



### Achieving excellence in M.A.T.H.S | 7 - 8 Dec 2017



# **MAV17 CONFERENCE PROCEEDINGS**

**54TH ANNUAL CONFERENCE** LA TROBE UNIVERSITY, BUNDOORA

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# 2017 Mathematical Association of Victoria Annual conference proceedings

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### Foreward

'Achieving excellence in M.A.T.H.S' is the theme of the 54th Annual Conference Proceedings of the Mathematical Association of Victoria, but what does excellence look like? Five themes - mindsets, assessing, targeting, hands-on and shifting the narrative have been chosen to investigate this further. Building on the work of Carol Dweck and Jo Boaler, several contributing authors and presenters show how having a growth mindset creates motivation and productivity in everything we do. Changing mindsets requires teachers to understand that everyone can learn mathematics and believe that this is achievable. Accurate assessment of what learners know and can do is critical in designing targeted teaching activities. Timely instructions and wise use of hands-on activities allow learners to experience and construct mathematics as taken-as-shared knowledge. Collectively, the teacher and learners work together in changing the narrative of mathematics to be one that is constructive, meaningful, and valuable and in a manner that can be playful and fun.

The collection of papers presented here represents a growing body of knowledge in mathematics education. Excellence in mathematics can be found from early years through to the secondary years and teacher education. It can be cultivated through learning from the past, building relationships and communication, and investigating new ways of doing mathematics and assessing mathematical knowledge. Every effort is aimed at making mathematics accessible to everyone. Excellence in mathematics is not restricted to having content knowledge of number and algebra, but also being able to reason geometrically and spatially.

On behalf of the editorial team, I would like to extend our heartfelt thanks to the authors for their commitment and contributions in sharing their insights of ways to improve the quality of teaching of mathematics across the educational community. I commend several first-time contributors as well as our regular, seasoned professionals who continue to support the Mathematical Association of Victoria. It has been a pleasure and privilege to work closely with such a committed team of authors.

I would also like to thank our reviewers for their dedication and advice. Their contribution and commitment to promote excellence in mathematics is greatly appreciated. Special thanks to the MAV conference organising committee, who work tirelessly to ensure that this conference and proceedings contribute to improving the learning of mathematics. This role in promoting excellence in mathematics cannot be over-emphasised.

I hope you enjoy this engaging and informative selection of papers.

Warmest regards, Rebecca Seah Chief Editor - RMIT University, Melbourne

#### THE REVIEW PROCESS

The Editors received 39 papers for reviewing. 8 for the Double blind review process, for which the identities of author and reviewer were concealed from each other. Details in the papers that identified the authors were removed to protect the review process from any potential bias, and the reviewers' reports were anonymous. Two reviewers reviewed each of the papers and if they had a differing outcome a third reviewer was required.

In addition we received 12 papers for the Peer review process, with one withdrawn before review and 19 Summary papers. These papers were reviewed by a combination of external reviewers and the editorial team.



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Double blind peer reviewed papers



### **Problems associated with learning to represent and define quadrilaterals**

Adrian Berenger, Tasos Barkatsas and Rebecca Seah, RMIT University

Geometric reasoning requires the skills of representing and defining quadrilaterals. In this paper, we are reporting on the results of an investigation into the ways students communicated their understanding of 2-dimensional geometric concepts using diagrams and keywords to describe quadrilaterals. The results showed that whenever Year 8 students used diagrams as part of their descriptions of shapes, mostly prototypical shapes were provided. Students also experienced difficulties discerning which properties could be used to define shapes. This paper provides insights into ways of analysing students' geometric reasoning.

#### INTRODUCTION

The current educational interest in science, technology, engineering, and mathematics [STEM] presents opportunities for richer connections of mathematics with other areas of learning. Having good spatial skills is a strong predictor of achievement and attainment in the STEM fields (Obara, 2013). Architects, artists, physicists, designers, doctors, and engineers employ spatial reasoning skills – an ability to visualise with the mind's eye (National Council of Teachers of Mathematics, 2013).

Geometric problems often require spatial reasoning (Owen & Outhred, 2006) and explicit instruction is critical in its successful application to reasoning and problem-solving (Gillies & Haynes, 2011). The lack of visual and spatial reasoning within the Australian Curriculum: Mathematics (Lowrie, Logan, & Scriven, 2012) denotes a geometry curriculum with an emphasis on memorising vocabulary and applying formulae (Seah, 2015). Most teachers are unlikely to have experienced explicit spatial education themselves and find it difficult to identify opportunities for explicit instruction in spatial reasoning (Diezmann & Lowrie, 2012; Liben, 2006).

Research indicates that many students have difficulties with (a) recognising geometrical shapes in non-standard orientation, (b) formulating accurate and complete definitions, (c) understanding the function or value of hierarchical classification of quadrilaterals, and (d) solving problems based on conceptual aspects of geometrical shapes (Elia & Gagatsis, 2003; Marchis, 2012). Students can be inhibited from engaging in tasks that require visual, logical, and deductive thought due to a lack of spatial and geometric reasoning ability.

#### THE VAN HIELE MODEL OF GEOMETRIC THOUGHT

The work of Pierre and Dina van Hiele provides a framework upon which geometry content may be planned, taught, and assessed. According to the van Hiele model (van Hiele, 1985), geometric thought develops through five distinguishable levels – *visual, analysis, abstraction, formal deduction,* and *rigour*. Progression is based on the principle that appropriate instruction is needed to move students through several qualitatively different levels of geometric thought and increasingly sophisticated levels of geometric understanding and reasoning (Battista, 2001). At *Level 1* (visualisation or recognition), students recognise features of shapes by appearance alone by comparison with known prototypes. Students conceive a shape as a whole not as the sum of its parts. At *Level 2* (analysis or descriptive), students focus on the relationship between parts of a shape and can recognise and name properties of a shape. Students are able to list properties but not discern the properties that are necessary or sufficient to describe shapes. At *Level 3* (abstraction, informal deduction, ordering or relational), students perceive relationships between properties and between figures. Logical implications and class inclusions are understood at an informal level. At *Level 4* (deduction or formal deduction), students can construct proofs and understand the role of definitions. They understand the meaning of necessary and sufficient conditions by deductive reasoning. At *Level 5* (rigour), students understand formal aspects of deduction as well as non-Euclidean systems at this level.

#### VISUALISATION

Visualisation is a process of generating a mental image, whether static or dynamic. The image generated as a mental construct depicts visual or spatial information (Presmeg, 2006). Visualisation involves constructing and transforming images of a spatial nature that are implicated in geometry. Essentially, visualisation is vital for communicating geometric



concepts both verbally and non-verbally at all levels of geometric reasoning (Battista, 2001).

In geometric reasoning, what is important is to have a sense that because a shape has certain properties, other properties must also be true (Cooke, 2007). It is important for students to be able to deduce facts by interpreting the geometric information that they 'see' in their minds (Fujita & Jones, 2006; Owens, 2003). A specific geometric diagram embodies the attributes of a class, providing students with prototypes. Prototypes in geometry are generalised representations having common visual characteristics and are useful for simple manipulations. However, they are limited references to geometrical concepts having internal constraints of organisation and do not support hierarchical, inclusive definitions (Presmeg, 2006). Students need to be able to explore shapes by 'seeing the parts' – a notion that Owens (2003) referred to as *disembedding*. Making sense out of a visual representation involves *re-seeing* it (Tartre, 1990). An image is no longer a 'picture in the head' but rather images are more abstract, malleable, less crisp, and are often segmented into parts.

