MAV Annual Conference 2018

Worthwhile CAS Calculator Use in This Year's 2nd Methods Exam?

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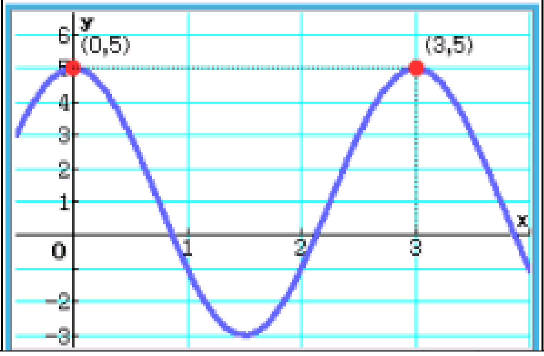
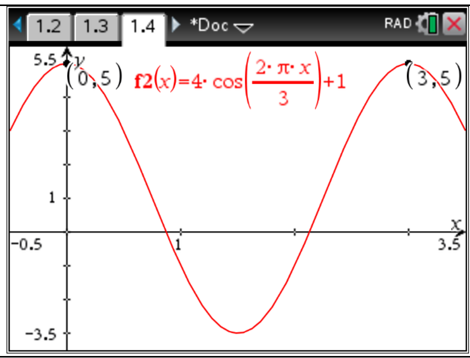
Multiple Choice analysis:

Question 1 (C)

Let $f: R \to R,\ f(x) = 4\cos\left(\dfrac{2\pi x}{3}\right) + 1$. The period of this function is:

Note: should be found by sight but a graph and good scale would assist.

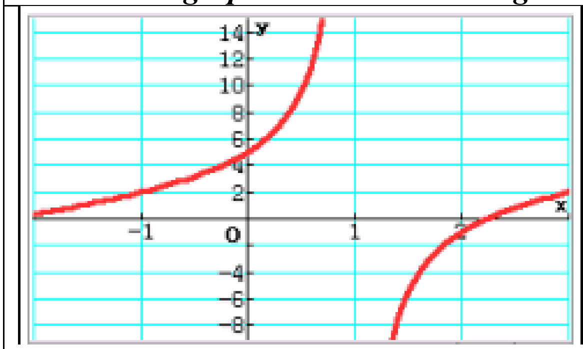


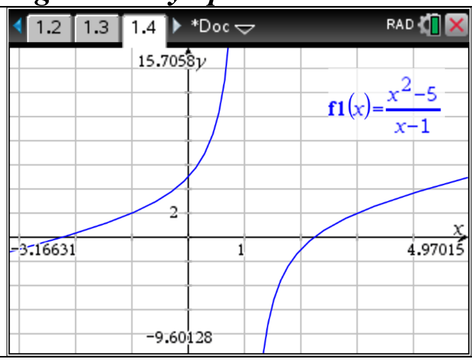


Question 2 (A)

The maximal domain of the function f is $R\backslash\{1\}$.

Note: Given graph is used to locate region corresponding to the asymptote.





Question 3 (D)

Consider the function $f: [a, b) \to R,\ f(x) = \dfrac{1}{x}$ where a and b are positive numbers. The range of f is:

Note: Could graph hyperbola and plot two randomly chosen points.

Question 4 (C)

The point $A(3, 2)$ lies on the graph of the function f. A transformation maps the graph of f to the graph of g, here $g(x) = \dfrac{1}{2} f(x-1)$. The same transformation maps the point A to the point P

The coordinates of the point P are

Note: not specifically calculator.

Question 5 (A)

Consider $f(x) = x^2 + \dfrac{p}{x},\ x \neq 0,\ p \in R$

There is a stationary point on the graph of f when $x = -2$. The value of p is

Note: good standard use of derivative functionality.

$\text{solve}\left(\frac{d}{dx}\left(x^2+\frac{p}{x}\right)=0 \mid x=-2, p\right)$ $\{p=-16\}$	$\text{solve}\left(\frac{d}{dx}\left(x^2+\frac{p}{x}\right)=0 \mid x=-2, p\right)$ $\quad p=-16$

Question 6 (D)

Let f and g be two functions such that $f(x) = 2x$ and $g(x + 2) = 3x + 1$. The function $f(g(x))$ is

Note: needs insight to set up using a transformation matrix as 'define' only allows a single character.

