse that means of calculating with the understanding: Finding the answer to a multiplicative ratio problem by carrying out repeated addition does not turn the problem into a repeated addition situation.

WHY DO WE PERSIST IN TEACHING MULTIPLICATION AS AN EXTENSION OF ADDITION?

One reason for the persistence in not treating multiplicative relations as being a very different way of seeing the world from additive relations is the commonly held belief that counting is the foundation of all number work. In the early years of school, counting is very much focused on additive relations. If you have two red counters and six blue counters, then students' attention is largely drawn to looking at the counters in ways that allow them to answer questions like, "How many counters are there altogether?" or "How many more blue counters are there than red counters?" It is not until later years that we might ask questions like, "How many times larger is the collection of blue counters than the collection of red counters?" or "How many times smaller is the collection of red counters than the blue collection?" Such questions arise more naturally in measuring contexts, working with continuous quantities: If the jug of water fills four glasses, then the relation between the jug and the glasses can be talked about both additively – How many glasses altogether make a full jug? – but also multiplicatively – How many times bigger is the jug than a glass? Some writers suggest that we have the foundations wrong and that early years mathematics should focus on continuous quantities and measuring (Schmittau, 2004). Curricula that have adopted this approach indicate that learning about fractions and algebra is more secure for students who have come to mathematics through measuring (Dougherty & Venenciano, 2007).

Another reason that we may be wedded to counting-as-foundational is the belief that additive reasoning is more 'natural' than multiplicative reasoning. That may simply be a result of some circularity – We teach additive reasoning prior to multiplicative reasoning, so students come to know additive reasoning before multiplicative reasoning. Research and cognitive science suggest that we may be just as 'hard-wired' to think multiplicatively as we are additively. For example, it is well established that when students are asked to put numbers on a number line, they often position the numbers closer together as the numbers get larger. Young students filling an empty number line from 0 to 10 will have the small numbers wider apart than the numbers close to 10, and older students display a similar propensity asked to put the multiples of 10 on a line marked 0 to 100. Then, 10 and 20 will be placed further apart than 80 and 90. It seems that students have to overcome this propensity of thinking that numbers get closer together as they get larger (This may still be a 'default' way of thinking even as adults, since it takes people longer to compare larger numbers than smaller ones, controlling for the time taken to read the number).

There is a logic to this. In multiplicative terms, quantities do get 'closer' together as they get larger. For example, imagine one person mixing one litre of cordial with two litres of water and another person mixing 10 litres of cordial with 11 litres of water. In additive relations terms, the difference between the amount of cordial and water in each case is one litre. Thinking about the multiplicative relationship between the cordial and water, then the 10:11 mix is much closer to a half-and-half concentration than the 1:2 mix is. Even young children can reason that the two mixes are not going to taste the same.

Dehaene (1999) theorises that this is accounted for by humans, and other animals, mentally storing and comparing amounts not as discrete quantities but through some sort of mental 'accumulator', which is like imagining two test tubes. Pouring one unit of liquid into one (metaphorical) test tube and two units of the same size into the other means that if the units are large enough, then it is easy to tell which test tube contains more, simply by looking. Imagine two other, identically sized test tubes. You now need to fill these with 10 and 11 units respectively, so the units are going to have to be smaller, making it a little more difficult to tell which tube holds the more liquid (hence the longer time needed to compare larger numbers) and why, again, 10 may, phenomenologically, seem closer to 11 than 1 does to 2.

This distinction between additive and multiplicative reasoning may only exist in the mind of the person making a comparison; it need not exist in two quantities being compared. For example, place value is commonly taught as an additive process – exchanging 10 units for one ten, and 10 tens for one hundred. Thinking of place value as a multiplicative process – scaling up the unit by a factor of 10, and then scaling up the 10 by a factor of 10 again to make a hundred – leads to a very different way of seeing the relationship between the 1 and the 10. This may have different implications for how learning progressed. For example, a different sense of where decimals come from is engendered. Rather than being seen as a type of fraction, arising from partition a one into 10 equal sub-parts, decimals can be understood as an extension of the place value system. A hundred can be imagined to be shrunk to one-tenth of its size,

giving a 10. This, in turn, is shrunk to one-tenth of its size, becoming a one. There is no need to stop this shrinking process at this point; the one can be shrunk to one-tenth of its size, giving rise to the first decimal place and so on. Thinking of place value as enlarging and shrinking (as opposed to adding and separating) is a key distinction, with the latter image being important in later years, and being able to be worked on from the beginning of introducing place value.

To move to the big idea of multiplicative reasoning as needing to be distinct from additive reasoning, we may need to change the models and images that we offer to students to embody mathematical ideas. One image that is showing increasing potential is the double number line.

THE DOUBLE NUMBER LINE AS A MODEL FOR MULTIPLICATIVE REASONING

There is growing evidence for the power of using the double number line to model multiplication in the lower secondary years to encourage students to move away from thinking additively. Küchemann et al. (2011) provide an example of the double number line (Figure 2).
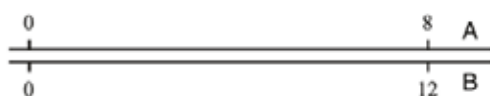
Figure 2. Double number line for × 1.5 mapping.

Working on this as a model for mapping any number onto Line *A* onto a corresponding number on Line *B* brings into awareness the general mapping (function) of ×1.5 (or $x \rightarrow 1.5x$ or $y = 1/5x$ and so on).

Presenting different contexts for the origins of this particular double number line can help students think about the functional relationship in different ways (National Council of Teachers of Mathematics, 2019). For example, one can consider the difference between the double number line representing the number of steps taken by an adult and a child walking along a straight path, and the same model representing the conversion rate of US $8 to NZ $12, thus raising awareness of the difference between working with discrete and continuous quantities. Moving between representations is also possible. Looking at the relationship between the double number line and a straight-line graph opens up further understandings about the nature of multiplicative reasoning. Similarly, scales on maps or rulers showing different units (e.g., metres and feet, centimetres and inches) provide further examples of double number lines showing multiplicative relations.

Küchemann et al. (2011) concluded that some students were able to work productively with the double number line, but there were students who had difficulty appreciating the multiplicative structure of the model.

Is this a developmental issue – that students have to have reached a level of understanding before being able to work productively with the double number line – or a consequence of limited experience of the model earlier in their schooling?

Younger students can benefit from working with double number lines. In a review of research in Japan about teaching whole-number multiplication to encourage proportional reasoning, Hino and Kato (2019) concluded that primary-aged children can work effectively with double number line representations and that the use of such representations "provides a means of scaffolding the learning of reasoning proportionally" (p. 135).

I have been working with contexts giving rise to double number lines with students as young as six years old, orally introducing problems like:

A mother frog is jumping along a straight path. To keep up, the baby frog has to take two jumps for every jump the mother makes. How many jumps does the baby frog make when the mother has made six jumps?

Modelling this situation with literal adult and child jumps, to make sure everyone has a sense of what is happening, students then produce informal representations. The linearity of the context means these representations are usually

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

presented on a single number line image with the smaller jumps drawn within the larger ones. The students then have no difficulty with my representing these on two parallel lines. While many have repeatedly counted in twos to arrive at an answer, subsequent discussion of extending the line in the imagination so that, say, mother had done 10 or 100 jumps leads many students to articulate the functional relationship that the numbers on the bottom line are twice those on the top.

CONCLUSION

Using the double number line as an introduction to the big idea of multiplicative reasoning being based in understanding ratio is the distinction made by researchers coming from the traditions of the Freudenthal Institute in Holland and the distinction between 'models of' and 'models for', ultimately leading to tools for thinking (e.g., Van den Heuvel-Panhuizen, 2002). The development of the big idea of multiplicative reasoning from this perspective is seen as moving from students seeing the double line as a model for what they might be working on informally, to constructing and using it for themselves as an external 'scaffold' for thinking about multiplicative relations, then to become a mental tool for thinking about such relations. Of course, the double number line is not the only model that students need to meet (Arrays, for example, provide different affordances for learning), but it does appear to be a particularly productive model to use.

As I hope that I have shown through this example of multiplicative reasoning, thinking about the big ideas underpinning mathematical structure and generality can be used to design teaching experiences that bring coherence and continuity to developing students' mathematical understanding.

REFERENCES

Dehaene, S. (1999). *The number sense: How the mind creates mathematics*. Oxford, England: Oxford University Press.

Hart, K. (1981). Ratio and proportion. In K. M. Hart, M. L. Brown, D. Kuchemann, D. Kerslake, G. Ruddock, & M. McCartney (Eds.), *Children's understanding of mathematics: 11-16* (pp. 88-101). London, England: John Murray

Hino, K., & Kato, H. (2019). Teaching whole-number multiplication to promote children's proportional reasoning: A practice-based perspective from Japan. *ZDM*, *51*(1), 125-137.

Küchemann, D., Hodgen, J., & Brown, M. (2011). Using the double number line to model multiplication. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 326-335). Rzeszów, Poland: University of Rzeszów.

Schmittau, J. (2004). Vygotskian theory and mathematics education: Resolving the conceptual-procedural dichotomy. *European Journal of Psychology of Education*, *19*(1), 19-43.

Van den Heuvel-Panhuizen, M. (2002). Realistic Mathematics Education as work in progress. In F.-L. Lin (Ed.), *Common sense in mathematics education: Proceedings of 2001 The Netherlands and Taiwan Conference on Mathematics Education* (pp. 1-42). Taipei, Taiwan: National Taiwan Normal University.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

From 'uh-oh' to 'ah-ha!'

Greg Oates, The University of Tasmania

FOSTERING INTUITION IN THE CLASSROOM

Inquiry-based learning, problem solving, challenging tasks, student-centred learning; active learning: These terms are commonly used for an approach to learning mathematics which sees the student develop their own understanding as they work through a mathematical problem, and such approaches are - quite rightly - widely promoted in our classrooms.

These approaches give students better chances to experience authentic mathematical experiences as they attempt to model real life problems and data. The exact nature of the inquiry can vary greatly, from a more traditional confirmation inquiry where the teacher frames questions for the student, usually with a specific pre-determined outcome, on to structured and guided inquiry with decreasing levels of scaffolding, through to open inquiry, where students formulate their own research question(s).

There are many rich examples of inquiry based learning resources in mathematics available for teachers and students, with varying levels of scaffolding. In the Australian context, there are, among others, reSolve, Maths300, and Maths Inside (all supported by AAMT).

The reSolve protocol makes explicit the importance of providing purposeful, inclusive and challenging tasks within a knowledge building culture which celebrates productive struggle and the confidence to take risks (http://resolve.edu.au/protocol).

Most teachers and resource developers would agree however that student's ability to engage with inquiry learning often requires them to take risks and to build on initial intuitions they may have about the problem.

The best problems are thus usually seen as being accessible to a range of abilities and levels, what is known as 'low floor, high-ceiling' tasks. But even a low floor does not always allow students to get started on the problem. Students can frequently doubt their intuitions and be afraid to follow them for fear of being wrong. We need to find ways of encouraging students to think intuitively, to have a productive disposition towards trying things out.

Following your intuitions down sometimes blind alleys is a natural part of authentic mathematical enquiry (think for example of Andrew Wile's path to proving Fermat's last theorem, as immortalised in the book and film by Simon Singh). There is a big difference between a 'wrong guess or thought', and misconceptions (often ingrained) and mathematical errors in say calculations or incorrectly applied procedures. Sadly, I feel students often see these collected under the same umbrella of 'wrong' and marked with a 'X'.

I try to build 'safe' classrooms, where students are not afraid to be wrong, where its safe to take risks, exactly because being wrong is often what leads to real learning. I believe that we should start this early and start out simple, but also that its never too late to start. A simple example I have used at many levels of the curriculum is a 'pattern-cards' game, where I first show students examples of a series of cards that indicate a pattern, but in the early stages at least the pattern they detect might be (deliberately) the wrong one. I do not claim that the game is itself original, I have seen many such examples elsewhere. However, what I do wish to highlight is the intent behind the approach. I use it to deliberately provoke intuitions which may be wrong, as much as addressing the algebra and patterns content strand within which this game is likely positioned.

PATTERN CARDS

There are of course infinite possibilities for the first two turnovers (see images on p. 8), as indeed there are for the third, although the latter is usually less obvious to students, wherein lies the opportunity for surprise: 8 is probably the most common suggestion for the third turnover, but it could easily be shown as 5 (repeating growing pattern of +1, +2, +1, +2..) or many other possibilities.

Figure 1. Starting cards: What could the next card be? Have a guess.

Figure 2. Now, what could the next card be? Make a prediction.

Figure 3. Second turn over: Were you right? What might the next card be now? Are you sure?

Figure 4. Third turn over: Do we know what is next now?

The question of whether students are then sure what the pattern is after the third turnover is interesting. Most people will say '10' for the next card (which is what I will turn over) and most will also be convinced that this is now the only possibility, so I leave this as a challenge to see if they can come up with an alternative that works.

The process I promote to the students is:

• Say what you think (or believe), make a prediction

• Convince yourself, test your answer

• Convince a friend or your teacher your answer is correct

• Can you prove it?

Once we have tried a few as a group, I usually encourage the students to design their own patterns, and see if they can fool their friends. The challenge is who can devise a pattern that needs the biggest number of cards before their partner gets the pattern. The level at which you use this game can be varied, by varying the conceptual complexity of the patterns, for example with simple repeating patterns in the early years; introducing the Fibonacci Sequence in the middle years, or geometric and arithmetic sequences in later years, or the degree to which you develop the structure of the pattern, in for example a verbal description such as doubling in lower years to formulating algebraic rules at higher levels.

Psychologically, questions or problems that lead students to intuitively wrong answers can be viewed as setting up cognitive dissonance, which can be a powerful motivator for change, if treated in a productive manner (cognitive dissonance is also viewed as somewhat traumatic, which reinforces the need for care in developing a safe classroom).

I first started collecting such problems after trying out two problems in a university foundation mathematics (transition) course in 1995. The problems I used then came from a 1991 article I found by Avital and Barbeau called 'Intuitively misconceived solutions to problems' (*For the Learning of Mathematics*, *11*(3)). They provoked intense reactions from my students, and ultimately led to my thinking about 'safe classrooms'.

There are many famous problems in the history of mathematics which testify to the frequent stumbling path of mathematical discovery, but which have ultimately become turning points in our mathematical knowledge, e.g the Königsberg Bridge problem which led to Euler's development of graph theory in 1736 (see the TEDEd talk *How the Königsberg bridge problem changed mathematics* by Dan Van der Vieren: http://www.youtube.com/watch?v=nZwSo4vfw6c). It is important that we give students a sense that this actually the way mathematics works, and provide opportunities for them to experience it for themselves.

For me there is nothing better than those 'Ah-ha' moments in the classroom. It's those moments when we see students' faces light up at the pleasure of solving a problem they have been struggling with. And there is nothing sadder than students' being too scared to try.

I encourage teachers to start the process of building safe classrooms early on, using tasks that are deliberately designed to go against students' intuitions or beliefs about mathematics they may previously have constructed, but with care to always make it clear to your students that being wrong can also be the right way to learn!

THE MATHEMATICAL ASSOCIATION OF VICTORIA

Myth-busting mathematics

Kate Smith-Miles, University of Melbourne

CHALLENGING STUDENT PERCEPTIONS THAT MATHS IS IRRELEVANT, BORING, AND TOO HARD

For mathematics teachers trying to inspire their students to engage and continue with mathematics, there are numerous challenges. Student attitudes towards mathematics are formed early (Dowker, Cheriton, Horton, & Mark, 2019), and it is sometimes hard to undo deeply engrained perceptions that maths is too hard, boring, and irrelevant to their future. But these perception issues must be challenged to ensure each student has equal opportunity for a strong career in a future workforce where high level numerical skills and analytical thinking will be essential and assumed (Finkel, 2019).

Calling out these perceptions as myths, and being equipped with arguments and counter-examples, is the best response for teachers at the front-line. My YouTube video 'Myth-busting Mathematics' (Smith-Miles, 2017) may provide some ideas for responding to such student perceptions in your classroom. Fundamentally, the belief that mathematics is irrelevant, boring, and a career path that is only suitable for brainy geniuses (and many other misconceptions) often stems from two main causes:

Lack of understanding of what mathematics really is, and therefore what it can do. A belief that some people have a 'maths brain' and others do not.

We should start by acknowledging that most people do not understand what mathematics really is. They think they know because they have been studying maths since before kindergarten when they learned basic ideas about numbers. Surely after more than a decade of studying a subject, you would be entitled to think you have a clear idea of what it is all about! Unfortunately, the foundational nature of the mathematics studied at school leads to a belief that mathematics is mostly useful for trivial 'everyday' applications, like making recipes using ratios, laying brick patterns using geometry, or figuring out how to place a ladder against a wall.

Certainly this message – that maths is important for everyday activities - is important for raising national numeracy awareness (although it is often counteracted by parents who unhelpfully point out 'I hated maths at school, dropped it as soon as I could, and have never needed it since!'). But we need to be careful that students don't believe that everyday applications of maths is all that mathematics can do.

Just as musical scales are preparation for playing great musical works, and grammar provides the foundations for literature, the kind of mathematics learned at school is foundational and mere preparation for something more powerful that most students do not ever see. How do we expose students to the real power and beauty of more significant mathematics, and inspire them to want to learn more, beyond the basic foundations of numeracy?

The future (well-paid) jobs for the current generation of students will demand greater numeracy skills, problem-solving and critical, analytical and logical thinking taught only by studying higher levels of mathematics. There is a depth and breadth of the field of mathematics that the school curriculum simply can't explore. We need students to realise that more advanced mathematics is important for tackling truly significant problems like managing spread of diseases, improving green energy, and designing new products and technologies that will revolutionise our lives (just like mathematics has done for centuries).

The challenges for curriculum design and pedagogy are to ensure the foundations are strengthened for all students (for everyday numeracy) while motivating a greater number of students to explore the more creative side of mathematics required for careers that build upon advanced mathematics training.

The movie *Hidden Figures* provides a great example to enable students to imagine how basic concepts explored at school (like parabolas and projectile motion) could be extended in more advanced form to solve real-world problems of great importance (like ensuring the safe return of astronauts). Certainly, not all school maths topics easily lend themselves to this kind of imagined extension. But students can be reassured that each foundational topic extends through the rich tree of mathematics, to enable some truly critical problems to be tackled.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Good summaries of such applications exist (e.g., https://mathigon.org/applications), but the simplest answer is this: mathematics is a language, and we use it to describe our world. With mathematics we can prove facts and estimate uncertainties. We can mathematically model a system (e.g., traffic networks, the human brain, stem cells, stock markets), then use mathematics to predict what would happen if we make changes, and use more mathematics to decide how to improve the system. The opportunities for positive impact – in the corporate world as well as for social good - are endless for those who speak this powerful language. They need to learn to speak it fluently!

With this approach we may be able to convince students that it's important for more people to study advanced mathematics, but many students are affected by their belief that they don't have a 'maths brain'. Hence we need to tackle that misconception too. Early negative experiences with mathematics can be detrimental to student perceptions of their maths ability, and while this can affect students of all genders, it is particularly prevalent amongst girls, who seem to disengage earlier and in greater numbers than boys (Li & Koch, 2017). There is no evidence that girls perform more poorly than boys in mathematics (O'Dea, Lagisz, Jennions, & Nakagawa, 2018), and yet the belief that girls' brains are not as 'hard-wired' for mathematics as boys' brains continues to persist in our society. Given this myth, it should not be surprising that there is evidence that girls are less confident than boys in their mathematics ability. Research conducted by the Australian Mathematical Sciences Institute has shown that this confidence can be changed with very simple intervention exercises (Li & Koch, 2017). The impact of 'maths anxiety' (which can be so easily passed down from parent to child) and use of mind-shift thinking to increase confidence should inform strategies around teacher training and influencing parental attitudes.

Even if we convince students that there is no such thing as a 'maths brain' – that all students can be taught to thinking mathematically and problem solve, just as all students can be taught to read and decipher and understand concepts and messages discussed in literature – we still need to tackle the perception that only super smart people should progress to study higher level mathematics. The Hollywood stereotype of the mathematician is unfortunate indeed: typically a male (often mentally-ill) genius who isolates themselves for many years to tackle an impossibly difficult maths problem that no-one else in the 'real world' really cares about or can understand (such as the portrait of John Nash in *A Beautiful Mind*). This is not a very attractive aspirational goal for most young people! There is an urgent need to communicate more effectively about what mathematics really is, its significance for the real world, and to showcase role models who are relatable, in order to change student perceptions and attitudes towards mathematics.

I will discuss such myth-busting in my keynote talk at MAV19 in December, along with classroom suggestions to help students see that – far from being a dusty and irrelevant subject – mathematics is an exciting evolving subject. Just as it has throughout history, mathematics evolves whenever society faces a challenge, be it an industrial revolution or wartime crises. Our society is facing many significant challenges over the coming decades, and new mathematics is being developed now to ensure solutions are found to pressing problems in food security, climate change, energy efficiency, traffic management, public health, to name just a few. How do we help students understand how maths can solve these challenges? That is our challenge as educators.

REFERENCES

Dowker, A., Cheriton, O., Horton, R., & Mark, W. (2019). *Educational Studies in Mathematics*, *100*(3), 210-230. doi:10.1007/s10649-019-9880-5

Finkel, A. (2019). *Renewing the signals, restoring the continuum*. Retrieved from http://www.chiefscientist.gov.au/2019/07/speech-renewing-the-signals-restoring-the-continuum/

Li, N., & Koch, I. (2017). *Gender report 2017: Participation, performance and attitudes towards mathematics*. Retrieved from https://amsi.org.au/publications/gender-report-2017-participation-performance-attitudes-towards-mathematics/

O'Dea, R. E., Lagisz, M., Jennions, M. D., & Nakagawa, S. (2018). Gender differences in individual variation in academic grades fail to fit expected patterns for STEM. *Nature Communications*, *9*, 1-8. doi:10.1038/s41467-018-06292-0

Smith-Miles, K. (2017, October 8). *Myth-busting mathematics* [Video file]. Retrieved from https://youtu.be/IgsDI2CaRg4

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Double-blind peer reviewed papers

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Parental perceptions of NAPLAN results: Who is telling the truth?

Linda Cranley, Gregory Hine, Christine Robinson, University of Notre Dame, Australia

Each year across Australia, students in Years 3, 5, 7, and 9 participate in the National Assessment Program – Literacy and Numeracy (NAPLAN). The NAPLAN results are reported nationally through summary and national reports, and an individual report is provided to parents and caregivers for each student. One dimension of the case study presented in this paper explored parents' perceptions of NAPLAN in mathematics, finding that there was confusion and concern over the discrepancy in the results between NAPLAN and school reports. This paper presents findings from a larger case study that explored teachers', parents', and students' concerns regarding NAPLAN. This paper presents those findings pertaining to parental perspectives of the discrepancies between school reports and NAPLAN results in mathematics and the effect these discrepancies have on the parent-school relationship.

INTRODUCTION

This paper presents findings that emanated from a broader qualitative study investigating the effects NAPLAN has had on the teaching and learning of mathematics. The findings presented in this paper pertain to parental perspectives of NAPLAN. Overall, the literature available suggests that parents are in favour of NAPLAN testing (Dulfer, Polesal, & Rice, 2012). However, there is also evidence to suggest that parents experience some anxiety and confusion in relation to NAPLAN. There are two main elements that fuel this anxiety and confusion: (1) media reports that exacerbate the high stakes nature of NAPLAN and (2) discrepancies between NAPLAN results and school reports. Dounay (2000) and Barksdale-Ladd and Thomas (2000) commented that the importance of standardised testing is often overstated in the media for the sake of increasing parents' interest during the implementation and reporting of high-stakes testing results. Dounay (2000) reported that parents became confused and concerned when comparing the results of school reports and standardised test results. Parents were perplexed in two ways; firstly, they questioned the performance of the school itself. Barksdale-Ladd and Thomas (2000) stated "describing and decrying failing test scores must lead parents to questioning their children's schools" (p. 386). The second form of confusion concerned the results students achieved in NAPLAN compared to school reports. In this study, the students' results in NAPLAN were below the national average and the school report had awarded the students above average marks. This caused the parents to question which results were accurate and ask, "Who is telling the truth?"

RESEARCH PURPOSE AND SIGNIFICANCE

The purpose of this research was to investigate how the teaching and learning of mathematics was affected by the implementation of national high-stakes testing through NAPLAN. Perspectives from students, classroom teachers and parents at one Western Australian Catholic primary school were gathered through interviews. Semi-structured interviews (Neuman, 2006) were employed along with researcher-generated field notes. Using these methods, the researcher wished to give individual children, their parents and their teachers a voice in sharing their experiences about how NAPLAN had affected their relationships with mathematics. In doing so, it was anticipated that following an analysis of gathered data and development of key findings, educators could better understand student, teacher and parent perceptions about NAPLAN testing which has been identified as an area of need (Belcastro & Boon, 2012). The focus of this paper is to examine at depth one of the sub-questions of the study, as outlined below.

KEY RESEARCH QUESTIONS

The key research question for this project was: What is the impact of high-stakes testing on the teaching and learning of mathematics in one Western Australian Catholic primary school?

Sub-questions

While there were three sub-questions developed from the key research question, this paper reports exclusively on the findings of the following sub-question: What is the impact of high-stakes testing on the understanding of mathematical teaching and learning for the parents/guardians of Year 3 and Year 5 students?

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

METHODOLOGY

CASE STUDY

This research was conducted through an intrinsic case study (Stake, 1995) where all data were collected from one triple-stream Western Australian Catholic primary school. A case study approach was chosen because the researcher wished to carry out a detailed investigation over a period of time within a particular context (Neuman, 2006). By involving students, parents and teachers from Year 3 and Year 5, the researcher was able to explore the extent to which high-stakes testing impacted the teaching and learning of mathematics for these participants. Specifically, the case study design enabled the researcher to discern similarities and differences through both Year 3 and Year 5 cohorts for students, teachers and parents.

PARTICIPANTS

The parents, children and teachers involved with the Year 3 and Year 5 classrooms were the key participants in this research. Six children from each year group were selected purposively by their classroom teacher prior to sitting NAPLAN. To ensure the holistic nature of the research, parents of participating students and the Year 3 and Year 5 classroom teachers were also interviewed. All interviews were conducted on an individual basis within each group of participants.

	Students	Parents	Teachers
Year 3	6	3	3
Year 5	6	3	3

Table 1. Number of participants.

The parent participants were chosen to discern the extent to which their experiences affected their understanding of NAPLAN and their relationship with both the teacher and the school. To allow for an appropriate commentary, the parents were interviewed shortly after NAPLAN results were disseminated.

METHODS

As this research dealt with personal experiences of participants involved in NAPLAN testing, it was important to use data collection methods that allowed the human elements of the research to be investigated. Consequently, the researchers used semi-structured, qualitative interviewing and field notes to collect data. Individual interviews were conducted face-to-face with key participants soon after the NAPLAN test had been administered (teachers and students) and after the results had been disseminated (parents). The interviews were recorded with a digital device and these were transcribed and analysed at a later date. The researcher also took field notes during the interviews to note any salient observations or emerging thoughts arising during the interviews. Doing so contributed to both the accuracy and trustworthiness of data collected (Babbie, 2016).

DATA ANALYSIS

All qualitative data collected from the student, parent and teacher participants were analysed according to an interactive framework offered by Miles, Huberman and Saldaña (2014). This framework comprises of four steps, namely: data collection, data reduction, data display, and conclusion drawing/verification. Within each of these components, the researcher executed the following operations: coding, memoing, and developing propositions. Miles and Huberman (1994) described codes as "tags or labels for assigning units of meaning to the descriptive or inferential information compiled during a study" (p. 56). Codes developed a posteriori by the researcher were attached to data gathered via interviews and field notes, and were selected from those data based on their meaning. Memoing was then used to synthesise coded data so that they formed a recognisable cluster of information anchored in one general concept. Finally, the researcher generated propositions about connected sets of statements regarding participants' perceptions, reflected on the findings, and drew conclusions about the study.

RESULTS

Within this study, parents of students in Years 3 and 5 were interviewed. Once data were collected and analysed three main themes emerged: (i) parents' understanding of NAPLAN, (ii) the confusion experienced by parents in the discrepancy of results between the school reports and NAPLAN results, and (iii) how the relationship between the school and parents was affected by NAPLAN.

PARENTS' UNDERSTANDING OF NAPLAN

The parents interviewed conveyed a comprehensive understanding of how NAPLAN was implemented, the components of the test and the purpose of NAPLAN. The parents also indicated an awareness they would receive an individual report of their child's performance in NAPLAN. One parent participant concisely outlined how the test was implemented:

> Everyone in Australia does the same test at the same time on the same day, everyone does it. And they give instructions on how to implement it and they keep it as uniform as possible. This is really for the government.

DISCREPANCY IN SCHOOL REPORTS AND NAPLAN RESULTS

A majority of parents expressed that they became disillusioned and confused by the disparity in the mathematics results their children achieved in their school reports and their child's reported achievements in NAPLAN. Many parents doubted the school report as it was evident in the NAPLAN results that the school at which the study was conducted, was below the national average in mathematics. Parents also expressed concern about their child's preparedness for secondary school. To illustrate one parent stated:

> So what worries me is, is our academic maths class really an academic maths class, when compared with the rest of Australia. Maybe our whole level is not right? Maybe we shouldn't have an academic's maths class if we're not up there with the academics. I don't want her going to high school thinking she's an A student, but when she gets there she's a C.'

This expressed confusion about the school's grading and reporting system for their mathematics program continued throughout the collection of data with another parent citing:

> My daughter didn't perform as well in maths as we thought she would. But I did speak to some other parents about it and they were surprised that their child didn't perform particularly well in maths. But my child sat below the school average, which I thought was very surprising as she is in the top maths group.

Parents articulated that before they received their child's NAPLAN results, they had faith in the school's ability to report and assess their child's performance in mathematics accurately, but once they had viewed the NAPLAN reports they became disillusioned and concerned about the level of mathematics that was taught in the school. One parent indicated that she rarely saw mathematics results from the school during the term. This parent's main confusion was exacerbated by her daughter being in the top mathematics class in her year level but having achieved results below the school and national average in NAPLAN.

RELATIONSHIP BETWEEN SCHOOL AND PARENTS

The third theme developed was how NAPLAN had affected the relationship between the school and the parents. This theme is closely connected to the previous theme as the discrepancy in results had far-reaching effects on the school-parent relationship. Initially parents were very supportive of the school and its relaxed attitude to NAPLAN. However, this attitude changed when the NAPLAN results were released. Some parents questioned their decision to send their child to this fee-paying Catholic school rather than the neighbouring government-funded state school which had generated better NAPLAN results in mathematics. One parent commented, "I started to think why I am paying for my child to go to this school when the school next door is doing better at NAPLAN, they are above the national average."

Most parents of Year 5 students indicated that they were very concerned about the poor results the students achieved in NAPLAN particularly as these test results are required as part of their child's application for entry into secondary school.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

One parent stated:

> Well, I just thought if NAPLAN doesn't mean anything, why are they doing that? They sorta [sic] tell you 'Don't worry about it, it's just for the national placing' but to me I think, I don't want to hand this into the teacher knowing that it's not a true evaluation of my child. I've got the primary school that doesn't worry about it but then the high school that does. So if the primary school doesn't worry about it, it's not giving my high school a true reading.

Parents also expressed frustration by the lack of information teachers were able to provide them regarding NAPLAN. Parents recalled feeling surprised that teachers were not provided with a diagnostic report of their child's performance in NAPLAN. For instance, one parent stated:

> I was surprised by my daughter's results so I went and spoke to her teacher but she said she doesn't get to see individual results, she only gets to see a year group summary, I felt it was a waste of time.

Parental frustration and confusion about NAPLAN clearly had an effect on the relationship between the school and the parent. Parents began to lose confidence in the school's ability to assess and report on their child's performance in mathematics.

DISCUSSION

In exploring the sub-question, What is the impact of high-stakes testing on the understanding of mathematical teaching and learning for the parents/guardians of Year 3 and Year 5 students?, three themes emerged: (i) parents' understanding of NAPLAN, (ii) the confusion experienced by parents in the discrepancy of results between the school reports and NAPLAN results, and (iii) how the relationship between the school and parents was affected by NAPLAN. Overall, the parents displayed a very sound understanding of NAPLAN, which is representative of existing literature (Dulfer et al., 2012).

A significant finding from this study was the confusion and concern that parents displayed in dealing with the discrepancy between the school report and NAPLAN results. Douney (2000) stated that many parents doubted the validity of standardised testing results when they do not correspond to the school grades. However, in this study, the opposite was found. Parents began doubting the validity of the school reports and in addition, expressed concern in the ability of the school to report accurately on their child's mathematical ability. The parents' discontentment and loss of confidence in the school led teachers to be faced with increasing perceived pressure from parents as teachers were forced to clarify the discrepancy. This finding led to the final emergent theme, which is the effect NAPLAN has had on the school-parent relationship.

There was evidence to indicate that the school-parent relationship was affected negatively by NAPLAN. Originally, parents responded favourably to questions about the general mathematical strategies teachers employed at the school and to the school's approach to the implementation of NAPLAN. Overall, parents were content that initially there was not much emphasis placed on the importance of NAPLAN by the school; however, this sentiment altered when discussing NAPLAN once the results were released. Parents expressed concern about the achievements from one cohort to the next; the main concern was that the mathematics results were declining and not improving. Parents then began to question the school's casual response to NAPLAN. The discrepancy in the school reports and NAPLAN results clearly affected the school-parent relationship as parents questioned teachers about the results and were not satisfied with the explanations they received (Barksdale-Ladd & Thomas, 2000). The use of NAPLAN reports as entry criteria into secondary schools also contributed to poor school-parent relationship: One Year 5 teacher stated, "It wasn't an issue when I first started NAPLAN, but since high schools have been asking for the data in the meetings, that's when I've seen a change in parents." It may be asserted that if the implementation of NAPLAN has adversely affected the relationship between schools and teachers, that NAPLAN as a tool itself requires further investigation from policy makers.

The final contributing factor that affected the school-parent relationship were media reports associated with NAPLAN. This finding accords with those of Freeman, Mathison, and Campbell-Wilcox (2015), who highlighted that the relationship between teachers, parents, and schools can be affected and reshaped by reports in the media.

CONCLUSION

This paper has presented the parental perceptions of NAPLAN that emanated from a larger qualitative study. What became clear in these findings was that parents displayed a sound understanding of NAPLAN and its implementation. Findings revealed that parents became disillusioned and concerned with the school's ability to report on their child's mathematical achievements as a discrepancy existed between the school report results in mathematics and the results their children achieved in NAPLAN. This discrepancy, along with media reports exacerbating the high-stakes nature of NAPLAN, had detrimental effects on the school-parent relationship.

Implications arising from these findings indicate that schools need to be more proactive in addressing any discrepancies between their own reporting of mathematics results and those achieved in NAPLAN. To maintain strong parent-school relationships, it is important for schools to articulate clearly how NAPLAN results can complement school reports and for schools to report more formatively on children's achievements and progress. Currently, schools do not receive an individual diagnostic NAPLAN report for each student. However, as a recommendation, parents could be invited to share their child's NAPLAN report with the teachers in order to address strengths and weaknesses. What is clear is that NAPLAN results need to complement school-based assessments in order to allay parental concerns and to assist children's progress through the mathematics curriculum. If they do not, parents will continue to be left wondering, 'Who is telling the truth?'

REFERENCES

Babbie, E. (2016). *The practice of social research* (14th ed.). Boston, MA: Cengage Learning.

Barksdale-Ladd, A., & Thomas, K. F. (2000). What's at stake in high-stakes testing: Teachers and parents speak out. *Journal of Teacher Education*, *51*(5), 384-397.