#### THE ROLE OF DEFINITIONS

Definitions in geometry help to classify shapes (de Villiers, 1998; Usiskin, Griffin, Witonsky, & Willmore, 2008). They serve the dual role of identifying a category to which a shape belongs, and indicating how it might be distinguished from other objects in that category. *Concept definitions* are definitions as a form of words used to specify that concept, and a *concept image* is the total cognitive structure that is associated with the concept (Fujita & Jones, 2006), including all the mental pictures and associated properties and processes (Tall & Vinner, 1981). Fischbein (1993) also defined the notion of a figural concept – a square, for example, is a concept as well as a geometric figure. 'One cannot think of a shape without knowing its definition, and one cannot know the definition without being able to construct the shape in intuition' (Heis, 2014, p. 608). However, many secondary teachers expect a one-way process for concept formation, that is, that '…the concept image will be formed by means of the concept definition and will be completely controlled by it' (Vinner, 1991, p. 71). Consequently, their students have a tendency to use partitional definitions (Heinze & Ossietzky, 2002) creating difficulties with including squares as rectangles, for example, as this 'new' information does not logically connect with what they have been previously taught.

#### MATHEMATICAL DISCOURSE

An analysis of student discourse is an important aspect in understanding students' interpretations of tasks, as well as their ability to communicate geometric concepts. Discourse concerns not only the patterns or regularities of language used in context, but also of the rules of human communicative actions. According to Sfard (2008), mathematical discourse is exhibited by four inter-related components. These are:

- Word use Shapes are described and defined in distinctly mathematical ways using *keywords*. Their usage reveals how a student sees and interprets that shape.
- **Visual mediators** Visual objects that are operated on as part of the communication process that help define shapes and their properties in a universal visual format are known as *visual mediators*.
- **Narratives** A sequence of expressions or statements used to frame descriptions of objects, either spoken or written, are subject to rejection or acceptance as deductive accounts of an endorsed consensus. These accounts are known as *narratives*.
- **Routines** Specific repetitive patterns characteristic of creating and substantiating narratives about shapes form *routines* of mathematical discourse.

The keywords and visual mediators give rise to the narratives and possible routines one might apply to mathematical practices as the '…taken-as-shared ways of reasoning, arguing, and symbolising established while discussing particular mathematical ideas' (Cobb, Stephan, McClain & Gravemeijer, 2001, p. 126). Convincing others requires connecting facts to construct logical arguments as endorsed mathematical discourse.

#### METHOD

Students in two Year 8 classes in an inner suburban school of Melbourne were involved in this study. Students in Class



A (16 boys and 9 girls) participated in a mainstream mathematics curriculum. Students in Class B (15 boys and 9 girls) were part of an accelerated mathematics program since Year 7. According to their mathematics teachers, both classes had studied geometry prior to the commencement of this study although the type of content and instruction were not provided.

The mathematics teachers provided two structured written tasks to students in both classes on two separate occasions. Task A: Properties of Squares and Rectangles required students to draw each shape, identify common properties, and provide a definition that would *include* squares as rectangles. Task B: Properties of Parallelograms and Rectangles required students to draw and list properties of a parallelogram, identify properties similar to rectangles, and provide a definition that would *include* rectangles as parallelograms. The purpose of these tasks was to promote student thinking and communication about geometric concepts in terms of independent and connected properties. Written work samples of students was the main method for generating data. Initial analysis involved classifying student use of *keywords* and *visual mediators*. Further analysis considered the construction of *narratives* used in the students' written discourse.

#### **FINDINGS**

The results from each task are quoted separately in terms of *keywords*, *visual mediators*, and *narratives* used by students. Student misconceptions about shapes are also reported. Since these two tasks were conducted on separate occasions, it is not possible to report on *routines* requiring well-defined discourse patterns.

#### TASK A: PROPERTIES OF SQUARES AND RECTANGLES

#### Keywords

Essentially, squares and rectangles are quadrilaterals with right angles. Consequently, this results in squares and rectangles having pairs of equal opposite sides. What separates them is that squares have sides of equal measure. Several other definitions for squares result from them lying at the bottom of the hierarchy for quadrilaterals (Usiskin, Griffin, Witonsky, & Willmore, 2008). This means that squares may also be defined by properties of other quadrilaterals. For example, because a square is also a parallelogram means that it has two pairs of equal opposite angles because a square is a rectangle means that its diagonals have the same length and bisect each other.

Keywords for a	% of responses	
square	Class A	Class B
quadrilateral	14.3	68.4
2-D	23.8	15.8
polygon	0	15.8
4 sides	14.3	0
4 even sides	33.3	0
4 equal sides	33.3	94.7
4 right angles	38.1	31.6
4 90o angles	33.3	68.4
corners	33.3	5.3
vertices	9.5	15.8
edges	19.0	10.5
parallelogram	0	26.3
parallel sides	33.3	36.9
angle sum	14.3	47.4

Keywords for a	% of responses	
rectangle	Class A	Class B
quadrilateral	14.3	57.9
2-D	9.5	15.8
polygon	0	0
4 sides	19.0	0
pairs even sides	14.3	0
pairs equal sides	19.0	31.6
4 right angles	28.6	26.3
4 90o angles	14.3	68.4
corners	33.3	5.3
vertices	9.5	10.5
edges	19.0	10.5
parallelogram	0	26.3
parallel sides	28.6	36.9
angle sum	14.3	42.1

Table 1. Keywords Used by Students for Squares and Rectangles



As indicated in Table 1, the keywords used by students to describe squares were comparable to keywords that they used to describe rectangles. Both squares and rectangles were also understood to be parallelograms by 26.3% of students in Class B. Students across both classes used *corners* and *vertices* even though they had used *angles*. Similarly, students unnecessarily used *edges* as well as *sides*.

#### **Visual Mediators**

The range of visual mediators used by students is provided in Table 2. All students in Class B provided an accurate diagram of a square with correct signifiers for right angles and equal side measures, and 89.5% provided accurate diagrams for rectangles. In comparison, 71.4% of students in Class A produced basic or incomplete diagrams.

	% of responses		
visual mediators for a square	Class A	Class B	
No diagram	0	0	
Incorrect diagram	0	0	
Basic shape (no signifiers)	52.4	0	
Correct shape with inaccurate signifiers	19.0	0	
Accurate shape (correct use of signifiers)	28.6	100	

Visual mediators for a	% of responses		
rectangle	Class A	Class B	
No diagram	0	0	
Incorrect diagram	0	0	
Basic shape (no signifiers)	47.6	0	
Correct shape with	23.8	10.5	
inaccurate signifiers			
Accurate shape	28.6	89.5	
(correct use of signifiers)			

Table 2. Types of Visual Mediation Used by Students for Squares and Rectangles

These proportions indicate that students in Class A had difficulties with understanding the significance of diagrams in communicating geometric concepts beyond basic shape outlines. It is conjectured that their inability to produce mathematically acceptable diagrams led to difficulties in using keywords to describe each shape accurately. Table 3 provides a range of diagrams used by students to represent squares and rectangles, and use of keywords and narratives to define them.