Question 7 (B)

Let $f : R^+ \to R, f(x) = k\log_2(x), k \in R$. Given that $f^{-1}(1) = 8$, the value of k is …

Note: standard substitution and solve

$\text{solve}(x=k\cdot\log_2(y) \mid x=1 \mid y=8, k)$ $\left\{k=\frac{1}{3}\right\}$	$\text{solve}(x=k\cdot\log_2(y) \mid x=1 \text{ and } y=8, k)$ $\quad k=\frac{1}{3}$

Question 8 (B)

If $\int_1^{12} g(x)\,dx = 5$ and $\int_{12}^{5} g(x)\,dx = -6$, then $\int_1^{5} g(x)\,dx$ is equal to

Note: Intuitive thinking to set up and solve; no calculator

Question 9 (C)

A tangent to the graph of $\log_e(2x)$ has a gradient of 2. This tangent will cross the y-axis at

Note: great use of calculator using two steps; solving a derivative and finding a tangent.

$\text{solve}\left(\frac{d}{dx}(\ln(2\cdot x))=2, x\right)$ $\left\{x=\frac{1}{2}\right\}$ $\text{tanLine}\left(\ln(2\cdot x), x, \frac{1}{2}\right)$ $2\cdot x-1$	$\text{solve}\left(\frac{d}{dx}(\ln(2\cdot x))=2, x\right)$ $\quad x=\frac{1}{2}$ $\text{tangentLine}\left(\ln(2\cdot x), x, \frac{1}{2}\right)$ $\quad 2\cdot x-1$

Question 10 (C)

The function f has the property $f(x + f(x)) = f(2x)$ for all non-zero real numbers x. Which one of the following is a possible rule for the function?

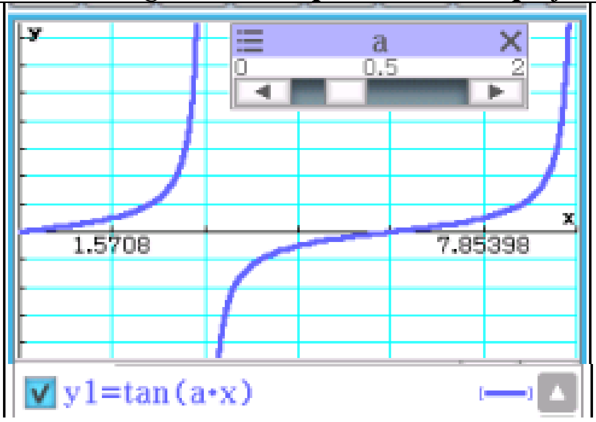
Note: standard comparison of function formats

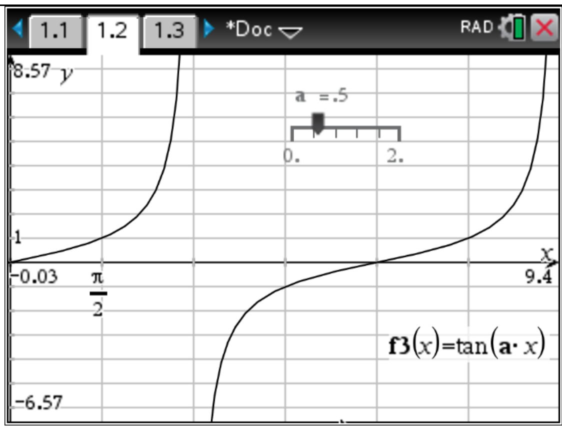
Define $f(x)=x$ done $\text{judge}(f(x+f(x))=f(2x))$ TRUE	Define $f(x)=x$ $\quad$ *Done* $f(x+f(x))=f(2\cdot x)$ $\quad$ true

Question 11 (C)

The graph of $y = \tan(ax)$, where $a \in R^{+}$, has a vertical asymptote $x = 3\pi$ and has exactly one x-intercept in the region $(0, 3\pi)$. The value of a is

Note: Using the slider option with a step of 1/6





Question 12 (E)

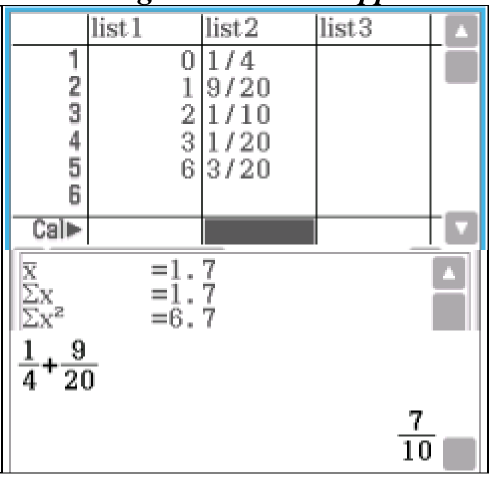
The discrete random variable X has the following probability distribution.