Belcastro, L., & Boon, H. (2012). Student motivation for NAPLAN tests. *Australian and International Journal of Rural Education*, *22*(2), 1-20.

Dounay, J. (2000). High-stakes testing is high-stress, too. *Education Digest*, *65*(9), 9.

Dulfer, N., Polesal, J., & Rice, S. (2012). *The experience of education: The impact of high stakes testing on school students and their families*. Retrieved from https://www.whitlam.org/publications/2017/10/17/an-educators-perspective-the-impacts-of-high-stakes-testing-on-school-students-and-their-families

Freeman, M., Mathison, S., & Campbell-Wilcox, K. C. (2006). Performing parent dialogues on high-stakes testing: Consent and resistance to the hegemony of accountability. *Cultural Studies Critical Methodologies*, *6*(4), 460-473. doi:10.1177/1532708606288647

Miles, M. B., & Huberman, M. A. (1994). *Qualitative data analysis: An expanded sourcebook* (2nd ed.). Thousand Oaks, CA: SAGE.

Miles, M. B., & Huberman, M. A., & Saldaña, J. (2014). *An expanded sourcebook: Qualitative data analysis* (3rd ed.). Thousand Oaks, CA: SAGE.

Neuman, W. L. (2006). *Social research methods: Qualitative and quantitative approaches*. Toronto, Canada: Pearson.

Stake, R. E. (1995). *The art of case study research*. Thousand Oaks, CA: SAGE.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Flipped learning: A mathematics experience for Year 9 boys

Penelope Kalogeropoulos, Monash University, and Angela Liyanage, Salesian College

Flipped learning is a pedagogical approach where students are introduced to the learning material before class. Classroom time is then used to deepen student understanding through peer discussion and collaborative tasks. This innovative pedagogical approach was implemented in mathematics lessons for a Year 9 class of all boys at a school in metropolitan Melbourne. Teacher reflections and feedback from students and parents were used to evaluate this approach. Students' engagement and autonomy have increased. Suggestions to improve the flipped learning approach have been identified by the teacher.

INTRODUCTION

Flipped learning is more prevalent due to innovations in educational technologies. Bishop and Verleger (2013) define the flipped classroom approach as one in which teachers design resources to provide direct computer-based individual instruction outside the classroom and interactive group learning tasks inside the classroom. Teachers who use flipped learning prepare and/or source pre-lesson materials such as prescribed readings, videos, online activities and screencasts for students to explore. The use of these pre-classroom resources allows face-to-face lessons to focus mainly on a collaborative working environment.

The flipped learning approach differs from the more traditional approaches through:

- Allocating homework tasks for new content learning, which prepares students for their upcoming mathematics lessons, in contrast to revision tasks.

- Online learning (e.g., visual presentations, readings, quizzes) is favoured over textbooks and worksheets.

- Class time focuses on facilitating learning rather than whole-class instruction.

(Straw, Quinlan, Harland, & Walker, 2015)

Although there is no single pedagogical model to implement flipped learning (Mercieca, 2018), the Flipped Learning Network (FLN, 2014) established by Bergmann and Sams (2012), describe the Four Pillars of FLIP that teachers should incorporate into their practice in order to engage in flipped learning. Table 1 provides a summary of what is involved in each of the four pillars.

Pillar	Characterised by
Flexible Learning Environment	• Spaces and time frames that permit students to reflect on their learning are established. • Educators are flexible in their expectations of student timelines for learning and in their assessment of student learning.
Learning Culture	• Instruction focuses on a learner centred approach. • Students are actively involved in knowledge construction as they participate in and evaluate their own learning.
Intentional Content	• Educators help students develop conceptual understanding and procedural fluency. • Concepts used in direct instruction are prioritised for learners to access on their own.
Professional Educator	• Teacher is available to provide real time feedback to individuals, small group and class discussion. • Ongoing formative assessment is conducted during class time to inform future instruction.

Table 1. Overview of Four Pillars of FLIP (FLN, 2014).

THE MATHEMATICAL ASSOCIATION OF VICTORIA

TECHNOLOGICAL TOOLS FOR FLIPPED LEARNING

An abundant number of technological tools are being used by teachers in flipped learning (e.g., Matific – an online mathematics teaching resource that focuses on improving core skills such as the four processes). Similarly, there are various approaches used such as the provision of direct instruction via the use of a prescribed reading or a video recorded lecture, to those that allow teachers to individualise learning according to student needs (Attard, 2019). The evolvement of new technology provides teachers with opportunities to prepare pre-lesson materials via multimedia and shift students from passive learning to active learning.

A CLASSROOM EXPERIENCE OF FLIPPED LEARNING

SETTING THE SCENE

Flipped learning was used (by Liyanage) to re-conceptualise traditional secondary mathematics instruction for Year 9 boys (48 students, 14-15 years old) and provide students with the opportunity to take responsibility for their own learning. When flipped learning was first introduced, many students found self-directed learning challenging. As their mathematics teacher, I explained to the students my rationale for flipping the classroom, my expectations on viewing the videos, taking notes and coming to class with an awareness of the content that we would be discussing and reinforcing in class. I chose to share with the students why I was going to flip the classroom so that students would be cognisant of my significant investment in the process, from filming and recording the videos to purchasing the equipment to produce the content.

EXPERIMENTING WITH RESOURCES

After trialling a number of different programs, the most straightforward method was to screen record using my iPad and then upload the videos to *Edpuzzle*. *Edpuzzle* allows the editing of videos, the incorporation of formative assessment into the videos and it prevents students from skipping the video. I initially uploaded my content to a private YouTube page accessible to the students. However, it became difficult to track which student had actually watched the video. Therefore, I transferred the videos to *Edpuzzle*, which offers teachers useful data and analytics on student progress per video. I have also established a class notebook on OneNote (Figure 1) where students can find set tasks for the lesson, extra material, basic class notes, and the relevant video links.
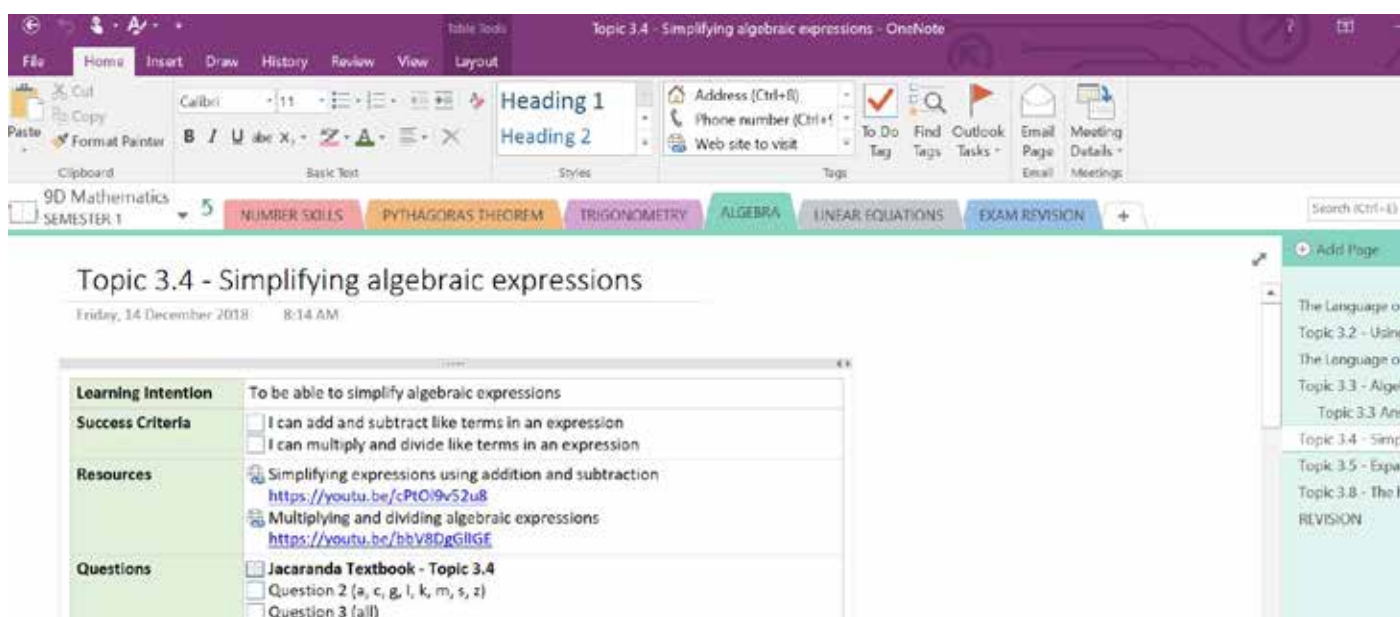

Figure 1. A snapshot of my class OneNote.

Implementing the Flipped Classroom

The implementation of a flipped classroom has required me to change the structure of my mathematics lessons. In the first 20 minutes of a lesson, the video content is reviewed and discussed, and a set of class notes is developed (Figure 2). The class notes are derived from the individual notes taken by the students (Figure 3). In the next 20-30 minutes, students work on collaborative tasks, while I work with students individually or in small groups. During double period of mathematics, I meet with students to focus on reviewing a certain concept. This provides me with an opportunity to intervene earlier when a student is experiencing difficulties. My role has somewhat shifted from being a teacher to a mentor. As students are working at their own pace, I am able to scaffold their learning according to their needs.
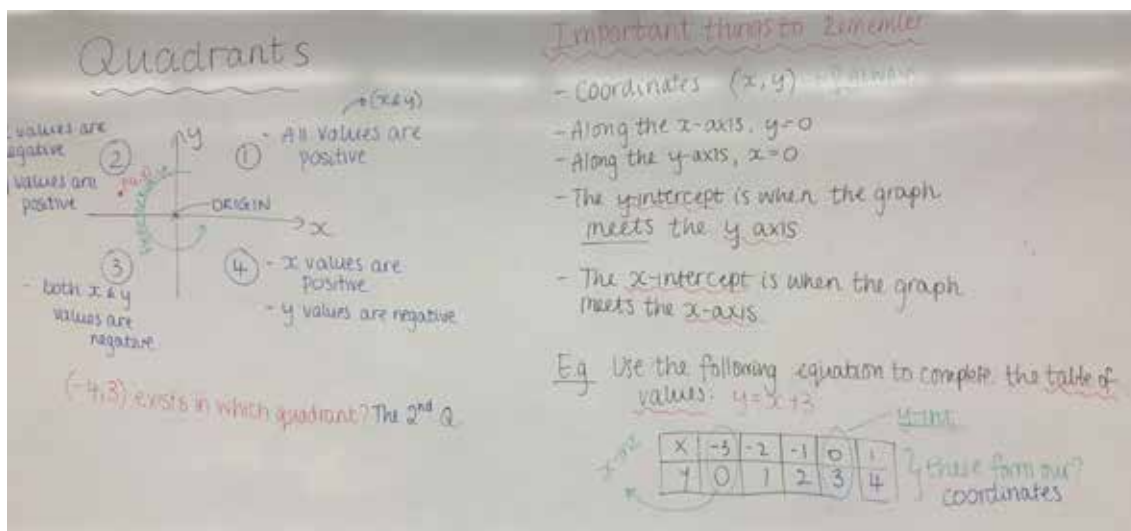

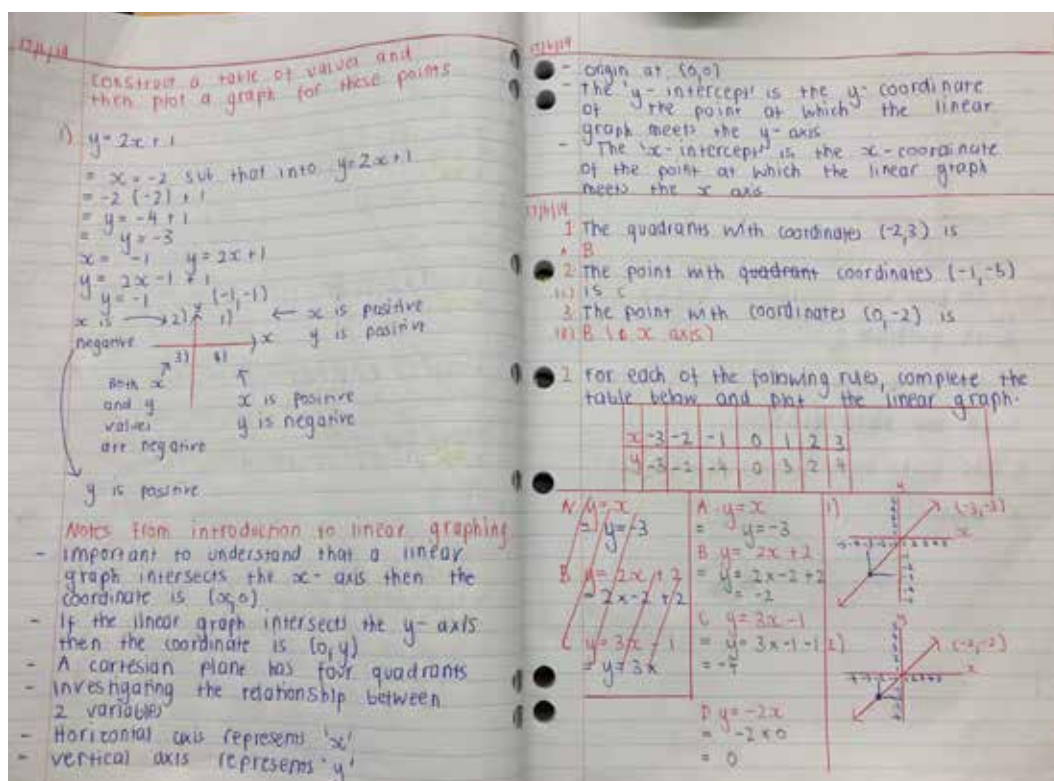Figure 2. Class notes developed by the students.

Figure 3. Notes that a student made when he watched the video at home.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

RESULTS

The effects of the flipped classroom were evaluated through my own teacher observations and through feedback from students and parents.

TEACHER OBSERVATIONS

Based on my own observations (Liyanage), students who previously lacked the confidence to volunteer in class were now regularly contributing to class discussions and were more receptive to learning. Since implementing a flipped classroom, I have increased collaborative exercises during class time which have promoted group work. Heterogeneous grouping of students allows the more capable students in mathematics to support the students who are finding the content challenging. The cooperative learning that occurs has enabled students to further develop their reasoning strategies and consolidate their understandings. My own teaching philosophy is one where I value rationalism and progress most (Bishop, 1988). Therefore, I design tasks which involve more peer-to-peer interactions so that students are required to rationalise their thought process with others and not just solve the question. I actively encourage students to engage in a growth mindset (Dweck, 2008) and provide opportunities for constructive feedback amongst peers. Classwork becomes challenging when students have not engaged with the pre-class activities. I commonly hear students ask their peers "Did you watch the video? She showed an example of how to work this out". Ironically, the students are now questioning each other about the completion of homework!

The main disadvantage of flipped learning has been the 'burnout' in developing the resources because recording the videos has been a time-consuming process. I have been reluctant to source outside material, choosing rather to create everything myself because the students have explained that my personalisation helps them to connect with what is done in class. Most of my allocated time for planning at school is now dedicated to developing the templates to film and what the students will be viewing at home (de Araujo, Otten, & Birisci, 2017). This has resulted in me doing more preparatory work after hours; for in-class tasks as well as filming the videos. I feel that I would benefit from receiving support in how to better utilise class time, to maximise student learning.

STUDENT FEEDBACK

Upon the completion of a unit on linear graphing, I asked students whether my *Edpuzzle* videos had supported them. Students described the videos as more "engaging" than the usual "confusing and tedious textbook exercises" (Student, 2019). Students believed that they understood the working out of mathematics problems better when they watched the *Edpuzzle* video of me modelling the process. Students have also reported increased satisfaction with the relevancy of materials provided, and greater engagement with, and autonomy over their learning compared with that experienced in the past.

Students have responded favourably to the videos that I have created because it felt like they were receiving one-on-one support. In producing the content, much of my attention is focused on producing quality content that is accessible to all, whilst also planning for educationally rich experiences in class.

PARENT FEEDBACK

Parents were also appreciative of the videos, and my endeavour to make their sons more independent in their learning. One of the students in my Year 9 class has dyslexia and recently, his mother informed me that since implementing flipped learning, she had noticed that her son had become much more independent in his approach to learning mathematics. He would regularly re-watch the videos to understand a new topic and he was less reliant on extra support to complete his coursework. For this student, offering him the opportunity to learn in a different way has reduced his "anxiety that has stemmed from his dyslexia" (Parent, 2019).

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

DISCUSSION

ADVANTAGES OF FLIPPED LEARNING

The classroom experience described above shows evidence of students actively engaging with their mathematics learning. The increased autonomy that allows students to access their learning resources at their own pace, also promoted a flexible learning environment.

Flipped learning supports students by decreasing the cognitive load required in a typical mathematics lesson. It provides more time for students to process the content of a mathematics lesson through a preview at home. Classroom lessons can then be designed to provide maximised opportunities for students to discuss their mathematical understandings and misconceptions.

Flipped learning is a similar model to the G.R.I.N. (Getting Ready In Numeracy) intervention program that aims to prepare students for their upcoming mathematics lesson through a pre-class tutoring session (Sullivan & Gunningham, 2011). Decreasing the cognitive load through G.R.I.N. has reported similar results to flipped learning in that pre-familiarity with the focus content and associated processes seem to engage students who otherwise may have been disinclined to participate in classroom discussions (Kalogeropoulos, Russo, Sullivan, & Klooger, 2020). Similar to the G.R.I.N. intervention program, the teacher noted that the students in flipped learning came to class with prerequisite knowledge upon which they could build. The teacher perceived this as benefiting students and allowing them to be more active in class. The students' interactions were also greater with one another. Flipped learning has the potential for differentiating the learning for individual students along with enabling them to develop autonomy over their learning, leading to mastery of the content demonstrated through successful completion of the assessment tasks (Muir, 2019).

DISADVANTAGES OF FLIPPED LEARNING

Whilst flipped learning is achieving success in relation to student engagement through access to technology for self-paced learning, it is important that the current challenges are also addressed. de Araujo et al. (2017) suggested that much of the teacher's planning time and attention in preparing for flipped learning is on the at-home activities rather than the in-class activities. Although the teachers' attention to the at-home resources is important because the home context is where the content delivery has shifted, the same attention needs to be given to the planning of the in-class context so that rich communication is fostered within and among students in mathematics. Furthermore, the teacher's role in class is to facilitate student learning in rich tasks that are closely related to the pre-lesson materials. Seeking or designing the appropriate resources are also a challenge in themselves.

Teacher burnout is also a concern due to the demands of creating content, monitoring blogs and other activities planned for flipped learning. Students may not complete the pre-class tasks and this will add more pressure to a teacher who will need to adjust planned lessons accordingly (Attard, 2019). Teachers must keep up to date with evolving technology and be able to filter and choose prepared resources that are appropriate for their students' learning context.

CONCLUDING THOUGHTS

Given the amount of experimentation and time needed to find appropriate resources, teachers would benefit from knowledge and support related to the creation of high-quality video and multimedia resources to promote effective mathematics teaching in flipped learning (de Araujo et al., 2017). Also, teachers could require support in how to better prepare for making effective use of the expanded in-class time. Teachers should consider the tasks they use to support collaboration as well as the quality and purpose of that collaboration (Herbel-Eisenmann, Steele, & Cirillo, 2013).

Flipped learning may shift a teacher's practice if the frequency of increased student self-confidence and engagement in mathematics learning prevails. A teacher can be described as the central force in flipped learning because teachers often initiate the change in practice themselves (de Araujo et al., 2017). How teachers conceive and enact flipped learning reflects their values in mathematics learning. For example, a teacher who values small group interaction with her/his students may aim to front-load the students with instructional content (prior to the lesson) and use class time to conference with students as reported in this case study.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

An area for future research is understanding teachers' values for flipped learning so that we can help to align their practise with instructional strategies that literature suggests might help them achieve these goals (de Araujo et al., 2017). For example, if a teacher values classroom experiences such as group discussions and student presentations, they could be directed to specific professional development on how to enact these in maximised classroom time. Lo and Hew (2017) emphasise the importance of carefully scaffolding the transition to flipped learning. This includes explaining the rationale of flipped learning and its potential benefits to both students and parents. Teachers should carefully choose online material for flipped learning to ensure it is appropriate for their students and the context. There appears to be considerable benefit in exploring flipped learning if it is valued by a teacher. This case study reported an increase in student engagement through flipped learning. Using technology has facilitated this pedagogical practice. However, a considerable amount of planning is required to ensure that flipped learning is implemented successfully.

REFERENCES

Attard, C. (2019). *Flipped learning in primary and secondary mathematics: Some lessons*. Retrieved from http://engagingmaths.com

Bergmann. J., & Sams, A. (2012). *Flip your classroom: Reach every student in every class every day*. Washington, DC: International Society for Technology in Education.

Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective on mathematics education*. Dordrecht, The Netherlands: Kluwer Academic.

Bishop, J., & Verleger, M. (2013, June). *The flipped classroom: A survey of the research*. Paper presented at the 120th American Society for Engineering Education Conference, Atlanta, GA.

de Araujo, Z., Otten, S., & Birisci, S. (2017). Mathematics teachers' motivations for, conceptions of, and experiences with flipped instruction. *Teaching and Teacher Education*, *62*, 60-70. doi:10.1016/j.tate.2016.11.006

Dweck, C. S. (2008). *Mindset: The new psychology of success*. New York, NY: Random House Digital.

Flipped Learning Network. (2014). *The four pillars of F-L-I-P*. Retrieved from http://www.flippedlearning.org/definition

Herbel-Eisenmann, B., Steele, M., & Cirillo, M. (2013). (Developing) teacher discourse moves: A framework for professional development. *Mathematics Teacher Educator*, *1*(2), 181-196. doi:10.5951/mathteaceduc.1.2.0181

Kalogeropoulos, P., Russo, J. A., Sullivan, P., Klooger, M., & Gunningham, S. (2020). Re-enfranchising mathematically-alienated students: Teacher and tutor perceptions of the Getting Ready In Numeracy (G.R.I.N.) program. *International Electronic Journal of Mathematics Education*, *15*(1), 1-17. doi:10.29333/iejme/5881

Lo, C. K., & Hew, K. F. (2017). A critical review of flipped classroom challenges in K-12 education: Possible solutions and recommendations for future research. *Research and Practice in Technology Enhanced Learning*, *12*(1), 1-22. doi:10.1186/s41039-016-0044-2

Mercieca, B. (2018). Flipping the mathematics classroom. In G. FitzSimons (Ed.), *Teachers creating impact: Proceedings of the 2018 Mathematical Association of Victoria Annual Conference* (pp. 30-35). Brunswick, Australia: Mathematical Association of Victoria.

Muir, T. (2019). Flipping the learning of mathematics: Different enactments of mathematics instruction in secondary classrooms. *International Journal for Mathematics Teaching and Learning*, *20*(1), 18-32. Retrieved from http://www.cimt.org.uk/ijmtl/index.php/IJMTL/index

Straw, S., Quinlan, O., Harland, J., & Walker, M. (2015). *Flipped learning research report*. Retrieved from https://www.nfer.ac.uk/media/2018/nesm01.pdf

Sullivan, P., & Gunningham, S. (2011). A strategy for supporting students who have fallen behind in the learning of mathematics. In J. Clark, T. Spencer, & S. Thornton (Eds.), *Mathematics: Tradition and (new) practices: Proceedings of the 34th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 719-727). Alice Springs, Australia: Mathematics Education Research Group of Australasia.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Geometry + technology = proof

Irina Lyublinskaya, Teachers College, Columbia University

Teaching mathematical reasoning and proof in high school geometry is one of the challenging tasks that teachers face today. Geometry Expressions™ developed by Saltire Software is a dynamic, constraint-based symbolic geometry software, which allows algebraic and figural representations of geometry to co-exist in an environment that fosters a holistic view of mathematics. In this paper, we illustrate how symbolic geometry can be used in a variety of ways to motivate, develop and realise plans for proof, and facilitate different proof techniques.

INTRODUCTION

Proof occupies an important place in school mathematics since experiences of proving develop students' analytic abilities and let them experience rigorous mathematical thinking (Usiskin, 1980). For more than a century, most geometry curricula in the United States have included opportunities for students to understand and conduct proofs. Yet observations of students' geometry thinking and problem solving have yielded a conflicting picture. Schoenfeld (1988) documented how students who have completed proof-based courses in geometry, still act as empiricists when solving geometric problems.

Technology can support the development of mathematical reasoning and understanding through geometric proof. Some educators argue that a more effective way to engage students in proving is to use dynamic geometry software (DGS), which opens up entirely new approaches to the teaching of proof (Gawlick, 2002; Olivero & Robutti, 2007). Educators argue that DGS is essential for building up associations between graphs, drawings, and other tools used for formal proofs, and plays an important role in enabling students to formulate deductive explanations, and provide a foundation for developing ideas of proof and proving (Jones, 2000; Zarzycki, 2004).

Another powerful tool is computer algebra systems (CAS), which have been recognised as being highly valuable for doing mathematics and potentially valuable for teaching and learning reasoning skills. CAS are already a standard tool in universities worldwide and have been adopted in high schools in various regions of the world: Austria, parts of Australia and New Zealand, and Germany (Flynn, 2003; Garner, 2004; Herget, Heugl, Kutzler, & Lehmann, 2000). While CAS have been widely adopted in some US universities, the technology has yet to make any significant mark in U.S. high schools.

Even though dynamic geometry software (DGS) and algebra computer systems (CAS) were introduced for the learning and teaching of secondary school mathematics in the early 1980s, their use over the last couple of decades has largely been separate. *Geometry Expressions* is the first of a new class of interactive symbolic geometry systems (Saltire Software, 2018). It is a mathematical tool that provides outputs in a symbolic form as algebraic expressions involving the input parameters. Integrating geometric and algebraic explorations could be a powerful toolkit for helping students develop reasoning skills using an inductive exploration-based approach (Lyublinskaya & Funsch, 2012; Lyublinskaya, Ryzhik, & Funsch, 2009; Majewski, 2007; Todd, 2008; Todd, Lyublinskaya, & Ryzhik, 2010).

This paper illustrates how this symbolic geometry system can be used in a variety of ways to motivate, develop and realise plans of proof, and to facilitate different proof techniques in high school geometry.

DESCRIPTION OF PROOF TECHNIQUES

There is no universal method of proof in the study of geometry. This paper outlines examples of five commonly used methods that appear frequently in elementary geometry courses, which are typically taught at high school level:

1. Geometric - based on visual representation,

2. Algebraic - based on investigation of various algebraic and trigonometric expressions,

3. Coordinate - based on introducing a system of coordinates and the use of formulas from coordinate geometry,

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

4. Vector - based on using operations with vectors, and

5. Transformations - based on both isometric (rigid) and non-isometric (non-rigid) transformations on a plane.

Each of these methods has its own features and application to proofs. For some problems, a particular method of proof is more effective than another method. Many problems can be addressed effectively using more than one method. There are problems in which a combination of methods is needed to complete the proof. Additionally, there are tools, such as construction and loci, whose use can be integrated within any of the described methods.

Proof, regardless of its method of analysis and argument, requires logical thinking and provides an excellent opportunity to strengthen students' thinking and logic skills. The student's choice of method depends on the nature of the problem and on the student's experience. Familiarity with a wide number of approaches and methods provides the optimum tools for the student learning geometry.

GEOMETRIC METHOD

The geometric method (sometimes called the synthetic method) is the principal method used in elementary geometry. This method cannot be formalised and is based on spatial thinking. The core of this method is to represent the problem as a "picture" or a diagram. The student then uses the diagram, properties, and characteristics of the figure(s) to derive a proof. *Geometry Expressions* enables students to investigate the properties of geometric figures (their measures, invariants, etc.) in the most general way. The software gives students an opportunity not only to formulate conjectures but also to visualise them. Consider the following example of the problem, in which students explore the relationship between the bisector of the exterior angle and the base of a triangle.

> Problem statement: Given isosceles triangle ABC, $AB = AC$. The angle bisector is constructed for the exterior angle of vertex A. What is relationship between the angle bisector and side BC?

This is a typical geometry problem solved geometrically without additional constructions. Students create a sketch and constrain an angle $\angle ABC = \alpha$. They establish the fact of parallelism between the angle bisector and the base of the triangle with the help of the software (Figure 1).
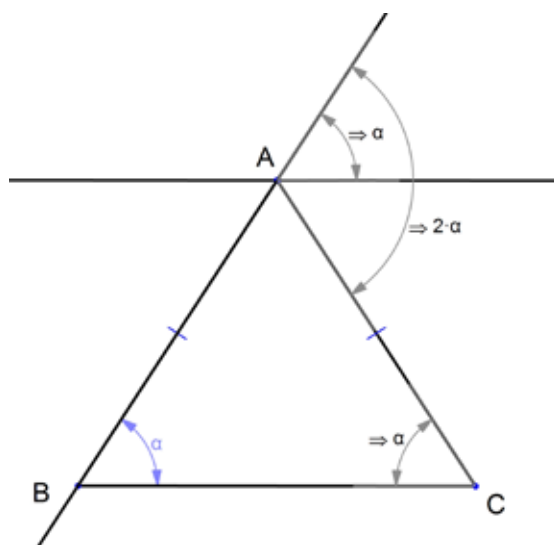
Figure 1. Diagram illustrating symbolic outputs for the problem
about an exterior angle bisector leading to the plan of proof.

One important technique in geometric proofs is the construction of auxiliary elements. These fall into two groups: the construction of additional elements "within" the figure and the construction of new elements that lie "beyond" the given. Students generally find the latter less obvious and, hence, more difficult. The following two examples illustrate the use of additional constructions.

1. Construction "within" the given figure: Prove that the angle inscribed in a circle is equal to half of the subtended (minor) arc (Figure 2).

Figure 2. The construction of radius *OC* helps to visualise the problem.
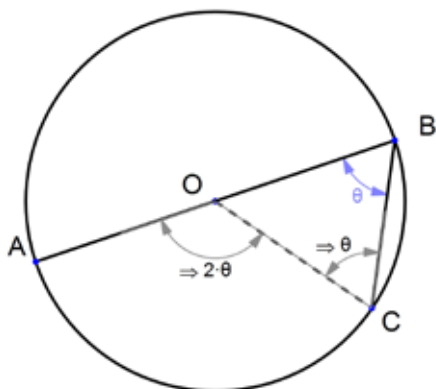
2. Construction "beyond" the boundaries of the figure: Prove that the median drawn to the hypotenuse of the right triangle is equal to half of the hypotenuse (Figure 3).
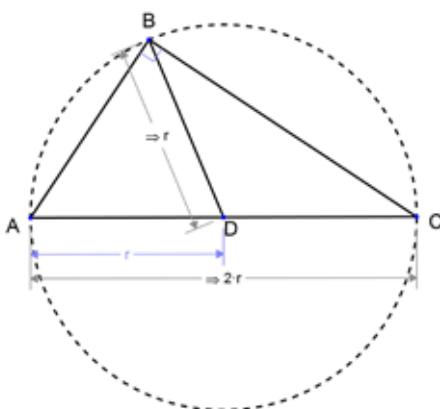

Figure 3. Construction of a circle with the center at the midpoint of the hypotenuse makes the result obvious, since the vertex of the right angle is on the circle.

ALGEBRAIC METHOD

Solving geometric problems often involves the use of formulas, since with the help of formulas we can express the measures of segments, angles, and areas. In particular, trigonometric formulas are used. Problems that utilise non-trivial algebraic operations will be classified under the algebraic method. The use of algebraic formulas in geometric problem solving could lead to the proof of an identity or inequality, to the solution of an equation or inequality, or to finding the extremum of a function. Consider the following example:

> Problem statement: In a triangle *ABC*, with fixed angle *BAC*, $AB + AC$ is constant. Among all such triangles, find the one with the largest area.

This is an optimisation problem that can be solved algebraically without additional constructions. After constraining the sides and the angle of the triangle to satisfy problem conditions, students can obtain a symbolic expression for the area (Figure 4).
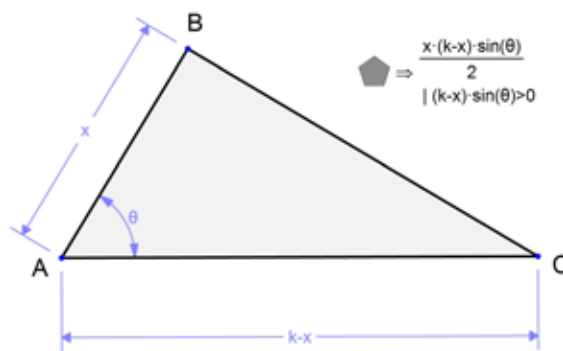
Figure 4. Symbolic output for the area of the triangle given two sides and an angle between them provides a possible plan for proof.

Since the angle is constant, it is sufficient to find the maximum of $x(k - x)$. Based on properties of a quadratic, the maximum is reached when $x = \dfrac{k}{2}$. Thus $AB = AC = \dfrac{k}{2}$, so the triangle with largest area is isosceles.

COORDINATE METHOD

The first step in using the coordinate method is to define a coordinate system. Of course, it makes sense to define it in the most convenient manner; that is, relating both the location of the origin and the orientation of the coordinate axes to the given figure(s) in order to facilitate the proof. For example, for a problem about a square, the origin may be advantageously placed at the square's centre with the axes parallel to the sides of the square. Discussion of the relative advantages of various placements and orientations of the coordinate system provides students with rich opportunities to reason logically and to communicate their thoughts. With the help of the coordinate method, we can tackle two types of problems:

1. If the given figures are defined by equations, then the relationships between the figures can be expressed in coordinates. The coordinate expressions are then interpreted geometrically. Problems of this type include finding the locus of a moving point and determining the shape of an obtained figure(s). Consider the following example:

> Problem statement: Given a line and two points where Point *A* is not on the given line and point *B* is on the given line. Point *C* is chosen so that *AB* and *BC* are equal and perpendicular. What is the locus of the point *C*, if the point *B* moves along the given line?

For solving this problem, it is most convenient to align the x-axis with the given line and to have point *A* on the *y*-axis. After completing the construction, students can dynamically explore positions of point *C* when point *B* moves along the *x*-axis. Students can verify their conjecture by generating the locus of point *C*. They can also find an implicit equation for the locus. The plan of the proof can be devised after they find coordinates of the point *C* and construct the additional segment *CD* (Figure 5).
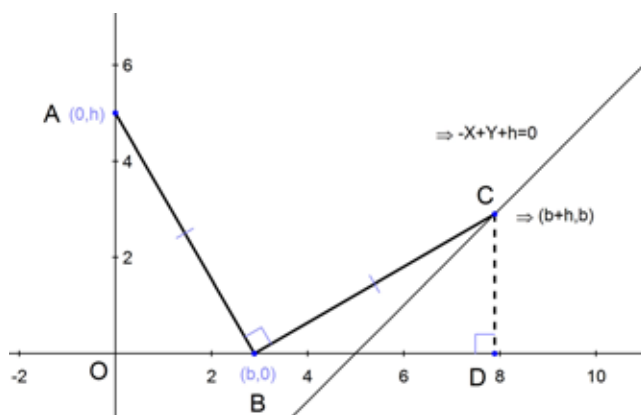

Figure 5. Symbolic outputs for the coordinates and locus of the point *C*.

Consider right triangles AOB and BCD. Given $AB = BC$, $\angle ABO = \angle BCD$ are acute angles with mutually perpendicular sides, thus $\triangle AOB \cong \triangle BCD$. Therefore $CD = OB = b$ and $BD = OA = h$. Since the difference between the coordinates of point C, $x = b + h$ and $y = b$, is constant, the locus is a line. The equation of the line is $y = x - h$.