Visual Mediators for Squ Rectangles	ares and Keywords (similar properties)	Narratives (inclusive definition)
Samuel (Class A)	· ·	
рени рения Бения Б. на, d	4 right angles	(not given)
(square) (rectangle)	)	
Max (Class A)		
	(not given)	(not given)
(square) (rectangle	)	
Ava (Class A)	quadrilateral right angles	that have 4 right angles
(square) (rectangle	)	
(square) (class B)	angles 90° 4 sides	that have all angles at $90^{\circ}$
Hannah (Class B)	opposite sides are equal quadrilateral 4 right angles $360^{\circ}$ A = L X W	shapes with only 4 right angles

Table 3. Visual Mediators, Keywords and Narratives Used for Squares and Rectangles

Angelo's (Class B) diagram of a square illustrated a higher level of geometric thinking consistent with van Hiele's *Level 3: Abstraction*, allowing him to reason that only one right angle needed to be signified for a square. This type of thinking depicted an understanding of the sufficient properties of squares. However, this is insufficient for defining rectangles requiring more than one right angle to be signified.

#### Narratives

Table 4 illustrates the range of endorsed narratives for squares and rectangles. These indicated stark differences between students in these classes. More than 50% of students in Class A had difficulties in providing a mathematically acceptable definition. Few students in Class B had some understanding of the hierarchy of quadrilaterals, being able to connect and order squares and rectangles in relation to parallelograms.

Definitions (Endorsed Narratives)	% of responses	
'Only squares and rectangles are quadrilaterals	Class A	Class B
that have four right angles $(90^{\circ})$ .	33.2	60.9
[that are also] parallelograms with right angles (90°)	0	10.5

Table 4. Definitions for Squares and Rectangles



#### **Misconceptions and difficulties**

The students that used signifiers tended to over-use them when drawing diagrams. In the case of Hannah (Class B), her diagrams showed further understanding that the diagonals of a square bisect each other at right angles. Students like Hannah used their diagrams to indicate all that they knew about shapes without discerning what was necessary and sufficient to define them. Several inconsistencies with matching visual mediators with lists of properties and accurate definitions were detected in many students' written work. For example, 15.8% of students in Class B used words such as *quadrilaterals*, but also used *polygons* or *2-D shapes* which were both redundant. Where students listed properties but were unable to discern which properties were necessary and sufficient to describe an object, is consistent with van Hiele's *Level 2: Analysis*.

Students had difficulties in articulating an inclusive definition for squares and rectangles (66.8% for Class A and 26.3% for Class B). Many students in both classes believed that rectangles must have a pair of opposite sides longer than the other pair. This was the case whether diagrams were included or not. It may be conjectured that students' previous learning experiences treated squares and rectangles as different shapes, and thereby hindered many of them from seeing squares as a subset of rectangles.

Ryan:	longer than a square
Andre:	2 long lines
Christopher:	uneven length of edges
Henrietta:	two sets of even sides

Some students had difficulties using the *parallel* concept effectively. Often, students wrote the term *parallel* in reference to squares and rectangles in isolation and without any narrative or visual mediators with signifiers. Several students also used parallel line signifiers even though these were redundant given that 4 right angles had already been indicated.

#### Task: Properties of Parallelograms and Rectangles

#### Keywords

An endorsed narrative for parallelograms and rectangles is that they are quadrilaterals with two pairs of parallel sides. Rhombuses are also included within this definition. Students exhibiting the ability to order properties is consistent with van Hiele's *Level 3: Abstraction* where logical implications and class inclusions are understood, signifying the geometric thinking required for understanding a hierarchy of quadrilaterals. The keyword *parallel* is embedded in the word parallelogram, providing students with an essential property for this shape and its definition. As indicated in Table 5, the keywords generated by students in Class B included *parallel* in their lists of properties in higher proportion than students in Class A.

Keywords	% of re	% of responses	
	Class A	Class B	
quadrilateral	13.6	31.6	
2-D	0	10.5	
4 sides	22.7	15.8	
4 lines	13.6	5.3	
pair equal sides	13.6	63.2	
4 corners	22.7	0	

Keywords	% of re	% of responses	
	Class A	Class B	
face	4.5	0	
opp sides eq.	9.1	36.8	
diagonal	0	42.1	
parallel	27.3	84.2	
cointerior	0	21.1	
angle sum	0	26.3	

Table 5. Keywords Used to Describe Parallelograms and Rectangles

Individual students in Class B indicated an understanding of terms such as *congruency* and *diagonal bisection*. Some students in Class B (36.8%) and Class A (9.1%) identified that parallelograms and rectangles have opposite sides of equal length, and one student in each class also stated that both shapes have equal opposite angles.



#### Visual mediation

For students in Class A who provided an accurate diagram of a parallelogram, 13.6% indicated parallelism. The other 18.2% indicated pairs of equal sides. The rest of the class continued to provide basic shapes without any signifiers. Table 6 summarises the types of diagrams used by students for parallelograms.

Keywords	% of responses		
	Class A	Class B	
No diagram	9.1	0	
Incorrect diagram	9.1	0	
Basic shape (no signifiers)	50.0	0	
Correct shape with inaccurate signifiers	0	21.1	
Accurate shape (correct use of signifiers)	31.8	78.9	

Table 6. Visual Mediators Used by Students for Parallelograms

Table 7 shows a sample of visual mediators, keywords and narratives used by students. Louie's use of right angles might suggest an inability to order properties necessary to facilitate an understanding of hierarchies of quadrilaterals. It is unclear why Louie would draw a rectangle – a specific parallelogram. Almost every other parallelogram drawn by students was in prototypical (horizontal) orientation as in Harriot's depiction.

Visual Mediators for Parallelograms	Keywords (similar properties)	Narratives (inclusive definition)	
Andre (Class A)	they have equal sides	[they] have equal sides	
Lucas (Class A)	tilted rectangle	[have] two short and two long – exactly the same length	
Louis (Class B) A rectangle is a parallelogram because it has 2 sets of parallel lines		[contain] at least 2 pairs of parallel lines	
Josephine (Class B)	$360^{\circ}2$ sets of parallel sides	[have] at least 2 equal angles	
Harriot (Class B)	A parallelograms' only difference to a rectangle is that it's corners aren't 90°. In all other ways, it is the same.	only have 2 pairs of equal AND parallel sides. No more, and no less. They also do not have any additional sides	

*Table 7. Samples of Visual Mediators, Keywords and Narratives Used by Students to Describe Parallelograms and Rectangles* 

#### Narratives

Where students provided a narrative about parallelograms and rectangles, these were based on pairs of parallel sides as shown in Table 8. Formulating a narrative for parallelograms and rectangles could also be based on pairs of equal opposite sides. Knowing that definitions involve necessary and sufficient conditions suggests an ability to reason and understand the hierarchical nature of quadrilaterals.



	% of responses	
Definitions (Endorsed Narratives)	Class A	Class B
<i>All parallelograms and rectangles</i> have two pairs of parallel sides.	0	41.7
have two pairs of equal parallel sides.	9.1	36.8

Table 8. Student Definitions for Parallelograms and Rectangles

Most interestingly, Angus (Class B) defined parallelograms and rectangles as shapes where 'diagonals meet in the middle'. If Angus meant that diagonals bisect each other, then his narrative is accurate and sufficient to define all parallelograms without having mentioned the necessary condition of parallel sides.

#### **Misconceptions and difficulties**

Students in Class A drew basic shape outlines without signifiers. Students in Class B tended to over-use signifiers in their diagrams. Even though every student in Class B included a diagram of a parallelogram with signifiers, these often indicated pairs of equal sides only, and further, several diagrams contained inaccuracies such as diagonals intersecting at right angles – true for rhombuses but not for all parallelograms.

Many students that provided definitions also included unnecessary information such as having parallel sides *and* having opposite sides equal. Students in Class A often stated that two sides needed to be longer, or described and depicted parallelograms as 'tilted' rectangles. These misconceptions are summarised in Table 9.