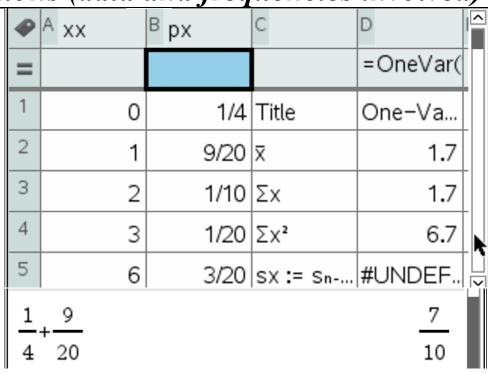
X	0	1	2	3	6
$\Pr(X = x)$	$\frac{1}{4}$	$\frac{9}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{3}{20}$

Let μ be the mean of X.

$\Pr(X < \mu)$ is

Note: using the statistical app and its related calculations (data and frequencies involved)





Question 13 (E)

In a particular scoring game, there are two boxes of marbles and a player must randomly select one marble from each box. The first box contains four white marbles and two red marbles. The second box contains two white marbles and three red marbles. Each white marble scores −2 points and each red marble scores +3 points. The points obtained from the two marbles randomly selected by a player are added together to obtain a final score. What is the probability that the final score will equal +1?

Note: tree diagram or similar with standard arithmetic skills required.

Question 14 (B)

Two events, A and B, are independent, where $\Pr(B) = 2\Pr(A)$ and $\Pr(A \cup B) = 0.52$ $\Pr(A)$ is equal to

Note*: *Probability table or Karnaugh Map, but no useful calculator

Question 15 (E)

A probability density function, f, is given by:

$$f(x)=\begin{cases}\frac{1}{12}\left(8x-x^3\right) & 0\le x\le 2\\ 0 & \text{elsewhere}\end{cases}$$

The median, m, of this function satisfies the equation

Note: good use of integral without using the solve function

$$\text{simplify}\left(\left(\int_0^m \frac{1}{12}\cdot(8\cdot x-x^3)\,dx=0.5\right)\times 48\right)$$

$$-m^4+16\cdot m^2=24$$

$$\text{rewrite}(-m^4+16\cdot m^2=24)$$

$$-m^4+16\cdot m^2-24=0$$

$$\text{expand}\left(\left(\int_0^m \left(\frac{1}{12}\cdot\left(8\cdot x-x^3\right)\right)dx=0.5\right)\cdot 48\right)$$

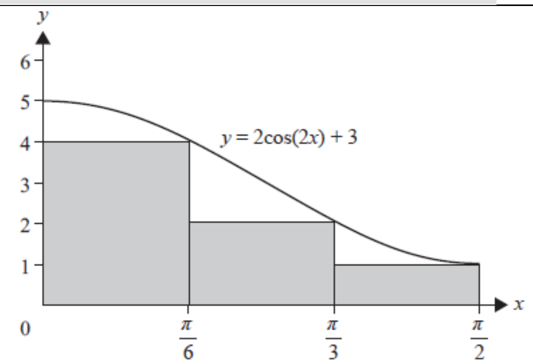
$$16\cdot m^2-m^4=24.$$

Question 16 (B)

Jamie approximates the area between the x-axis and the graph of $y = 2\cos(2x) + 3$, over the interval $\left[0,\frac{\pi}{2}\right]$, using the three rectangles shown.

Jamie's approximation as a fraction of the exact area is

Note: could use a summation function and an integral



$$\frac{\frac{\pi}{6}\left(\sum_{n=1}^{3}\left(2\cos\left(2\times\frac{\pi}{6}\times n\right)+3\right)\right)}{\int_0^{\frac{\pi}{2}} 2\cos(2x)+3\,dx}$$

$$\frac{7}{9}$$

$$\frac{\frac{\pi}{6}\cdot\sum_{x=1}^{3}\left(2\cdot\cos\left(2\cdot\frac{\pi}{6}\cdot x\right)+3\right)}{\int_0^{\frac{\pi}{2}}\left(2\cdot\cos(2\cdot x)+3\right)dx}\qquad \frac{7}{9}$$

Question 17 (C)