2. If the given points are defined by their positions, then the relationships between given figures are found using equations. Problems of this type include finding distances between figures, establishing collinearity of points (or its absence), and using the distance formula to verify whether one of the distances is equal to the sum of the other two. Consider the following example:

> Problem statement: Given a square $ABCD$. Point K is on the side AB and point L is on the side CD, so that $AK = CL$. Will the line KL contain the centre of the square O?

For solving this problem, it is convenient to define a system of coordinates with origin O at the centre of the square. Let the length of the side of the square be $2a$. Then coordinates of the point K are $(-a, -b)$ and the coordinates of point L are (a, b) (Figure 6).
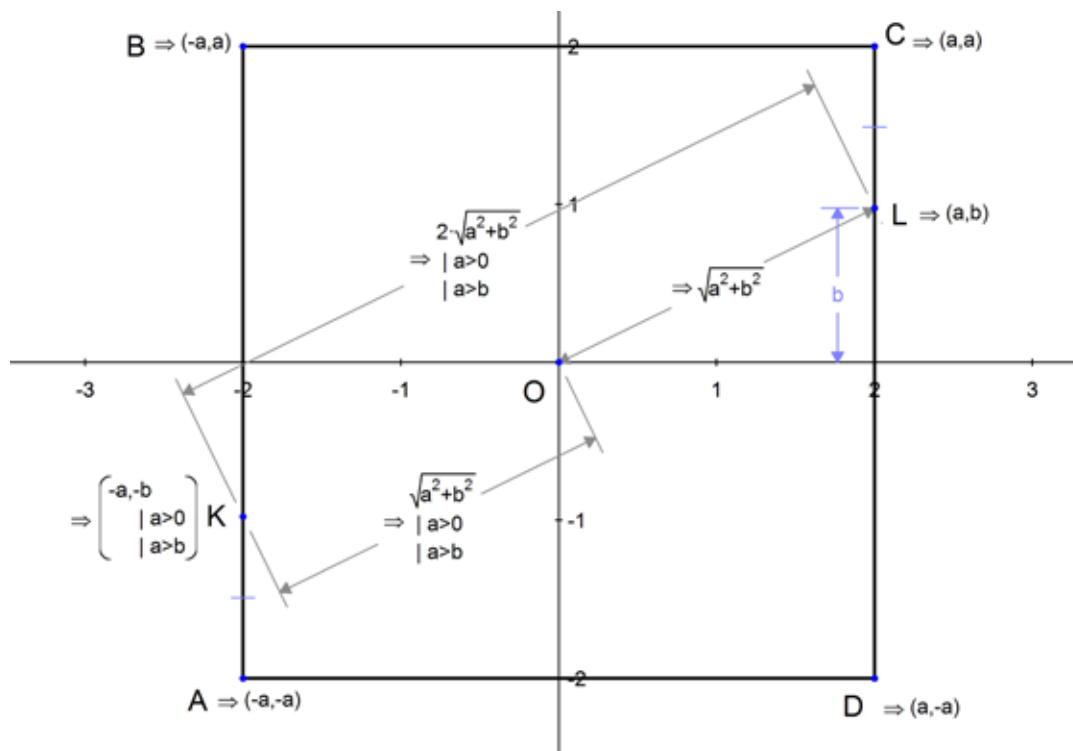
Figure 6. Coordinates and distance calculations guide students to plan the proof.

Students can generate expressions for distances to verify that $KL = OK + OL$. Proof will then involve using the distance formula to confirm software outputs and therefore proving $O \in KL$.

VECTOR METHOD

The vector method can be used to solve problems of various types, such as finding measures and determining the relative position of lines and points. Proofs by this method generally involve three steps:

1. Interpret the problem statement in terms of vectors by introducing vectors that correspond to the given information.

2. Obtain a solution of the problem in a vector form by using the properties of vectors.

3. Translate the vector-form solution back to the original terms of the problem.

Consider an example of a problem where students are asked to investigate whether it is possible to construct a triangle from the three medians of a given arbitrary triangle. The necessary and sufficient condition for constructing a triangle from three segments is given by the triangle inequalities. For this particular problem, it is very difficult to verify the triangle inequalities for the medians. Thus, this problem is easier to solve by vector method. In vector terms, the necessary and sufficient condition for constructing a triangle is equivalent to the condition that the sum of three non-collinear vectors is equal to the zero-vector (Figure 7).

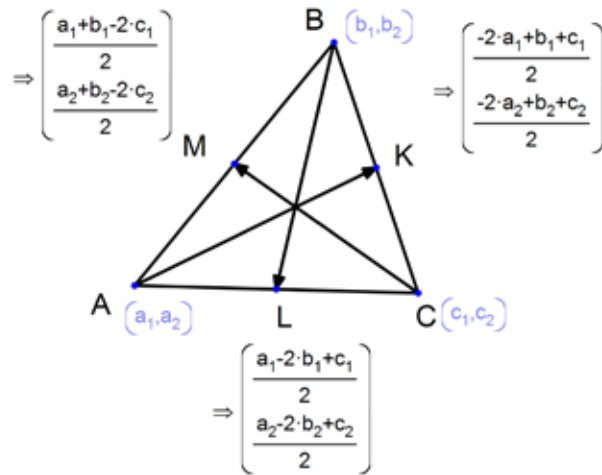*Figure 7.* Analysis of symbolic expressions for the vector coordinates defined based on medians provides basis for proving the existence of a triangle.

Adding the coordinates of all three vectors, we see that result is a zero-vector:

$$\overrightarrow{AK} + \overrightarrow{BL} + \overrightarrow{CM} = \begin{bmatrix} \dfrac{-2a_1 + b_1 + c_1}{2} \\ \dfrac{-2a_2 + b_2 + c_2}{2} \end{bmatrix} + \begin{bmatrix} \dfrac{a_1 - 2b_1 + c_1}{2} \\ \dfrac{a_2 - 2b_2 + c_2}{2} \end{bmatrix} + \begin{bmatrix} \dfrac{a_1 + b_1 - 2c_1}{2} \\ \dfrac{a_2 + b_2 - 2c_2}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

TRANSFORMATIONS METHOD

The core of this method is to transform the given figure into another one, which leads to a clearer solution. Consider a famous problem – finding the shortest distance between two points with reflection of the path from a line. This problem illustrates an application of geometry in physics, where the shortest path corresponds to the path that light travels from one point to another when reflected from a mirror. This problem is solved by the application of a transformation, specifically a reflection.

Problem statement: Points *A* and *B* are on the same side of a given line. Find a point *C* on the line, such that *AC* + *CB* is smallest.

In this problem, we need to have a fixed line. For simplicity, we will use the *x*-axis as the given line. Another possibility is to draw a line and constrain its equation to $y = $ const (for example, $y = 0$). In the latter case, the system of coordinates does not have to be displayed. Either is done only to create a fixed horizontal line in the diagram. Similarly, constraining the coordinates of points A and B only fixes the location of these points (Figure 8).
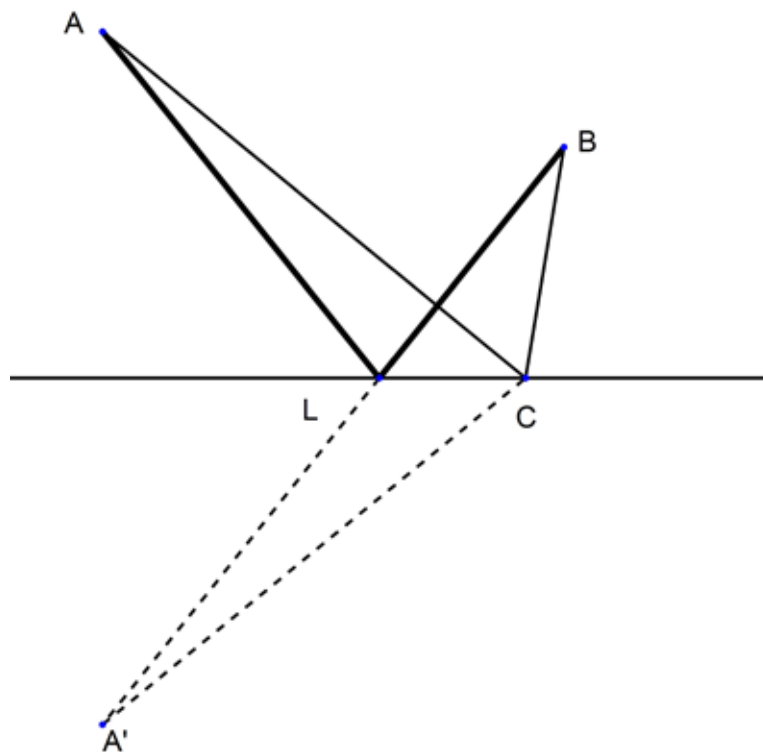
Figure 8. Use of reflection suggests an approach for the proof about
position of the point *C* when distance *AC + CB* is minimal.

CONCLUSIONS

Proof is in the heart of geometry. It would not be too much of a stretch to suggest that proof is what geometry is "about" – relegating lines, planes, pyramids, and perpendiculars to the status of the raw material on which we practice the art of proof. The premise of this paper is that *Geometry Expressions* software can alter how students reason about geometry. So, the first step is to acquire a degree of fluency with *Geometry Expressions*. Students need to understand the underlying concept of this constraint-based software. Failure to grasp the nature of this basis will likely be the source of much confusion and discouragement. You don't need to know everything but you do need a working knowledge of the principal features of the software.

This paper illustrated various techniques and methods for mathematical proof. It also displayed the different ways *Geometry Expressions* can be used. For some problems, the software is used to provide input that suggests a conjecture. For others it may suggest an approach to developing a proof. In a few examples, the symbolic representation generated by the software is an element of the proof. We hope that this paper challenged the reader to reflect on the role that computing technologies such as *Geometry Expressions* could play in the future of mathematics education.

REFERENCES

Flynn, P. (2003). *Using assessment principles to evaluate CAS-permitted examinations*. Paper presented at the 2003 Symposium on Computer Algebra in Mathematics Education.

Garner, S. (2004). The CAS classroom. *Australian Senior Mathematics Journal*, *18*(2), 28-42.

Gawlick, T. (2002). On dynamic geometry software in the regular classroom. *International Reviews on Mathematical Education*, *34*, 85-92.

Herget, W., Heugl, H., Kutzler, B., & Lehmann, E. (2000). Indispensable manual calculation skills in a CAS-environment. In V. Kokol-Voljc et al. (Eds.), *Exam questions and basic skills in technology-supported mathematics teaching* (pp. 13-26). Hagenburg, Austria: BK Teachware.

Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics*, *44*, 55-85.

Lyublinskaya, I., & Funsch, D. (2012). Geometry + technology = proof. *The Mathematics Teacher*, *105*(6), 448-454.

Lyublinskaya, I., Ryzhik, V., & Funsch, D. (2009). *Developing geometry proofs with Geometry Expressions*. Beaverton, OR: Saltire Software.

Majewski, M. (2007). A symbolic approach to teaching geometry. *The Electronic Journal of Mathematics and Technology*, *1*(1), 1933-2823.

Olivero, F., & Robutti, O. (2007). Measuring in dynamic geometry environments as a tool for conjecturing and proving. *The International Journal of Computers for Mathematical Learning*, *12*, 135-156.

Saltire Software. (2018). *Geometry Expressions* (Version 3.3) [Computer software]. Tigard, OR: Saltire Software.

Schoenfeld, A. (1988). When good teaching leads to bad results: The disasters of "well-taught" mathematics courses. *Educational Psychologist*, *23*(2), 145–166.

Todd, P. (2008). Geometry Expressions - A dynamic symbolic geometry environment. *International Journal of Continuing Engineering Education and Life-Long Learning*, *18*(5/6), 666-677.

Todd, P., Lyublinskaya I., & Ryzhik, V. (2010). Symbolic geometry software and proof. *International Journal of Computers for Mathematics Learning*, *15*(2), 151-159.

Usiskin, Z. (1980). What should not be in the algebra and geometry curricula of average college-bound students? *The Mathematics Teacher*, *73*, 413-424.

Zarzycki, P. (2004). From visualizing to proving. *Teaching Mathematics and its Applications*, *23*(3), 108-118.

Addressing misconceptions about mass and volume

Heather McMaster, University of Sydney

Upper primary school students often find it difficult to differentiate between mass and volume. In this article, possible reasons are given as to why this is difficult. It is recommended that activities in primary school, integrate mass and volume (from the mathematics curriculum) with the properties of materials (from the science curriculum) so students are better prepared for secondary school. Ideas for such activities are listed.

INTRODUCTION

In the Australian primary mathematics curriculum, students learn to measure 'mass' and 'volume'. Children come to school with intuitive understandings of measurement attributes (McDonough, Cheeseman, & Ferguson, 2013) however many students fail to be able to differentiate between mass and volume in the upper primary and early high school years (Senzamici & McMaster, 2019). This confusion could stem from the mathematics and science curricula being taught in isolation from each other and a failure to provide instruction that explicitly connects new meanings to students' prior knowledge and the understandings they have gained through everyday experiences. Very similar but somewhat different meanings of 'weight' and 'capacity' in everyday life, and the relationship between different units of measurement in the metric system, add to the confusion.

DEFINITIONS OF MEASUREMENT ATTRIBUTES

'Mass' is defined in the mathematics curriculum as "how much matter is in a person, object or substance" (Australian, Curriculum, Assessment and Reporting Authority [ACARA], n.d.). The term 'matter' is complex, so the term 'stuff' is commonly used in primary science educational literature (Skamp, 2005). The 'Chemical Sciences' strand of the primary science curriculum is primarily concerned with properties of materials and substances (chemically pure materials) while the 'Measurement and Geometry' strand of the mathematics curriculum is mainly concerned with measuring objects.

People generally understand 'mass' to be synonymous with 'weight' because in everyday life, both are measured in the same units (grams, kilograms, and tonnes). Weight, however, is measured in Newtons and refers to the amount of gravitational force acting on matter. Weight and mass are directly related on earth where gravity is the same. An astronaut has the same mass on the earth as on the moon, however he weighs about six times less on the moon so he can jump about six times higher.

'Volume' is an amount of three-dimensional space. The primary science curriculum does not mention the word 'volume' but states that gases "take up space" (ACARA, n.d.). The mathematics curriculum introduces and defines the term 'volume' but only in relation to solid objects: "The volume of a solid is a measure of the space enclosed by the solid. For a rectangular prism, Volume = Length × Width × Height" (ACARA, n.d.). Three-dimensional space can be measured in cubic units or in litres (e.g., the size of a suitcase). However, the volume of solids is generally measured in cubic units and the volume of pourable substances, such as potting mix, is usually measured in litres or its derivatives. A millilitre is defined as a cubic centimetre. Capacity is defined as "how much a container will hold" (ACARA, n.d.).

SOURCES OF CONFUSION BETWEEN MASS AND VOLUME

'Weight' and 'density' can be confused because both are described in terms of 'heaviness' (Senzamici & McMaster, 2019). Weight is the heaviness of an object whereas 'density' is a property of the material an object is made of. The heaviness of a substance or material is its density (meaning its mass in relation to its volume), while the heaviness of an object is its weight (which is directly related to its mass).

Children will often assume that one object must be heavier than another based on the stuff from which it is made, and not realise that the volume of the object also needs to be considered (Senzamici & McMaster, 2019). A simple way of describing the density of an object is to say that it is heavy or light 'for its size'. 'Density', however, is not mentioned in either the mathematics curriculum or the science curriculum in primary school.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Some children think that mass and volume are the same thing because they are both measures of an amount of material. If 'mass' and 'volume' are referring to the same non-compressible material at the same temperature, this is not a problem because mass is then directly proportional to volume. In cooking recipes, the ingredients are often stated in cups (an informal measure of volume) because measuring an ingredient with a cup is easier than weighing it on kitchen scales (which requires getting the tare correct). The weight of flour in a cup, however, differs depending on whether it has been sifted or compacted.

Although the mathematics curriculum expects students to understand capacity as a volume, the everyday meaning of the word 'capacity' applies to both weight and volume. "How much a container will hold" (ACARA, n.d.) could be measured in grams. Based on their research, Ho and McMaster (2019) suggest that the term 'interior volume' replace the word 'capacity' so students recognise that in measuring the capacity of a container, they are measuring an amount of three-dimensional space and not the weight of a pourable material that the container can hold (i.e., the filling).

The use of differently-named attributes and units of measurement for different states of matter could also be contributing to the confusion between mass and volume. Commonly, solid materials (e.g., ice) are sold by weight and liquids (e.g., water) are sold by volume. Students therefore generalise that the difference between a gram and a millilitre is whether the matter being measured is a solid or a liquid. Viscous materials such as sauces and honey are sometimes sold by weight and sometimes sold by volume.

Students confusing mass and volume in the upper years of primary school could also be related to them being introduced to formal units within the metric system. They see the metric system of measurement as a system in which all units are related to each other. Having been told that a cubic centimetre equals a millilitre, it's reasonable for them to also equate a millilitre to a gram. Historically a gram was defined as the mass of water at its greatest density (i.e., at 4ºC), so students may think this relationship holds for all liquids and possibly for all solids as well.

Only measuring water at school (or measuring common supermarket items such as milk or juice which are composed largely of water) reinforces the misconception that a millilitre weighs a gram, as does the use of plastic centicubes manufactured to weigh one gram so they can be used to measure both mass and volume.

Yet another confusion has arisen in recent years. Different measurement attributes are usually measured using different measuring instruments, however today you can buy a set of electronic kitchen scales and change the units from grams to millilitres of water or milk by just pressing the 'units' button.

THE AUSTRALIAN MATHEMATICS AND SCIENCE CURRICULA

The primary school Australian Curriculum science and mathematics outcomes related to materials, mass and volume, are listed in Table 1. Notice that the science curriculum is concerned with the properties of materials and the mathematics curriculum is concerned with the measurement of objects. The mass of an object depends on its volume and the density of the material from which it has been made. Beginning at the Foundation year, students could be talking about the heaviness of materials (meaning density) in science lessons and the heaviness of objects (meaning mass) in mathematics lessons. The attributes of mass and volume are not specifically mentioned in the primary school science curriculum until Year 5 when, in the 'Chemical Sciences' sub-strand, the outcome is "Solids, liquids and gases have different properties and behave in different ways" (ACARA, n.d., ACSSU077). Under this outcome is the elaboration: "observing that gases have mass and take up space, demonstrated by using balloons or bubbles".

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Year level	Science curriculum outcome	Mathematics curriculum outcome
Foundation	Objects are made of **materials** that have observable properties (ACSSU003)	Uses direct and indirect comparisons to decide which is longer, **heavier** or **holds more**, and explain their reasoning using everyday language (ACMMG005)
1	Everyday **materials** can be physically changed in a variety of ways (ACSSU018)	Measure and compare the lengths and **capacities** of pairs of objects using uniform informal units (ACMMG019)
2	Different **materials** can be combined for a particular purpose (ACSSU031)	Compare and order several shapes and objects based on length, area, **volume** and **capacity** using appropriate uniform informal units (ACMMG037) Compare the **masses** of objects using balance scales (ACMMG038)
3	A change of state between **solid and liquid** can be caused by adding or removing heat (ACSSU046)	Measure, order and compare objects using familiar metric units of length, **mass** and **capacity** (ACMMG061)
4	Natural and processed **materials** have a range of physical properties that can influence their use (ACSSU074)	Use scaled instruments to measure and compare lengths, **masses**, **capacities** and temperatures (ACMMG084) Compare objects using familiar metric units of area and **volume** (ACMMG290)
5	**Solids, liquids and gases** have different observable properties and behave in different ways (ACSSU077)	Choose appropriate units of measurement for length, area, **volume**, **capacity** and **mass** (ACMMG108)
6	Changes to **materials** can be reversible or irreversible (ACSSU095)	Convert between common metric units of length, **mass** and **capacity** (ACMMG136) Connect and **volume** and **capacity** using their units of measurement (ACMMG138)

Table 1. Outcomes in the F-6 Australian Curriculum: Science and the Australian Curriculum: Mathematics that involve 'mass', 'volume', and properties of materials. Words related to mass, volume, and materials are bolded.

THE NATIONAL NUMERACY LEARNING PROGRESSION

The National Numeracy Learning Progression (ACARA, 2017) has the sub-element 'Understanding Units of Measurement'. This progression has nine levels. The ability to explain the difference between mass and volume occurs late in this progression, at Level 7. The other pairs of attributes mentioned as being difficult to distinguish between, are "area and perimeter" and "volume and capacity".

These pairs of attributes that children find confusing are all attributes that are siloed in the early years of the mathematics curriculum. Because 'capacity' (measured by filling containers with a liquid) has unstructured units, it is taught earlier than the volume of solids. To understand the cubic units for the volume of a solid, students need to understand their row, column and layer structure. The measurement of perimeter and the measurement of area are also taught initially as separate topics because understanding the structure of the formal units of area (two dimensions) requires an understanding of units of length (one dimension). The attributes of mass and volume are similarly taught initially as separate topics. The measurement of mass and capacity/volume are taught earlier than the volume of a solid, because their formal units are unstructured. Mass does not even appear in an outcome of the mathematics curriculum after Year 6.

CHILDREN'S UNDERSTANDING OF DENSITY

The attributes of mass and volume are related by the concept of 'density'. An understanding of the properties of materials requires an appreciation of this concept. Based on their research in science education, Smith, Wiser, Anderson,

and Krajick (2006) claim that children should be taught about properties of materials before being taught about the volume of objects. Having understood density, it follows that objects made from the same type of stuff (irrespective of their volume) have the same density.

Because many children in primary school have already heard of the word 'density' and have some intuitive understanding of it (Senzamici & McMaster, 2019), its meaning could be made more explicit before students reach secondary school. Kloos, Fisher, and van Orden (2010) found that children are better able to differentiate 'density' from 'mass' and 'volume' in tasks where 'density' is made more salient.

Density is never mentioned in the mathematics curriculum and may not be mentioned in science lessons until Year 8, when they learn about the particle theory of matter: "Properties of the different states of can be explained in terms of the motion and arrangement of particles" (ACARA, n.d., ACSSU151). Senzamici and Ho (2019) found that giving primary school students a formal definition of density as a ratio of mass to volume is of little help to them. However, Kloos et al. (2010) found that children could still appreciate the meaning of 'density' when this attribute is made salient to them. Senzamici and Ho (2019) recommend age-appropriate engagement with density, rather than its complete exclusion from mathematics and science teaching in primary school. Such engagement could help students distinguish between *heavy* as a description of mass and *heavy* as a description of the density of a type of material. It may also help them differentiate between the attributes of 'mass' and 'volume'.

SOME ACTIVITIES INVOLVING BOTH MASS AND VOLUME

Below are some suggested activities for helping students differentiate between 'mass' and 'volume'. They integrate the topics of mass and volume from the mathematics curriculum with the topic of materials from the science curriculum (see Table 1).

ACTIVITY 1 – TWO TYPES OF 'HEAVY'

- Give students a variety of materials to sort as 'heavy material' or 'light material':

    Heavy materials - plasticine, play-dough, clay, heavy wood, heavy stone

    Light materials - lightweight foam clay (e.g., rainbow foam), cork, polystyrene foam, sponge, cotton wool, balsa wood, pumice stone.

- Choose two different materials (one heavy and one light) from which to make pairs of objects with the same shape and size.

- Using a beam balance, compare the mass of same-sized and same-shaped objects made from different materials.

- Discuss the objects using the words 'heavy object' and 'light object'.

- Describe some objects as being 'heavy for their size' or 'light for their size'. A 'light object' can also be 'heavy for its size' because it is small but made of heavy material.

ACTIVITY 2 – KEEPING MASS THE SAME

- Give students a lump of playdough and a lump of lightweight foam clay to put on each side of a beam balance.

- Ask them to remove some of the playdough from the balance until the play-dough and foam clay have the same mass.

- Change the shape of each material and weigh them again. Discuss how the mass doesn't change if you change the shape of the material. Similarly show that if you split the material on one side of the balance into smaller pieces, the total mass remains the same.

- Discuss how the same mass of different materials can have different volumes. Heavier material will have less volume for the same mass.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

ACTIVITY 3 – KEEPING VOLUME THE SAME (NON-COMPRESSIBLE MATERIAL)

- Give students a small rectangular prism (a block) and centicubes (plastic or wooden). Ask them to find the approximate volume of the block using the centicubes.

- Discuss whether the mass of the block will be the same as the mass of the centicubes.

- Ask them to use the same number of centicubes to make a different shaped object.
  Discuss whether the volume changes when the dimensions change.

- Give students a lump of playdough to make into an object. Tell them that playdough (like the centicubes) cannot be compressed.
  Discuss whether the volume changes when the dimensions change.

ACTIVITY 4 – THE DENSITY OF FLUIDS

- Give students a cylindrical container. Tell them you have three substances (e.g., chocolate sauce, orange drink and honey). Ask them to draw what they think will happen if you put the same weight of each substance into the cylinder.

- Discuss whether there will be layers of each substance or if they will be mixed up.

- Give them access to the substances to weigh and put them into the cylinder in whatever order they choose. Tell them that the group with the most accurate weighing, the cleanest placing of substances in the cylinder and the clearest layering, wins a prize.

- Judge the cylinders. Discuss why the heights of the different layers are different.

By using two or more different materials (hence having different densities) within the same activity, discussion includes the attribute of mass and the attribute of volume. As with the attributes of area and perimeter, investigating two attributes simultaneously could better enable students to distinguish between them.

REFERENCES

Australian Curriculum, Assessment and Reporting Authority. (n.d.). *The Australian Curriculum F-10*. Retrieved from https://www.australiancurriculum.edu.au/f-10-curriculum

Australian Curriculum, Assessment and Reporting Authority. (2017). *National numeracy learning progression*. Retrieved from https://www.australiancurriculum.edu.au/resources/national-literacy-and-numeracy-learning-progressions/pdf-documents/

Ho, A., & McMaster, H. (2019). Is 'capacity' volume? Understandings of 11 to 12-year-old children. In G. Hine, S. Blackley, & A. Cooke (Eds.). *Mathematics education research: Impacting practice – Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 356-363). Perth, Australia: Mathematics Education Research Group of Australasia.

Kloos, H., Fisher, A., & van Orden, G. C. (2010). Situated naïve physics: Task constraints decide what children know about density. *Journal of Experimental Psychology General*, *139*(4), 625-637.

McDonough, A., Cheeseman, J., & Ferguson, S. (2013). Young children's emerging understandings of the measurement of mass. *Australasian Journal of Early Childhood*, *38*(4), 13-20.

Senzamici, L., & McMaster, H. (2019). The heaviness of objects and heaviness of a material kind: Some 11 to 12-year-old children's understandings of mass and density. In G. Hine, S. Blackley, & A. Cooke (Eds.). *Mathematics education research: Impacting practice – Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 636-643). Perth, Australia: Mathematics Education Research Group of Australasia.

Skamp, K. (2005). Teaching about stuff. *Primary Science Review*, *89*, 20-22.

Smith, C., Wiser, M., Anderson, C. W., & Krajick, J. (2006). Implications on research for children's learning for standards and assessment: A proposed learning progression for matter and atomic molecular theory. *Measurement: Interdisciplinary Research and Perspectives*, *4*(1/2), 1-98.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Open-ended tasks in mathematical methods

Trang Pham, Methodist Ladies' College

Mathematical reasoning is the ability to analyse, prove, evaluate, explain, infer, justify and generalise. Open-ended tasks provide opportunities for students to develop the ability to make choices, interpret, formulate, model and investigate problems, and communicate solutions effectively and efficiently. Open-ended tasks are often designed to be fun so that students are more engaged in primary schools. However, many students seem to struggle with the openness of these tasks at the secondary level. This paper will explore a way to introduce open-ended questions at the start of Year 11 Mathematical Methods through an investigation project. Hence, it will help students to become familiarised with the format of these questions, and that they will learn there is often more than one correct answer as well as more than one strategy to solve the problem.

INTRODUCTION

According to the assessment of the study design, which now has an accreditation from 2016 to 2021, for Units 1 and 2 Mathematical Methods, "teachers should use a variety of learning activities and assessment tasks that provide a range of opportunities for students to demonstrate the key knowledge and key skills in the outcomes" (Victorian Curriculum and Assessment Authority [VCAA], n.d.-b, pp. 36 & 41).

Mathematical Methods teachers are quite comfortable with and good at assessing students using tests/assignments/exams. However, not everyone is comfortable with preparing an assessment task such as a mathematical investigation. These tasks provide students an opportunity to explore mathematically in greater depth and breadth, resulting in a wide range of student responses, which may not be possible to complete under timed conditions such as an examination. Furthermore, these tasks require students to apply mathematical processes in non-routine contexts, make a choice, and tackle open-ended questions. They need to utilise a variety of techniques to explore, investigate, interpret, and draw valid conclusions.

WHAT DOES A MATHEMATICAL INVESTIGATION LOOK LIKE?

Let's start by comparing a closed question and an open-ended question.

1. Sketch a graph of $f(x) = 4(x-3)^2 - 1$, showing all key features.

2. Investigate a family of parabolas that has a turning point at (3, -1) which has zero, one, or two x-intercepts. What generalisations can be made?

In the first question, there is only one graph with fixed key features: an open upwards U-shaped curve known as a parabola, a turning point at (3, -1), x-intercepts of 2.5 and 3.5, and a y-intercept of 35. While in the second question, students will soon learn that there is a wide range of responses.

From the above example, it is clear that students need to understand the difference between a closed question and an open-ended question. However, students often struggle to navigate open-ended questions as they do not know what would be the best approach to take. They do not know where to start, what information they need to provide and therefore they are not confident in their final answer. Not only that, students do not realise that an open-ended question can produce multiple solutions depending on their methodology. Thus, a possible class activity is to teach students how to change a closed question into an open-ended one that assesses similar skills but in more depth. Students would need to familiarise themselves with command key terms such as comment, compare, contrast, construct, describe, deduce, explain, explore, interpret, investigate, model, justify, and verify.

This paper provides two samples. The first sample is written according to the advice given from the study design, which is suitable for Unit 1 Mathematical Methods. The second sample is an activity which helps students to recognise the difference between closed and open-ended questions. For open-ended questions, do students use trial and error methods to solve the problem? Do students know how to apply strategic problem-solving skills to solve the problem for an efficient and elegant solution(s) which has met all the given criteria?

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

SAMPLE 1 – INVESTIGATING GRAPHS OF POWER FUNCTIONS

This task is designed for students to investigate different types of graphs of power functions and their transformations. Students would benefit from watching a short video on Hyperbola graphs before they do this investigation in order to familiarise themselves with the basic features of these graphs.

It is important that students understand the key features of graphs, which include:

*   Axial intercepts – usually given in the coordinate pair form

*   End point(s) – label point(s) and 'empty/solid' dot(s)

*   Turning points – given in the coordinate pair form

*   Asymptotes – draw a 'dashed' line and label with their equations

Consider the function $f(x) = x^n$, where $n \in N$ and $n \in \left\{ -2, -1, \frac{1}{3}, \frac{1}{2} \right\}$

RECTANGULAR HYPERBOLA GRAPHS

When $n = -1$ the function $f(x) = x^n$ becomes $f(x) = x^{-1}$ which can also be written as $f(x) = \frac{1}{x}$.

This is the rule for a rational function called a rectangular hyperbola. It is also called a right hyperbola, or hyperbolic graph.

What happens if:

• $x = 0$? Since division by zero is not defined, hence 0 must be excluded from the domain.

• $y = 0$? Since there is no number whose reciprocal is zero, hence 0 must be excluded from the range.

Thus, this graph has two asymptotes, namely $x = 0$ and $y = 0$, which are perpendicular to each other. The asymptotes are key features of the graph of a hyperbola.

Component 1

This component refers to introducing the context through specific cases or examples (VCAA, n.d.-a).

Question 1

a. Sketch the following graphs and clearly indicate all key features.

    i. $f(x) = \frac{1}{x}$

    ii. $g(x) = \frac{1}{x+2} + 1$

b. State the transformations required to map the graph of $y = f(x)$ onto the graph of $y = g(x)$.

c. The graph $y = af(x - b) + c$ has an asymptote at $x = 2$ and an x-intercept of 4.

    i.   Explain why $c \neq 0$.

    ii. State the relationship between a and c.

    iii. If $-5 \leq c \leq 5$ and $c \neq 0$, investigate all possible values of a.

    iv. Choose 2 different values of c from part ciii, hence find the corresponding values of a and sketch these 2 graphs on a different set of axes, showing all key features.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Question 2

Use your CAS calculator to explore the following functions. What are the similarities and differences between these graphs?

    a. $f(x) = \dfrac{1}{x}$

    b. $f(x) = \dfrac{1}{x-1}$

    c. $f(x) = \dfrac{1}{x+2}$

    d. $f(x) = \dfrac{1}{3-x}$

    e. $f(x) = \dfrac{1}{x} - 2$

    f. $f(x) = 2 - \dfrac{1}{x}$

    g. $f(x) = \dfrac{1}{x+2} + 1$

    h. $f(x) = \dfrac{1}{1-x} + 1$

Component 2

This component refers to considering the general features of the context (VCAA, n.d.-a).

Question 3

The graph of $y = f(x)$ where $f(x) = \dfrac{1}{x}$ has transformed into the graph of $y = af(x - b) + c$ where $a, b, c \in R$ and $a \neq 0$.

Choose four different sets of values of the parameters a, b and c. Explain why you have chosen these values.

a. In each of the cases in part a, sketch the graphs of these functions on a different set of axes, clearly indicating all key features.

b. Choose two cases from part a, state the transformations required to map the graph of $y = f(x)$ onto the graph of $y = af(x - b) + c$.

c. Investigate how the values of $a, b,$ and c relate to the locations of the asymptotes and the quadrants to which the graphs of $y = af(x - b) + c$ are in. Draw a conclusion about the key features of these graphs.

Component 3

This component refers to the "variation or further specification of assumption or conditions involved in the context to focus on a particular feature or aspect related to the context" (VCAA, n.d.-a).

Question 4

Consider the function $y = af(x - b)^n + c$, where $a, b, c \in R$ and $a \neq 0$, $n \in N$ and $n \in \left\{ -2, -1, \dfrac{1}{3}, \dfrac{1}{2} \right\}$.

a. Investigate the graphs and their names of the above function for combinations and ranges of values of the parameters $a, b, c,$ and n.

b. What generalisations can be made?

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

SAMPLE 2 – INSIGHT INTO CLOSED VS. OPEN-ENDED QUESTIONS AND STUDENT CHOICE ASPECTS

SCENARIOS 1 TO 5

The number of Tails, X, when a coin is tossed n times has the following probability distribution table.

x	a	b	c	d	e	f	g	h	i
$X=x$	j	k	l	m	n	p	q	r	s

Scenario 1

The coin is *fair* and tossed 6 times.

a. Find the values of the unknown from the above probability distribution table.

b. Construct a scatterplot using the values found from the probability distribution table.

c. Find and interpret the following:

   i. E(X)　　　　　ii. SD(X)　　　　　iii. $\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$

Scenario 2

The coin is *fair* and tossed 8 times.

a. Find the values of the unknown from the above probability distribution table.

b. Construct a scatterplot using the values found from the probability distribution table.

c. Find and interpret the following:

   i. E(X)　　　　　ii. SD(X)　　　　　iii. $\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$

Scenario 3

The coin is *fair* and you choose a number of times it can be tossed (state this value).

a. What would you expect the scatterplot to be? How is it different/similar to Scenarios 1 and 2?

b. How would the following change?

   i. E(X)　　　　　ii. SD(X)　　　　　iii. $\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$

Scenario 4

The coin is *fair* and tossed n times.

a. Find in terms of n

   i. E(X)　　　　　ii. SD(X)

b. How would the scatter plot and $\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$ change if n is increasing?