	% of responses	
Misconceptions	Class A	Class B
parallelograms are tilted rectangles	22.7	4.8
parallelograms and rectangle have longer sides	18.2	0

#### Table 9. Misconceptions for Parallelograms and Rectangles

An explanation why students in Class B did not demonstrate the same misconceptions at the same rate as students in Class A might be as a result of the correct use of visual mediation for parallelograms. Their attention was not drawn to side lengths due to the use of diagrams of parallelograms with parallel side signifiers.

Other less common misconceptions were observed in some students' written work in Class A, attributed to the use of prototypical diagrams of shapes. For example,

Anthony: ...4 angles, two bigger and two smaller

A proportion of students in Class A understood the significance of diagrams in mediating their understanding about shapes yet listed memorised facts rather than using their diagrams to help with the reasoning required to produce inclusive and sufficient definitions.

#### CONCLUSION AND IMPLICATIONS

Students in Class B demonstrated a more sophisticated repertoire of geometric language that they used to generate descriptions of shapes, and were able to order properties, moving toward sufficient definitions. There was some alignment between visual mediators and narratives used by these students indicating that their concept images of certain shapes coincided with their concept definitions resembling a more 'formal' concept definition – one accepted by the mathematical community at large (Tall & Vinner, 1981).

Interpretations of their own diagrams however, often led students to incorrect definitions. For example, when students indicated on a diagram of a rectangle, two pairs of equal opposite sides, this, in turn, was incorrectly understood that



both pairs of sides must be of different lengths. It could be conjectured that this is due to a preference for prototypical examples of shapes limiting students' abilities to identify relevant definitional properties (Heinze & Ossietsky, 2002).

Student written work illuminated several aspects to the teaching and learning of 2-dimensional shapes. Tasks that ask students to list as many facts they know about a particular shape are commonplace. However, whilst listing known or memorised facts does indicate geometric thinking at one level, it potentially limits some students' ability to then discern which facts or properties are necessary to define a particular shape. A tendency to develop an exhaustive list leads students to believing that all these properties are necessary to define the shape. Where students list the necessary properties about a particular shape it is likely to produce more precise definitions. Students need to be made aware that it is the minimum number of keywords and therefore the least number of properties required in order to define a shape economically (de Villiers, 1998), and that the definition goes hand-in-hand with the image of the shape (Heis, 2014). To avoid such tasks means that students are unable to understand and appreciate the function or the value of a hierarchical classification of quadrilaterals.

Reasoning about 2-dimensional shapes requires an alignment between the essential properties of a shape needed to define it as well as to provoke its image. The use of prototypical examples is a necessary starting point for identifying key properties of shapes. However, students' preference for prototypical shapes as shown in this study was a significant barrier to being able to define shapes accurately, and leading to misconceptions about them. Teachers need to be aware that deficits in student thinking are strongly affected by prototypical representations often depicted in textbooks (Heinze & Ossietzky, 2002). Vincent and Stacey (2008) argued that geometry problems in Australian eighth-grade mathematics textbooks often involved stating concepts requiring low cognitive demand. Students need to be taught how to use diagrams in order to take advantage of the many features that a diagram represents, and how these features are interpreted and inter-related (Dreyfus, 1991). Students need exposure to a variety of examples and counter-examples of shapes including shapes in non-typical orientation. This may help to reduce the incidence of, for example, rotated squares being re-named as diamonds, or rectangles being defined as made up of horizontal and vertical lines.

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### **The four proficiency strands plus one?** Productive disposition and the Australian Curriculum: mathematics

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The Australian Curriculum: Mathematics (AC:M) (ACARA, 2016) defines four proficiency strands. However, the work of Kilpatrick, Seward and Findell (2001), upon which Australia's proficiency strands are framed, includes a further proficiency – productive disposition. Productive disposition relates to students' propensity to persevere and to perceive mathematics as worthwhile. We argue for the inclusion of productive disposition in Australian mathematics education, as reflective of the importance of affect in mathematics leaning. Upon close inspection, the AC:M and the overarching General Capabilities, especially Personal Capability, offer insufficient direction in regards to developing productive disposition in students. For productive disposition to be fostered in students, teachers themselves must have productive disposition towards mathematics and the teaching of mathematics. Teachers must believe that they and all students can learn and improve in their ability to learn mathematics; that mathematics is intelligible, and that they as teachers can improve their teaching and understanding through effort. Addressing the productive disposition of mathematics teachers and students may be a means of improving mathematics achievement in Australia.

#### A CASE FOR PRODUCTIVE DISPOSITION

Thomson, Wernert, O'Grady and Rodrigues (2017), in their analysis of Australia's 2015 Trends in International Mathematics and Science Study (TIMSS) results, highlighted two key areas of concern for Australian mathematics education and achievement. The first is that Australia did not significantly increase the percentage of Year 8 students achieving the 'Advanced' benchmark standard over the 20-year researched period, and linked to this, the number of students falling below the 'Low' benchmark did not change during the period 1995-2015. This suggests that any changes to mathematics education during this time, e.g. the *Australian Curriculum: Mathematics*, have not increased the mathematical performance of Australian secondary students.

Also released in 2016, the Organisation for Economic Cooperation and Development (OECD) report on the Program for International Student Assessment (PISA) findings, *Low performing students: Why they fall behind and how to help them succeed*, highlighted two key foci for improvement in mathematics education. The first relates to students with less than average positive attitudes towards school and learning. Those with less than average perseverance, motivation and self-confidence are more likely to be the low performers in mathematics, and reading and science.

The second key finding relates to teachers with low expectations for themselves (and their students), who are frequently absent from school and have low morale; these teachers are most often linked to low performing students. Teachers' beliefs about their students' ability to succeed and their own ability to influence student learning are associated with improved student mathematics achievement (Archambault, Janosz, & Chouinard, 2012). Data from PISA 2012 and 2015 showed that better teacher-student relationships were associated with greater engagement and learning, which in turn were associated with higher performance (OECD, 2014; OECD, 2017).

As shown below in Figure 1, behaviour, affect and cognition are interconnected. The Figure shows the most relevant relationships (OECD, 2016). In regards to improving academic performance, investments made in both improving perseverant behaviours and affect and in combating mathematics anxiety, are strongly encouraged. These behavioural and affective changes require changes in belief and self-belief. It is important to note that these suggested changes are not solely student directed; but rather also encouraging the development of such cognitive, affective and behavioural strategies in educators.



Figure 1. OECD (2016) Low performing students: Why they fall behind and how to help them succeed.



A student's belief that they possess the skills and knowledge required to solve mathematical problems (self-concept) and that, as a result, they are able to solve mathematical problems (self-efficacy) underpins both engagement in learning and performance in school. Boaler (2015) championed the idea of a mathematics mindset, building on the work of growth mindsets proposed by Dweck (2006). The cultivation of a mathematics mindset requires increasing students' confidence in their knowledge, skills and ability to apply this to a wide range of problems as well as impressing that success is due to hard work, not innate ability. Shunck and Pajares (2009) highlighted that a growth mindset in mathematics reduces anxiety and is associated with greater intrinsic motivation. An increase in a student's intrinsic motivation is likely to result in a greater investment of time and effort, leading to a positive cycle of learning behaviour (OECD, 2016). Jablonka (2013), however, found that students in many countries described mathematics as boring and unrealistic.

The interconnectedness of behaviour, affect and cognition, highlighted above, and its effect on achievement, is not solely student focussed. The importance of such cognitive, affective and behavioural strategies in educators must also be considered, as well as understanding how these factors may be developed and sustained over time.