The turning point of the parabola $y = x^2 - 2bx + 1$ is closest to the origin when

***Note*: finding a turning point and then a minimum distance.**

$$\text{define }\ cts(a,b,c)=a\left(x+\frac{b}{2a}\right)^2+\text{simplify}\left(\frac{4ac-b^2}{4a}\right)$$

done

$$cts(1,-2b,1)$$

$$(x-b)^2-b^2+1$$

$$\text{solve}\left(\frac{d}{db}\left(b^2+(-b^2+1)^2\right)=0,b\right)$$

$$\left\{b=0,\ b=\frac{-\sqrt{2}}{2},\ b=\frac{\sqrt{2}}{2}\right\}$$

$$\text{completeSquare}\left(x^2-2\cdot b\cdot x+1,x\right)$$

$$(x-b)^2-b^2+1$$

$$\text{solve}\left(\frac{d}{db}\left(b^2+\left(-b^2+1\right)^2\right)=0,b\right)$$

$$b=\frac{-\sqrt{2}}{2}\text{ or }b=0\text{ or }b=\frac{\sqrt{2}}{2}$$

Question 18

Consider the functions $f: R^+ \to R, f(x) = x^{\frac{p}{q}}$ and $g: R^+ \to R, g(x) = x^{\frac{m}{n}}$ where p , q , m and n are positive integers, and $\frac{p}{q}$ and $\frac{m}{n}$ are fractions in simplest form.

If $\{x: f(x) > g(x)\} = (0,1)$ and $\{x: f(x) > g(x)\} = (1,\infty)$, which of the following must be false?

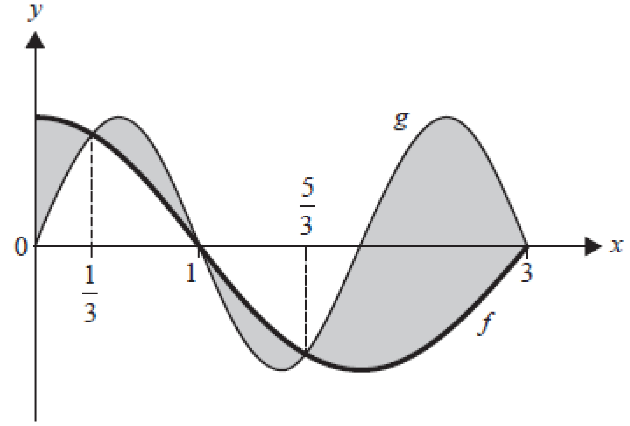
Note: nothing of significance with calculator apart from viewing power functions ….

Question 19

The graphs $f: R \to R, f(x) = \cos\left(\frac{\pi x}{2}\right)$ and $g: R \to R, g(x) = \sin(\pi x)$ are shown in the diagram below.

An integral expression that gives the total area of the shaded regions is

Note: no useful functionalty on calculator



Question 20

The differentiable function $f: R \to R$ is a probability density function. It is known that the median of the probability density function f is at $x = 0$ and $f'(0) = 4$.

The transformation $T: R^2 \to R^2$ maps the graph of f to the graph of g, where $g: R \to R$ is a probability density function with a median at $x = 0$ and $g'(0) = -1$.

The transformation T could be given by

Note:. no useful functionalty on calculator.

Extended Answer analysis:

Question 1.
Consider the quartic and part of the graph of $y = f(x)$ below.
1a. Find the coordinates of the point M, at which the minimum value of the function f occurs.

$\text{fMin}(3\cdot x^4+4\cdot x^3-12\cdot x^2, x, -\infty, \infty)$ $\{\text{MinValue}=-32, x=-2\}$	$\text{fMin}(3\cdot x^4+4\cdot x^3-12\cdot x^2, x, -\infty, \infty)$ $x=-2$ $3\cdot x^4+4\cdot x^3-12\cdot x^2 \mid x=-2$ -32

1b. State the values of $b \in R$ for which the graph of $y = f(x) + b$ has no x-intercepts.

1c. Part of the tangent, l, to $y = f(x)$ at $x = -\frac{1}{3}$ is shown below. Find the equation of the tangent l.

$\text{tanLine}\left(3\cdot x^4+4\cdot x^3-12\cdot x^2, x, -\frac{1}{3}\right)$ $\frac{80\cdot x}{9}+\frac{41}{27}$	$\text{tangentLine}\left(3\cdot x^4+4\cdot x^3-12\cdot x^2, x, \frac{-1}{3}\right)$ $\frac{80\cdot x}{9}+\frac{41}{27}$

1d. The tangent l intersects $y = f(x)$ at $x = -\frac{1}{3}$ and at two other points.