Scenario 5

The coin is *biased* and tossed 5 times.

a. If VAR(X) is to be between 0.5 to 1.1, investigate the missing values from the probability distribution table.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

b. Hence or otherwise, investigate the possible values of:

   i. E(X)   ii. SD(X)

c. What would you expect the scatterplot to be? Discuss how the scatterplots change if:

      the coin is tossed n times and n is increasing.

      the probability of getting a tail on a single trial changes.

d. Investigate whether it is possible to have a probability distribution table if VAR(X) = 3. How do you know? Explore the possible values of variance.

Scenario 6

A spinner is numbered from 0 to 5 and each of the six numbers has a different chance of coming up.

The probability of the numbers 1 and 4 coming up are 0.125 and 0.086, respectively.

If the VAR(X) is to be between 1 to 3:

a. Investigate a possible probability distribution table.

b. Use your probability distribution table to find the following and discuss your results.

   i. E(X)   ii. SD(X)   iii. $\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$

REFLECT ON YOUR INVESTIGATION

- Discuss the similarities and differences between these questions.

- Which question(s) would be considered "closed/open-ended question(s)? Justify your answer.

- What are the key command terms in these questions? What are the expectations of these terms?

- How do you take a "closed" question, and turn it into an "open-ended" question that assesses similar skills but in more depth? Can you give some examples?

REFERENCES

Victorian Curriculum and Assessment Authority. (n.d.-a). *Advice for teachers – Mathematical Methods*. Retrieved from https://www.vcaa.vic.edu.au/curriculum/vce/vce-study-designs/mathematicalmethods/advice-for-teachers/Pages/Units3and4AssessmentTasks%E2%80%8B.aspx

Victorian Curriculum and Assessment Authority. (n.d.-b). *Mathematical Methods study design*. Retrieved from https://www.vcaa.vic.edu.au/curriculum/vce/vce-study-designs/mathematicalmethods/Pages/Index.aspx

Beyond the arithmetic operation: How the equal sign is introduced in Chinese classrooms

Jiqing Sun, Deakin University

The relational understanding of the equal sign is an essential foundation for algebra. In recent decades, researchers have documented that many students have a narrow conception of the equal sign, viewing it as a one-directional 'show result' symbol. On the other hand, Chinese primary students commonly understand the equal sign relationally, which is the focus of this paper. Four features of the Chinese approach will be discussed: a) introducing the equal sign before traditional arithmetic operations, b) an instructional sequence that is in line with RME theory, c) the way of drawing an equal sign, and d) the emphasis of 'two-sides' sense. This research contributes to providing an effective starting point of developing students' relational understanding of the equal sign, which could be adapted to the Australian Curriculum.

BACKGROUND

Forty years ago, Kieran (1981) highlighted that many students view the equal sign as a 'show result' symbol, rather than an indication of the equivalence of two sides. With this misconception, many students have difficulties in understanding equations such as '3 = 3', and '3 + 2 = 4 + 1' (Kieran, 2004). Nowadays, the persistence of this narrow conception is still widely documented (e.g., Carpenter, Franke, & Levi, 2003; McNeil, Fyfe, & Dunwiddie, 2015; Pang & Kim, 2018; Schwarzkopf, Nührenbörger, & Mayer, 2018). Carpenter et al. (2003) called for the development of the relational understanding of the equal sign (i.e., that it indicates a bidirectional equivalent relation of two sides).

This understanding is a fundamental concept in students' transition from arithmetic to algebra (e.g., Blanton et al., 2015; Carpenter et al., 2003; Pang & Kim, 2018). Without it, students will encounter difficulties learning algebra. Filloy and Rojano (1989) showed students with a narrow equal sign conception were not able to understand and solve equations with unknowns on both sides (e.g., $2x + 1 = x + 3$), since they thought the equal sign should always be followed by an answer rather than another unknown. There has been a great deal of research exploring instructional strategies to foster students' relational understanding of the equal sign. This study contributes to this growing research by investigating the Chinese approach to teaching this concept.

LITERATURE REVIEW

Carpenter et al. (2003) suggested a series of number-sentence activities that could broaden students' equal sign conception. For instance, 'true/false' number sentences have been shown to be beneficial in extending students' conception of the equal sign. By evaluating related number sentences as true or false (e.g., 3 + 5 = 8, 8 = 3 + 5, 8 = 8, 3 + 5 = 5 + 3), Carpenter et al. (2003) showed that students' misconception could be challenged. This kind of task could create a cognitive conflict between students' intuitive knowledge (e.g., considering unconventional equations such as 8 = 8, 3 + 5 = 3 + 5 as true) and their narrow conception of the equal sign (e.g., considering other unconventional equations such as 8 = 3 + 5 as false). Therefore, it can provide students with an opportunity to further examine their conception of the equal sign. This instructional approach is evidenced to be effective by other algebra researchers (e.g., Knuth, Stephens, Blanton, & Gardiner, 2016; Molina & Ambrose, 2006).

Alternatively, others (e.g., McNeil et al., 2015) proposed a teaching approach based on modifying students' arithmetic operation practice. As researchers (McNeil, 2008; Seo & Ginsburg, 2003) showed, when students practice arithmetic operation skills, questions are generally presented in a traditional form such as '1 + 2 = _'. Schwarzkopf et al. (2018) explained that this traditional form of practice might lead to students thinking the equal sign always requires an answer based on computation. Therefore, this form of practice could reinforce and activate students' one-directional 'show result' conception of the equal sign, and therefore hinder the emergence of the relational (bidirectional) view (McNeil, 2008). McNeil et al. (2015) suggested that teachers and textbooks should provide students with more exposure to non-traditional forms of arithmetic operation practice, such as '_ = 1 + 2' and '3 = 1 + _'. Thus, the emergence of a bidirectional view of the equal sign will not be impeded by a one-directional calculation pattern. In fact, McNeil et al. (2015) demonstrated that modified arithmetic practice (non-traditional forms) improved students' relational conception of the equal sign.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Based on the literature reviewed, I consider that both aforementioned approaches (number sentence comparison and non-traditional arithmetic operations) are similar, although their starting points are different. Both approaches address students' one-directional conception of the equal sign by exposing them to a wide range of non-traditional equation forms. Hence, I argue that presenting non-traditional equations to challenge students' pre-conceptions could be a foundational way to develop relational understanding of the equal sign at an early stage.

While students' narrow conception of the equal sign is common, this difficulty is not universal. Li, Ding, Capraro, and Capraro (2008) showed that Chinese primary students commonly understand the equal sign relationally. Li et al. (2008) identified factors that contribute to Chinese students' relational understanding, such as the presentation of the concept of equality in Grade 1 before students encounter the arithmetic operation. With a range of realistic contexts and everyday language such as "the same as" and "more than", teachers put emphasis on comparing two quantities. The formal equal symbol is introduced afterwards. Li et al. (2008) suggested that these instructions can help students to transition from a concrete context to the formal mathematical symbol, and build students' knowledge based on prior experiences. I further explore the Chinese approach from an early algebra educator's perspective, so as to inform Australian mathematics classroom instruction.

RESEARCH DESIGN

METHODOLOGY

This qualitative case study is inspired by Clarke's (2001) complementary accounts methodology. According to Clarke (2001), for a comprehensive understanding of students' learning activity, perspectives from researchers, teachers, and students need to be investigated tso that the possible bias from one data source can be avoided through triangulation.

A lesson plan (Figure 1) to introduce the equal sign from a Chinese official teachers' guidebook was analysed. I made connections between the teaching sequences described in the lesson plan and the factors extracted from the existing literature that support students' relational understanding of the equal sign. Furthermore, I reviewed a teacher's reflective journal, which documented the teacher's comments on the lessons that were designed in accordance with the lesson plan in Figure 1. Due to ethical restrictions, I did not have an opportunity to conduct student interviews.

DATA SOURCE

The lesson plan used in this study is from a teachers' guidebook Teachers' Instruction Guidebook, by the Primary and Secondary School Teaching and Learning Research Group, JiangSu (PSSTLR). PSSTLR JiangSu is a research body under China's Ministry of Education, so this guidebook can is an official educational document.

The teacher's reflective journal was provided by one teacher who publicly shared it in a professional development workshop. This teacher works in a kindergarten in the city of Wuxi, JiangSu province.

RESULTS AND DISCUSSION

LESSON PLAN

Figure 1 (p. 54) shows the English version of the lesson plan.

The concept of equality and the equal sign are presented in the kindergarten curriculum, and students learn formal arithmetic operations such as '1 + 2 = 3' in Grade 1. This is consistent with the finding by Li et al. (2008) that students in China encounter the concept of the equal sign prior to the formal arithmetic operation. Following Li et al. (2008), I tried to gain further insight into the importance of this setting. As discussed earlier, a common misconception of the equal sign is that it is a one-directional 'show results' symbol. This misconception could be triggered and reinforced by the traditional arithmetic operation (McNeil, 2008; McNeil et al., 2015). In the lesson plan, at the start, the equal sign is presented without any arithmetic operations. Thus, the students' first encounter with the equal sign is not muddled by a 'left to right' operation; rather, students can clearly see that it indicates an equivalence between two sides. Hence, the equal sign is presented as representing a bidirectional equivalence relationship before students see it embedded in an arithmetic operation. With this approach, the risk that students view the equal sign as a one-directional symbol could be minimised.

Objective:

1. Learn to use '=' or '≠' to indicate the quantity relationship between two sets of numbers.
2. Be able to think about the problems proactively, be able to use the appropriate language to present the results of the activity.

Preparation:

1. 3-4 cards with concrete objects, one '=' card and one '≠' card
2. Children's graphic book

Activity:

1. Recognising the equality, understanding how to represent the equivalent relationship between two sets of numbers.
   a) Teacher shows students two cards with the same number of fruits, for instance, six apples and six pears. Let students count the numbers of apples and pears, and let them decide whether they have the same number of fruit.
   b) Teacher asks students what symbols they can put between two cards so other people can clearly see that they have the same number. After that, you can introduce the equal sign by emphasising two short horizontal lines must be at the same length.
2. Recognising the inequality, understand the inequality and how to represent it.
   a) Teacher shows students cards with different numbers of fruits, asks students whether two cards have the same number now. If two sides have different numbers, asks students what symbols they can put to indicate the two sides have different numbers. After that, showing students the sign '≠'.
   b) Teacher shows students several pairs of cards with a mixture of equivalent and non-equivalent relationships, and then asks students what sign they should put in-between and why.
3. Practicing "the left side and the right side is equivalent" in the children's graphic book.

Figure 1. English version of lesson plan.

The instructional sequence to introduce the formal equal sign in this lesson plan is notable. While showing two cards that depict an equal number of fruits, the teacher asks students whether the two cards show the same number. This question is in accordance with Li et al. (2008), who found the conception of equality was developed based on students' own experiences in a realistic context in Chinese classrooms. According to the Realistic Mathematics Education (RME) theory (Van den Heuvel-Panhuizen & Drijvers, 2014), a context that students can experientially make sense of is an effective starting point for students' formal mathematics learning. After students make a comparison, the teacher asks students what they can put between these two cards so other people can clearly see that the quantities on them are the same. In this step, it is possible that students use various self-invented symbols that indicate the equivalence, meaning they may not necessarily use the formal equal sign. However, as long as students use symbols to represent the concept of 'the same as' or 'equal number', they step beyond the concerte experience into formal mathematics representation. This process is in line with transition activities from informal to formal mathematics suggested by RME theory. The lesson plan requires teachers to use the phrase such as "between these two cards". This phrase emphasises bidirectionality to students, which is helpful to reinforce a relational conception of the equal sign.

In the final step of the lesson, teachers introduce the formal equal sign to students, and they need to emphasise that when drawing this symbol, the two horizontal lines must be the same length. By stressing this detail, a 'two sides' sense is further pressed in students' conception.

After instruction, the lesson plan requires students to complete a quiz as shown in Figure 2 (p. 55).

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Figure 2. Quiz.

In this quiz, students fill the unknown numbers either on the left or the right side. By completing this quiz, students can see that the equal sign is bidirectional. Furthermore, this quiz is a kind of prototype of formal mathematics questions such as '5 = _' and "_ = 6". According to Matthews, Rittle-Johnson, McEldoon, and Taylor (2012), this type of question could probe whether students possess a rigid 'show results' conception of the equal sign.

In summary, the 'two sides' sense is ubiquitously highlighted in this lesson to introduce the equal sign in Chinese kindergarten classrooms, which is a major factor that contributes to Chinese students' relatively strong bidirectional view of the equal sign.

TEACHER'S REFLECTIVE JOURNAL

Figure 3 shows the excerpts from a teacher's reflective journal.

In the beginning, students could easily compare whether two sides have the same number of fruits. For example, if one side has nine apples, and the other side also has nine apples, I asked students why they have the same number, students told me because the left side has nine, the right side has nine as well, so two sides have the same number. In contrast, when the number of apples on two sides are different, students could tell me they have different numbers by counting. This activity is not difficult for students.

In the next step, I let students put a symbol between two cards to shwo other people that two sides have the same number clearly. Students could use a variety of their own symbols to show that two sides have the same number. Some students did not quite understand what symbol meant, so they used word "the same amout of" to represent this relationship. Some students used the symbol " ⟷ ". There are also some students had already known to use the formal equal sign, so it was believed that they had learnt the equal sign before. In this process, although many students used their own symbols to represent the equlviance of two sides, they understood the symbol put in-between needs to be used to represent that two sides had the same number.

When I told them that they could use '=' to represent the equivalence, especially I emphasised that two short horizontal lines must be drawn at the same length, students can understand why this sign can be used to represent the equivalence. Students considered it was very approriate that an equivalent relationship was represented by two short horiontal lines with equal length.

In terms of the quiz in the children's graphic book, about 90 percent of students in my class can correctly do the task. It menifested that most students have a 'two-sides' sense about the equal sign, because they know it did not matter whether they need to fill something on the right side or the left side.

Finally I would like to say, the emphasis of 'two-side' relationship is everywhere in this lesson plan, I think it is most important thing when introducing equal sign.

Figure 3. Excerpts from teacher's reflective journal.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

As shown, this teacher considers this instructional approach to be effective. The teacher commented that students had no difficulty in evaluating whether two cards show the same number of fruits. This evidences that students' everyday experience is an accessible starting point for formal mathematical learning. When the teacher required students to put a symbol between two cards, she found that students inserted different symbols and even words such as "the same" or "the same number". This is consistent with RME theory, as students might use informal strategies to solve problems. The teacher commented that although they were not able to use a formal equal sign, students clearly understood that they had to put something indicating that the numbers on the two sides were equivalent. Interestingly, the teacher also noted that a small number of students chose a formal equal sign, which means they might have learnt this at home.

The teacher also highlighted that it is helpful to stress that the two horizontal lines must be of equal length when introducing the equal sign. As the teacher stated, students thought it made sense to draw two short lines of equal length to represent the equivalent relationship of two sides, so they quickly accepted this symbol. As such, the formal equal symbol can be easily understood, which is in line with RME theory.

After the presentation of the content, the teacher used the quick quiz (Figure 2) to test students' understanding of the equal sign. The teacher reported that out of 35 students in the class, about 90% of them correctly provided an equivalent number of objects on the left or right side. As mentioned, students' success with this type of question could be an indicator that their conception of the equal sign has evolved beyond the one-directional 'show results' level (Matthews et al., 2012). Finally, the teacher concluded that "the emphasis of 'two-side' relationship is everywhere in this lesson plan, it is the most important thing when introducing equal sign". This statement echoes my analysis of the lesson plan presented above.

CONCLUSION AND LIMITATIONS

Several features of the Chinese approach to introduce the equal sign can be concluded. First, introducing the equal sign before the formal arithmetic operations can reduce the interference of the 'show results' conception. Second, there is a ubiquitous emphasis on the 'two-side' equivalence, which stresses a bidirectional conception of the equal sign. Furthermore, the instructional sequence starts from students' own experience and informal solutions, and continues to formal mathematical symbols. Finally, the way of drawing an equal sign (i.e., two short horizontal lines with the same lengh) is highlighted.

The Chinese approach has implications for Australian mathematics classrooms. In fact, there have been some activities designed by Australian primary teachers to develop the bidirectional sense of the equal sign. For instance, the dice comparison game by Russo (2016) and balance beam activity by Perso (2005) showed positive results. Following from these studies, I suggest that it is still possible that Australian mathematics educators can borrow ideas from the Chinese approach. For example, in the current Australian curriculum, number sense (e.g., ordering numbers) is introduced before formal addition and subtraction (Australian Curriculum, Assessment and Reporting Authority, n.d.). I recommend that when teaching number sense, teachers let students compare numbers and take this opportunity to introduce the equal sign. By doing this, the one-directional misconception elicited by traditional arithmetic operations could be further reduced. Moreover, highlighting the way of drawing the equal symbol is also helpful to reinforce the bidirectional conception.

In terms of limitations, due to ethical restrictions, I was unable to conduct student or teacher interviews. More insights could be gained if more students and teachers' voices were included.

REFERENCES

Australian Curriculum, Assessment and Reporting Authority. (n.d.). *Australian Curriculum: Mathematics*. Retrieved from https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/

Blanton, M., Stephens, A., Knuth, E., Gardiner, A. M., Isler, I., & Kim, J.-S. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education*, *46*(1), 39-87. doi:10.5951/jresematheduc.46.1.0039

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Clarke, D. (2001). Complementary accounts methodology. In D. Clarke (Ed.), *Perspectives on practice and meaning in mathematics and science classrooms* (Vol. 25, pp. 13-32). Berlin, Germany: Springer Science & Business Media.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, *9*(2), 19-25. Retrieved from https://flm-journal.org/

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*(3), 317-326. doi:10.1007/BF00311062

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator*, *8*(1), 139-151. Retrieved from http://tme.journals.libs.uga.edu/index.php/tme/index

Knuth, E., Stephens, A., Blanton, M., & Gardiner, A. (2016). Build an early foundation for algebra success. *Phi Delta Kappan*, *97*(6), 65-68. doi:10.1177/0031721716636877

Li, X., Ding, M., Capraro, M. M., & Capraro, R. M. (2008). Sources of differences in children's understandings of mathematical equality: Comparative analysis of teacher guides and student texts in China and the United States. *Cognition and Instruction*, *26*(2), 195-217. doi:10.1080/07370000801980845

Matthews, P., Rittle-Johnson, B., McEldoon, K., & Taylor, R. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education*, *43*(3), 316-350. Retrieved from https://www.nctm.org/publications/journal-for-research-in-mathematics-education/

McNeil, N. M. (2008). Limitations to teaching children 2 + 2 = 4: Typical arithmetic problems can hinder learning of mathematical equivalence. *Child Development*, *79*(5), 1524-1537. doi:10.1111/j.1467-8624.2008.01203.x

McNeil, N. M., Fyfe, E. R., & Dunwiddie, A. E. (2015). Arithmetic practice can be modified to promote understanding of mathematical equivalence. *Journal of Educational Psychology*, *107*(2), 423-436. doi:10.1037/a0037687

Molina, M., & Ambrose, R. (2006). Fostering relational thinking while negotiating the meaning of the equal sign. *Teaching Children Mathematics*, *13*(2), 111-117. Retrieved from https://www.nctm.org/publications/teaching-children-mathematics/

Pang, J., & Kim, J. (2018). Characteristics of Korean students' early algebraic thinking: A generalized arithmetic perspective. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5-to 12-year-olds* (pp. 141-165). New York, NY: Springer.

Perso, T. (2005). Algebra outcomes for your K–9 class. In M. Coupland, J. Anderson, & T. Spencer (Eds.), *Making mathematics vital: Proceedings of the Twentieth Biennial Conference of the Australian Association of Mathematics Teachers* (pp. 209-216). Adelaide, Australia: Australian Association of Mathematics Teachers.

Russo, J. (2016). Using picture story books to discover and explore the concept of equivalence. *Australian Primary Mathematics Classroom*, *21*(2), 26-31.

Schwarzkopf, R., Nührenbörger, M., & Mayer, C. (2018). Algebraic understanding of equalities in primary classes In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5-to 12-year-olds* (pp. 195-212). New York, NY: Springer.

Seo, K.-H., & Ginsburg, H. P. (2003). "You've got to carefully read the math sentence…": Classroom context and children's interpretations of the equals sign. In A. J. Baroody & A. Dowker (Eds.), *Studies in mathematical thinking and learning. The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 161-187). Mahwah, NJ: Lawrence Erlbaum Associates.

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2014). Realistic mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 521-525). New York, NY: Springer.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Peer reviewed papers

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

VCAA mathematics annotated work samples F-10

Crystal Afitu and David Leigh-Lancaster, Victorian Curriculum and Assessment Authority

Annotated student work samples provide a way of seeing examples of work that students do when they demonstrate achievement of aspects of standards with respect to tasks related to the curriculum. The VCAA has developed and published a new set of annotated student work samples during 2019 and aims to progressively develop and publish others in 2020. This paper provides an introduction to these new resources and looks at how they could be used to support the implementation of the curriculum, including how schools might use a similar approach to develop complementary school-based annotated student work samples.

INTRODUCTION

Annotated student work samples are one of a range of resources that education systems develop to support teachers in the implementation of the curriculum and assessment of student learning. Providing student samples for educators on aspects of the Victorian Mathematics F-10 Curriculum assists with supporting teacher clarity with Content Descriptions and Achievement Standards (Adie & Wills, 2014). Enriching teacher vocabulary with regards to the Victorian Curriculum through sample annotations allows for clarity with the content being taught and effective feedback to students on their learning (Ball, 2010). Annotated student work samples have previously been developed for the Curriculum and Standards Framework II (Victorian Curriculum and Assessment Authority, 2000), the Victorian Essential Learning Standards (Victorian Curriculum and Assessment Authority [VCAA], 2007), and the Australian Curriculum: Mathematics F-10 (Australian Curriculum, Assessment and Reporting Authority, 2014).

This new set of annotated student work samples, of which the first batch has been recently published by the VCAA (VCAA, 2019), have been developed for the Victorian Curriculum: Mathematics F-10 (VCAA, 2016b) to be indicative of the thinking and writing students do when working on specifically developed tasks designed to elicit responses that demonstrate achievement of aspects of the standards at a given level of the curriculum.

The annotations were constructed using moderated observations, experience and judgments of teachers across multiple schools and classes. All students undertook the same task integrated within the school planned teaching and learning sequence of activities in mathematics for that level of the curriculum. The first batch, published at the beginning of Term 3 in 2019, covers a sample for each of Levels 3, 5, and 7 from Number and Algebra, and a sample for Level 9 from Measurement and Geometry. The second batch, planned for publication in the latter part of Term 4, aims to complement the 2019 examples and cover a sample for each strand at Levels 3, 5, 7, and 9. Subsequent similar work could also be developed for corresponding annotated student work samples across each of the strands for Levels F, 1, 2, 4, 6, 8, and 10.

SUMMARY OF RESOURCES

Annotated student work samples have been published on the VCAA website in the following structure:

Level	Strand	Sub-strand	Activity
Level 3	Number and Algebra	Number and place value	What is my place?
Level 5	Number and Algebra	Patterns and algebra	What is missing?
Level 7	Number and Algebra	Number and place value, Real numbers	Over and under estimates
Level 9	Measurement and Geometry	Pythagoras and trigonometry	When three sides work

Table 1. Term 3, 2019 publications.

THE MATHEMATICAL ASSOCIATION OF VICTORIA

Level	Strand	Sub-strand	Activity
Level 3	Measurement and geometry	Measurement	What time is it?
	Statistics and probability	Statistics	What is your favourite?
Level 5	Measurement and geometry	Geometry	Shape up!
	Statistics and probability	Probability	What are the chances?
Level 7	Measurement and geometry	Measurement	Finding areas?
	Statistics and probability	Statistics	Mean, median and mode
Level 9	Number and algebra	Patterns and algebra	This expression is the same as that expression when…
	Statistics and probability	Probability	Matching digits

Table 2. Term 4, 2019 publications.

THE STRUCTURE OF ANNOTATED STUDENT WORK SAMPLES

The annotated student work samples have been developed to demonstrate student achievement with regards to the Victorian Curriculum.

The samples include:

• Lesson overview

• Links to the Victorian Curriculum content descriptors and achievement standards

• Links to the NAPLAN minimum standards for teacher reference

• Annotated student work samples describing the learning the student has demonstrated

• Where to next? Guidance as to which curriculum areas students may need to review to further develop their understanding of the curriculum.

AN EXAMPLE

The template design of the annotated work samples has been developed in a simple to read format that can be easily accessed and interpreted by a range of people. The blank tasks can be downloaded by teachers for students to complete, should they wish to compare their students' results with the published samples. The published samples demonstrate a range of typical student responses to the tasks that enable them to demonstrate the various ways a response could be given at that level. References to NAPLAN's minimum standards have been made at Levels 3, 5, 7, and 9 so that teachers can identify expected student achievement within that 'Band Level'.

Teachers involved in the project participated in professional learning workshops on writing annotations aligned with the Victorian Curriculum and collecting work samples. Over several workshops and ongoing moderation sessions across a range of schools, they developed annotations relying on their experience and expertise with support from the VCAA.

Templates were designed to include direct hyperlinks to relevant aspects of the curriculum. They can be downloaded via PDF or Microsoft Word document on the VCAA website.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Level 5 – Number and Algebra

Overview

Task name	What is missing?
Learning intention	To find unknown quantities in a sentence
Duration	30 minutes

Links to Victorian Curriculum

These work samples are linked to Level 5 of the Mathematics curriculum

Extract from achievement standard

Students solve simple problems involving the four operations using a range of strategies including digital technology. They estimate to check the reasonableness of answers and approximate answers by rounding ... They find unknown quantities in number sentences ...

Relevant content descriptions

- Use estimation and rounding to check the reasonableness of answers to calculations (VCMNA182)
- Solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental, written strategies and appropriate digital technologies (VCMNA183)
- Solve problems involving division by a one digit number, including those that result in a remainder (VCMNA184)
- Use efficient mental and written strategies and apply appropriate digital technologies to solve problems (VCMNA185)
- Use equivalent number sentences involving multiplication and division to find unknown quantities (VCMNA193)

Student work samples – Unknown quantities

These work samples were created by students working at Level 5. Evidence of student achievement has been annotated.

Victorian Curriculum links

Solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental, written strategies and appropriate digital technologies (VCMNA183)

Solve problems involving division by a one digit number, including those that result in a remainder (VCMNA184)

Use efficient mental and written strategies and apply appropriate digital technologies to solve problems (VCMNA185)

Use equivalent number sentences involving multiplication and division to find unknown quantities (VCMNA193)

Find the missing numbers in the following number sentences. Explain and show your thinking in the space below.

$$84 + \boxed{49} = 133$$

Partitions numbers to calculate answer

Adds 16 to get from 84 to 100, then adds 33 to get to 133

$$84 + \boxed{16} = 100 + \boxed{33} = 133$$

Identifies 16 and 33 as numbers to be added, which gives the answer 49

$$84 + \boxed{49} = 133$$

Uses a diagram to show skip counting as an efficient mental strategy, skip counting by 10 to 124

Continues by adding 5 to make 129, then 4 to make the required amount of 133

$$40 + 9 = 49$$

THE MATHEMATICAL ASSOCIATION OF VICTORIA

Figure 1. Level 5 Number and Algebra annotated student work samples published on the VCAA website.

UNDERSTANDING THE SAMPLES – AN EXTRACT

The annotated student work samples are developed for ease of use by a range of members of the community. The samples may be beneficial for parents, teachers, whole schools and community members who wish to view samples of work reflecting a particular aspect and level of the Victorian Curriculum. It is important to note the various ways that students have demonstrated their understanding to allow those using the samples to identify achievement.

In the student excerpts extracted from published samples, the VCAA has published various ways a student can demonstrate correct responses. These images were extracted from Level 5 Number and Algebra, and display correct responses and strategies to calculate multiplication problems.

LEVEL 5 NUMBER AND ALGEBRA

Content Descriptors:

"Solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental, written strategies and appropriate digital technologies" (VCAA, n.d., VCMNA183)

"Use equivalent number sentences involving multiplication and division to find unknown quantities" (VCAA, n.d., VCMNA193)

Achievement Standard:

"Students solve simple problems involving the four operations using a range of strategies including digital technology. They estimate to check the reasonableness of answers and approximate answers by rounding… They find unknown quantities in number sentences."

THE MATHEMATICAL
ASSOCIATION OF VICTORIA
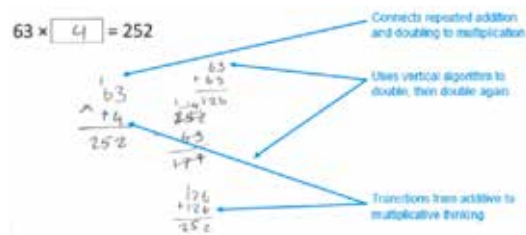
Figure 2. Using repeated addition in multiplication.

Figure 3. Using repeated addition and multiplicative thinking.
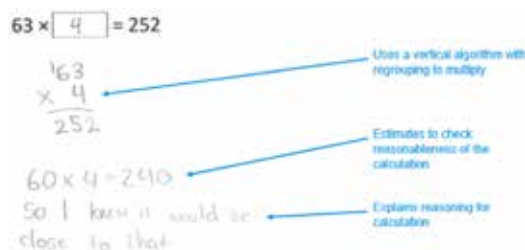
Figure 4. Vertical algorithm and explanation.

Figure 5. Using 'guess and check' and a vertical algorithm.

In the images, the students have demonstrated the correct answer using a range of strategies. They have all demonstrated the requirements of the Content Descriptors and the Achievement Standards from the Level 5 Victorian Curriculum Mathematics. Annotations were published in a separate colour and can be directly linked to the evidence of student learning to allow the reader to easily identify aspects of student achievement.

HOW COULD ANNOTATED STUDENT WORK SAMPLES BE USED IN SCHOOLS?

There are several ways that the annotated student work samples could be used in schools, such as:

- providing individual teachers with access to a continuum of examples across the strands, including those corresponding to topics and levels for which they may not have taught

- as a basis for faculty/department/PLC/PLT discussions about how students demonstrate achievement of learning

- as a context for discussion about moderation in teacher judgment of student achievement, and possible directions for

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

further student learning

- linking them as resources to course plans

- identifying areas for professional learning

- as a stimulus to developing a school or network-based collection of annotated student work samples

- as a student sample within the classroom 'what a good sample looks like', setting standards with students

- assist with developing success criteria/ "I can" statements with students

While the task on which the annotated student work samples are based are short tasks, they can be used as activities within a regular teaching and learning program - blank versions of the tasks have been provided for this purpose. Schools and teachers may decide to incorporate these tasks as learning activities at specific times/stages when developing and planning teaching and learning programs, possibly in conjunction with the VCAA curriculum mapping templates (VCAA, 2016a) and/or the mathematics sample program (VCAA, 2017). They could also be used to provide additional information that informs assessment and reporting.

WHERE TO NEXT?

The annotated student work samples contain a 'where to next' section for teachers to determine which aspects of the curriculum a student may need to review and consolidate or advance towards a new learning goal. Connections have been made between content descriptors across the levels above and below the produced sample to enable teachers to identify the aspects of the curriculum that relate to the activity.

The benefit of having this included in the samples is to enable teachers to differentiate their teaching and apply new learning goals for the student/s based on the response they gave in the task. The annotated student work samples can be accessed and edited using Microsoft Word. This enables teachers to use the format to create their own annotated student work samples for student portfolios and evidence of student learning. The tasks can also be edited and reproduced to allow teachers to use them to make connections between student learning and the samples produced by the VCAA.

REFERENCES

Adie, L., & Wills, J. (2014). *Using annotations to inform an understanding of achievement standards*. Retrieved from https://pdfs.semanticscholar.org/8dfc/00c35709dd3cecfd53f19d158d2c00e2964d.pdf

Australian Curriculum, Assessment and Reporting Authority. (2014). *Mathematics work samples portfolios*. Retrieved from https://www.australiancurriculum.edu.au/resources/work-samples/mathematics-work-samples-portfolios/

Ball, C. E. (2010). Annotation an effective device for student feedback: A critical review of the literature. *Nurse Education in Practice*, *10*(3), 138-143. doi:10.1016/j.nepr.2009.05.003

Victorian Curriculum and Assessment Authority. (2000). A*nnotated work samples*. Retrieved from https://webarchive.nla.gov.au/awa/20090505225008/http://pandora.nla.gov.au/pan/99103/20090505-1614/csf.vcaa.vic.edu.au/asw/asma.html

Victorian Curriculum and Assessment Authority. (2007). *Mathematics assessment maps (annotated student work samples)*. Retrieved from https://webarchive.nla.gov.au/awa/20121205175941/http://pandora.nla.gov.au/pan/129125/20121206-0015/vels.vcaa.vic.edu.au/support/maps/maths/index.html

Victorian Curriculum and Assessment Authority. (2016a). *Mathematics: Curriculum mapping templates*. Retrieved from https://www.vcaa.vic.edu.au/curriculum/foundation-10/resources/mathematics/Pages/default.aspx?Redirect=1

Victorian Curriculum and Assessment Authority. (2016b). *Victorian Curriculum F-10: Mathematics*. Retrieved from https://victoriancurriculum.vcaa.vic.edu.au/mathematics/curriculum/f-10

Victorian Curriculum and Assessment Authority. (2017). *Victorian Curriculum F-10: Mathematics sample program*. Retrieved from https://victoriancurriculum.vcaa.vic.edu.au/mathematics/curriculum/f-10

Victorian Curriculum and Assessment Authority. (2019). *Teaching and learning resources: Work samples*. Retrieved from https://www.vcaa.vic.edu.au/curriculum/foundation-10/resources/mathematics/Pages/default.aspx?Redirect=1

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Creating links between Year 7 mathematics and science

Rochelle Dickson, Mentone Girls' Secondary College

STEM-related careers are currently receiving a lot of attention, as we know that these fields are some of the fastest-growing ones. Schools need to develop curriculum and programs to foster the skills in these associated disciplines and the ability to use these skills to solve problems and communicate solutions. It is often difficult to know where to start when faced with such a daunting concept and an already full curriculum. We have begun by creating links between Year 7 mathematics and science. In this paper, activities to link the content areas of statistics and ecosystems, and algebra and forces have been explored.

INTRODUCTION

It has been claimed that we, as a society, are on the cusp of a fourth industrial revolution (Marr, 2018). The growth of various technologies is increasing at an unprecedented pace. For economies to survive in such a world, citizens must be knowledgeable and adaptable. It has been suggested that many of the jobs of the future have not yet been invented and that the best way to prepare young people for this uncertain future is to equip them with the flexibility and critical thinking skills to adapt to a continuously changing workplace and world (Marr, 2018).

In recent years, there has been growing interest in the area of STEM education. It is a rising field in education with a great deal of government support. STEM education covers the specific knowledge and skills found in the science, technology, engineering and mathematics disciplines. It also encompasses the interrelationship between these areas, allowing learning to be delivered in an integrated way, encouraging a deeper engagement with each discipline (State Government of Victoria, 2018). With this integration, STEM education becomes a mechanism through which students can develop 21st century skills such as collaboration, communication, and creative and critical thinking.

In recent years, mathematics and science participation and achievement in senior years in Australia has declined. In 2015, only one in ten Year 12 students completed an advanced mathematics subject and a rising proportion of high-achieving students were choosing not to study mathematics in their final year of secondary education (Timms et al., 2018). We, as educators and as a society, need to take action to ensure that the young people of Australia are prepared to meet the significant challenges and opportunities of the coming years. By incorporating STEM activities, we can hope to increase student engagement with mathematics through linking various key mathematical skills to real-world applications.