Beswick (2011, 2007) claimed that it is teachers' beliefs about the nature of teaching and learning mathematics that matter most to student learning. Higher levels of mathematical knowledge in students is related to teacher belief regarding the value of productive struggle as well as the importance of building upon and justifying student understanding and ideas, and fostering inquiry in mixed ability settings (Beswick, 2011). This aligns to the problem solving nature of mathematics and a learner-focussed approach to mathematics teaching and learning (Beswick, 2005). Kilpatrick, Swafford & Findell (2001) stated that teachers must believe that they (the teacher) and all their students can learn mathematics and that both teacher and student can improve their skills and understanding of the subject through effort. This requires a positive attitude towards mathematics, demonstrated by the teacher and encouraged in students, as well as effort to combat or manage negative temporary emotional responses (states) or to foster positive states. The effort to encourage positive emotional states within the mathematics classroom links to the broaden and build theory; the cultivation of positive emotion allows for the broadening of thought- action repertoire (Fredrickson, 2001); and to growth mindset (Dweck, 2006); both components of positive education.

Teacher beliefs and attitudes matter, leading to learning environments that impact students' affect and achievement in discernible ways (Beswick, 2005; 2012). Perceptions of teacher affective support, for example active listening and respect, are associated with increased academic effort and achievement (Sakiz, Pope & Hoy, 2012). PISA results from 2012 and 2016 emphasise the importance of student-teacher relationships in school engagement and performance, in part, due to the fostering of a belonging mindset for students (OECD, 2014, 2017). Chatzistamtiou, Dermitzaki and Bagiatis. (2014) found that teachers with high self-efficacy beliefs regarding mathematics teaching demonstrate more enthusiasm and effort in planning lessons and in reaching their goals, are flexible and persistent when facing challenge, and have high academic expectations of their students.

Students of committed, resilient teachers are more likely to achieve positive outcomes (Hwang et al., 2017). Bajorek, Gulliford and Taskila (2014), in studying 300 teachers in the United Kingdom over the school year, found that the teachers who displayed higher levels of resilience and wellbeing were the most effective teachers, as determined by student SAT scores.

Effective mathematics teaching may be impacted by a teacher's anxiety towards mathematics, or anxiety regarding the teaching of mathematics with anxious teachers having lower expectations for students' mathematical achievement (Mizala, Martinez & Martinez, 2015). There is also concern that teacher anxiety may engender anxiety in students (Beilock, Gunderson, Ramirex & Levine, 2010). Stoeher's (2017) synthesis of the literature showed that confident and competent mathematics teachers are vital for student engagement and achievement in mathematics. A teacher who displays these characteristics can be said to have productive disposition. That is, he/she has a view of mathematics as 'sensible, useful and worthwhile [and] coupled with a belief in diligence and one's own efficacy' (Kilpatrick et al., 2001, p. 26.). Productive disposition in teachers is transmissible to students through: elucidating, where possible, the usefulness of mathematics; showing mathematics to be sensible and worthwhile in wide contexts (not just for passing tests or for future employment); willingness to give assistance and use failures as powerful teaching moments; and to embody life-long learning in mathematics (that as teachers we can always learn new approaches to solving or representing problems from our students) (Boaler, 2015).



#### THE PLACE FOR PRODUCTIVE DISPOSITION IN THE AUSTRALIAN CURRICULUM: MATHEMATICS

*The Australian Curriculum: Mathematics* includes the proficiency strands, adapted from the recommendations of Kilpatrick, Seward and Finell (2001), described in their seminal work on the notion of mathematical proficiency (the integral skill set that sit above mathematical content). The aim of the elaborations of the proficiencies in the AC:M was to inform both teaching and assessment, with the National Curriculum Board (2008) stating that, 'There are specific topics for which understanding is critical ... and others for which standards for fluency will be specified. Expectations for proficiency in problem solving ... and reasoning ... will also be specified noting that these are central to ensuring a futures orientation to the curriculum' (p. 4). This approach, and the proficiencies themselves, differ from the interconnectedness aspired to by Kilpatrick et al. (2001).

Kilpatrick et al. (2001), stated that the keys to teacher proficiency in mathematics teaching are effectiveness and versatility: effectiveness in terms of assisting students to learn worthwhile content; and versatility in terms of working effectively with a range of students, environments and content. They defined mathematical proficiency as tightly interwoven threads of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition; highlighting that proficiency cannot be achieved without all of these components working, and being taught, interdependently. Kilpatrick et al. (2001) stated that mathematical proficiency was as, if not more, important for teachers of mathematics than for students.

*The Australian Curriculum: Mathematics* (ACARA, 2016) elucidates four proficiency strands and three content strands. The proficiency strands are: understanding; fluency; problem solving; and reasoning. These four proficiency strands share characteristics with the five interdependent strands described by Kilpatrick et al. (2001), as shown in Table 1.

Proficiencies	Kilpatrick et al. (2001)	Australian Mathematics Curriculum v 8.2 (ACARA, 2016)
Understanding	Integrated and functional comprehension of mathematical content and ideas	Know, adapt, connect & represent mathematical concepts
Fluency	Ability to recall and use mathematical knowledge flexibly, accurately and efficiently	Recall definitions, facts and procedures and efficiently calculate answers
Problem Solving	Formulate, represent and solve problems	Make choices, design, interpret, formulate and model problems
Reasoning	Logically consider relationships & justify methods and solutions as appropriate to the task.	Analysing, proving, adapting, explaining, inferring, justifying and generalising.
Productive Disposition	An attitude that mathematics is both useful and worthwhile.	

Table 1. Comparison of Proficiency Strands

Unlike the other four proficiency strands, productive disposition concentrates on the affective characteristics of teaching and learning, rather than the cognitive aspects. Guy, Cornick and Beckford (2015) described the affective characteristics of productive disposition as motivation, autonomy, desire to seek assistance, and willingness to expend effort to learn. Students with productive disposition are autonomously motivated to learn (Deci & Ryan, 2008), display a growth mindset to their understanding and skills in mathematics (Boaler, 2015; Dweck, 2006) and see mathematics as worthwhile.

Despite research indicating the importance of productive disposition, this proficiency strand is not an explicit component of the AC: M. It may be argued that the overarching General Capability of Personal and Social Capability addresses productive disposition in mathematics. The sub elements of Personal Capability address both self-awareness and self-management. Of the eight sub-elements, two relate to emotion (recognise emotions and express emotions appropriately). Whilst emotional regulation is not strictly a component of productive disposition, it is a factor of developing a growth



mindset towards mathematics (Boaler, 2015). Three of the sub-elements (recognise personal qualities and achievements; develop self-discipline and set goals; and work independently and show initiative) relate to motivation, which is a component of productive disposition that addresses 'belief in diligence and one's own efficacy' (Kilpatrick et al., 2001, p 26). The remaining three sub-elements (understand themselves as learners; develop reflective practice; become confident, resilient and adaptable) align with the key ideas of growth mindset, in that 'students with a growth mindset work and learn more effectively, displaying a desire for challenge and resilience in the face of failure' (Boaler, 2013, p 143).