State the x-values of the two other points of intersection. Express your answers in the form $\frac{a \pm \sqrt{b}}{c}$, where a, b and c are integers.

$\text{solve}\left(3\cdot x^4+4\cdot x^3-12\cdot x^2=\frac{80\cdot x}{9}+\frac{41}{27}, x\right)$ $\left\{x=-\frac{1}{3}, x=\frac{-\sqrt{42}}{3}-\frac{1}{3}, x=\frac{\sqrt{42}}{3}-\frac{1}{3}\right\}$	$\text{solve}\left(3\cdot x^4+4\cdot x^3-12\cdot x^2=\frac{80\cdot x}{9}+\frac{41}{27}, x\right)$ $x=\frac{-(\sqrt{42}+1)}{3}$ or $x=\frac{-1}{3}$ or $x=\frac{\sqrt{42}-1}{3}$

1e. Find the total area of the regions bounded by the tangent l and $y = f(x)$. Express your answer in the form $\frac{a\sqrt{b}}{c}$ where a, b and c are positive integers.

$\text{simplify}\left(\int_{\frac{-\sqrt{42}}{3}-\frac{1}{3}}^{\frac{\sqrt{42}}{3}-\frac{1}{3}} \frac{80\cdot x}{9}+\frac{41}{27}-(3\cdot x^4+4\cdot x^3-12\cdot x^2)\,dx\right)$ $\frac{784\cdot\sqrt{42}}{135}$	$\int_{\frac{-(\sqrt{42}+1)}{3}}^{\frac{\sqrt{42}-1}{3}} \left(\frac{80\cdot x}{9}+\frac{41}{27}-(3\cdot x^4+4\cdot x^3-12\cdot x^2)\right)dx$ $\frac{784\cdot\sqrt{42}}{135}$

Let $p: R \rightarrow R, p(x) = 3x^4 + 4x^3 + 6(a-2)x^2 - 12ax + a^2, a \in R$
f. State the value of a for which $f(x) = p(x)$ for all x.
g. Find all solutions to $p'(x) = 0$, in terms of a where appropriate

$\text{solve}(6\cdot(a-2)=-12, a)$ $\{a=0\}$	$\text{solve}\left(\frac{d}{dx}\left(3\cdot x^4+4\cdot x^3+6\cdot(a-2)\cdot x^2-12\cdot a\cdot x+a^2\right)=0, x\right)$ $x=-(\sqrt{1-a}+1)$ or $x=\sqrt{1-a}-1$ or $x=1$

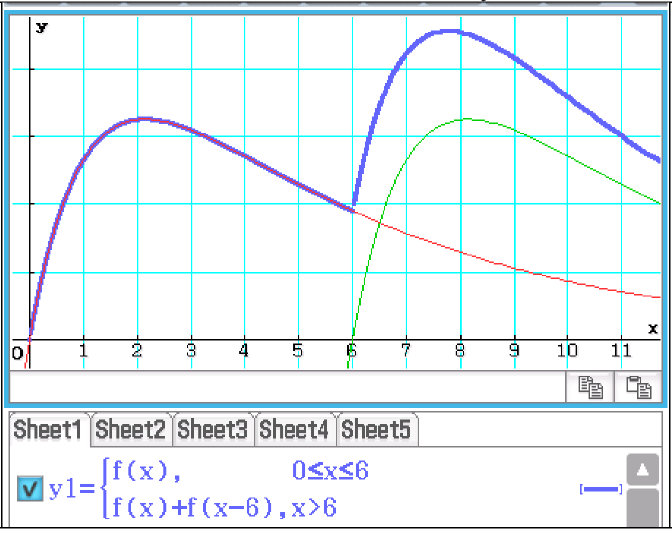
hi. Find the values of a for which p has only one stationary point.