CREATING LINKS BETWEEN MATHEMATICS AND SCIENCE

While most educators recognise the need to incorporate STEM education and 21st century skills into the classroom, many ask how we can do this effectively. With an already crowded curriculum and, often, fixed structures and timetables in place in schools, it is hard to know where to begin this journey. At Mentone Girls' Secondary College, one way we began was by creating links between mathematics and science at the Year 7 level.

The Victorian Curriculum remains structured around domain-based learning, despite the inclusion of the general capabilities. The challenge for educators is to choose what to include from each domain and how to integrate aspects of each discipline within the already crowded curriculum. It is important that students do not see mathematics education in isolation. While mathematics can, and should, be studied for its own inherent value, mathematics should also be seen as a tool for problem solving, and for understanding and interpreting the world (Timms et al., 2018).

English (2016) notes that while many have lauded the benefits of integration and an inter-relationship between STEM disciplines, this relationship thus far seems inequitable. Often the focus is on the areas of Science and Technology and with the integration of these subjects, we run the risk of increasing achievement in some areas at the expense of meaningful learning in others. Shaughnessy (2012) takes this notion further, suggesting that, "STEM education must involve significant mathematics for students. Otherwise, the M in STEM is silent. If we are going to promote STEM education, as mathematics teachers we must make the mathematics transparent and explicit." (p. 324).

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

In establishing connections between mathematics and science at a Year 7 level, we have been explicit in detailing the content covered from both curriculum areas and emphasising the many relationships between the disciplines. Following are descriptions of two of the relationships between mathematics and science that can be explored at the Year 7 level.

EXPLORING STATISTICS AND ECOSYSTEMS

There is a growing use of data in many fields including education, industry, business, and government. Not only is this an area of high job growth, but in a world of 'fake news', where information is readily accessible and statistics are used to convince an audience, students must be able to correctly interpret and analyse the information given to them. They must become discerning consumers of information, and much of this is related to an understanding of data and how it can be presented to either inform or mislead the public (Education Council, 2015).

The mathematics study of statistics can be linked to the study of ecosystems. The relevant Victorian Curriculum descriptors can be found below.

Level 7 Mathematics content descriptors (Victorian Curriculum and Assessment Authority [VCAA], n.d.-a)

• Identify and investigate issues involving numerical data collected from primary and secondary sources

• Construct and compare a range of data displays

• Calculate mean, median, mode and range for sets of data. Interpret these statistics in the context of data

Level 7 and 8 Science content descriptors (VCAA, n.d.-b)

• Interactions between organisms can be described in terms of food chains and food webs and can be affected by human activity

• Collaboratively and individually plan and conduct a range of investigation types, including fieldwork.

• Construct and use a range of representations including graphs, keys and models to record and summarise data from students' own investigations and secondary sources.

• Communicate ideas, findings and solutions to problems including identifying impacts and limitations of conclusions and using appropriate scientific language and representations.

While visiting a marine sanctuary students identified the organisms present and collected data, later returning to school to analyse this data using the knowledge gained from their mathematics and science classes. Prior to the excursion, in mathematics classes students were taught some basic information relating to graphical methods of displaying data, calculations of measures of centre and spread, and how to analyse and interpret data for a particular context. Meanwhile in science they explored the concept of ecosystems by considering abiotic and biotic factors, the interactions between organisms, how to construct food chains and webs, as well as possible impacts that humans may have on ecosystems.

Upon arrival at the marine sanctuary, students needed to work collaboratively to determine which data they wished to collect and to determine a timeline and task distribution to ensure that all data collection was completed. Students were required to:

• Identify the abiotic and biotic factors affecting the ecosystem.

• Choose one abiotic factor (such as air temperature or soil moisture level) and measure it at three different times.

• List all organisms observed.

• Use quadrat sampling at three locations to sample organisms found upon the rocky platform and categorise their findings by individual organism, kingdom, or another method of their choice.

• Measure the leaf lengths of various coastal plants (to later be compared to inland plant leaf lengths).

• Identify signs of human impact on the ecosystem.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

In the following week at school, students then chose the appropriate form of graphical display, constructed graphs with and without the use of technology, calculated mean, median, mode and range, analysed and compared their data. They also created food webs for the organisms observed and considered the impact of human intervention upon the food web created and on the ecosystem in general.

Finally, all pertinent information was summarised on a poster, clearly identifying particular trends and communicating their findings to other members of the school. Two such posters can be seen below.
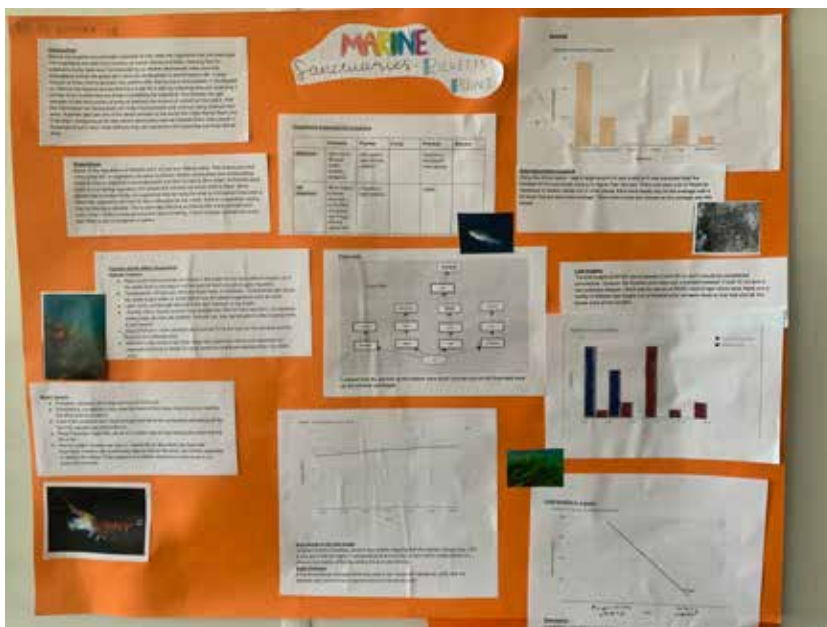

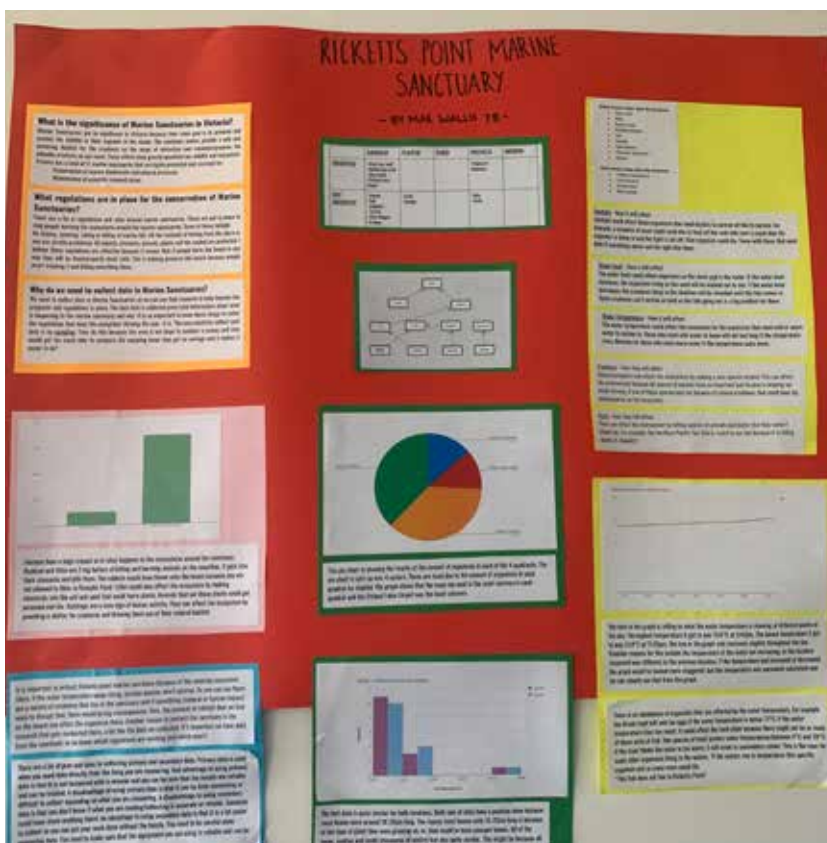
Figure 1. A student's final poster.

Figure 2. Another student's final poster.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Many of our students were highly engaged with this process and could use statistical information and graphs to analyse their results. While some had difficulties selecting the most appropriate format in which to present different data types, many were able to both select an appropriate graphical format and explain the reasoning behind their choice. Students were able to compare and use measures of centre to draw conclusions relating to their data.

INVESTIGATING EQUATIONS AND FORCES

The Victorian Curriculum for Level 7 and 8 Science (VCAA, n.d.-b) includes the content descriptor:

• Change to an object's motion is caused by unbalanced forces acting on the object.

The Level 7 mathematics curriculum (VCAA, n.d.-a) includes the content descriptors:

• Introduce the concept of variables as a way of representing numbers using letters

• Create algebraic expressions and evaluate them by substituting a given value for each variable

• Solve simple linear equations

The lesson created integrates these concepts by investigating the forces acting on a wooden block in motion, and the coefficient of friction. As this lesson was aimed at Year 7 students, the equations of motion were relatively simple.

The lesson embraced the study of mechanics usually introduced only in VCE Specialist Mathematics, which includes the study of equations of motion developed from force diagrams (VCAA, 2015). It allowed students to explore these concepts to show how the forces acting on a body can be represented through variables and equations. Prior to commencing this lesson, students had been studying forces in Year 7 science and had learnt how to construct force diagrams and describe forces such as gravity, the normal reaction and friction. Through their work in mathematics classes students had developed their understanding of pronumerals, substitution and evaluation of formulae and solving simple equations using inverse operations or backtracking.

Through the investigation into the coefficient of friction, students were introduced to the concept of weight force, where weight equals mass multiplied by gravity. They needed to construct force diagrams, resolve forces parallel and perpendicular to the horizontal plane and measure the effect of friction by pulling a wooden block along a variety of surfaces using a spring balance. Throughout this activity, we were explicit in relation to the aspects of mathematics and science that students were utilising, assisting them to recognise the connections between these two areas.

Learning Activity/ Questions to the Class	Student Responses and Opportunities for Further Teaching	Curriculum Connections
Suppose we are dragging a wooden block along some carpet.		
Draw a diagram and label all forces acting on the wooden block.	Some students labelled the weight force simply as gravity. This presented a good opportunity to speak about how weight force will change due to the mass of the object, by asking questions such as "Is it harder to lift some objects up than others?" and "Why is this the case when the force of gravity remains constant?"	Science: Identify the forces acting upon a body in motion and label the forces appropriately.

Mathematics: Use pronumerals to represent variables. |

Think about the moment when the block is not moving. What does this mean about the forces?	Students were able to say that the forces were balanced, but many needed prompting to think about how this could be displayed algebraically. The idea that 'balanced' is synonymous to 'equal' was clear to some students but not to all, despite the balance method of solving an equation having been taught previously.	Science: Consider when forces are balanced and unbalanced. Mathematics: Set up simple equations.
Think about when the block is moving at a constant speed. What does this mean about the acceleration of the block and the forces acting on it?	The idea that forces can be balanced when an object is moving was quite new. This allowed for the exploration of the concept of acceleration and could also lead to a brief discussion of Newton's Laws.	Science: Consider when forces are balanced and unbalanced.
Write an equation to represent the vertical forces acting on the block.	Through identifying the forces and assigning appropriate pronumerals, most were able to determine $R = mg$. Where R is the Normal Reaction force, m is the mass of the object and g acceleration due to gravity.	Science – Consider when forces are balanced and unbalanced. Mathematics: Set up simple equations.
Write an equation to represent the horizontal forces acting on the block.	Most were able to determine $f = P$, where f is the frictional force resisting motion and P is the pulling force.	Science: Consider when forces are balanced and unbalanced. Mathematics: Set up simple equations.
The friction force is calculated using the formula $f = \mu R$, where μ is the coefficient of friction (related to how rough the surface is) and R is the normal reaction force.		
Calculate the value of R for your block.	Students discussed how to calculate R. They needed to come up with a method to calculate the mass of their chosen wooden block.	Mathematics: Substitute values into formulae and evaluate expressions. Science: Select and use appropriate equipment.
You will now experiment on at least three different surfaces to determine the coefficient of friction. Choose three different surfaces. In each case, use a spring balance to pull the wooden block along the surface. Record the force when the block is in equilibrium (moving at a constant rate). For each surface record the frictional force three times and calculate the average.	It was important the students pull the spring balance at a constant speed in order to use their balanced equations. This ensured that they would calculate kinetic friction rather than static friction. They chose a variety of surfaces including the science benches, the linoleum floor, carpet, concrete and grass.	Mathematics: Calculate the mean of a set of data. Science: Select and use appropriate equipment. Increase the reliability of results.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Use the formula $f = \mu R$. Substitute your value of f and solve the equation to calculate the coefficient of friction for each surface.	Many students had trouble solving these equations. While still a simple linear equation, the inclusion of decimals (resulting from their experiments) lead to some confusion. This offered a great opportunity to reinforce the language of inverse equations. While results would not have been accurate, the size of the coefficients did increase according to the apparent roughness of each surface.	Mathematics: Substitute values into formulae and solve simple linear equations.
If the wooden block is moving, we can use the formula $F = ma$ to calculate the acceleration of the block. Choose one of the surfaces that you have already investigated and complete the steps below. Assume that you now pull the wooden block with a force of 2N.		
Use the formula $F = ma$ and solve the equation to calculate the acceleration of the block	The concept of F as the net force acting on the body needed to be introduced, leading to the idea that in this case $F = 2 - f$. Once students grasped this idea, they were usually able to substitute values into the formula and solve the equation to determine acceleration.	Mathematics: Substitute values into a formula and solve simple linear equations. Science: Consider the meaning of acceleration.
Investigate how changing the pulling force will change the acceleration of the block. i. Choose at least three different values for the pulling force. ii. Substitute into the formula and solve the equation to calculate the value of a in each case. iii. Write a sentence to summarise what you found out.	Investigating the effect of a parameter in this way is now part of School Assessed Coursework in Unit 3 and 4 mathematics. This allows students an early introduction to this concept. The idea could be extended further, with students instead investigating how altering the mass of the block could affect the acceleration on their chosen surface.	Mathematics: Substitute values into a formula and solve simple linear equations. Investigate the effect of a parameter on a particular situation. Science: Consider the meaning of acceleration.

Overall, students responded positively to this activity and were able to build connections between mathematics and science. The difficulty experienced by students solving equations involving decimal numbers suggests that our emphasis on introducing equation solving using natural numbers may not ultimately be beneficial to student understanding. When conducting experiments or dealing with real-world situations students must begin to realise that it is unlikely that all solutions will be whole number values, and should be able to interpret and work with all real numbers.

CONCLUSION

Mathematics, while beautiful in its own right, can often be most useful in its ability to further the understanding and analysis of scientific concepts. Building connections between mathematics and science is a useful way for schools to begin the STEM journey, but to fully embrace the power of STEM education the next step is to integrate the other elements such as engineering and technology. This allows students the freedom to choose a real-life problem to solve, using various discipline-based skills as well as the 21st century skills of critical and creative thinking, communication and collaboration. In Australia, most educators are only just beginning to see the possibilities and determine the logistics of creating open-ended experiences for students.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

REFERENCES

Education Council. (2015). *National STEM school education strategy*. Retrieved from http://www.educationcouncil.edu.au/site/DefaultSite/filesystem/documents/National%20STEM%20School%20 Education%20Strategy.pdf

English, L. D. (2016). STEM education K-12: Perspectives on integration. *International Journal of STEM Education*, *3*(3), 1-8. doi:10.1186/s40594-016-0036-1

Marr, B. (2018). *The 4th industrial revolution is here – Are you ready?* Retrieved from https://www.forbes.com/sites/bernardmarr/2018/08/13/the-4th-industrial-revolution-is-here-are-you-ready/#2bbbacc5628b

Shaughnessy, J. M. (2013). Mathematics in a STEM context. *Mathematics Teaching in the Middle School*, *18*(6), 324-324. doi:10.5951/mathteacmiddscho.18.6.0324

State Government of Australia. (2018). *About STEM education in Victoria*. Retrieved from https://www.education.vic.gov.au/about/programs/learningdev/vicstem/Pages/about.aspx

Timms, M., Moyle, K., Weldon, P., & Mitchell, P. (2018). *Challenges in STEM learning in Australian schools*. Camberwell, Australia: Australian Council for Education Research.

Victorian Curriculum and Assessment Authority. (n.d.-a). *Victorian Curriculum – Mathematics*. Retrieved from https://victoriancurriculum.vcaa.vic.edu.au/mathematics/curriculum/f-10

Victorian Curriculum and Assessment Authority. (n.d.-b). *Victorian Curriculum – Science*. Retrieved from https://victoriancurriculum.vcaa.vic.edu.au/science/curriculum/f-10

Victorian Curriculum and Assessment Authority. (2015). *Mathematics study design*. Retrieved from https://www.vcaa.vic.edu.au/Documents/vce/mathematics/2016MathematicsSD.pdf

Connecting probability

Heather Ernst and Anna Morton, Federation University Australia

Probability is an important part of our lives. Learning probability increases relevance and engagement in mathematics, however students' understanding of it is often mixed with misconceptions. This article aims to share ideas about teaching probability from the current literature and our experiences as teachers of primary, secondary and university students. The problems of learning probability are summarised. Ideas for teaching probability are described, along with how it can be connected to other mathematics topics, the proficiencies, and cross-curriculum areas. An example of probability incorporating measurement is detailed, explaining how teachers can develop concepts to prepare younger students for probability in Year 11 and 12 Mathematics.

THE PROBLEM WITH PROBABILITY

Probability is an important part of our lives. Understanding and using probability concepts can improve our financial decision making, sport, health, and even what we wear each day (Kahneman, 2012), but it is last on our list of topics in the curriculum, and often taught last in the year if we have time. There are a number of factors that could explain and contribute to this situation. Firstly, school mathematical concepts can be generally demonstrated by the use of physical objects, however probability cannot be modelled with certainty, due to randomness (Gomez-Torres, Batanero, Diaz, & Contreras, 2016).

Secondly, students confuse the three types of probability: frequentist/experimental, classical/theoretical and subjective probability (Batanero, 2014; Truran, 2016). Teachers need to clarify which probability they are using. While experimental probability will tend towards the theoretical probability in the long term, it cannot be relied on with a smaller sample. Subjective probability is still a factor confusing students' learning. For example, even adult conversations include ideas such as lucky numbers, dice, colours and throws.

Thirdly, a number of misconceptions surround the learning of probability (Ang & Shahrill, 2014; Fischbein & Schnarch, 1997). The representativeness misconception assumes a sample will contain the same proportion of outcomes as the population. The gambler's fallacy assumes outcomes are affected by previous outcomes. The conjunction fallacy suggests combined events are more likely than the individual events. Equiprobability bias views random events as equally likely. The frequency of the misconceptions of the effect of sample size and the availability heuristic, which is based on personal experience, both increased in older students, from Grade 5 to college students (Fischbein & Schnarch, 1997). Equally, probability is full of paradoxes and can seem counter-intuitive (Kahneman, 2012) as shown by examples such as the Monty Hall problem, which involves a game show where the outcome seems to improve if you change your mind (Edwards, 2012); the birthday problem, where the chance of a friend with a birthday on the same day seems usually high; and the chance of a baby being a boy seeming to depend on the day of the week the brother is born (Guan, 2011; Taylor & Stacey, 2014).

Finally, the probability curriculum develops slowly in quantity and complexity over curricular Years 1 to 10, however it increases quickly in Years 11 and 12, specifically in Mathematical Methods. Level 1 uses everyday language and events, building to Year 7, with assigning probabilities and single-step equally-likely outcomes, then in Year 10, develops into students working with up to three-step experiments, with and without replacement and conditional statements. Year 11 and 12 Mathematical Methods leaps ahead to include many new abstract concepts: combination, discrete and continuous probability functions, binomial and normal distributions, and the recently introduced topics of statistical inference, sampling and confidence intervals. This increase in content and complexity of probability could be better supported by increasing the content in Years 7 to 10, by integrating probability into other areas of the curriculum. The new complex concepts of sampling and confidence intervals can be introduced in a practical, concrete format before Years 11 and 12. Thus this paper first sets out several ways to connect probability into other topics within mathematics and cross-curriculum areas and then details a measurement example.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

CONNECTING PROBABILITY

The following ideas for connecting and integrating probability into various curriculum areas are drawn primarily from the literature, and our experience in teaching primary, secondary and pre-service teachers.

Students' learning of probability can be hampered by their poor fraction, decimal and percentage skills, but these topics can be used to initiate conversations about chance, and vice versa. For example, when students draw coloured fraction pies as part of the fractions topic, they can then be made into spinners. Spinners are important as they can be designed so the probabilities are not equally likely, which help overcome a common misconception (Callingham, 2000). For example in Figure 1, in the spinner on the left, red is twice as likely as blue or green. The yellow section, in two spinners on the right, can be used to show one third equals two sixths.

Figure 1. Spinners showing fractions and probabilities.

Using probability language accurately by all teachers in any class can increase understanding. For example, "The chance of a visitor tomorrow is very likely" and "It will probably rain tomorrow". Another idea is that as a teacher calls names off icy-pole sticks to answer questions (on any topic), they could also ask, "What is the chance of calling on a student who is absent?"

When constructing 3D shapes in geometry, students could experiment as to the likelihood of rolling the shape onto particular sides. How many tosses would be needed to find the experimental probability? With spinners, dice, cards and marbles the theoretical probability can be calculated and compared to the experimental probability, but with unusual shapes, using experimental probability is the only way. It is important to include shapes and experiments with uneven probability, to address the *equally likely* misconception. Dropping drawing pins to see the chance of the pin landing point-up is another example of uneven experimental probability (Gomez-Torres et al., 2016).

Games to increase fluency like Bingo, and matching cards are engaging and can be further developed to support the understanding of probability. For example, frequency and probability can be used when the students make their own Bingo boards. Nisbet and Williams (2009) use the example of multiplication Bingo, where the students investigate which numbers are likely answers when they design their own Bingo boards. The probability can be highlighted from other games like Greedy Pig, snakes and ladders, or matching cards. Arcade games, both electronic and cardboard (Leavy & Hourigan, 2015), can be used to compare theoretical and experimental probability, using premade games, and the games the students design themselves.

As playing card and board games are being replaced by electronic games as entertainment, using these tried and true probability games as learning activities are more important than ever: dice, cards, spinners, and marbles in a jar. Teachers can use these engaging games to teach interpersonal skills as well as a variety of mathematical concepts. Popular current trends can be used to advantage. Supermarket plastic tokens are examples. Is the probability of collecting each plastic toy the same? How can you tell? How many toys would you need to collect the whole set? What is the cost of the groceries involved? (Attard, 2015).

Newspaper reports on environmental issues were used as engagement prompts by Watson and English (2015) to link current issues with probability and statistics. Students subsequently planned, implemented, and reported on surveys of their peers on environmental issues, and compared them to Census At School data. The follow-up discussion on sample size, survey techniques, and the interpretation of results was important to develop a deeper understanding of the statistics, probability and the cross-curriculum issue.

Gambling is one area of probability that teachers are hesitant to teach, at the risk of encouraging it by providing engaging gambling activities that arguably may result in attraction to gambling and possible addiction. Victorian Responsible Gambling Authority (2016), with the support of MAV, has created a valuable resource for teachers of cross-curriculum units of work for Years 9-12 to support teachers of numeracy, literacy, health and humanities teachers in this important topic. In the numeracy section, guided inquiries use Excel simulations to investigate overall trends rather than wins. Five lessons spanning gambling with cards, pokies, sports betting and horse racing, explain the concepts and terminology behind gambling and then provides simulations with cards, coins and spreadsheets. Betting situations are role-played with and without betting agencies, to discover the long term expected winnings. Structured worksheets are provided for teachers, with the detailed dynamic colour-coded spreadsheets and graphs supplied. Reflection questions conclude each lesson to clarify the key points of the lessons, including the important point that the only long-term winners in sports gambling are the bookmakers.

The use of technology and coding can be linked to probability. Lukac and Engel (2010) describe how an Excel spreadsheet can be used to simulate dice rolls, and combined dice rolls, where the data is collected and displayed in many ways. The Excel instructions are included in this article (Lukac & Engel, 2010). Electronic games can be used for probability, as Mills (2016) demonstrated with Minecraft. A wolf in the Minecraft computer game has a probability of 1/3 of being tamed with a bone, and the students asked how many bones would be needed for various tasks.

Having briefly set out several ideas, we now turn to a more detailed discussion of an example of teaching probability using measurement.

CONNECTING PROBABILITY TO MEASUREMENT

Whenever we measure anything, we can incorporate probability. Watson and Wright (2008) use the measurement of a student's arm span to introduce variation. The length of a student's arm span is difficult to measure, partly because bodies and arms move, influencing the accuracy of measurements. The accuracy and confidence in measurements were discussed and then graphed using the computer software *TinkerPlots* (Konold & Miller, 2011). This computer software is flexible, intuitive, and easy to use and interpret. Several graphing activities included: the variation of arm span of one student, of a whole class, between class comparisons, and finally a comparison of arm span and heights using line regression. This idea could have used CAS calculators instead of *TinkerPlots*. Variation around measurements will now be explored without the use of technology.

PACE LENGTH AND FITBITS

Fitbits have been a trend in our schools, with students involved in setting step targets for their health programs. Fitbits can be calibrated to convert the steps taken to meters or kilometres travelled, but how far is a pace? Measuring one pace is difficult. Students take bigger and smaller steps depending on the surrounds and how fast they are moving. After discussion, several measurements of pace length of one student were taken under different conditions, so a discussion on accuracy and confidence levels would occur. The ten measurements were displayed on a Post-it note graph (Figure 2). Students made statements like, "The probability of a step being bigger than 80 cm is 0.2". Using mathematical symbols: Pr (Pace of Ashley > 80) = 0.2. The most appropriate average was discussed, mean, mode, or median. The Fitbit could now be calibrated, and the measurement or Health/PE activity resumed.

The average pace lengths of a class of students were compared and graphed. The students' names on Post-it notes constructed a physical, personal graph (Figure 2). Statistics and probabilistic language were both used. Students described "six out of twenty students had a pace less than 80". The teacher wrote: The probability of a random student's pace length of less than 80 cm is 6/20 = 3/10 = 0.3. Pr (Pace length from the class ≤ 80) = 0.3.

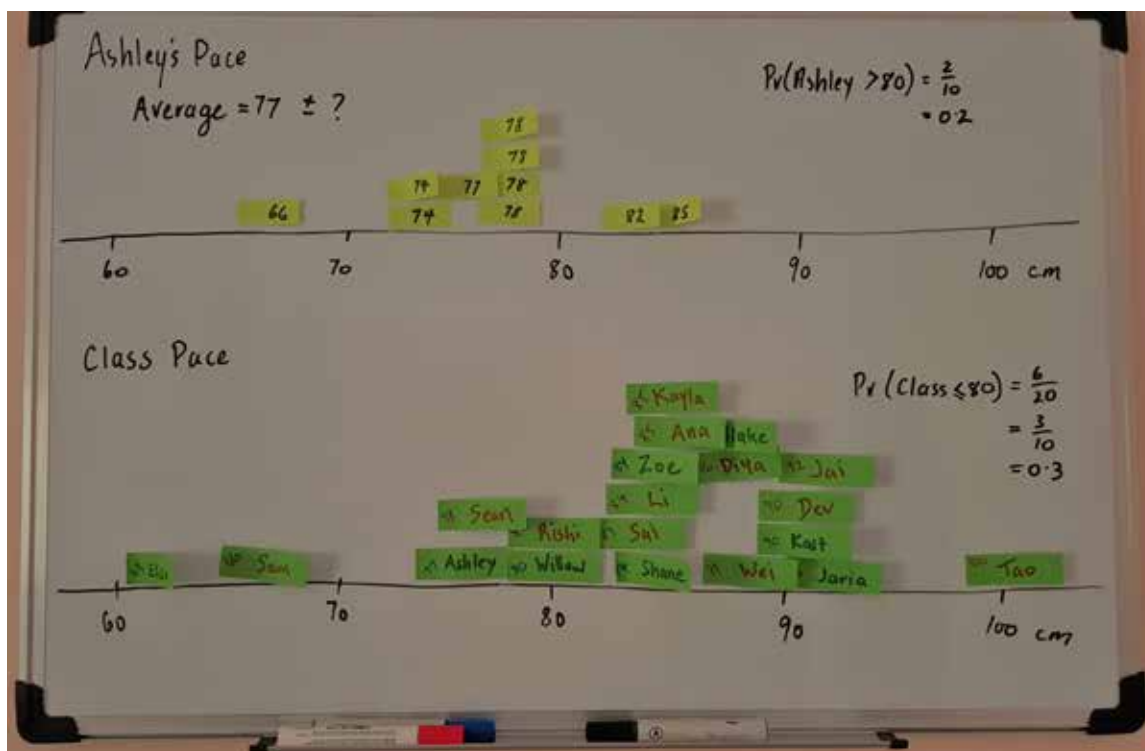THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Figure 2. Post-it note graphs, to contrast variation of a student compared to the variation of a class.

These examples of variation of one student's pace length over several trials, compared to the variation of a class of students, helped informally explain the concepts of variation, sampling and confidence intervals needed for Years 11 and 12 Mathematical Methods. Post-it notes as data points had the advantage of being able to be moved during the difficult task of selecting scales on the axis. Students initially see the graphs as bar graphs, then histograms, and with guidance can see probability distributions, where the area under the graph represents the probabilities.

CONCLUSION

There are many engaging learning activities in the literature incorporating probability, including how to avoid misconceptions. Teachers need not fear this important topic. Probability can be integrated into many areas of the mathematics curriculum and also incorporated into cross-curriculum areas. Activities both with and without technology, teacher or student-designed, can support learning in mathematics, reasoning and interpersonal skills. Consistent use of accurate language by all teachers will support the understanding of probability and it is important to include the concepts of chance and probability into incidental classroom activities throughout all levels of schooling. Mathematics teachers can include the Year 11 and 12 concepts of probability in a concrete but informal way, to support the students' future understanding.

ACKNOWLEDGEMENTS

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

REFERENCES

Ang, L. H., & Shahrill, M. (2014). Identifying students' specific misconceptions in learning probability. *International Journal of Probability and Statistics*, *3*(2), 23-29.

Attard, C. (2015). *Woolworths and dominoes: How parents can turn marketing ploys into mathematical learning opportunities.* Retrieved from https://engagingmaths.com/2015/03/22/woolworths-and-dominoes-how-parents-can-turn-marketing-ploys-into-mathematical-learning-opportunities/

Batanero, C. (2014). Probability teaching and learning. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 491-496). Dordrecht, The Netherlands: Springer.

Callingham, R. (2000). Come in spinner! *Australian Primary Mathematics Classroom*, *5*(1), 9-14.

Edwards, G. (2012). Snippet: The Monty Hall problem. *Vinculum*, *49*(2), 20.

Fischbein, E., & Schnarch, D. (1997). The evolution with age of probabilistic, intuitively based misconceptions. *Journal for Research in Mathematics Education*, *28*(1), 96-105.

Gomez-Torres, E., Batanero, C., Diaz, C., & Contreras, J. M. (2016). Developing a questionnaire to assess the probability content knowledge of prospective primary school teachers. *Statistics Education Research Journal*, *15*(2), 197-215.

Guan, T. E. (2011). Think of a number: Adapting the birthday problem for the classroom. *Australian Mathematics Teacher*, *67*(4), 22-25.

Kahneman, D. (2012). *Thinking, fast and slow*. London, England: Penguin.

Konold, C., & Miller, C. (2011). *TinkerPlots: Dynamic data exploration (Version 2.0)* [Computer software]. Emeryville, CA: Key Curriculum Press. Retrieved from http://www.srri.umass.edu/tinkerplots/

Leavy, A., & Hourigan, M. J. T. S. (2015). Motivating inquiry in statistics and probability in the primary classroom. *Teaching Statistics*, *37*(2), 41-47.

Lukac, S., & Engel, R. (2010). Investigation of probability distributions using dice rolling simulation. *Australian Mathematics Teacher*, *66*(2), 30-35.

Mills, P. (2016). Minecraft and mathematics. *Vinculum*, *53*(1), 15.

Nisbet, S., & Williams, A. (2009). Improving students' attitudes to chance with games and activities. *Australian Mathematics Teacher*, *65*(3), 25-37.

Taylor, W., & Stacey, K. (2014). Gardner's two children problems and variations: Puzzles with conditional probability and sample spaces. *Australian Mathematics Teacher*, *70*(2), 13-19.

Truran, J. (2016). What is the probability of...? *Australian Mathematics Teacher*, *72*(3), 59-60.

Victorian Responsible Gambling Foundation. (2016). *Resources for teachers*. Retrieved from https://responsiblegambling.vic.gov.au/reducing-harm/schools/resources-teachers/

Watson, J., & English, L. D. (2015). Expectation and variation with a virtual die. *Australian Mathematics Teacher*, *71*(3), 3-9.

Watson, J., & Wright, S. (2008). Building informal inference with TinkerPlots in a measurement context. *Australian Mathematics Teacher*, *64*(4), 31-40.

THE MATHEMATICAL ASSOCIATION OF VICTORIA

The use of manipulatives in high school

Vicky Kennard, Australian Mathematical Sciences Institute

In my role as an Outreach Officer for the Australian Mathematical Sciences Institute (AMSI), I have been privileged to have worked in many classrooms across Australia, in both primary and secondary schools. I have also taught in the UK. My experiences have shown me that although many classrooms and schools are well stocked with interesting mathematical equipment of various types, they are rarely used outside of the early primary years. In this paper and presentation, I want to explore why this is so and what could we be doing in high school to make mathematics more accessible and engaging for students.

If you were to walk into a Prep classroom during a mathematics or numeracy lesson, you would probably see counters, MABs, Cuisenaire rods, etc. being used. As we move up the year levels, the number of manipulatives (or concrete materials) being employed in classrooms seems to decrease. There may still be dice or cards when we are doing probability but that is about all. There seems to be a general perception that manipulatives are for younger, or weaker older, students. It is rare to see them being used consistently in a secondary classroom. Uribe-Flórez and Wilkins (2010) found that there was a significant difference in usage depending on grade level, with very little use in higher primary grades and almost no use in high school.

Van de Walle, Karp, and Bay-Williams (2013) define manipulatives as "any object, picture, or drawing that represents the concept or onto which the relationship for that concept can be imposed. Manipulatives are physical objects that students and teachers can use to illustrate and discover mathematical concepts, whether made specifically for mathematics (e.g. connecting cubes) or for other purposes (e.g. buttons)" (p. 24). According to this definition, a manipulative is any object or representation that can help develop a concept or relationship. This is a very wide definition and will include pen and paper. When we use the term, in relation to teaching, we tend to picture equipment made specifically for the classroom or other physical objects. Today we are seeing the introduction of virtual manipulatives, which are representations of the physical objects made for computers and tablets. These can be 'manipulated' with a mouse or by touch. If we accept this wide definition, then we can see manipulatives being used throughout the curriculum, though it may only be a calculator, paper and pen, or the occasional use of dice in the higher grades.