The General Capabilities, however, as the name suggests, have been written to apply to all learning areas. Absent from the description of Personal Capability is the essence of productive disposition that is particular to mathematics – the acknowledgement and belief that mathematics is sensible, useful and worthwhile. This belief is deeper than aligning mathematics with numeracy outcomes specifically, which state that 'the effective use of mathematics [is] to meet the general demands of life at school and at home, in paid work, and for participation in community and civic life' (MCEETYA Benchmarking Task Force 1997, p 4). Furthermore, mathematics must be seen to be sensible, useful and worthwhile in and unto itself so that students and teachers can set goals that are personal, timely and meaningful. The lack of connection of this general capability to mathematics and hence to the mathematical proficiency of productive disposition is also evident from the summary of where personal capability is found within the Australian Curriculum. Where subjects are listed from highest to lowest in terms of the proportion of content descriptions tagged, mathematics is second last. Indeed, closer inspection of all elaborations within the content strands of the secondary school mathematics curriculum reveals no links to Personal and Social Capability.

Interestingly, two of the sub-elements of Personal Capability focus upon the role of emotion. Martinez-Sierra and Garcia-Gonzalez (2017) stated that there is limited research regarding student emotion in mathematics education, beyond emotional responses to problem solving. Frenzel, Pekrun & Goetz (2007) found that the emotions most commonly reported in mathematics classes were enjoyment, anxiety, anger and boredom. Larkin and Jorgensen's (2015) research, involving primary school students, found that positive emotions were reported when the mathematics had a real-world link and purpose. This finding connects to a key part of productive disposition- the belief that mathematics is useful and worthwhile. This again highlights that the education for personal capability is insufficient to develop students' productive disposition in mathematics.

Students' positive emotional response to mathematics also occurs when they are positive about the support that they receive from their teacher (Martinez-Sierra & Garcia-Gonzalez, 2017). Villavicencio and Bernado (2016) claimed that teachers should value students' positive emotions and work to create opportunities that allow for these emotions to be experienced and sustained. For example, pride, associated with successful cognitive and motivational processes, is a positive emotion that enhances positive self-belief that is linked to learning (Duckworth & Yeager., 2015).

Beswick and Callingham (2014) highlighted two areas of concern in regards to emerging teachers' productive disposition. Firstly, primary preservice teachers often fear and dislike mathematics. Similarly, Kalder and Lesik (2011) described the unease experienced by primary preservice teachers. Secondary preservice teachers are less likely than mathematics teacher educators to regard problem solving as inherent to mathematics, but more likely to do so than primary pre-service teachers (Kalder & Lesik, 2011). This may serve to compartmentalise problem-solving into content-specific problems, such as a measurement problem, rather than teaching and assessing it as an overarching skill and knowledge set of the curriculum (Harkness & Noblitt, 2017).

Developing and maintaining productive disposition in mathematics teachers is important for improving student enjoyment of and outcomes in mathematics. As an affective characteristic, it can be difficult to teach and assess. Further investigation into how productive disposition of mathematics teachers and students of mathematics can be improved and maintained is arguably warranted.



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# An interactive statistical sampling activity with chocolate

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We will present an activity that engages students with key statistical ideas – including sampling, bias, variability and estimation – using virtual blocks and real chocolate. The activity provides an experiential foundation for building understandings of important statistical concepts, produces data that are meaningful to students, and it is a learning task that allows for multiple entry and exit points. Furthermore, it can be tailored for year levels 8-12 and class sizes from 20 upwards. The activity is easy to implement, supported by good documentation, and freely available on the web.

#### TASKS IN STATISTICS

Statistics has become increasingly important in the modern world, and this is reflected in the Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2015). The Australian Curriculum: Mathematics includes statistics and probability as one of its three content strands from foundation through to year 10. Moreover, the focus is not solely on procedural proficiency, but rather the content is explored or developed over the four proficiency strands: understanding, fluency, problem-solving and reasoning. Statistics topics also appear in all the senior mathematics subjects, and range from interpretation of numerical measures to statistical inference for means and proportions.

Interactive classroom activities have long been used to engage students with statistics (Gnanadesikan, Scheaffer, Watkins & Witmer, 1997), and it is recognised that carefully designed sequences of activities can help students improve their reasoning and understanding (Garfield & Ben-Zvi, 2007). What makes a good statistical activity? Many of the principles of mathematical task design also apply to statistics, such as offering multiple entry and exit points, approaching the unknown from what is known to the students (Ahmed, 1987) and offering the potential for mathematically (or statistically) fruitful activity (Mason & Johnston-Wilder, 2006; Gunn, 2011). If you are working with a system's view of learning (Gunn, 2011) then it is also important for the activity to be grounded in a rich experiential context and to provide elements of surprise to disrupt existing knowledge structures. The resulting cognitive conflict can lead to learning (Watson, 2002). A further consideration when designing learning tasks is that teachers in schools are often time-poor and would benefit from tasks that are easy to implement and well resourced. That is, tasks that are quick to set up, which employ easy-to-use technology, and for which supporting resources are available, such as pre-prepared task sheets, prompt questions, possible extension activities, and suggestions on how to incorporate the task into the existing curriculum.

Sampling is a foundational statistical concept, and various classroom activities have been developed to address it. For instance, Gnanadesikan et al. (1997) describe a sampling activity using random rectangles, which was based in turn on an older activity with a box of rocks by William Cochran. A similar activity is described in the Guidelines for Assessment and Instruction in Statistics Education (GAISE) report (Franklin, Kader, Mewborn, Moreno, Peck, Perry & Scheaffer, 2007, pp. 52-54). Dyck & Gee (1998) describe an activity using chocolate M&Ms to simulate the sampling distribution of the sample mean. In the Australian context, the *TIMES Modules* (Australian Mathematical Sciences Institute, 2016) and the *Islands in Schools Project* (Baglin, J., Bulmer, M, MacGillivray, H., Dunn, P., Marshman, M., Huynh, M., & Hart, C., n.d.) provide resources and activities for teaching sampling which are aligned to the Australian Curriculum, and resources such as *Mind on Statistics Australian and New Zealand Edition* (MacGillivray, Utts & Heckard, 2013) provide further examples and activities suitable for the Australian context.

In this paper we will present a classroom sampling task designed to engage students with key statistical ideas using virtual blocks and real chocolate. The use of chocolate acts as 'a hook' (Foster, 2015), drawing students into engagement with the task. Furthermore, it is a disruption to their existing belief about 'what constitutes statistics'. On a more subtle level, the use of chocolate engages the students' sense of smell, thereby enhancing the environmental stimulation provided by the task and enriching the experiential foundation upon which the task is laid. The role of the senses in developing memory and learning is well documented in the research literature (Stoffers, 2011; Wilmes, Harrington, Kohler-Evans & Sumpter, 2008). The task and its supporting resources have been designed in consideration of the demands placed on teachers, by providing online documentation and a setup program which configures the necessary



technology for the task with one click. In the next section we give a brief outline of the task, and in the final section we discuss some ways that the task could lead to engagement with important statistical concepts.

#### THE CHOCS AND BLOCKS ACTIVITY

Students are shown a tray of irregularly shaped pieces of chocolate (Figure 1). They are told that there are 100 pieces on the tray, and are given the chance to look at the pieces for themselves (and to smell them, though not to touch – because we want to eat the chocolate later!). Their task is to estimate the average weight of a piece of chocolate on the tray. The most accurate estimate will earn first pick of chocolate from the tray.



Figure 1. Tray of chocolate pieces of irregular shapes and various sizes.

Initially, they are not given any more guidance than this, except that they can ask the teacher questions – which the teacher may or may not be able to answer – to help them come up with a strategy for estimating the average weight. The teacher will evade most questions, such as 'what is the total weight of all the pieces?', with answers such as 'I can't remember, did you bring some scales?' but can reveal the following information, if asked the right questions:

- there are 100 pieces on the tray;
- the weight of each piece of chocolate is known, and students can find out the weights of individual pieces later in the class;
- each student can find out the weights of 10 pieces from the tray, but they must specify which 10.