$\text{solve}(1-a<0, a)$

$\{a>1\}$

hii. Find the minimum value of p when $a = 2$	
fmin$(3\cdot x^4+4\cdot x^3+6(a-2)\cdot x^2-12ax+a^2 \mid a=2, x, -\infty, \infty)$ $\{\text{MinValue}=-13, x=1\}$	
hiii. If p has only one stationary point, find the values of a for which $p(x) = 0$ has no solutions	
solve$(3\cdot x^4+4\cdot x^3+6\cdot(a-2)\cdot x^2-12\cdot a\cdot x+a^2=0 \mid x=1, a)$ $\{a=-\sqrt{14}+3, a=\sqrt{14}+3\}$	
Question 2. A drug, X, comes in 500 milligram (mg) tablets. The amount, b, of drug X in the bloodstream, in milligrams, t hours after one tablet is consumed is given by the function $b(t) = \frac{4500}{7}\left(e^{\left(-\frac{t}{5}\right)} - e^{\left(-\frac{9t}{10}\right)}\right)$ **a.** Find the time, in hours, it takes for drug X to reach a maximum amount in the bloodstream after one tablet is consumed. Express your answer in the form $a\log_e(c)$, where $a, c \in R$	
simplify$\left(\text{fmax}\left(\frac{4500}{7}\left(e^{-\frac{t}{5}}-e^{-\frac{9t}{10}}\right), t, -\infty, \infty\right)\right)$ $\left\{\frac{\ldots 35^{\frac{1}{35}}\cdot 108^{\frac{1}{5}}-500\cdot 108^{\frac{1}{35}}\cdot 48^{\frac{1}{5}})}{\ldots}, t=\frac{10\cdot\ln\left(\frac{9}{2}\right)}{7}\right\}$	$\text{fMax}\left(\frac{4500}{7}\cdot\left(e^{\frac{-t}{5}}-e^{\frac{-9\cdot t}{10}}\right), t, -\infty, \infty\right)$ $t=\frac{10\cdot\ln\left(\frac{9}{2}\right)}{7}$
b. Find the average rate of change of the amount of drug X in the bloodstream, in milligrams per hour, over the interval [2, 6]. Give your answer correct to one decimal place. **c.** Find the average amount of drug X in the bloodstream, in milligrams, during the first six hours after one tablet is consumed. Give your answer correct to the nearest milligram.	
define $\text{ar}(a,b)=\frac{f(b)-f(a)}{b-a}$ done define $f(t)=\frac{4500}{7}\left(e^{-\frac{t}{5}}-e^{-\frac{9t}{10}}\right)$ done ar(2, 6) −33.4837811	$\frac{1}{6-0}\cdot\int_0^6\left(\frac{4500}{7}\cdot\left(e^{\frac{-t}{5}}-e^{\frac{-9\cdot t}{10}}\right)\right)dt$ 255.850312778

ci. On the graph above, sketch the total amount of drug X in the bloodstream during the first 12 hours after Tablet 1 is consumed.
cii. Find the maximum amount of drug X in the bloodstream in the first 12 hours and the time at which this maximum occurs. Give your answers correct to two decimal places



$\text{fMax}(f(t)+f(t-6),t,0,12)$ $\quad t=7.77900191605$

$f(t)+f(t-6)|t=7.77900191605$ $\quad 455.81643906$

$$\text{define } f(t)=\frac{4500}{7}\left(e^{-\frac{t}{5}}-e^{-\frac{9t}{10}}\right)$$

done

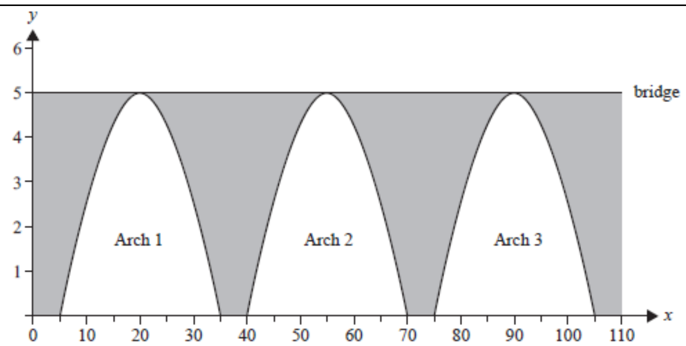
$\text{fmax}(f(t)+f(t-6),t,0,12)$

$\{\text{MaxValue}=455.8164391, t=7.779\}$

Question 3
c. The area above ground level between the arches and the bridge is filled with stone. The stone is represented by the shaded regions shown in the diagram below. Find the total area of the shaded regions, correct to the nearest square metre.

$$\int_0^{110} 5dx-3\times\int_5^{35} 5\sin\left(\frac{(x-5)\pi}{30}\right)dx$$

263.5211024



A second bridge has a height of 5 m above the ground at its left-most point and is inclined at a constant angle of elevation of $\frac{\pi}{90}$ radians, as shown in the diagram below. The second bridge also has three arches below it, which are identical to the arches below the first bridge, and spans a horizontal distance of 110 m.
Let x be the horizontal distance, in metres, from the left side of the second bridge and let y be the height, in metres, above ground level.
d. State the gradient of the second bridge, correct to three decimal places

$\tan\left(\frac{\pi}{90}\right)$

0.03492076949

P is a point on Arch 5. The tangent to Arch 5 at point P has the same gradient as the second bridge.
e. Find the coordinates of P, correct to two decimal places.