Piaget, in his theory of constructivism, suggests that students construct knowledge and meaning from their experiences (Harlow, Cummings, & Aberasturi, 2006). Based on this pedagogical theory, the use of manipulatives in the classroom is thought to provide students with a physical experience of a concept from which they can then make meaning. Constructivist theory is also the basis of the Concrete, Representational, and Abstract (CRA) model (sometimes called the CPA – Concrete, Pictorial, and Abstract model). When using this approach manipulatives (concrete objects) are used to help students to visualise and acquire a physical experience of a concept. Then, as their knowledge and familiarity with the concept increases, they can move away from the physical object to making a representation, normally a picture, of the objects. These drawings/diagrams will become more sophisticated and abstract as the knowledge deepens and then students can be introduced (or will move naturally) to abstract or symbolic representations.

Using algebra tiles as an example: Using the tiles themselves is the concrete stage. The students spend time familiarising themselves with different tiles and how to arrange them to solve problems. They can then move on to a representational stage. This can be a diagram, photograph, or a more abstract drawing:
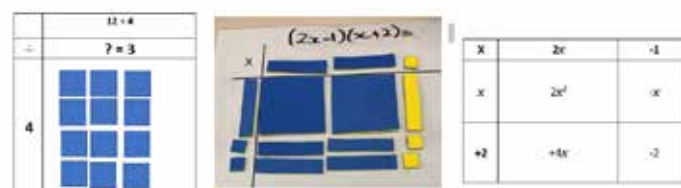

Figure 1. Example of algebra tiles.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Not all students will move along this continuum at the same pace and it should not be seen as a linear progression. Some students may enter at the representational, or even abstract, phase and they may move back to the representational or concrete phase at different points. This model of instruction would be familiar to (most) lower-primary teachers and to teachers of special needs students but may not be so familiar to others.

With problem-solving and reasoning being encouraged in the curriculum we need to give our students more tools for exploring ideas and ways to express their thinking. It is my contention that by removing manipulatives at an early stage in schooling and, therefore, restricting the tools we make available to students in the classroom, we are limiting the ways they can explore and communicate. By associating manipulatives with the younger years, we are also branding them as 'babyish' and as toys to be played with. We encourage students to move to the more abstract modes of thinking and visualising problems before they are ready. Similarly, counting using fingers, which neuroscience has now shown is so important for developing basic numeracy, is discouraged at an early age.

The research on the use of manipulatives, especially in older grades, does not seem to be conclusive as to their efficacy. There are many variables associated with their use and so it is difficult to compare and aggregate findings. Many teachers will say, anecdotally, that the use of a manipulative does improve the understanding of concepts especially when a concept is first introduced. But this is not always corroborated by the research. My contention is that the reason for this inconclusive research finding is that teachers are not using manipulatives with older years in the most effective way.

As with all tools they can be used well or not so well. From the research, here are some general tips for using manipulatives:

• When a new manipulative is introduced it is important to allow the students time to explore it for themselves – to play! This stage is important because you do not want the students to just copy what you the teacher, do. This would be the equivalent of just teaching the algorithm with no understanding. The aim of using the manipulative to develop understanding. (Van de Walle et al., 2014)

• When the teacher uses the manipulative to teach a concept, the teacher needs to make the mathematics very explicit. The same manipulative may be used to explore different concepts, at various times, so the teacher needs to be very clear on what the purpose for that lesson is.

• The manipulatives should be easily and freely accessible. Often, they are stored in cupboards or resource rooms. The teacher will bring out a specific manipulative for a particular activity and then they will disappear. By having them readily available, a culture can develop where the students use them as, and when, they feel the manipulatives will be of use. Students may also use them in very innovative ways. Students should feel comfortable in using physical objects to help them explore and communicate ideas.

• Virtual manipulatives should be introduced after students have had a chance to 'play' with the physical manipulative equivalent. The physical nature of a manipulative is important and so virtual equivalents, although useful, do not have quite the same initial effect.

• If students have been using manipulatives in class and have not yet moved confidently to a representational or abstract mode, then allow the students to use the manipulatives during tests. Remember the calculator is a manipulative too!

Using a manipulative can also help to engage students who find mathematics too abstract and struggle to make sense of the concepts and symbols used. They add a sense of excitement and change to the normal classroom routine. However, it is important that they are not viewed just as toys with no serious purpose. In a study (Moyer, 2001), teachers perceived using manipulatives as 'fun' and used them as a reward, rather than 'real' mathematics. For a manipulative to be effective, it should be perceived as a serious part of the learning process.

Teacher mindset and experience of using manipulatives is another important factor in the success or otherwise of manipulative use (Moyer & Jones, 2004). Encouraging students to use manipulatives to explain their thinking is a good formative assessment technique that can uncover misconceptions and knowledge gaps. There are some disadvantages to using manipulatives. These include classroom management issues such as the time taken to distribute and collect them,

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

storage and availability of class sets. Some of these can be overcome with familiarity and classroom routine.

It is important that the teacher scaffolds the learning, as "teachers need to bridge the manipulatives to the representational and then abstract understanding in mathematics so that students internalize their understanding" (Furner & Worrell, 2017, p. 21). The teacher cannot expect that all students will make the transition from the concrete to the abstract without assistance and guidance.

Using manipulatives is not a guarantee that the students will 'see' the concept. There needs to be guidance and explicit connections made by the teacher and they must be thoughtfully used and introduced (Baroody, 1989). In the high school setting particularly, not all students will feel the need or feel comfortable with using a concrete, physical object in mathematics. This is especially true when students have already developed the mindset that they are 'babyish'. Students may also be able to move straight to the representational, or even the abstract phase. Teachers need to demonstrate, and believe, that using manipulates can help develop understanding in some topics and support students in constructing understanding. Studies have shown (Suydam & Higgins, 1977) that it is the way that manipulatives are used in the classroom and the attitude of the teachers involved that make the most difference to their effectiveness.

With a greater emphasis on STEM and problem solving in the curriculum being able to model, represent and explain ideas are skills that are becoming more important. Using manipulatives can be useful in developing these 21st century skills.

In conclusion, I believe that there is a place for the use of manipulatives across the mathematics curriculum and across all year levels and abilities. They are a powerful way for students to develop a deeper understanding of concepts and are engaging for students who find mathematics difficult and distant from their life experiences. We as teachers need to educate ourselves on how to use them in the classroom and how to manage their use by students.

Below is a list of some common manipulatives and how they could be used. I will be exploring some ways to use these manipulatives in a high school context in my session at the conference.

- Blocks: There are many different types of blocks available from simple, cube building blocks to snap together cubes. These are normally multi-coloured and approximately 2 cm cubes. These have so many uses but generally are used for counting, addition and subtraction, measurement, area, and patterning.

- Cuisenaire rods: These are rods of different lengths and colours. They can be used for counting, addition and subtraction, multiplication and division, fractions, ratio and patterning.

- Simple abacus: There are two rows of ten beads, five of each of red and white. They can be used for counting, modelling numbers in different ways, number facts to 10 or 20, one more than, one less than, doubles, near doubles etc.

- Pattern blocks: These are six two-dimensional shapes in different colours. They can be used for exploring shape properties, symmetry, transformations, and fractions.

- Geometric solids: These are three-dimensional models that can be investigated, measured, and sorted.

- Counters: These range from simple, one colour, round counters to teddy bears and frogs. They can be used for counting, addition and subtraction, multiplication and division, fractions, ratio and patterning.

- Two-colour counters: These have a different colour on each side of the counter. They can be used as normal counters but also for introducing directed numbers and probability.

- Clocks: These are many different types of clocks available. Using a clock with only the hour hand is a good way to introduce analogue time.

- Sorting circles: These are circles that can interlock to form a Venn diagram. They can be used to explore sorting by different attributes and probability.

- Geoboards: These are boards with pegs that can be used with rubber bands to explore shape, area, perimeter, transformations, symmetry, and congruency.

- Mirrors: These can be used to explore transformations, symmetry, and congruency.

- Rulers: These come in many different lengths and units. They can be used to explore measurement and unit conversion.

- Dice: There are many different dice available. These can be used to explore probability and data generation.

- Balance: There are different styles available, either hanging along the length or with a bucket at the end. These can be used for exploring mass and balancing equations/number sentences.

- AngLegs: These are plastic, snap together rods of different lengths and colours, with a snap-on protractor. They can be used for polygons, perimeter, area, angle measurement, side lengths, etc.

- Algebra tiles: These are two coloured tiles of different shapes and sizes: a small square, a rectangle, and a large square. They can be used for counting, addition, subtraction, multiplication, division, fractions, ratio, operations with directed numbers, algebraic equations, operations with linear and quadratic expressions, and linear and quadratic equations.

There are many more manipulatives that I have not included in this list, such as cards and number lines, that can be simply made. I have also not included the many games that have a strong mathematical component.

REFERENCES

Baroody, A. J. (1989). Manipulatives don't come with guarantees. *The Arithmetic Teacher*, *37*(2), 4-5.

Furner, J. M., & Worrell, N. L. (2017). The importance of using manipulatives in teaching math today. *Transformations*, *3*(1), 1-25.

Harlow, S., Cummings, R., & Aberasturi, S. M. (2006). Karl Popper and Jean Piaget: A rationale for constructivism. *Educational Forum*, *71*(1), 41–48.

Moyer, P. S. (2001). Are we having fun yet? How teachers use manipulatives to teach mathematics. *Educational Studies in Mathematics*, *47*(2), 175-197.

Moyer, P. S., & Jones, M. G. (2004). Controlling choice: Teachers, students, and manipulatives in mathematics classrooms. *School Science and Mathematics*, *104*(1), 16-31.

Suydam, M. N., & Higgins, J. L. (1977). *Activity-based learning in elementary school mathematics: Recommendations from research*. Columbus, OH: Education Resources Information Center Clearinghouse for Science, Mathematics, and Environmental Education.

Uribe Flórez, L. J., & Wilkins, J. L. (2010). Elementary school teachers' manipulative use. *School Science and Mathematics*, *110*(7), 363-371.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2013). *Elementary and middle school mathematics: Teaching developmentally* (8th ed.). Upper Saddle River, NJ: Pearson.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

From additive to multiplicative thinking: Key ideas for teaching

Mayamiko Malola and Max Stephens, The University of Melbourne

Multiplicative thinking is a central pillar of mathematical learning in the primary and middle years. It forms the basis for understanding numerous topics such as proportions, patterns, ratios, fractions, and percentages. The development of algebraic thinking also relies on students having developed sound multiplicative thinking. However, the transition from additive thinking to multiplicative thinking is a challenge for many learners in the middle years, and is not a simple one-step process. The focus of this paper is to enhance teachers' pedagogical content knowledge that supports students' transition from additive to multiplicative thinking. It uses three perspectives of multiplicative thinking, which are isomorphism of measures, measure of units, and multiplicative actions independent of addition, to suggest four steps for teaching that progressively move students from additive to multiplicative thinking.

THE CONCEPT OF MULTIPLICATIVE THINKING

Multiplicative thinking represents learners' mental adaptive processing of multiplicative concepts by using different methods and approaches in various mathematical problem contexts (Singh, 2012). Multiplicative thinking allows learners to successfully grapple with mathematical problems across different topics that require understanding and application of multiplicative ideas. Researchers such as Brown and Quinn (2006) suggest that success in topics such as proportions, patterns, fractions, measurement, rates, percentage, as well as the development of algebraic and statistical thinking and understanding some complex issues in the society rely on multiplicative thinking. Studies by Siemon, Breed, Dole, Izard, and Virgona (2006) and Singh (2012) reveal that many conceptual stumbling blocks that learners encounter in the elementary and even middle school curriculum are related to poor understanding of multiplicative thinking. Askew et al. (2019) emphasise that multiplicative thinking is a critical area within primary mathematics and that strengthening attainment in this aspect of mathematics would create a more secure foundation for subsequent mathematics.

PERSPECTIVES OF MULTIPLICATIVE THINKING

Singh (2012) has presented three perspectives of multiplicative thinking and the cognitive structures necessary to understand it. These perspectives are isomorphism of measures, the measure of units, and multiplicative actions independent of additive ideas. They suggest the different ways in which children acquire and develop the knowledge of multiplicative thinking. Figure 1 shows five questions from the Butterfly House task (Scaffolding Numeracy in the Middle Years [SNMY], Siemon et al., 2006) that will be used to examine each of these perspectives and how teachers can use this task to progressively move learners from additive to multiplicative thinking.
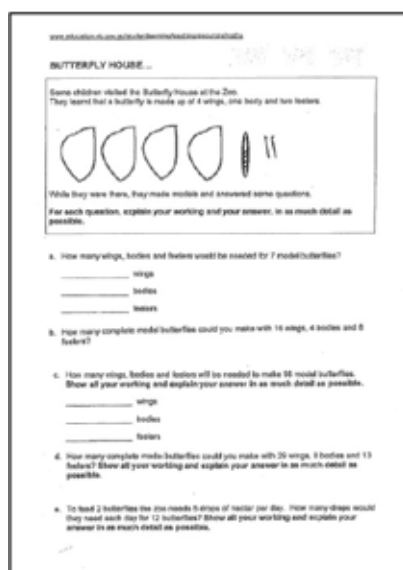

Figure 1. Butterfly house task (SNMY, 2006).

THE MATHEMATICAL ASSOCIATION OF VICTORIA

ISOMORPHISM OF MEASURES

Isomorphism directs our attention to a relationship that remains the same between different units irrespective of the size of units. This perspective is represented by a structure consisting of the relationship between two measure units, for example, distance and time, and items purchased and their cost (Singh, 2012), or relationship between minutes and hours. It describes large number of situations in ordinary and technical life (Vergnaud, 1983). In addition, it involves proportional relationship between the two units involved. However, Singh (2012) maintains that this perspective can be applied and represented in different problem contexts at different levels of abstraction. With reference to the Butterfly House task, this perspective would address the relationship among the number of bodies, wings and feelers required to make a specified number of butterfly models. Given that one butterfly model requires one body, two feelers, and four wings, the information is represented in Table 1 below, and shows how students can use it find the number of wings, feelers, and bodies required to make any number of butterfly models.

Model(s)	Body(ies)	Feelers	Wings
1	1	2	4
2	2	4	8
3	3	6	12
4	4	8	16
5	5	10	20
6	6	12	24
7	7	14	28

Table 1. Using proportional relationship to solve multiplication problems.

Table 1 uses Question 1a where students are told that each model butterfly requires one body, two feelers and four wings and asked to find the number of bodies, feelers and wings required to make seven models. They use the proportional relationship with the number of bodies, feelers and wings increasing at a constant rate with a constant increase in the number of models. Seven models will require 7 bodies, 14 feelers, and 28 wings. The seven-step approach, shown in Table 1, may be the starting point for many students. However, the isomorphism diagram (see Table 2) proposed by Vergnaud (1983) can be used to solve problems in a more effective and time-efficient manner.

1 model	1 body	2 feelers	4 wings
x 7	x 7	x 7	x 7
7 models	?	?	?

Table 2. Showing the application of proportional relationship.

Table 2, which refers to Question 1a, shows that instead of having to write the number of models from 1 to 7, students can find the relationship between 1 and 7. They find the relationship by establishing the arithmetic operation that has been used to get a 7 using 1. Students multiply the 1 by 7 to get 7. The 7 becomes the multiplier for the number of bodies, feelers and wings required to form one model butterfly in order to find the number of bodies, feelers and wings required to form 7 model butterflies.

In some cases, understanding the relationship between the initial unit and the number of units provided in the question is important for effective problem solving in relation to multiplicative thinking. For instance, let's look at question (1d) where students are provided with different numbers of wings, bodies, and feelers. How many butterfly models can they make?

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

1 model	1 body	2 feelers	4 wings
X?	X8	X6 = 12	X7 = 28
? models possible	8 bodies available	13 feelers available	29 wings available

Table 3. Finding number of models given number of bodies, feelers, and wings.

Table 3 presents question 1d where students are given 8 bodies, 12 feelers, and 29 wings, and required to find the number of models that can be formed. Looking at the multiplying factors we see that with 29 wings we can make 7 models, 13 feelers can make 6 models and 8 bodies can make 8 models. The smallest multiplying factor should inform us the number of complete butterfly models we can produce given 8 bodies, 13 feelers, and 29 wings. In this case, the smallest multiplying factor of all units is 6. Students can make 6 a maximum of butterfly models since the number of feelers in this case determines the maximum number of models we can make.

THE MEASURE OF UNITS

This perspective involves measure of units where students build on prior knowledge of multiplication and model the situations based on their schema (Singh, 2012). He suggests three levels of competency that students go through progressively from pre-multiplying scheme (pure repeated addition) to more abstract multiplication. These schemes are: (1) pre-multiplying scheme, (2) iterative multiplication scheme, and (3) scalar functional operator. In our view, these schemes are important milestones in the transitioning of learners from additive to multiplicative thinking and are described below.

Pre-multiplying scheme

This procedure uses additive strategy that is based on establishing the relationship within the given units and extending it to find unknown units or the units in question (Singh, 2012).

Body	Feelers	Wings
1 model - 1 body	1 model -2 feelers	1 model - 4 wings
1 model - 1 body	1 model - 2 feelers	1 model - 4 wings
1 model - 1 body	1 model - 2 feelers	1 model - 4 wings
1 model - 1 body	1 model - 2 feelers	1 model - 4 wings
1 model - 1 body	1 model - 2 feelers	1 model - 4 wings
1 model - 1 body	1 model -2 feelers	1 model - 4 wings
1 model - 1 body	1 model - 2 feelers	1 model - 4 wings
7 model - 7 bodies	7 models - 14 feelers	7 models - 28 wings

Table 4. Pre-multiplying scheme using butterfly task question 1(a).

Table 4 refers to Question 1a where students specify the number of bodies, feelers and wings required to make 7 models. Students write down the requirements for one model 7 times, each with one body, two feelers, and four wings. The total number of bodies, feelers, and wings is obtained by adding up the components.

Body	Feelers	Wings
1 body - 1 model	2 feelers - 1 model	4 wings - 1 model
1 body - 1 model	2 feelers - 1 model	4 wings - 1 model
1 body - 1 model	2 feelers - 1 model	4 wings - 1 model
1 body - 1 model	2 feelers - 1 model	4 wings - 1 model
4 bodies - 4 models	8 feelers - 4 models	16 wings - 4 models

Table 5. Pre-multiplying scheme using butterfly task question (1b).

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

In Table 5, which refers to Question 1b, students were given 4 bodies, 8 feelers, and 16 wings and asked how many models can be formed. Table 5 shows that students repeatedly write down the requirements (bodies, feelers, and wings) to form one model Butterfly until all the bodies, feelers, and wings provided in the question are used to form complete four models. While this strategy might not be very efficient, it is a good starting point for learners in lower primary school and prepares them for a smooth move to iterative multiplication.

Iterative multiplication

This scheme involves repeated distribution of a quantity over another (Singh, 2012).

Body	Feelers	Wings
1 model - 1 body	1 model - 2 feelers	1 model - 4 wings
2 models - 2 bodies	2 models - 4 feelers	2 models - 8 wings
3 models - 3 bodies	3 models - 6 feelers	3 models - 12 wings
4 models - 4 bodies	4 models - 8 feelers	4 models - 16 wings
5 models - 5 bodies	5 models - 10 feelers	5 models - 20 wings
6 models - 6 bodies	6 models - 12 feelers	6 models - 24 wings
7 models - 7 bodies	7 models - 14 feelers	7 models - 28 wings

Table 6. Iterative scheme using question (1e).

Unlike the pre-multiplying scheme in Table 5, each row in Table 6 contains the complete specification of bodies, feelers, and wings for any given number of models. This scheme allows students for example to specify the number of bodies, feelers, and wings required for 14 models (by doubling the last row) or by 70 models by multiplying the last row by 10. This is a big advantage over previous pre-multiplying scheme and a bridge to the third scheme.

Scalar functional operator

The scalar functional operator (Askew, 2018) involves proportionality that emanates from repeated addition. The scheme fosters students' ability to move from fundamental iterative scheme that we discussed above to a more abstract level of understanding in multiplicative thinking (Singh, 2012). The relationship between the given units and the units in question was explored and used as the basis to solve the problem. Using the butterfly task question (1e), instead of repeatedly adding 2 butterflies and 5 drops each time respectively to obtain the number of drops for 12 butterflies, the scalar functional operator allows such a problem to be approached as follows:

2 butterflies need	5 drops of nectar
X6	X6
12 butterflies need	?

Table 7. Scalar function.

The approach shown in Table 7 requires students to be able to abstract the result of many iterations to be captured by a known multiplication fact (Singh, 2012). The capacity to compress repeated additions into a single multiplication is instrumental in effective and logical transition from additive to multiplicative thinking. Development and understanding of multiplication facts and their application in solving diverse problems are also critical. With reference to question 1e, figuring out the relationship between the 2 butterflies initially provided in the question and the 12 butterflies in question is an important step. Having to add $2 + 2 + 2 + 2 + 2 + 2 + 2$ to get the 12 signifies heavy reliance on repeated addition. However, knowing that one can multiply the 2 by 6 (2 x 6) to get the 12, demonstrates a clear move towards multiplicative thinking. Since we have increased the number of butterflies six times, the corresponding number of drops of nectar should also be increased six times (i.e., 5 x 6). However, students will need to have a developed understanding of multiplication facts in order to find answers to 2 x 6 and 5 x 6. This is a remarkable step in the transition process as it builds on learners' prior knowledge of additive thinking.

MULTIPLICATIVE ACTIONS INDEPENDENT OF ADDITIVE IDEAS

This third perspective involves a higher level of abstraction in solving multiplicative problems without drawing on additive thinking. Actions and images are analysed using strategies such as partitioning and splitting to aid the solution (Singh, 2012). We view this perspective as the core purpose of all efforts on multiplicative thinking. Students need to effectively use multiplication actions without relying on additive ideas. Vergnaud (1983) referred to such students as multiplicative thinkers. The following solution strategies illustrate this level of multiplicative thinking.

98 model butterflies		
Bodies	Feelers	Wings
98 x 1	98 x 2	98 x 4

Table 8. Multiplication independent of addition.

With 98 model butterflies and required to find the respective number of bodies, feelers, and wings required. Table 8 shows how students can directly multiply the number of models (98) by the number of bodies, feelers, and wings respectively required for each model butterfly.

SUMMARY AND IMPLICATIONS FOR TEACHING

In this summary section, we present four steps that teachers can follow in order to progressively move learners from additive to multiplicative thinking.

PRE-MULTIPLYING

Multiplicative thinking in children relies on additive thinking. Additive thinking moves from a simple count all to additive strategies that involve combining and skip counting. These refinements of additive strategies need to be carefully cultivated if students can transition more confidently to multiplicative procedures. In their instruction, teachers need to be aware that students' informal knowledge of multiplication has basis on the idea of repeated addition. We suggest that teachers in Grades 1 and 2 should start with counting strategies before delving on addition. Teachers in latter grades should focus on activating students' informal knowledge of addition with emphasis on repeated addition. This will lay a good foundation for the second step.

ITERATION

This second step in the teaching of multiplicative thinking requires teaching activities to emphasize the development of the idea of groups of equal size, number of groups, and the total quantity. The teacher may involve learners in varied activities including the use of concrete materials, group work, and carefully selected mathematics games in order to facilitate students' conceptualisation of the idea of groups. For example, using the nursery rhyme "Baa Baa Black Sheep", students can draw a picture of 3 bags of wool for every sheep and ask how many bags of wool there will be for 5 black sheep. This problem can be investigated using a variety of iterative methods starting with count all, but all involving grouping and skip counting.

PROPORTIONAL RELATIONSHIPS

Teaching and learning activities should focus on establishing and developing understanding of proportional relationships between and among quantities as the sizes, numbers of groups and total quantity changes. For example, how many bags of wool would there be for 20 black sheep? The goal is to take learners to the final step where they use multiplication ideas in multiplicative situations without having to use additive thinking.

MULTIPLICATION INDEPENDENT OF ADDITIVE THINKING

Teachers should aim to take learners to a point where they can deal with multiplicative situations without having to think additively. Viewing multiplication as scaling rather than adding is a key aspect of thinking that need to be developed. Teaching activities in the classroom should focus on exploring all possible strategies that can be used to solve

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

multiplication problems without the use of additive strategies. For example, older students can consider the number of bags of wool for 100, 500, and other numbers of sheep. In this way, students will become competent multiplicative thinkers.

In suggesting these four steps, it is helpful to take note of an "overlapping waves" theory that suggests that mathematical solution strategies do not replace one another in a linear fashion with less sophisticated strategies disappearing as new strategies are acquired. Instead, they wax and wane emerging from co-existing patterns in individuals. The steps suggested in this article may not represent the only way of fostering the transition from additive to multiplicative thinking, but they do suggest that young students should be introduced quite early to situations that foster multiplicative thinking even when students are still using additive procedures. Supporting students' transition from additive to multiplicative thinking is a vital task for the teaching of mathematics in the early and middle years. Building teachers' capacity in this area is fundamental to children's continuing mathematical development.

REFERENCES

Askew, M. (2018). Multiplicative reasoning: Teaching primary pupils in ways that focus on functional relations. *The Curriculum Journal*, *29*(3), 406-423. doi:10.1080/09585176.2018.1433545

Askew, M., Venkat, H., Mathews, C., Ramsingh, V., Takane, T., & Roberts, N. (2019). An intervention's impact on foundation phase learners' understanding. *South African Journal of Childhood Education*, *9*(1), 1-10. doi:10.4102/sajce.v9i1.622

Brown, G., & Quinn, R. (2006). Algebra students' difficulty with fractions: An error analysis. *Australian Mathematics Teacher*, *62*(4), 28-40.

Siemon, D., Breed, M., Dole, S., Izard, J., & Virgona, J. (2006). *Scaffolding numeracy in the middle years – Project findings, materials, resources. Final report submitted to Victorian Department of Education and Training and the Tasmanian Department of Education*. Retrieved from https://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/assessment/Pages/scaffoldnum.aspx

Singh, P. (2012). Multiplicative thinking and learning. In N. M. Seel (Ed.), *Encyclopedia of the sciences of learning* (pp. 2,389-2,392). doi:10.1007/978-1-4419-1428-6_531

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landa (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127-174). New York, NY: Academic Press.

Chance, data, and the gambling issue

Robert Money, Retired secondary teacher, and Ian Lowe, Educational consultant

Simulations and data sources related to gambling can be used to cover all the Year 9/10 probability and statistics curriculum. Students learn that random numbers are unpredictable and that the longer they 'play the pokies' the less likely they are of 'breaking even'. Real data sources on gambling losses – locally and beyond – provide opportunities for meaningful data analysis, representation and interpretation. Mathematics teachers can do their bit in helping students develop healthy attitudes toward gambling.

INTRODUCTION

Australia leads the world for gambling losses - $1,251 per adult in 2016-17 (Queensland Government Statistician's Office, 2018). This is 75% more than for second-placed Singapore and more than double that for any other country (The Economist, 2017). Our adolescent generation is the first to be exposed to saturation marketing of gambling, particularly online betting - and mobile apps mean gambling is more accessible than ever before. On the Victorian Responsible Gambling Foundation (VGRF, 2019) website, Miller reports that "one in five adolescents play social casino games" and "one in ten young people gamble online".

Gambling and mathematics education in Victoria first came together in Lovitt and Clarke (1988) *The Mathematics Curriculum and Teaching Program* where, in the rationale for engagement with social issues the authors argued, "if mathematics is to prepare pupils for the real world, then part of its presentation should address issues from the real world" (p. 89). The education authorities came on board when copies of Tout, Santburn, Kindler, Bini, and Money's (2011) *Consumer Stuff! Maths: A Resource for Teaching and Learning Numeracy*, with a section on gambling, were distributed to all government secondary schools. Following this initiative, the Mathematical Association of Victoria has been successfully funded to trial curriculum resources dealing with the gambling issue in particular because:

- The mathematical aspects of gambling fit closely with the mathematics curriculum, so do not need to be seen as an additional burden on the crowded curriculum.

- Mathematics is studied by most of each year level cohort, with consequent opportunities and responsibilities for mathematics teachers.

A VCAL version of these resources is now available on the VRGF website (2019). The Year 9/10 version includes units focussing on electronic gaming machines ('pokies') and sports gambling (Money, Lowe, & Smith, 2015). Teachers' notes and worksheets are available but this paper concentrates on a selection of the key spreadsheets that are central to the learning activities involved.

LESSONS USING THE SPREADSHEETS

The spreadsheets described below are recommended for use in lessons that include introductory hands-on activities and worksheets containing related sample data. Class discussion can follow the predict-observe-explain framework and, where appropriate, can diverge into a discussion of student attitudes to gambling. Key messages about gambling (particularly on the pokies and on-line gaming) are:

- 'Chance has no memory' – independent events, random numbers

- 'The longer you bet, the less chance there is of breaking even'

- Australians lose too much on gambling. There are other ways of spending money.

BETTING ON THE CARDS

Four students, representing hearts, diamonds, spades, and clubs, bet $1 each on the suit of the next card. If the payout to the winner is $3, rather than the expected $4, then the students have their first experience of the hard truth of commercial

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

gambling, namely that the long term expected return to the gambler is less than 100%. The key equation is:

Expectation = Price x Payout = 75% in this instance

Students enter data in the three yellow cells in the 'Choose the suit' (sampling without replacement) spreadsheet (Figure 1) and commence rapid simulation. Discussion of the results can teach students that:

- Relative frequencies are variable – although they eventually get closer to the theoretical probability (1/4 in this case).

- 'Chance has no memory.': The suit of the next card (or the underlying random number) is independent of previous results.

- With an expectation of 75%, the longer you play this game, the less chance there is of making any profit.
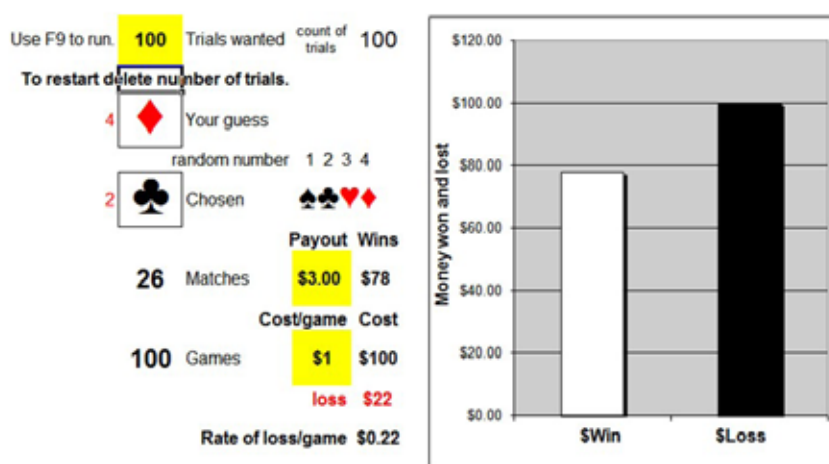
Figure 1. One hundred trials of guess the suit, with a payout of $3.

POKIES SIMULATION

Students are shown the random numbers that determine the outcomes on the electronic gaming machines ('pokies') that inhabit local hotels and clubs. With the 'house margin' set at the typical 10% (Punter's expectation = 90%), spreadsheet simulations show that an initial amount of $10 might be exhausted after 100 $1 bets, more so if you are lucky. At 10 bets per minute, the price of $10 would give approximately 10 minutes of 'excitement'. Sharing data from this simulation will give a skewed distribution for students to predict-observe-explain the centres and spread.
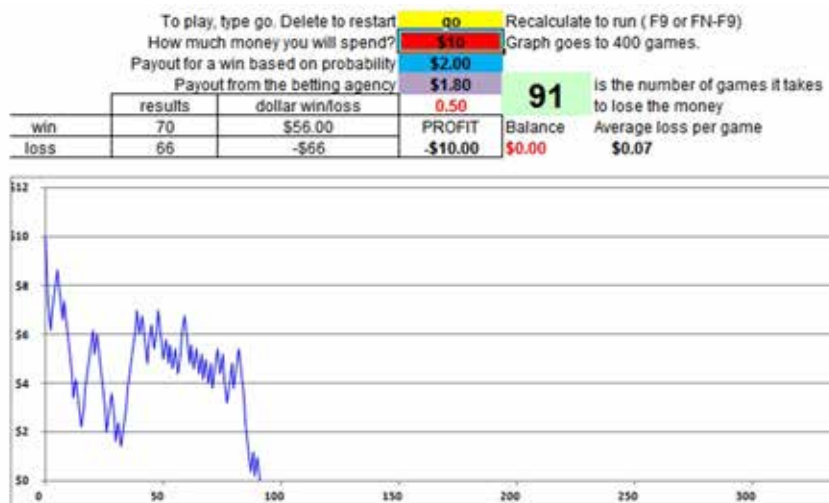

Figure 2. Taking longer than expected to lose $10.

Comparing results for 10 bets, 100 bets, and 1,000 bets, students will see that, despite variability, the longer you bet the less chance there is of breaking even. Some Year 10 students might estimate these chances through calculating the standard deviation for a sample of 1,000 bets at an expectation of 90%.

Figure 3. Less percentage spread gives no wins for 1,000 bets.

POKIES STATISTICS

Student statistical investigations can be based on data drawn from three key sites:-

- Victorian Commission for Gambling and Liquor Regulation (2018)

- Australian gambling statistics (Queensland Government Statistician's Office, 2018)

- Victorian Responsible Gambling Foundation (2019)

Of the many questions students might investigate some are:

- How does spending on pokies in Victoria and other states compare with spending on other forms of gambling?

- How much per adult is spent on different forms of gambling?

- Where is the most money spent on 'pokies – in metropolitan areas or in the country?

- Who spends the most? How does spending correlate with socio-economic disadvantage?

- How many pokies machines are there in your Local Government Area and how much do they take from customers?

Use of the VCLGR gambling spreadsheets will test student skills in selecting and sorting data, summarizing with measures of centre and spread and in representing their summaries using the available choices of graph. Coordination with students' studies of health and human relationships might allow students to also research

- What is problem gambling?

- What are the health and human relationship risks associated with problem gambling?

- How could you assist a friend or relative who had a gambling problem?

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

PAYOUTS VS. PROBABILITIES

Advertised football payouts provide real data for student analysis. If payouts of Brisbane $1.65 vs. Melbourne $2.25 represent chances of winning, then the ratio $\frac{1}{1.65} : \frac{1}{2.25}$ represents the odds of the two teams winning.

These two fractions add up to more than 1, so the probability of Brisbane winning is

$$\frac{1}{1.65} \div \left(\frac{1}{1.65} + \frac{1}{2.25} \right) = \frac{2.25}{1.65 + 2.25} = 0.577$$

From this, Expectation = payout x probability = 1.65 x 0.577 = 95.2%, typical of commercial sports betting agencies that plan to take their 5% share of all bets placed.

Various sports gambling simulations (See Figure 5) can tease out this relationship between payouts, probabilities and expectation – involving just two outcomes (win or lose), three outcomes (win lose or draw) or more outcomes (a horse race). Using real payout data, the sum of the reciprocals of the payouts will inevitably be more than 1 – and the reciprocal of this sum will be the expectation, consequently less than 100%. Repeated simulations will produce eventual overall losses.