Once most of the class have found out that they will be able to obtain the weights of specific pieces, the teacher directs the students to the 'Chocs and Blocks' website. The website shows a virtual representation of the tray of chocolate, with 100 blocks of irregular sizes and shapes (Figure 2). The teacher tells the students that their task now is to choose 10 pieces, whose weights they will be given. Students choose pieces by clicking or touching a block. Once they have chosen 10 blocks, the website tells them the weights of each chosen block and the mean and standard deviation of the sample. The website prompts them for their name and then submits the results to an online spreadsheet hosted by Google Docs. The spreadsheet can be displayed on a projector so the class can see their data as they come in.





Figure 2. Screenshots of Chocs and Blocks website showing virtual blocks (left) and of an online spreadsheet showing students' results as they arrive (right).

Once everyone's results are recorded on the spreadsheet, a graphical display of the sample means can be produced, such as a dotplot or stem-and-leaf plot (see Figure 3 for an example). The teacher can lead an investigation of the distribution of sample means. What was the 'average' estimate that people obtained? How 'spread out' are the estimates? Are there any interesting features, e.g., 'clumps' of estimates at either extreme, and if so, what might explain them? The teacher might focus on some particular data points, such as those at the extremes or near the centre, and ask the corresponding students what strategy they used for choosing their blocks. The teacher might interrogate the assumptions implicit in the strategy – for instance, a common strategy of choosing '5 large and 5 small' blocks assumes that the distribution of block weights is symmetric. What would happen to the resulting estimate if the distribution were *not* symmetric? The teacher can then reveal the true mean. Many students will be surprised to discover that they overestimated the mean by a considerable amount! The students with the estimates closest to the true mean can be rewarded with chocolate and asked to describe their strategies to the group.



*Figure 3. Dotplot of students' sample means from the first phase of the activity.* 

For the next phase of the task, students are directed to click on a hidden link at the bottom of the page, which switches the website to random sampling mode. The website now lets students generate a random sample of 10 blocks, and again record the data to a spreadsheet. Once every student has generated a random sample, the random sample means can be displayed graphically (comparative dotplots or back-to-back stem-and-leaf plots are suitable here to show both distributions side-by-side, see Figure 4). The class can see that the random sampling method has produced estimates much closer to the true mean of 33.5, and with less variability, than the judgement sampling method. The teacher might now lead a comparison of the two distributions. Finally, the chocolate is shared around at the end of class.





Figure 4. Comparative dotplot of sample means from judgement samples (top) and random samples (bottom).

We have given a brief outline of one possible realisation of the task, but we refer the reader to the Mathematics and Statistics Learning Centre website (n.d.) for further documentation, including a suggested running sheet for a 50 minute class, and to Gunn & Morphett (2017) and Gunn (2011) for a discussion of some theoretical principles underlying the activity and for a case study of the integration of the activity into an undergraduate introductory statistics course.

The Chocs and Blocks website (Figure 2) allows students' data to be recorded directly to an online spreadsheet, but this requires specially configured spreadsheets and web forms and a customised link to the Chocs and Blocks website. To make the setup process easier, we have created a Chocs and Blocks setup program, which automatically creates all the necessary spreadsheets, forms and web links with a single click. The only requirement is that the user has a Google user account. The setup program and further documentation about the Chocs and Blocks activity and setup program can be found at.

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Figure 5. Screenshot of the Chocs and Blocks setup program.

#### ENGAGING WITH KEY STATISTICAL IDEAS

We describe some potential ways that the Chocs and Blocks activity can engage students with statistical concepts. References to relevant content descriptors of the Australian Curriculum: Mathematics (ACARA, 2016) are given in brackets.

*Sampling methods and bias*. The activity requires students to perform two sampling methods – one in which they choose what they think (informally) is a representative sample, and one in which the sample is chosen randomly. Students are thus engaged in thinking about sampling throughout the activity (ACMSP206). The discussion of strategies, their assumptions, and possible consequences for the resulting estimates, brings to light the pitfalls of some sampling methods (ACMEM131), and the comparison of the choice and random sample means illustrates the value of random sampling for obtaining unbiased estimates (ACMMM172).

*Linking to real-world statistical practice*. The chocolate is important to engage students and to situate the activity in familiar experiences, but it results in a somewhat artificial scenario, which may seem separated from real-world statistical practice. This could be addressed by having students apply the knowledge gained from Chocs and Blocks to other contexts. For instance, in a follow-up activity students could be presented with a scenario such as 'a farmer wants



to estimate the average weight of his cows. How could you help him estimate it?', or students could generate their own real-world sampling scenarios.

Sampling variability and sampling distribution. The distribution of random sample means obtained from the second phase of the activity shows the variability of random samples drawn from the same population (ACMMM173, ACMSM137). The distribution of random sample means can be compared with the distribution of block weights, which has mean 33.6, standard deviation 20.5 and positive skew (the full set of block weights can be downloaded from the Mathematics and Statistics Learning Centre website (n.d.). As predicted by theory, the distribution of random sample means is approximately normal in shape, with the same mean as the population of weights and standard deviation reduced by a factor of  $\frac{1}{\sqrt{n}}$  (ACMSM138). For a deeper investigation we might ask why the (theoretical) distribution of sample means is *approximately*, but not exactly, normal. Some points to consider here include the sample size, the shape of the distribution of weights in the population, and the sampling method (sampling without replacement verses sampling with replacement).

Additional summary statistics and their distributions. Once a student has selected their 10 blocks or generated their random sample, the website displays the weights of each selected block as well as the sample mean and sample standard deviation. This data is also recorded in the online spreadsheet (although most columns are hidden by default to avoid cluttering the display). Further summary statistics such as median, quartiles or inter-quartile range could be calculated from the block weights, either by hand or with the use of technology (ACMSP171, ACMSP248, ACMSP278). If the class's medians (quartiles, IQR...) are aggregated, this could lead to an investigation of the distribution of sample median (quartiles, IQR, ...) and a comparison with the distribution of the sample mean.

*Confidence intervals*. Using their random sample mean and standard deviation, students could calculate a confidence interval for the population mean (ACMSM141). These could be aggregated and used to investigate the proportion of confidence intervals that contain the true mean, and its relation to the confidence level (ACMSM142). This line of investigation flows naturally into an illustration of a Binomial random variable – the number of CI's that capture the population mean (ACMMM147, ACMMM148).

*Decision making and inference*. If we treat the two sets of sample means as two populations, we may lay the foundations for the kind of thinking that underlies statistical inference. One possible pathway to inferential thinking is the following follow-up activity. The class is told that:

## 'Jenny did the Chocs and Blocks activity yesterday and wrote down that her sample mean was 46.0. However, now she can't remember if this was her judgment sample mean, or her random sample mean. Can you help her?'

One way to approach this is to examine the distribution of judgment sample means and random sample means, and to observe that (based on the data from Figure 4) judgment samples with means around 46 were rather common, whereas there were very few random samples with means around 46. It is very unlikely that Jenny would obtain a random sample estimate around 46, so the more plausible explanation is that Jenny's estimate was from her judgment sample. This kind of thinking – assume a model and then see if the data is consistent with the model – is the essence of statistical inference. The reasoning could be made more quantitative by looking at the percentage of choice or random samples from the populations which gave estimates of at least 46. Alternately, students familiar with the Central Limit Theorem could use the normal distribution to estimate the probability that Jenny would obtain a random sample mean of at least 46. By doing this, students have effectively conducted a hypothesis test, and the resulting probability is a p-value.