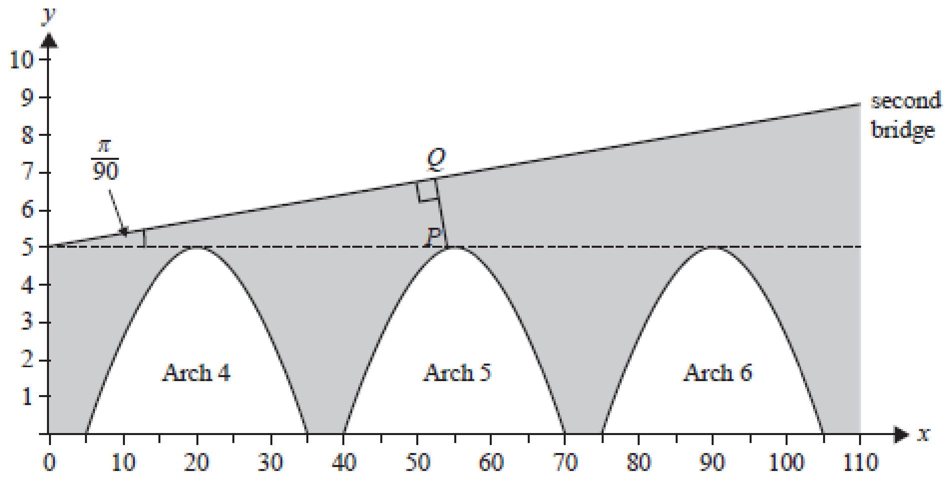
$$\text{solve}\left(\frac{d}{dx}\left(5\cdot\sin\left(\frac{(x-40)\cdot\pi}{30}\right)\right)=\tan\left(\frac{\pi}{90}\right),x\right)|40\le x\le 70$$

$\{x=54.36264844\}$

$$5\cdot\sin\left(\frac{(x-40)\cdot\pi}{30}\right)|x=54.36264844$$

4.988867464

f. A supporting rod connects a point Q on the second bridge to point P on Arch 5. The rod follows a straight line and runs perpendicular to the second bridge, as shown in the diagram on page 18.
Find the distance PQ, in metres, correct to two decimal places.



$$\text{solve}\left(y-4.988867464=\tan\left(\frac{\pi}{90}\right)\cdot(x-54.36264844)\mid x=0, y\right)$$

$$\{y=3.090481949\}$$

$$\text{solve}\left(\cos\left(\frac{\pi}{90}\right)=\frac{5-3.090481949}{d}, d\right)$$

$$\{d=1.910681987\}$$

Question 4.

Doctors are studying the resting heart rate of adults in two neighbouring towns: Mathsland and Statsville.
Resting heart rate is measured in beats per minute (bpm).
The resting heart rate of adults in Mathsland is known to be normally distributed with a mean of 68 bpm and a standard deviation of 8 bpm.

a. Find the probability that a randomly selected Mathsland adult has a resting heart rate between 60 bpm and 90 bpm. Give your answer correct to three decimal places.

normCDf(60, 90, 8, 68)
0.8383649828

normCdf(60,90,68,8)   0.838364921851

The probability that a randomly chosen Mathsland adult has a slow heart rate is 0.1587.

b. Find the probability that a random sample of 16 Mathsland adults will contain exactly one person with a slow heart rate. Give your answer correct to three decimal places

binomialPDf(1, 16, 0.1587)
0.1900904256

For random samples of 16 Mathsland adults, $\hat{P}$ is the random variable that represents the roportion of people who have a slow heart rate.

c. Find the probability that $\hat{P}$ is greater than 10%, correct to three decimal places.

$$\Pr(p>0.1)=\Pr\left(p>\frac{1.6}{16}\right)$$

$$\Pr(p>0.1)=\Pr(p>0.1)$$

binomialCDf(2, 16, 16, 0.1587)
0.746927897

For random samples of n Mathsland adults, $\hat{P}n$ is the random variable that represents the proportion of people who have a slow heart rate.