SAMPLING WITHOUT REPLACEMENT – QUINELLA SIMULATION

The 'quinella' situation can be introduced through choosing a pair from cards numbered 1 to 8. The 'Stawell Gift" simulation then asks students to test for a fair payout on a bet that Runners 7 and 8 will fill the first two places in a race with 8 appropriately handicapped runners (8 equal chances). Repeated simulations with different payouts and some theoretical analysis will be required before the student is convinced that a fair payout would be $8 x 7/2 = $28.
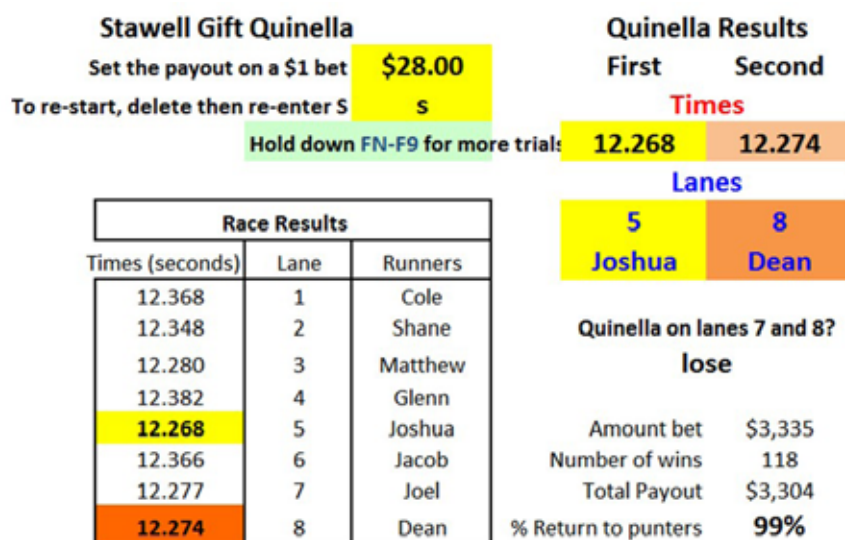

Figure 4. This student has chosen the correct payout.

A multi-bet simulation pays out only for a win on both of two games. The two individual payouts are multiplied to give the multi-bet payout, so with bookies' margins of 10% the expectation is reduced to 90% x 90% = 81%, an overall 'bookies' margin of 19%. Students can use tree diagrams to analyse this two-step event and link their understanding of multiplication of probabilities to reinforce what they already know about the link between expectation, payouts and probabilities. They should also conclude that multi-bets are not a good idea.

Figure 5. Students enter yellow box data, and then simulate the multi-bet on teams A and C.

A combination-bet simulation provides the added complication of money back if the second of the two events does not result in a win. Students need careful tree-diagram analysis to explain the underlying probabilities.

Figure 6. A combination bet with fair payouts.

CONCLUSION

These spreadsheets, together with associated hands-on activities, worksheets, and teacher's notes, have been trialled over the last three years (Money et al., 2015). Their wider use is encouraged for the following reasons:

- Mathematics teachers are uniquely placed to take on the responsibility of tackling Australia's gambling addiction in their classrooms. No extra time is involved in doing so, since the gambling context can be used to cover all aspects of Year 9 and 10 probability and statistics.

- Year 9/10 students are not allowed to gamble (It is illegal.) but the online gaming they play, with its addictive attraction and 'lucky loot boxes' is in many ways equivalent. They are at the right age to be confronted with the issue.

- A whole-school approach, involving teachers of English and Health, is also recommended – and more easily implemented prior to the restricted requirements of VCE studies.

REFERENCES

The Economist. (2017). *Australians spend more on gambling than people anywhere else*. Retrieved from https://www.economist.com/asia/2017/03/16/australians-spend-more-on-gambling-than-people-anywhere-else

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Hanss, D., Mentzoni, R., Delfabbro, P., Myrseth, H., & Pallesen, S. (2014). Attitudes toward gambling among adolescents. *International Gambling Studies*, *14*(3), 505-519.

King, D., & Delfabbro, P., (2019). Video game monetization (e.g., 'loot boxes'): A blueprint for practical social responsibility measures. *International Journal of Mental Health and Addiction*, *17*(1), 166-179.

Lovitt, C., & Clarke, D. (1988). *The mathematics curriculum and teaching program* (Vol. 1). Canberra, Australia: Curriculum Development Centre.

Miller, H. (2017). *Gen bet: Has gambling gatecrashed our teens?* Melbourne, Australia: Victorian Responsible Gambling Foundation.

Money, R. (2016). Sports gambling. In W. Widjaja, E. Yook-Kin Loong, & L. A. Bragg (Eds.), *Proceedings of the Mathematical Association of Victoria 53rd Annual Conference* (pp. 149-150). Melbourne, Australia: Mathematical Association of Victoria.

Money, R., Lowe, I., & Smith D. (2015). Biggest loser gambling curriculum project. In N. Davis, K. Manuel, & T. Spencer (Eds.), *Mathematics – Lead, Link: Proceedings of the 25th Biennial Conference of the Australian Association of Mathematics Teachers* (pp. 88-93). Adelaide, Australia: Australian Association of Mathematics Teachers.

Queensland Government Statistician's Office. (2018). *Australian gambling statistics 34th edition summary tables 2016–17*. Brisbane, Australia: Author.

Tout, D., Santburn, M., Kindler, L., Bini, M., & Money, R. (2011). *Consumer stuff! Maths: A resource for teaching and learning numeracy*. Melbourne, Australia: Consumer Affairs Victoria.

Victorian Commission for Gambling and Liquor Regulation. (2018). *Find venue data*. Retrieved from https://www.vcglr.vic.gov.au/i-want/find-venue-data

Victorian Responsible Gambling Foundation. (2019). *VCAL numeracy unit*. Retrieved from https://responsiblegambling.vic.gov.au/reducing-harm/schools/resources-teachers/

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Powerful conceptual models

Claire Moriarty, Phoenix P-12 Community College

Conceptual models (also commonly called analogies) provide a scaffold for deep conceptual learning. They allow students to access their intuition and transfer learning across topics. The best conceptual models not only promote deep understanding but also can be used across several areas of mathematics. A few such models are examined in this paper.

INTRODUCTION

There are three main types of knowledge in mathematics; factual, procedural and conceptual (Willingham, 2009). The focus here is on the development of conceptual knowledge or understanding, which is one of the hardest types of mathematics to help students develop (Willingham, 2009), and one that many weaker students lack (Boaler, 2018-2019). One of the chief difficulties in developing new conceptual understandings is that "new concepts must be built on something that students already know" (Willingham, 2009, p. 18). While some concepts can be built on students' pre-existing understanding of precursor areas of mathematics, at other times we can use their experiences in areas outside of mathematics to help further their understanding.

WHY CONCEPTUAL MODELS?

Conceptual models are very similar to what is often referred to in the literature as analogies. However, I have used the term "conceptual models" to highlight their importance in concept development and, perhaps owing to my background as a physics teacher, to the common use of the term 'model' in science as a way of explaining something unfamiliar through seeing the similarities in behaviour with something familiar (e.g., wave model of light).

Whatever term is used to describe them, the use of conceptual models is widespread in mathematics education (Richland, Holyoak, & Stigler, 2004; Sarina & Namukasa, 2010; William, 1997). They can range from the everyday to the fantastical to the highly abstract. Some conceptual models refer to areas outside of mathematics and others to the similarities with areas of mathematics that are already familiar with students. This linking with the already familiar, whether within mathematics or outside it, is crucial to the success of conceptual models.

Although models can be used in other ways, the focus here is on the use of models to develop deep conceptual understanding. It has been found that analogies or models are an excellent way to develop what can be very abstract ideas (Richland et al., 2004; William, 1997; Willingham, 2009). Analogies or models are particularly relevant to mathematics, which deals at its core with "abstract structure in which underlying relationships remain the same but the object slots can be filled in varying ways" (Richland et al., 2004, p. 39). Hence the very act of drawing parallels between the model and the mathematics can help students think in mathematical ways. Moreover, a 'powerful' conceptual model can lead to transfer of learning between topics and see the connections between different areas of mathematics (Richland et al., 2004; Willingham, 2009).

CRITERIA FOR A GOOD CONCEPTUAL MODEL

There are a number of different published criteria for a good conceptual model or analogy. However, there are commonalities between them.

1. The model must be something that is familiar to the students. In order to make connections with prior learning and existing concepts the model must be already present within those existing concepts (William, 1997; Willingham, 2009).

2. The model must be clearly on display. Although it is possible to teach the meaning of equality with a set of balance scales, the model becomes much more powerful if there is an actual set of scales (or at least a picture of one) on display. The model becomes even more powerful if the equal parts are placed or drawn on either side of the scale (Richland et al., 2004; Willingham, 2009).

THE MATHEMATICAL ASSOCIATION OF VICTORIA

3. The model must closely match and support the mathematics concepts required. A poor match can result in misconceptions developing, or the interference of irrelevant prior learning. For example, using a model for adding negative numbers as adding chilli or ice cubes to soup has poor match – Most students will know that chilli and ice do not have opposite effects on temperature (Barton, 2018; William, 1997). In addition, the best teaching with models makes the match explicit – in what ways does the model match the mathematics, and in what ways does it not (Richland et al., 2004; Willingham, 2009)

4. The best models can be used across multiple topics (William, 1997; Willingham, 2009). Indeed, in Willingham's (2009) article on the cognitive science of mathematics learning, his first recommendation is to find a model (analogy) that can be used across topics: "Using the same analogy across topics makes it much clearer to students how those topics relate to one another" (p. 19).

DANGERS OF CONCEPTUAL MODELS

Using conceptual models is not risk-free. A model that is unfamiliar to the students or is too abstract has limited (if any) value (William, 1997). If the limits of the model are not explicitly stated, some students may over generalise and apply the model in inappropriate situations (Sarina & Namukasa, 2010) or apply similarities to the situation that are not mathematically valid.

SOME EXAMPLES OF CONCEPTUAL MODELS

UNWRAPPING A PASS-THE-PARCEL: AN EXAMPLE OF POOR RANGE

Under some circumstances, a conceptual model with a limited range may be unavoidable and does not necessarily mean the model is not useful (William, 1997). Indeed, I would argue that the place of conceptual models as scaffolds means that the importance of the model fades with time and familiarity, meaning that consecutive models, each leading to increasingly sophisticated mathematics, may be of significant benefit.

Take the example of solving simple multi-step equations. One model used here is that of a pass-the-parcel. Preparing a game of pass-the-parcel starts with a single present in the middle and wrapping it in layers from the inside to the outside. Playing the game then means unwrapping the layers from the outside to the inside. The analogy lies in "building" and "unbuilding" an equation (and already we are mixing our metaphors!). When "building" an equation we start from the unknown and "wrap" it in operations from the inside first, but in "unbuilding" we must "unwrap" the outside layers first.

Advantages

If sufficient prior work is done to develop skills in inverse operations and writing expressions in "good" algebra, this model offers students a concrete representation of what can otherwise be a very abstract procedure.

Limitations

This conceptual model is packed with flaws. It is not helpful unless we have already addressed how to undo a specific operation, and unless substantial work has already been done in "building" equations and correctly expressing them algebraically. It applies only to equations where the unknown appears once (although I have found it surprisingly useful for helping students to identify why equations with the unknown in two places must be treated differently – We no longer have a single present). In addition, students often require extra support to identify which is the outside operation; I often resort to yet another analogy of "in the same room", "across the corridor", "downstairs", and "across the street". As such, I would not classify this as a particularly powerful conceptual model, particularly with such a limited range. However, it is still useful in helping students in those challenging early steps of solving multi-step equations.

THE SUBMALLOON: A BETTER EXAMPLE

An area with an abundance of available conceptual models is directed number (Barton, 2018). Common models include the height of land above and below sea level, games of gain and loss, temperatures above and below zero (William, 1997). As is commonly pointed out, all of these analogies are limited in terms of their range. Most are useful for developing a sense of the scale of negative numbers and their place on a number line. Some can also be useful for adding

or subtracting positive numbers to get a negative result (e.g., debts), adding directed numbers (e.g., games where points are won or lost) or even finding the magnitude of the difference between two numbers (e.g., What is the difference in temperature between Falls Creek at -5°C and Melbourne at 8°C?).

The submalloon model (Figure 1) can extend to adding and subtracting positive and negative numbers. The model can resolve a thorny issue in simplifying algebraic expressions, but even this does not address multiplying and dividing directed numbers. The submalloon model is my own adaptation of two separate resources. The name "submalloon" is taken from the "10ticks" resource (Fischer Educational, 2004); however, the usage of it here is significantly different. The "balloons and weights" concept is adapted from "The Integer Angel" worksheet (Watts, 2016). The submalloon model is a magical vehicle that can travel both in the air and under the water. It is accompanied by a vertical number line that is drawn on the board and that students copy or paste into their workbooks. The submalloon has both balloons and weights attached. The balloons symbolise positive numbers and the weights symbolise negative numbers.
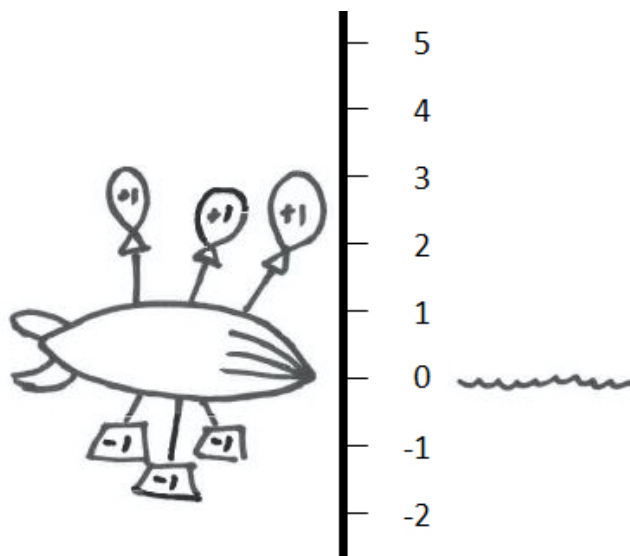

Figure 1. An introductory visual for the submalloon model.

Consider the expression –3 + (–2).

The first number represents our starting point: 3 meters below sea level. The operation is addition, so we are going to add something. The second number represents what we are going to add: 2 weights. The crucial concept is "If we are adding weights, which way will we go – up or down?" The journey of the submalloon is then drawn on the number line as shown below.
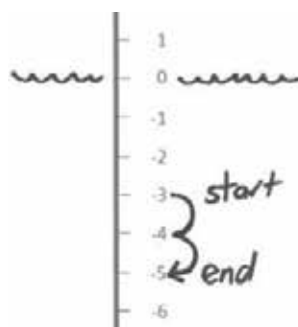

Figure 2. The journey of the submalloon for the expression –3 + (–2).

Consider next the expression 4 – (–2).

We start 4 metres above sea level, and this time we are subtracting weights, so we will move up. The journey is then drawn on the number line as shown below.
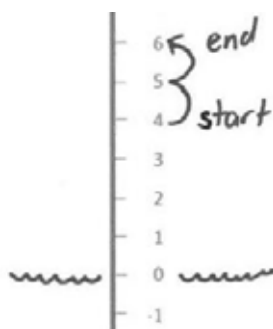

Figure 3. The journey of the submalloon for the expression 4 – (–2).

Once students are familiar and comfortable with the model, we can point out that adding balloons and subtracting weight has the same effect (adding a positive number and subtracting a negative number) and that subtracting balloons and adding weights have the same effect (subtracting a positive number and adding a negative). Hence, we start to work towards a rationale for the later practise of replacing the signs.

Advantages

This model provides a strong visual model for adding and subtracting directed numbers and a conceptual strategy for reasoning through such problems. It is not the only model to do so. However, it is one of the few that offers a conceptual framework for a common shortcut – that adding a negative number is equivalent to subtracting a positive (both cause the submalloon to move downwards), and that subtracting a negative number is equivalent to adding a positive (both cause the submalloon to move upwards). On the surface, this may seem trivial but is a significant result when it comes to simplifying expressions in algebra.

First, consider the following problem:

$$\textit{Simplify}:\quad 5x + 6y - 3x + 2y$$

A common point of confusion is what to do with the subtraction. What does it belong to? Students have been told for some time that, although it is possible to change the order in an addition, it is not possible to change the order in a subtraction.

Through prior exposure to the submalloon model, students should be reasonably familiar with the idea that adding a negative has the same result as subtracting a positive. Therefore, we can adjust this question to:

$$\textit{Simplify: } 5x + 6y + (-3x) + 2y$$

Since all our operations are now additions, we can happily rearrange this to

$$5x + (-3x) + 6y + 2y$$

and simplify by collecting the like terms.

Next, consider the problem:

$$\textit{Find the coefficient of g in the expression } 3f + 2k - 4g$$

As mathematicians, we know the correct answer is –4. However, for many students, this is a conceptual leap. Until now, we have expected them to read this as "subtract 4 lots of g", with the operation being separate from the operand. However, having established that subtracting 4 lots of g is equivalent to adding –4 lots of g, and that we really need to

think of the expression this way, it starts to make more sense why we expect our answer of –4.

Limitations

While the submalloon model can be extended to algebra, it does not extend to multiplying and dividing with negative numbers, a topic I usually tackle simply by exploring patterns. Another limitation is it can reinforce a common misconception in physics that exerting a force will cause a movement for a limited time and then stop. As I teach both mathematics and science, I am always explicit with this when introducing the model, saying, "It's bad science but it's good mathematics".

THE ARRAY/AREA MODEL: FROM MULTIPLICATION AND DIVISION, THROUGH TO FACTORS AND MULTIPLES, AND EXPANSION AND FACTORISATION

One of the most versatile models that I have come across is the array or area model. This is more model than an analogy, as we are referring to another area of mathematics rather than an experience outside of mathematics. Here, multiplication or division problems are presented as an array of objects. In the early stages of multiplication this is an effective visual representation that can reinforce utilise subitising and skip counting, as well as illustrating the inverse relationship between multiplication and division.
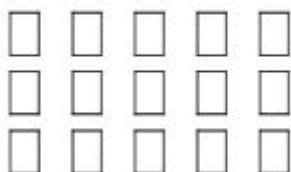
Example: 3×5

Figure 4. Array model for 3×5.

The model is also easily extended to calculating the area of a rectangle. This application of the same model reinforces both the concept of area as the amount of flat space a shape takes up (or the number of square tiles needed to cover a shape) and also the relationship between area and multiplication for a rectangle.
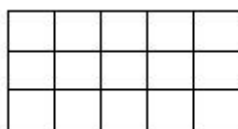

Figure 5. Area model for 3×5.

Once the concept of area has become very familiar, the model can be extended to mental strategies for multiplication or division (Boaler, 2018-2019). This is particularly useful for addressing a persistent issue in mental strategies, the oversimplification of a two-digit multiplication problem (Day & Hurrell, 2018). Using the area model, it is easy to illustrate why 21 × 34 is not equal to 20 × 30 + 1 × 4.


So 21 × 34 = 600 + 30 + 80 + 4 = 714

Figure 6. Area model for 21 × 34.

Once we have established that the area model is a useful representation of multiplication on division problems, we can then use it to explore expanding and factorising in algebra. Consider the simple expansion problem:

Expand $x(3x+2)$

This has traditionally been taught in a procedural manner. We explain that we need to multiply everything inside the brackets by what's outside. But questions arise from students year after year, such as "but the order of operations means that I'm supposed to do the brackets first". However, once students are familiar with the area model for multiplication, and breaking up the area to make the multiplication more efficient, this process becomes much more intuitive.
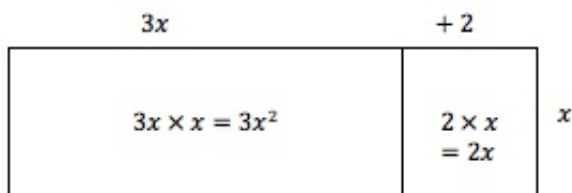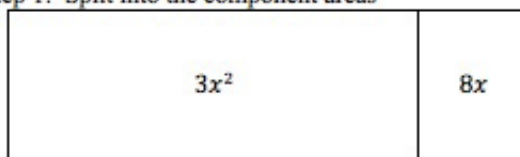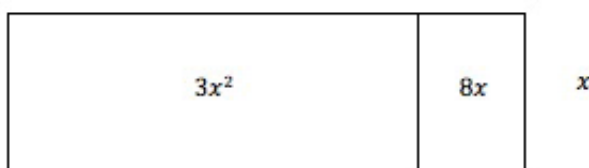

So $x(3x+2) = 3x^2 + 2x$

Figure 7. Area model for expanding $x(3x+2)$.

With some prior work around finding common factors, the model can be used to factorise expressions as well. The procedure is simply performed in reverse. When presented with the total area, we split it into the component areas, then look for a common factor to tell us a possible side length.
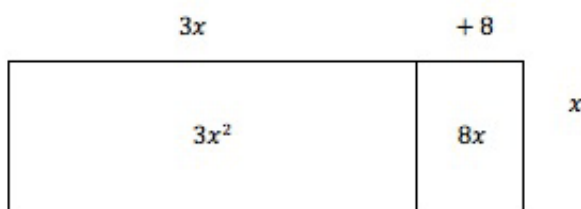

So $3x^2 + 8x$ factorises to $x(3x+8)$

Figure 8. Area model for factorising $3x^2 + 8x$.

A natural follow-up is more complex expansions, such as with two brackets:
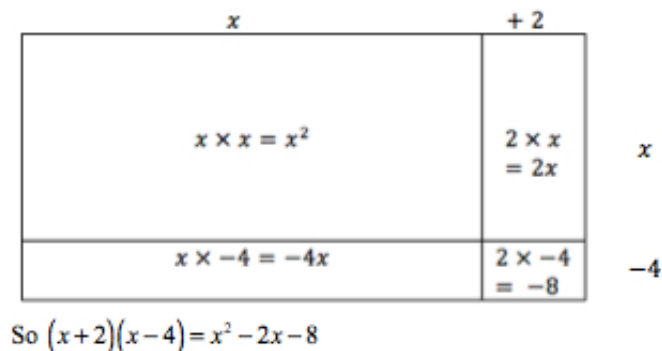

So $(x+2)(x-4)=x^2-2x-8$

Figure 9. Area model for expanding $(x+2)(x-4)$.

Finally, after some work with expansion patterns, the model can be used for factorising quadratics.

Step 1: Place the x^2 and constant terms
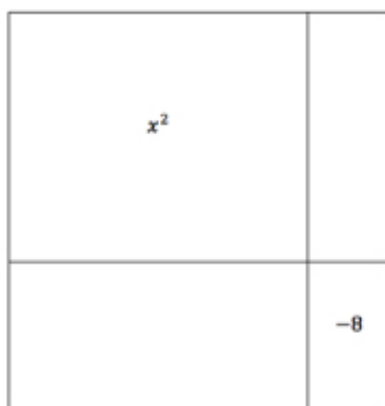
Step 2: Find two coefficients for x that multiply to the constant term and add to the coefficient of the x term. Place them in the other two rectangles. The order doesn't matter.
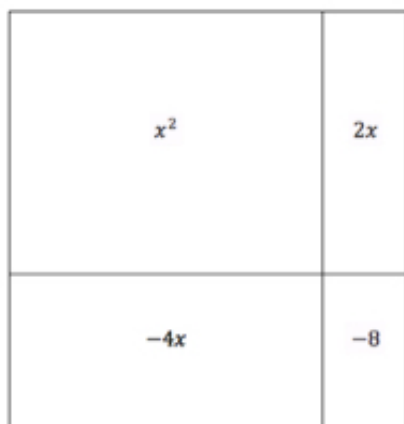
Step 3: Find a common factor, which might be the length of each row, then find the corresponding lengths of the columns.
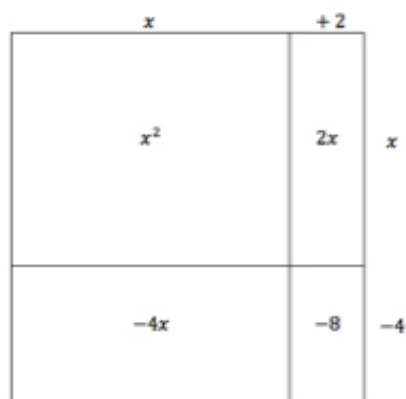
So $x^2-2x-8=(x+2)(x-4)$

Figure 10. Area model for factorising x^2-2x-8.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Limitations

The fact that the analogy is simply to another area of mathematics means that there are fewer outside influences that can interfere with the accuracy of the analogy. However, it does mean that teachers need to ensure that students are very familiar with arrays and area as representations for multiplication before extending the model to more abstract ideas. Similarly, students should be familiar with the model at each stage before it is extended further. Finally, in using the model for factorising, some students may become confused by the similarity in the representation between expanding and factorising. Familiarity with inverse operations and that the same array can represent both multiplication and division may help here.

CONCLUSION

Conceptual models can make abstract areas of mathematics relatable and intuitive and can lead to the development of deep understanding. As such, any well-chosen conceptual can be beneficial. However, using conceptual models that apply across different topic areas can lead to an understanding of the interconnectedness of mathematics and an appreciation of the links between topics.

REFERENCES

Barton, C. (2018). *How I wish I'd taught maths*. Melton, England: John Catt Educational.

Boaler, J. (2018-2019). Developing mathematical mindsets: The need to interact with numbers flexibly and conceptually. *American Educator*, *42*(4), 28-33.

Day, L., & Hurrell, D. (2018). *Process over product: It's more than an equation*. Paper presented at the 55th Annual Conference of the Mathematical Association of Victoria.

Fischer Educational. (2004). *Educational worksheets*. Retrieved from http://www.10ticks.co.uk

Richland, L. E., Holyoak, K. J., & Stigler, J. W. (2004). Analogy use in eighth-grade mathematics classrooms. *Cognition and Instruction*, *22*(1), 37-60.

Sarina, V., & Namukasa, I. K. (2010). Nonmath analogies in teaching mathematics. *Procedia Social and Behavioural Sciences*, *2010*(2), 5738-5743.

Watts, J. (2016). *Grade 7 mathematics. Unit 2: Integers*. Retrieved from https://docplayer.net/21520762-Grade-7-mathematics-unit-2-integers-estimated-time-15-hours.html

William, D. (1997). *Relevance as MacGuffin in mathematics education*. Paper presented at the British Educational Research Association Conference, York, England.

Willingham, D. T. (2009). Is it true that some people just can't do math? *American Educator*, *33*(4), 14-19.

When is a square a triangle?

Chris Wetherell, St Francis Xavier College, Australian Maths Trust, and Australian National University

Given how busy the curriculum can sometimes feel, students in the mathematics classroom could be forgiven for the impression that all they do is lurch from one topic to the next without ever having time to appreciate the beauty and interconnectedness of the things that they are learning, or to stop and ask themselves what else there might be to find out. The purpose of this article is three-fold: 1) to draw attention to the ways that even the simplest of mathematical tools can quickly give rise to very deep questions, 2) to highlight how the solution of such problems can often draw upon a variety of seemingly disconnected branches of mathematics, and 3) to illustrate how answering one question naturally leads to other questions.

FIGURATE NUMBERS

Armed with little more than the basic building blocks of whole numbers and addition, one can very quickly begin to pose a variety of sophisticated mathematical questions. Picture the scene. A keen student playing with counters at their desk notices that 12 of them can be stacked in three rows of four, or even two rows of six, and asks themselves the natural question: Can all numbers be arranged in a similar way? Suddenly the concept of non-rectangular numbers (also known as primes) is discovered and the questions just keep on rolling in…Well, this is the dream, anyway.

So-called figurate numbers arise very naturally when arranging counters on a desk. The square numbers are those where the number of counters in each row is the same as the number of rows. The trapezoidal numbers are those where each new row gets one more counter than the last, and a special case is the triangular numbers where there is a single counter in the first row. The hexagonal numbers are those that describe the outlines of ever-increasing hexagons (not to be confused with the centred hexagonal numbers where all the gaps are also filled in). And so on. See Figure 1 for some examples.

One hopes that the keen student then begins to ask things like: Is there a rule for the square numbers? (Yes, easy, n^2.) Is there a rule for the triangular numbers? (Yes, not quite so easy, $\frac{1}{2}n(n+1)$.) Is there a rule for the trapezoidal numbers?

(Not really, but there is a rule for the non-trapezoidal numbers, so maybe that's just as good?) Is there a rule for the hexagonal numbers? (Yes.) Is there a reason all hexagonal numbers are also triangular? (Yes.)
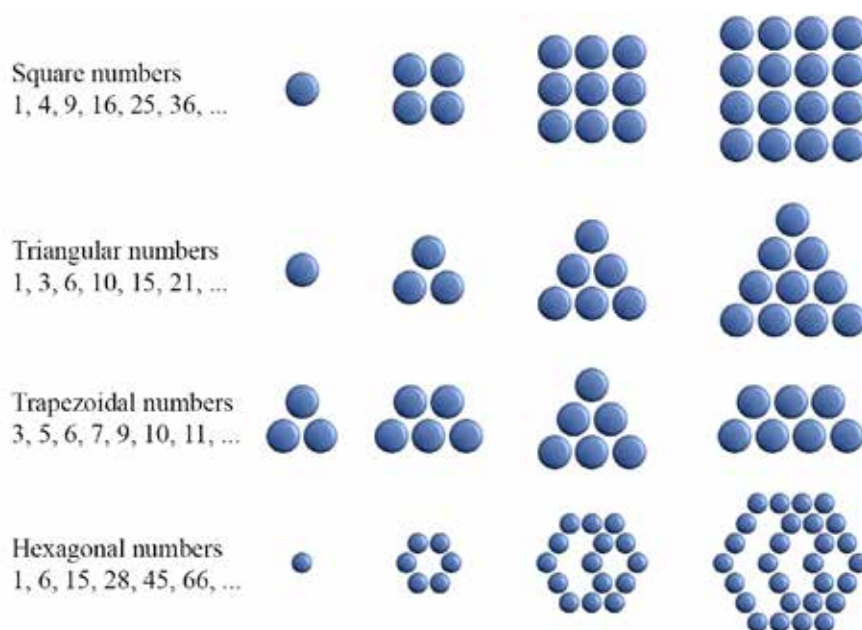

Square numbers
1, 4, 9, 16, 25, 36, …

Triangular numbers
1, 3, 6, 10, 15, 21, …

Trapezoidal numbers
3, 5, 6, 7, 9, 10, 11, …

Hexagonal numbers
1, 6, 15, 28, 45, 66, …

Figure 1. Examples of figurate numbers.

How about: Is there a good reason why 36 is both square and triangular? Are there any other numbers like this? Figure 2 shows one way in which 36 counters on a desk can be rearranged to establish that it is in both lists, but for the moment, this feels more like good luck than good mathematics. It certainly gives no insight into how we might hope to find other examples, or perhaps how to prove that there are no others (apart from the trivial case of the number one).
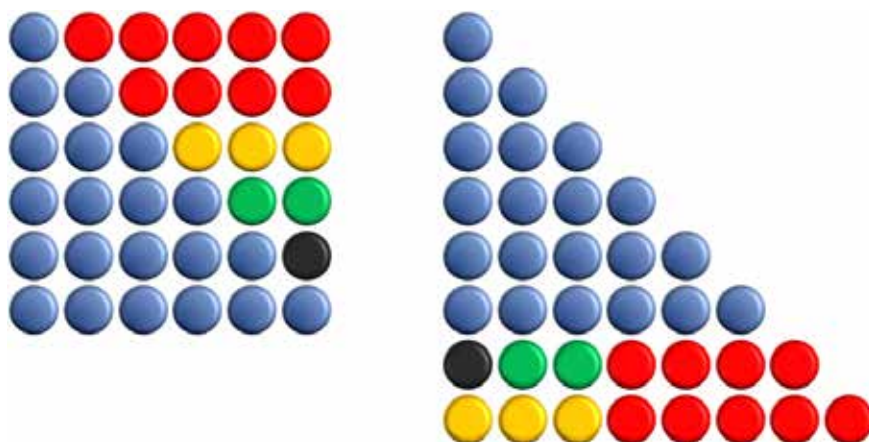

Figure 2. Thirty-six is square and triangular.

FROM COUNTERS TO EQUATIONS

It is a typical feature of mathematics that the tools required to solve a problem are often much more sophisticated than the tools required to pose it. Famous examples include the Four Colour Theorem (i.e., any map can be coloured using at most four colours without needing to double up for neighbouring regions) and Fermat's Last Theorem (i.e., while there are certainly many examples of Pythagorean triples, being integers that satisfy $a^2 + b^2 = c^2$, no such solutions can be found if the indices are increased from 2 to 3 or higher). Both problems eluded mathematicians for centuries and both eventual solutions required the development of new branches of mathematics – and not without controversy.

So it is with finding square triangular numbers. The problem arises very naturally from playing with counters on a desk or dots on a page, and it is very hard to imagine how its solution could rely on such esoteric concepts as the irrationality of $\sqrt{2}$ and constructing the sequence of best rational approximations to it.

A brute force approach to showing that 36 is not the only example is to extend both lists and see what happens:

Square: **1**, 4, 9, 16, 25, **36**, 49, 64, …, 1089, 1156, **1225**, 1296, 1396, …

Triangular: **1**, 3, 6, 10, 15, 21, 28, **36**, 45, …, 1128, 1176, **1225**, 1275, 1326, …

We notice that 36 is the 6th square number and 8th triangular number and, 1225 is the 35th square number and 49th triangular number. This is useful, but we would be listing for a very long time before we stumbled upon the next example, namely 41,616. A more systematic approach is needed.

The nth square number is n^2, since it forms an $n \times n$ rectangular array of dots. The nth triangular number is $T_n = \frac{1}{2}n(n+1)$, since two such triangles can be rearranged into an $n \times (n + 1)$ array of dots. Analysing why 36 is in both lists we see that

$$36 = 6^2 = (2 \times 3)^2 = 2^2 \times 3^2 = 4 \times 9 = (\tfrac{1}{2} \times 8) \times 9 = \tfrac{1}{2} \times 8 \times (8+1) = T_8$$

For 1225 we have

$$1225 = 35^2 = (5 \times 7)^2 = 5^2 \times 7^2 = 25 \times 49 = (\tfrac{1}{2} \times 50) \times 49 = \tfrac{1}{2} \times 49 \times (49+1) = T_{49}$$

The similarity between the two calculations is striking, but the key difference is also worth highlighting: for 36, the larger square factor 9 is one *more* than twice the smaller square factor 4, whereas for 1225, the larger square factor 49 is

one *less* than twice the smaller square factor 25. In any event, we end up with a pair of consecutive integers in the second last step, so the calculation matches the required form for a triangular number, as anticipated.

Summarising the above, the square numbers 6^2 and 35^2 are also triangular precisely because

$$6 = 3 \times 2 \quad \text{and} \quad 3^2 = 2 \times 2^2 + 1,$$
$$35 = 7 \times 5 \quad \text{and} \quad 7^2 = 2 \times 5^2 - 1.$$

Hence, our search for other such examples has reduced to finding positive integers a and b satisfying

$$a^2 = 2b^2 + 1 \text{ or } a^2 = 2b^2 - 1.$$

For then the square number $(ab)^2$ is also triangular since

$$(ab)^2 = a^2 b^2 = a^2 \times \frac{a^2 \pm 1}{2} = \tfrac{1}{2}a^2(a^2 \pm 1) = T_{a^2} \text{ or } T_{a^2-1}.$$

Conversely, via the Fundamental Theorem of Arithmetic, that positive integers have a unique representation as a product of primes, one can show that finding such solutions a and b is the only way to construct numbers that are simultaneously square and triangular, so we can be sure that we won't miss any examples with this approach. To see this, suppose the nth square equals the mth triangular number, so $n^2 = \tfrac{1}{2}m(m+1)$, or equivalently $2n^2 = m(m+1)$. Note that m and $m + 1$ have no common prime factors; otherwise, such a prime would also be a factor of their difference, which equals 1. Hence, for the two prime factorisations of $2n^2$ and $m(m + 1)$ to match, one of m or $m + 1$ must be a perfect square, equal to some number a^2 say, and the other must be double a perfect square, equal to some number $2b^2$. Hence $a^2 = 2b^2 \pm 1$ as claimed.

Requiring that the solutions a and b take only integer values brings us into the world of Diophantine equations, named for third-century mathematician Diophantus of Alexandria who pioneered the study of such things. Without this restriction, we can interpret the corresponding equations $x^2 = 2y^2 \pm 1$ as a pair of non-rectangular hyperbolas which happen to pass through integer-valued points (1, 0), (1, 1), (3, 2), and (7, 5), corresponding to the redundant solution 0, trivial solution 1 and interesting solutions 36 and 1225, respectively; see Figure 3. The question becomes: Are there any other points on these graphs for which both coordinates are positive integers? There is a limit to how much we can learn from a graphical approach or traditional algebraic methods, so something else is required.
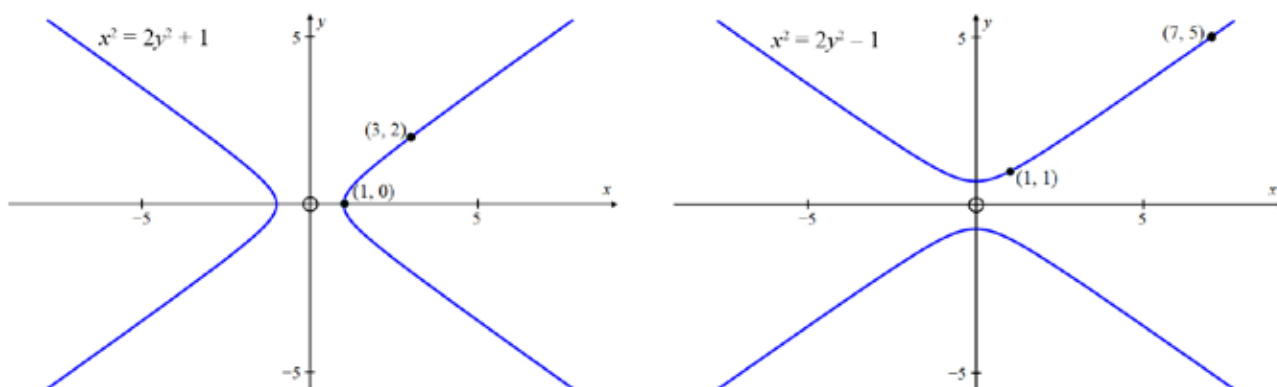

Figure 3. Integer-valued points on hyperbolas.

PELL'S EQUATION AND RATIONAL APPROXIMATIONS

The equations $a^2 = 2b^2 \pm 1$ are an example of the general Pell's equation, or Pell-Fermat equation, $x^2 - ny^2 = 1$ (AMT, NMSS). While equations of this form are named in honour of Englishman John Pell and Frenchman Pierre de Fermat of

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

the seventeenth century, their history goes back almost a thousand years earlier to Indian mathematician Brahmagupta who first solved the case $n = 2$ in the seventh century.

The key observation is this: as we look for larger and larger values of a and b then, relative to the size of the other terms, the $+1$ or -1 in the equation becomes more and more insignificant. Therefore, $a^2 \approx 2b^2$. Rearranging and taking square roots, we have

$$\frac{a}{b} \approx \sqrt{2}.$$

However, it has been known since at least the time of Pythagoras in the sixth century BCE that $\sqrt{2}$ is irrational, which is to say that it cannot be written exactly as one integer divided by another. On the other hand, the rational numbers are *dense* in the reals, meaning that wherever you look on the real line you will find them in abundance. Hence we can, in principle, find fractions $\frac{a}{b}$ as close to $\sqrt{2}$ as we like. The only problem now is to decide how best to calculate such fractions.

One simple method for finding a rational approximation of any number is to truncate its known decimal expansion, such as $\sqrt{2} = 1.414213562...$ (how these decimal digits can be calculated in the first place will be discussed in the last section). For example, truncating after 3 decimal places we have $\sqrt{2} \approx 1.414 = \frac{1414}{1000} = \frac{707}{500}$ and we might, therefore, hope that $a = 707$ and $b = 500$ gives a suitable solution. However, $a^2 = 707 = 499{,}849$ while $2b^2 = 500{,}000$: close, but not close enough. Attempting to truncate later in the expansion gives similarly poor results; in some sense, the accuracy that we gain by using more decimal places is outweighed by the rate at which the numerator and denominator of the corresponding fraction are growing. We do not just need more accurate approximations; we need ones that are as good as possible relative to the size of the numerator and denominator involved. This is the only way to ensure that we get the smallest possible difference of 1 or -1 in the equation $a^2 = 2b^2 \pm 1$.

CONTINUED FRACTIONS

Enter continued fractions. Just as every real number can be represented by a, possibly infinite, decimal expansion, every real number can also be represented by a, possibly infinite, continued fraction. The basic algorithm is this: subtract the integer part, find the reciprocal, repeat. For a rational number, this amounts to repeatedly converting improper fractions to mixed numerals, as in the following example:

$$\frac{48}{13} = 3 + \frac{9}{13} = 3 + \frac{1}{\frac{13}{9}} = 3 + \frac{1}{1 + \frac{4}{9}} = 3 + \frac{1}{1 + \frac{1}{\frac{9}{4}}} = \mathbf{3} + \frac{1}{\mathbf{1} + \frac{1}{\mathbf{2} + \frac{1}{\mathbf{4}}}}$$

Using a more compact notation, we write $\frac{48}{13} = [3;1,2,4]$, reading the entries in bold in the last expression above (The numerators are always equal to 1 so do not need to be recorded). Since the denominators of the improper fractions involved at each stage continually decrease, one can argue that every rational number has a continued fraction representation that terminates after a finite number of steps.

More interesting is the case of irrational numbers, which must have an infinite representation. For example,

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{\ddots}}}}} = [3;7,15,1,292,\ldots].$$

Truncating this representation after one term gives the familiar approximation $[3;7] = \dfrac{22}{7}$. After two terms we have $[3;7,15] = \dfrac{333}{106}$; after three, we have $[3;7,15,1] = \dfrac{355}{113}$, and so on. The claim is that these are the best possible rational approximations for their 'size'; that is, each one is closer to π than any other rational number with the same or smaller denominator. Compare this with the crude decimal truncation method: for example, $3.14 = \dfrac{314}{100} = \dfrac{157}{50}$ is a worse approximation than $\dfrac{22}{7} = 3.142...$ even though the latter has a much smaller denominator.

THE SOLUTION

It turns out that certain special real numbers have a continued fraction representation that is, eventually, periodic: precisely those that are solutions of quadratic equations with integer coefficients. This is certainly the case for $\sqrt{2}$, which is a solution of $x^2 - 2 = 0$, but curiously, it is more effective to consider the number $1 + \sqrt{2}$ first. For convenience, we let $x = 1 + \sqrt{2}$ and then reverse the process of completing the square to find a quadratic equation of which x is a solution:

$$x = 1 + \sqrt{2}$$
$$x - 1 = \sqrt{2}$$
$$(x-1)^2 = 2$$
$$x^2 - 2x + 1 = 2$$
$$x^2 - 2x - 1 = 0.$$

It happens that the constant in this quadratic equation is , which is very good news (if this is not the case for a given real number, then the situation is rather more complicated, but there are ways around it). Next, we rearrange the equation to find an expression for in terms of itself, substitute that same expression into itself, and continue this process indefinitely:

$$x^2 = 2x + 1$$
$$x = 2 + \frac{1}{x} = 2 + \cfrac{1}{2 + \cfrac{1}{x}} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{x}}}$$
$$= \cdots$$
$$= 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}} = [2;2,2,2,\ldots].$$

Finally, simply subtract 1 to deduce that $\sqrt{2} = x - 1 = [1;2,2,2,\ldots]$. There are certainly some finer details that are being glossed over here, such as whether the concept of an infinite continued fraction is even well-defined, but it is possible to prove that the method above is mathematically rigorous.

Next, we consider the rational approximations found by truncating this expression after a number of terms. For example, $[1;2] = \dfrac{3}{2}$ and $[1;2,2] = \dfrac{7}{5}$. Recognise anything? Yes! If $\dfrac{a}{b} = \dfrac{3}{2}$, then $a = 3$ and $b = 2$, which corresponds to the known square triangular number $(ab)^2 = 36$. Similarly, for $\dfrac{a}{b} = \dfrac{7}{5}$ we retrieve the known example $(ab)^2 = 1225$.

Table 1 summarises how this method can be used to find the first handful of square triangular numbers, highlighted in the second last column. The last column indicates which triangular number each example corresponds to, alternating between positions a^2 and a^2-1.

length	continued fraction	$\dfrac{a}{b}$	ab	$(ab)^2$	T_{a^2} or T_{a^2-1}
0	[1]	$\dfrac{1}{1}$	1	1	T_1
1	[1;2]	$\dfrac{3}{2}$	6	36	T_8
2	[1;2,2]	$\dfrac{7}{5}$	35	1225	T_{49}
3	[1;2,2,2]	$\dfrac{17}{12}$	204	41,616	T_{288}
4	[1;2,2,2,2]	$\dfrac{41}{29}$	1189	1,413,721	T_{1681}
5	[1;2,2,2,2,2]	$\dfrac{99}{70}$	6930	48,024,900	T_{9800}

Table 1. Square triangular numbers via truncated continued fractions.

GENERATING THE SEQUENCE MORE EFFICIENTLY

Recalculating truncated continued fractions from scratch each time quickly becomes tiresome, so we employ a shortcut to speed up the process. By way of example, suppose we know that $[1;2,2,2]=\dfrac{17}{12}$, then we can calculate the next term of the sequence, $[1;2,2,2,2]$, without redoing every part of the calculation:

$$[1;2,2,2,2] = 1 + \cfrac{1}{[2;2,2,2]} = 1 + \cfrac{1}{1+[1;2,2,2]} = 1 + \cfrac{1}{1+\dfrac{17}{12}} = \cdots = \dfrac{41}{29}$$

Generalising this approach, suppose the sequence of approximations is $\dfrac{a_1}{b_1}, \dfrac{a_2}{b_2}, \dfrac{a_3}{b_3}, \cdots, \dfrac{a_n}{b_n}, \cdots$. Then, given a known term $\dfrac{a_n}{b_n}$, the next in the sequence can be calculated as follows:

$$\dfrac{a_{n+1}}{b_{n+1}} = [1;\underbrace{2,2,\ldots,2}_{n}] = 1 + \cfrac{1}{[\underbrace{2;2,\ldots,2}_{n}]} = 1 + \cfrac{1}{1+[1;\underbrace{2,\ldots,2}_{n-1}]} = 1 + \cfrac{1}{1+\dfrac{a_n}{b_n}} = \cdots = \dfrac{a_n + 2b_n}{a_n + b_n}$$

Comparing numerators and denominators, what we now have is a co-dependent pair of sequences $\{a_n\}$ and $\{b_n\}$ which can be defined recursively as follows:

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} \quad \text{for } n \geq 1$$

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

This pair of sequences can be implemented very easily in a spreadsheet, as seen in Figure 4.

	A	B			A	B
1	1	1		1	1	1
2	=A1+2*B1	=A1+B1		2	3	2
3				3	7	5
4				4	17	12
5	copy down			5	41	29
6				6	99	70

Figure 4. Numerator and denominator sequences via a spreadsheet.

Alternatively, we can write the recursive definitions in matrix form as

$$a_1 = 1 \qquad a_{n+1} = a_n + 2b_n$$
$$b_1 = 1 \qquad b_{n+1} = a_n + b_n \qquad \text{for } n \geq 1$$

which it follows that

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In fact, it is possible to find all square triangular numbers by calculating the powers of the square matrix above and simply reading off the product of the entries in the first column, as seen in Table 2.

n	1	2	3	4	5
$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n$	$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$	$\begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix}$	$\begin{bmatrix} 17 & 24 \\ 12 & 17 \end{bmatrix}$	$\begin{bmatrix} 41 & 58 \\ 29 & 41 \end{bmatrix}$
product	1	6	35	204	1189
product^2	1	36	1225	41,616	1,413,721

Table 2. Square triangular numbers via matrices.

Proving that the above sequence or matrix formulations do indeed give infinitely many legitimate solutions to the equations $a^2 = 2b^2 \pm 1$, and hence infinitely many square triangular numbers, is a nice exercise in applying the principle of mathematical induction. The base case $a_1 = b_1 = 1$ being clear, suppose that we have such a pair of solutions $(a, b) = (a_k, b_k)$, for some $k \geq 1$, and construct the next pair of terms $(A, B) = (a_{k+1}, b_{k+1})$ in the respective sequences. From the recursive definitions we have

$$A = a + 2b, \quad B = a + b.$$

Then

$$A^2 = (a+2b)^2$$
$$= a^2 + 4ab + 4b^2$$
$$= 2a^2 + 4ab + 2b^2 - (a^2 - 2b^2)$$
$$= 2(a+b)^2 - (\pm 1) \qquad \text{(by inductive assumption)}$$
$$= 2B^2 \mp 1.$$

That is, (A, B) is also a solution, but with the role of $+1$ and -1 swapped as expected.

OTHER CONNECTIONS

Having now satisfactorily answered the question of which numbers are simultaneously square and triangular, that's the end of the story, right? No, of course not. We might ask: Is there actually a way to find them by analysing the rearrangement of counters in Figure 2? Yes! Surprisingly, though, reformulating the problem in this way gives rise to only the positive version of Pell's equation, $a^2 = 2b^2 + 1$. Hence, all square triangular numbers can be retrieved, in a quite different way, from every second term of the sequence of best rational approximations to $\sqrt{2}$.

Can we find numbers that are simultaneously square and pentagonal, say? Yes! But this time we need to analyse the continued fraction for $\sqrt{6}$.

Can we easily approximate other square roots without having to find their continued fraction representation? Yes! For example, in the matrix formulation above, simply change the 2 in the top-right corner to any other number N and the corresponding fractions will converge to $\sqrt{N}$. On a scientific calculator type `7 =`, then `(ans+7)÷(ans+1)=`, then repeatedly hit equals and watch the numbers converge.

How can we be sure that this process always converges to the correct value? The recursive process above can be described as $x_{n+1} = f(x_n)$, where $f(x) = \dfrac{x+N}{x+1}$. The study of chaos and dynamical systems provides two simple steps to decide what happens in the long-run to any sequence described by a function $f(x)$ in this way: 1) if the sequence has a limit, it must be a so-called fixed-point, namely a solution of $f(x) = x$; 2) if $\left| f'(x) \right| < 1$ at that solution, then it is a stable fixed-point and the sequence will converge to it.

How does the efficiency of this process compare to other algorithms, such as applying Newton's Method? Well, this is where things really start to get interesting.

CONCLUSION

Not everything discussed in this article is a ready-made activity to be slotted seamlessly into the average classroom, although the enthusiastic teacher might be inspired to adapt various aspects of the discussion into an appropriate teaching point or investigation. More significantly, though, is the philosophy of how to think about mathematics and mathematical problem solving: Ask deep questions, think about ways to connect what you already know, never be satisfied with an answer. Importantly, never worry too much if the answer is beyond your grasp – There is always more to learn! This mindset is applicable in all topics and at all stages of the curriculum, for teachers and students alike.

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

REFERENCES

Boyer, C. B. (1991). *A history of mathematics*. New York, NY: John Wiley and Sons.

Cut the Knot. (2018). *Triangular numbers that are also square*. Retrieved from https://www.cut-the-knot.org/do_you_know/triSquare.shtml

Di Pasquale, A., Do, N., & Matthews, D. (2014). *Problem solving tactics*. Canberra, Australia: Australian Mathematics Trust.

Hutchinson, J. (2006). *An introduction to contemporary mathematics*. Canberra, Australia: The Australian National University.

Larson, R. E., Hostetler, R. P., & Edwards, B. H. (1994). *Calculus*. Toronto, Canada: D. C. Heath and Company.

Sloane, N. J. A. (2018). *Online encyclopedia of integer sequences, A001110*. Retrieved from https://oeis.org/A001110

Wikipedia. (2018). *Pell's equation*. Retrieved from https://en.wikipedia.org/wiki/Pell%27s_equation

Wolfram MathWorld. (2018). *e continued fraction*. Retrieved from http://mathworld.wolfram.com/eContinuedFraction.html

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Summary papers

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Using the context of the Olympic Games to enhance the learning of mathematics

Pam Hammond

Teachers aim to embed school mathematics in relevant and engaging contexts. The Olympic Games provides an ideal opportunity for students to experience the use of maths content that they are engaged with in the classroom, to events beyond the classroom in sporting and other contexts, as well as making connections across mathematics content and other curriculum areas. In this paper, you will see how most aspects of mathematics and the proficiencies can be included as part of a unit/program focusing on the Olympics.

WHY USE THE OLYMPIC GAMES IN YOUR PROGRAM?

Embedding school mathematics in relevant and engaging contexts for students is advocated in the Australian Curriculum – Mathematics (Australian Curriculum, Assessment and Reporting Authority, n.d.). What could be more relevant and engaging for students than an international sporting event of such diversity as the Olympic Games? There will be approximately 205 nations participating in the Olympic Games in Japan from 24th July to 9th August. The countries range from the world's largest populations, such as China and India with populations of around 1.5 and 1.3 billion, to some of the world's smallest populations such as St. Helena, a 308 square kilometre island in the Atlantic, with a population of 8,000.

SPORTS AT THE OLYMPIC GAMES

There will be 33 different sports and 339 different events: aquatics, archery, athletics, badminton, basketball, boxing, canoeing, cycling – road, cycling – track, equestrian events, equestrian – dressage, fencing, football, golf, gymnastics, handball, hockey, judo, karate, pentathlon, rowing, rugby, sailing, shooting, skateboarding, surfing, table tennis, taekwondo, tennis, triathlon, volleyball, weightlifting, and wrestling.

The sport and mathematics connection provides a wonderful opportunity for students to experience the use of the mathematics at school to the society. This event is a rich context for the application of mathematics not related to sport, for example the exploration of the countries competing in terms of populations, area, team size in relation to population etc. Virtually every aspect of mathematics could be included as part of a unit focussing on the Olympic Games, as well as the potential to integrate this with all learning areas across the curriculum.

MATHEMATICS AND THE OLYMPIC GAMES

All mathematics domains and topics can be linked to the Olympic Games. Here are some suggestions.

Statistics and Probability

- Predict how many countries participating in the Olympics are represented by school families. Survey and graph nationalities of students in the school/year level/class.

- Students develop questions to collect data on favourite sports/games, sports played by students, and their favourite event in the Olympic Games. Explore ways of representing data.

- Students predict and justify the number of medals certain countries will win. Keep a medal tally over the duration of the Games. Compare at the end and discuss.

- Students collect personal data during sport (running, jumping, swimming, goals, shots on goal, etc.) and represent graphically comparing these with elite athletes.

- Investigate the schedule of events (available on the Tokyo Olympic website).

Measurement and Geometry

- Estimating, then measuring lengths (100 m, 200 m, 400 m, 1,500 m, long/high jump, shot put, javelin records), time/

measure themselves, compare with Games' records

- Students estimate the height/length of the Games record jumps

- Dimensions and volume of the Games swimming and diving pools

- Distance travelled and time taken for athletes to travel to Japan (include time zones)

- Shapes of different arenas and the reasons for these shapes, also field event areas

- Design of equipment – shapes involved

- Designing logos and flags – link to fractions

- Location of Tokyo and countries on a world map (include longitude and latitude)

Number and Algebra

- Ordinal number can be explored when young students are performing in a sports day or having races with toys (There are picture story books to use as stimulus)

- Tally medals; combinations of gold, silver, and bronze if there are 'x' medals

- Order numbers of athletes in each country's team; compare populations of countries

- Explore decimals by comparing results of competitors in an event (timing, height/length/mass); comparing winners with past events; comparing with students' results;

- Investigate scoring of different events

- Numbers of volunteers/officials needed in the various sports

- Explore catering quantities by using local data (e.g., canteen/catering outlet)

- Explore waste management by investigating this at school level first

- Speed of athletes/swimmers/cyclists – How can this be determined?

- Investigate fitness measures (pulse rate, heart rate at rest and when exercising)

- Investigate gearing on bikes

Links to the mathematics proficiency strands can be made using this context. There is also the possibility of developing integrated units, such as 'Plan a day at the Olympic Games' or 'Plan a trip to Japan to attend the Games', with students being personally involved in the decision making as an individual, or planning as a group.

REFERENCES

Australian Curriculum, Assessment and Reporting Authority. (n.d.). *Australian Curriculum: Mathematics*. Retrieved from https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/

Australian Olympic Committee. (n.d.). *Australian Olympics*. Retrieved from http://corporate.olympics.com.au/

International Olympic Committee. (2019). *Tokyo 2020*. Retrieved from https://www.olympic.org/tokyo-2020

Hammond, P. (2014). Mathematics and the Commonwealth Games. *Prime Number 29*(3), 5-7.

Mathematics and mathematical literacy teachers' comments on mathematics and mathematical literacy solution strategies: An issue of a formula

France Machaba, University of South Africa

INTRODUCTION

Officially, mathematics and mathematics literacy (ML) are two separate learning areas in South Africa. Learners from Grade 10 onwards take one or the other, but not both. This means that there is the potential that by the time learners reach Grade 11, they would have acquired different kinds of knowledge and problem solving skills depending on which of these they take. This paper, therefore, attempts to explore teachers' comments on mathematics and ML learners' solution strategies. Teachers' data is analysed in relation to learners' data analysis because learners' responses reflect how they have been taught (Machaba, 2018). How teachers teach mathematics and ML is also a reflection of how they perceive the two learning areas and how they have been inducted into mathematics and ML, and who inducted them.

MATHEMATICS AND MATHEMATICAL LITERACY

In the South African context, mathematics has been defined within the Curriculum and Assessment Policy Statement of the Further Education and Training phase as "mathematical problem solving that enables us to understand the world (physical, social, and economic) around us, and most of all, to teach us to think creatively" (Department of Basic Education [DBE], 2011, p. 8).

The fact that mathematics is "about problems in the physical, social and economic world" (DBE, 2011, p. 8) suggests that it appeals to the everyday context and to the context of mathematics itself. It can also be viewed as a practice that constitutes skills or practices such as "problem solving, observing patterns and generalizing to understand physical, social and economic world" (DBE, 2011, p. 8). It is apparent that, in this definition, mathematics is a hybrid subject (Parker, 2006), an integration of everyday and school-based mathematical knowledge, but skewed towards school mathematical knowledge. The everyday context in this instance could be interpreted as a vehicle to understand school mathematics.

TASK 4 ANALYSIS

In the main study, four tasks were administered to both mathematics and ML learners. Four mathematics teachers and four ML teachers were chosen for interviews because they are specialists in mathematics and ML. An in-depth interview was conducted with each teacher.

Task 4

Thandi washes her dishes by hand three times daily in two identical basins. She uses one basin for washing the dishes and the other for rinsing the dishes. Each basin has a radius of 30 cm and a depth of 40 cm, as shown in Figure 1.
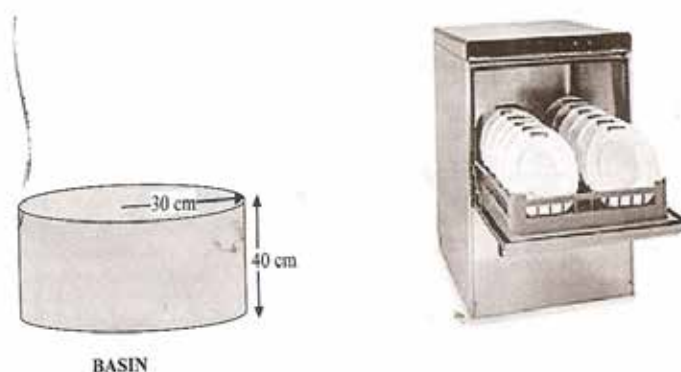

Figure 1. Task 4 question.

Thandi is considering buying a dishwasher that she will use to wash the dishes daily. Calculate the volume of one cylindrical basin in cm³.

FINDINGS AND CONCLUSIONS

In a test, I was not expecting the concept of volume to be constructed; that surely should have happened earlier in the learning phase, in which case a constructivist approach to developing the rule is clearly sensible to facilitate the development of conceptual understanding (Machaba, 2016). However, one would expect an understanding of the application of the formula because conceptual understanding would have been developed through an investigative way (Boaler, 1997) of teaching concepts such as volume. However, most teachers seem to suggest that learners will just substitute numbers without understanding the concept of volume. Some of the teachers' concern seems to be not only about the fact that learners are often given formulae in tests, but also about how they teach the concept of volume itself. Others seem not support the idea of providing formulae to learners, but with an understanding that the concept should have been understood during the teaching process.

Thus, I argue that during the teaching phase both mathematics and ML, teachers have to teach the concept of volume practically so that learners could develop a conceptual understanding rather than just substituting numbers in the formula, which lead to procedural knowledge. Though two subjects are preparing students for different future pathways – M for the options of engineering and medicine, and ML for a pathway that requires less calculus and complex mathematical understanding – all teachers have to emphasise the conceptual understanding of concepts such as volume.

REFERENCES

Boaler, J. (1997). *Experiencing school mathematics: Teaching styles, sex and setting*. Buckingham, England: Open University Press.

Department of Basic Education. (2011a). *Curriculum and assessment policy statement 10–12 (General): Mathematical literacy*. Pretoria, South Africa: Department of Education.

Department of Basic Education. (2011b). *Curriculum and assessment policy statement 10–12 (General): Mathematics*. Pretoria, South Africa: Department of Education.

Machaba, F. M. (2016). The concepts of area and perimeter: Insights and misconceptions of Grade 10 learners: Original research. *Pythagoras*, *37*(1), 1-11.

Parker, D. (2006). Grade 10-12 mathematical curriculum reform in South Africa: A textual analysis of new national curriculum statement. *African Journal of Research in Mathematics, Science and Technology Education*, *10*(2), 59-73.

Developing early maths concepts through play

Leah O'Neill, Back-to-Front Maths

Young students are active participants in their learning, particularly when engaged in play contexts, but how do we best use play to teach mathematical concepts and ensure deep understanding? This paper will explore a method for play-based mathematics learning suitable for young students. A simple planning structure that follows a sequence of small group play-based activities and whole class discussions is used to develop intended mathematical concepts and proficiencies.

IMPORTANT CHARACTERISTICS OF YOUNG LEARNERS

Provision of appropriate learning experiences for young students must take account of their characteristics. Young learners commonly learn by observing, examining and interacting with their environment. They are generally resilient and are capable of making and acting on choices, expressing their opinions and describing their understanding in a variety of verbal and non-verbal ways.

Young students learn best when they are actively involved in exploring, manipulating and constructing their environment in a way that is meaningful to themselves. They use a scientific approach to make inferences, then experiment and test their ideas, using their observations to develop broader and deeper understanding of their world.

PLAY AND LEARNING IN MATHEMATICS

A connectionist approach to learning mathematics can be used to structure mathematical learning experiences that begin with students engaging in free play with materials or within contexts purposefully selected by the teacher. The teacher's role is to extend this play to focus on the intended learning using challenging questions and problems to draw out student ideas and solutions. Focused questioning and explicit teaching of the intended concept and proficiencies follow with the goal of promoting problem solving and extending student thinking. Finally, the teacher assists students to make links with other materials, contexts and concepts.

LEARNING INTENT

When planning a unit of work, the teacher first needs to consider the concept that is to be the learning focus along with the mathematical proficiencies to be developed. Clarity regarding intentions assists the teacher to choose appropriate contexts for the learning and provide opportunities for the development of deep understanding of a concept by avoiding broad but superficial attention to too many topics.

Specifically planning for the development of mathematical proficiencies provides a focus for interactions between teacher and students. Using this focus, the teacher is able to encourage students to develop dispositions necessary for becoming effective users of mathematics. Teachers model focus dispositions and mathematical proficiencies, encouraging students to adopt them through their reactions to what students do and say.

VOCABULARY

Young learners are continually expanding their vocabulary and this includes words and phrases with mathematical meaning. The planning process, therefore, requires focused attention on the vocabulary to be developed throughout the unit. Through discussion with the teacher, students become familiar with mathematical terms and descriptions and begin using them within the context of the play and during subsequent learning activities.

CHILD-INITIATED, SPONTANEOUS PLAY

The teacher's role here is to choose materials or a play context that will provide opportunities to focus students' attention on the chosen learning intentions. Materials that are flexible and engaging will encourage students to play and to interact with other students in a way that is meaningful to them. As the students play, the teacher observes and asks questions that focus their attention on the intended mathematical concept (e.g., What did you use? How many did you use? What do you have most of?).

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

TEACHER-INITIATED, PLANNED PLAY

During subsequent play activities, the teacher further encourages the development of the intended concept by asking questions and encouraging students to wonder about something related to it (e.g., Can you make soup with only two ingredients? Which ingredient did you have more of? How could you be sure?). Observation of student responses will help identify the extent of student understanding and design questions that will extend and challenge them.

CHALLENGE

Challenging problem-solving questions need to be unfamiliar and require students to think about mathematical ideas that are not routine or haven't been taught yet (e.g., I have __ objects in my collection. [Choose a number larger than theirs.] Do you have more or fewer than I have? Can you make your collection have the same number as mine? What did you do?). For extra challenge, add and/or take away from your collection and note whether the student can adjust theirs to take account of the change.

EXPLAIN AND PRACTISE

During this phase, students share their strategies and listen to other students' explanations. Focused teaching occurs when the teacher re-explains the strategies that students used, simplifying and clarifying the ideas and modelling the intended vocabulary. Students are then encouraged to solve similar problems in order to practise choosing and using strategies they have learned.

CONNECT AND GENERALISE

Opportunities to extend the mathematical thinking are then sought to help students make links with other concepts and contexts. More structured materials like ten frames, counters, or cubes can be introduced at this point and questions asked that would extend and deepen students' understanding of the concept (e.g., I have four blue counters on my ten frame. I want to have nine. How could I make it into nine using these yellow, green or red counters?). Provide fewer than five of each colour if you want to encourage more two partitions.

REFERENCES

Fluckiger, B., Dunn, J., & Wheeley, E. (2015). *Age-appropriate pedagogies for the early years of schooling: Foundation paper*. Retrieved from https://earlychildhood.qld.gov.au/earlyYears/Documents/age-appropriate-pedagogies-foundation-paper-summary.pdf

Inquiry-driven mathematics planning: Using themes to link curriculum

Tim Teague and Stan Grazotis, Bell Park North Primary School

In-depth and hands-on planning by developing the interconnections throughout the mathematics curriculum, driven by inquiry and STEM themes, is fun, engaging, and allows students to work within their 'zone' and utilise their strengths to support their learning. It is imperative that teachers seek opportunities to plan for and develop these interconnections. By utilising themes and allowing for a collaborative and inclusive approach to planning, student engagement improves, professional practice develops through the development of skillsets, and teaching certainly becomes increasingly differentiated.

INTRODUCTION

Mathematics is fun. Technology can be engaging. Inquiry fosters play. Themes provide links. Combining these elements together provides a strong foundation to foster student engagement, allowing students to enjoy the process, not just focus on outcomes. Teachers can teach 'through' a task, not simply 'to' a task. Teachers must have sufficient curriculum knowledge to support the facilitation of lessons, particularly the rich conversations driven by students, drawn from rich, meaningful, and engaging tasks involving themes and technology. As the Australian Curriculum, Assessment and Reporting Authority (n.d.) stipulates, "Technologies are an essential problem-solving toolset in our knowledge-based society." These elements of the planning process allow teachers to focus on developing their skill set as well as their professional knowledge whilst also providing a 'localised' context to the curriculum. Pollari, Salo, and Koski (2018) assert that an important element of the Finnish education model is, "the education providers, usually the local education authorities and the schools themselves, draw up their own curricula within the framework of the national core curriculum, which also allows some local characteristics and emphases in the curricula" (p. 2).

THEMES

Themes are not a new idea in education or a new concept when designing inquiry-driven planning. When utilising inquiry to underpin a term of planning, the remainder of the curriculum makes more sense and interconnects. We utilise a two-year rotation of inquiry themes, ensuring the availability of the breadth of the inquiry curriculum and the opportunity to consistently integrate and reinforce mathematical concepts and proficiencies. Themes provided a level of engagement and connectedness that has allowed students to inherently learn at their point of need. More importantly, developing these themes within a localised context ensures a unit's relevance, longevity, and value. Engagement certainly goes to another level with the incorporation of technology, which is always utilised to complement learning processes, never as the main learning objective.

STEM AND TECHNOLOGY

At Bell Park North Primary School, we have implemented a framework that is utilised across Years 2-6, incorporating the Fab Lab philosophy to support learning processes and make connections. Fab Labs are "communities, services, places, and – especially – technologies" (Menichinelli et al., 2017, p. 33). Menichinelli et al. (2017) suggest that schools are inclusive, with both tangibles, such as technology, and the intangibles, such as human interaction. Hence, we have utilised VR headsets, Spheros, 3D printing, and Makey Makey as technologies to complement learning processes and provide opportunities to link the entire curriculum. STEM and the use of technology are overarching, binding student learning. By utilising different technologies within each theme, students are presented with many examples of how technology complements the learning process. As such, we are mitigating several layers of reasons why teachers do not differentiate – time, resources, limited preparation, concerns with classroom management, and the lack of knowledge to support differentiation (Roberts & Inman, 2015). Differentiation allows students to work through an interconnected curriculum.

'THROUGH', NOT 'TO', AND CREATIVE THINKING

Critical and creative thinking and the relevant curriculum allow teachers to teach 'through', not 'to' (Vingerhoets, n.d.).

THE MATHEMATICAL
ASSOCIATION OF VICTORIA

Teachers need to have the appropriate level of knowledge about the curriculum and assists with developing a skillset to engage learners at their point of need. Teaching 'through' an activity ensures that differentiation is happening and that natural links throughout the curriculum are made, which assists teachers with the 'Where to?' question and allows students to contextualise the curriculum and units of work. Sullivan (2016) supports this concept, stating that students can "process multiple pieces of information, with an expectation that they make connections between those pieces, and see concepts in new ways… choose their own strategies, goals, and level of accessing a task". Letting students take control over their learning can be challenging. The concept of 'through' provides authentic learning opportunities and can improve engagement via productive use of student time, stimulating the brain, and creating fairness and equity (Roberts & Inman, 2015). This form of differentiation ensures that missed opportunities are kept to a minimum.

CONCLUSION

Planning is an exciting opportunity to reflect student interests and make connections. The use of themes, technology, and conversations enable classrooms to be engaging, exciting, and experimental environments. Critical and creative thinking is incorporated into all classrooms. By utilising themes, we can make the curriculum work for us, not the other way around.

REFERENCES

Australian Curriculum Assessment and Reporting Authority. (n.d.). *Australian Curriculum: Digital technologies*. Retrieved from https://www.australiancurriculum.edu.au/f-10-curriculum/technologies/digital-technologies/rationale/

Menichinelli, M., Bosque, C., Troxler, P., Raspanti, C., Schaub, A., & Neves, H. (2017). *Fab lab: Revolution field manual*. Salenstein, Switzerland: Braun.

Pollari, P., Salo, O.-P., & Koski, K. (2018). In teachers we trust: The Finnish way to teach and learn. *Inquiry in Education*, *10*(1), 1-16.

Roberts, J. L., & Inman, T. F. (2015). *Differentiating instruction* (3rd ed.). Moorabbin, Australia: Hawker Brownlow.

Sullivan, P. (2016). *Starting learning sequences from an initial challenge*. Retrieved from http://mawainc.org.au/wp2/wp-content/uploads/2016/06/MAWA_15_no_photos.pdf

Vingerhoets, R. (n.d.). *Rob Vingerhoets educational consultancy*. Retrieved from http://robvingerhoets.com.au/maths/

THE MATHEMATICAL
ASSOCIATION OF VICTORIA