d. Find the least value of n for which $\Pr\left(\hat{P} > \frac{1}{n}\right) > 0.99$

solve(invBinomialCDf(0.01, x, 0.1587)=2, x) {x=39}	invBinomN(0.01,0.1587,1) 39

Every year at Mathsland Secondary College, students hike to the top of a hill that rises behind the school. The time taken by a randomly selected student to reach the top of the hill has the probability density function M with the rule $M(t) = \begin{cases} \frac{3}{50}\left(\frac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3} & t \geq 0 \\ 0 & t < 0 \end{cases}$ where t is given in minutes.

e. Find the expected time, in minutes, for a randomly selected student from Mathsland Secondary College to reach the top of the hill. Give your answer correct to one decimal place.

Students who take less than 15 minutes to get to the top of the hill are categorised as 'elite'.

f. Find the probability that a randomly selected student from Mathsland Secondary College is categorised as elite. Give your answer correct to four decimal places.

$\int_0^{\infty} t \times \frac{3}{50}\left(\frac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3} dt$ 44.64897557	$\int_0^{15} \frac{3}{50}\left(\frac{t}{50}\right)^2 e^{-\left(\frac{t}{50}\right)^3} dt$ 0.02663875848

Question 5.

Consider functions of the form $f: R \to R, f(x) = \frac{81x^2(a-x)}{4a^4}$ and $h: R \to R, h(x) = \frac{9x}{2a^2}$ where a is a positive real number.

a. Find the coordinates of the local maximum of f in terms of a.

b. Find the x-values of all of the points of intersection between the graphs of f and h, in terms of a where appropriate.

$\text{solve}\left(\frac{d}{dx}\left(\frac{81\cdot x^2\cdot(a-x)}{4\cdot a^4}\right)=0, x\right)$ $\left\{x=0, x=\frac{2\cdot a}{3}\right\}$ $\frac{81\cdot x^2\cdot(a-x)}{4\cdot a^4} \mid x=\frac{2\cdot a}{3}$ $\frac{3}{a}$	$\text{solve}\left(\frac{81\cdot x^2\cdot(a-x)}{4\cdot a^4}=\frac{9\cdot x}{2\cdot a^2}, x\right)$ $\left\{x=0, x=\frac{a}{3}, x=\frac{2\cdot a}{3}\right\}$

c Determine the total area of the regions bounded by the graphs of $y = f(x)$ and $y = h(x)$.

$$\int_0^{\frac{a}{3}} \frac{9\cdot x}{2\cdot a^2} - \frac{81\cdot x^2\cdot(a-x)}{4\cdot a^4} dx + \int_{\frac{a}{3}}^{\frac{2a}{3}} \frac{81\cdot x^2\cdot(a-x)}{4\cdot a^4} - \frac{9\cdot x}{2\cdot a^2} dx$$

$$\frac{1}{8}$$

Consider the function $g:\left[0,\frac{2a}{3}\right]\to R, g(x)=\frac{81x^2(a-x)}{4a^4}$, where a is a positive real number. **d.** Evaluate $\frac{2a}{3}\times g\left(\frac{2a}{3}\right)$ **e.** Find the area bounded by the graph of g^{-1}, the x-axis and the line $x=g\left(\frac{2a}{3}\right)$ **f.** Find the value of a for which the graphs of g and g^{-1} have the same endpoints. **g.** Find the area enclosed by the graphs of g and g–1 when they have the same endpoints

$\frac{2a}{3}\times\frac{81\cdot x^2\cdot(a-x)}{4\cdot a^4}\mid x=\frac{2a}{3}$ 2	$\int_0^{\frac{2a}{3}}\frac{3}{a}-\frac{81\cdot x^2\cdot(a-x)}{4\cdot a^4}dx$ 1
$\text{solve}\left(\frac{3}{a}=\frac{2\cdot a}{3},a\right)$ $\left\{a=\frac{-3\cdot\sqrt{2}}{2},a=\frac{3\cdot\sqrt{2}}{2}\right\}$	$4\int_0^{\frac{a}{3}}\frac{9\cdot x}{2\cdot a^2}-\frac{81\cdot x^2\cdot(a-x)}{4\cdot a^4}dx\mid a=\frac{3\cdot\sqrt{2}}{2}$ $\frac{1}{4}$

Very helpful, as long as I know how to use the CAS.