#### ACKNOWLEDGEMENTS

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# The International Lexicon Project: giving a name to what we do

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Different communities, speaking different languages, employ different naming systems to describe the events, actions and interactions of the mathematics classroom. The International Lexicon Project aims to identify the professional vocabularies available to middle school mathematics teachers in Australia, Chile, China, the Czech Republic, Finland, France, Germany, Japan and the USA to describe the events of the mathematics classroom. Local teams of researchers and experienced teachers in each country used a common set of classroom videos to stimulate recognition of familiar terms describing aspects of the middle school mathematics classroom.

Any reform initiative must take as its starting point the existing professional vocabulary by which mathematics teachers conceptualise their practice. This paper reports the characteristics, structure and distinctive features of this Australian Lexicon and illustrates these with specific examples. Selective comparison with other lexicons will be used to provide insight into features of the mathematics classroom foregrounded or ignored in the lexicons of different countries.

The International Lexicon Project has the potential to enrich the professional vocabulary of mathematics teachers around the world. Many terms familiar to Chinese or Czech teachers, for example, will offer Australian teachers novel and challenging perspectives on their practice. As a consequence, Australian teachers will gain access to sophisticated classroom practices named by teachers using languages other than English.

#### INTRODUCTION

Our lived experience, including our teaching, is mediated by our capacity to name what we see and what we do,

'We see and hear . . . very largely as we do because the language habits of our community predispose certain choices of interpretation,' (Sapir, 1949, p. 162).

Indeed, Marton and Tsui (2004) similarly argue that categories express the social structure, as well as create the need, for people to conform to the behavior associated with these categories. That is, teachers' classroom activity is shaped by those practices they are able to name; their behaviour most likely aligns with this familiar construction of named practice. Identifying the professional vocabulary of Australian mathematics teachers provides insight into teachers' conceptions of the mathematics classroom and the activities and practices with which they are engaged and value. Any reform agenda must take this existing professional vocabulary into consideration.

A professional language of the teaching community had been identified as lacking (Lortie, 1975). More recently, studies have concurred that teachers' professional language remains underdeveloped (Lampert, 2000; Grossman et al., 2009). Connell (2009) argued that this may be related to the few opportunities provided for teachers to work and discuss with colleagues the practices of the classroom. This is not the case in other communities. Japan has a strong tradition of educators and teachers discussing research lessons and, accordingly, has developed a sophisticated technical vocabulary by which teachers can discuss their practice. Similarly, China also has a well-articulated structure for classroom practice (Lampert, 2000).

Bhatia (2006) argues that studies of professional practice would assist us in gaining a more informed and comprehensive view of the lived experience of professionals and of the language employed by professionals to describe, represent, interpret and theorise. *The International Lexicon Project* is an international study of a particular aspect of the professional practice of the teaching profession: the vocabulary that the profession draws on to name its practices. The International Lexicon Project seeks to identify, document and compare the professional 'lexicon', 'the vocabulary of a person, language, or branch of knowledge' (Stevenson, 2015), of middle school mathematics teachers in nine communities worldwide: Australia, Chile, China, the Czech Republic, Finland, France, Germany, Japan and the USA. The documentation of these lexicons and their publication will expand the constructs available for identifying the characteristics of accomplished practice. Certainly, if a classroom teacher's conception of the classroom is constructed around those activities and practices they can name then it may follow that they are less likely to engage in practices that are named by other communities (and not ours) will give Australian



teachers the opportunity to consider other possibilities of classroom practice. We also suggest that a shared language of practice is one of the hallmarks of a profession, mediating access to and participation in the profession. As such, the identification of the lexicon of the community of Australian middle school mathematics teachers contributes to the establishment of the professional identity of these teachers.

#### THE RESEARCH DESIGN

#### **RESEARCH QUESTION**

The following research question has driven the research of the International Lexicon Project outlined in this paper:

#### What are the terms that teachers use to describe the phenomena of the middle school mathematics classroom?

Each national lexicon represents an immediate and significant product of this work. For the purposes of this project, a country's lexicon includes the following elements: a term, its description, and illustrative examples and non-examples (something that might be confused with the term). The lexicon, as represented by this collection of elements, documents the vocabulary used by teachers to describe the phenomena of the middle school mathematics classroom.

#### **RESEARCH METHOD**

#### **Research Team**

The local research team in each of the countries was stipulated to include junior and senior researchers and at least two experienced teachers of middle school mathematics. The Australian team included three academics and three high school teachers of mathematics, two of whom had taught extensively at years seven and eight, whilst the third was a more recent member of the profession.

#### **Stimulus Package**

Each team contributed video material, time-stamped transcripts as well as supporting material related to one lesson of year eight mathematics. These lessons were re-packaged as 'three-ups', that is, three camera angles with a time-code and subtitles all visible in one viewing window (see Figure 1). A stimulus package of nine lessons, one from each team, was constructed and distributed to each local research team for project-wide use. These nine lessons, one each from Australia, Chile, China, the Czech Republic, Finland, France, Germany, Japan and the USA, presented a variety of classroom settings and instructional approaches both familiar and unfamiliar to the research team members.



Figure 1. The video material on the left was re-packaged as a 'three-up' on the right.



#### **Identifying Terms**

Each team began with the prompt *'What do you see that you can name?'* and recorded their responses on a standardized template. This prompt was intentionally crafted so that restrictions were not placed on what could be identified. For example, one researcher might name an activity a student is engaging in whilst another might name some aspect of the classroom environment. The nine videos (one from each country) were intended to stimulate thinking about the possible terms of the lexicon, however, it was not necessary that each term recorded had to be present in the video stimulus material. In other words, identifying and naming one classroom phenomenon present in the video, may call to mind another classroom phenomenon. Both terms would be recorded for possible inclusion in the lexicon. The essential point was to record single words or short phrases that are familiar to at least two-thirds of middle school mathematics teachers with an agreed meaning.

Each of the Australian team members viewed the videos independently and came to team meetings with possible terms for inclusion in the lexicon. Each of the candidate terms was carefully considered and team consensus was required for the inclusion of a term. In cases where there was not agreement, it was up to the team member to argue that the term was familiar to teachers. This accorded particular authority to the teacher members of the research team in determining whether these terms were to be included in the first draft of the lexicon.

#### VALIDATING THE DRAFT LEXICON

Operational definitions were developed for each of the terms in the draft lexicon. These included descriptions, examples and non-examples. A sample of operational definitions of terms currently employed by teachers (and endorsed by the teaching community) is given in Figure 2.

Differentiating (Differentiation)	Any action in which instruction is modified, adapted or varied to cater for student differences.	<ul> <li>For example:</li> <li>The teacher groups students for instruction according to their achievement on a pre-test of the content to be taught.</li> <li>The teacher provides different feedback to students according to her judgement of their emotional or motivational state.</li> <li>Non-example:</li> <li>All students complete the same set of questions regardless of experience with the topic.</li> </ul>
Giving Feedback	The provision of information (typically evaluative) in response to the actions of a student or teacher;	<ul> <li>For example:</li> <li>The teacher makes an evaluative comment on a student's solution to a problem.</li> <li>Non-example:</li> <li>The teacher gives a student a direction that is clearly not in response to any action by the student.</li> </ul>
(use of a) Hook	The engaging introduction of a topic or sub-topic that captivates students' attention.	<ul> <li>For example:</li> <li>The teacher introduces polyhedra with a video detailing how a soccer ball, a truncated icosahedron, is made.</li> <li>Non-example:</li> <li>The teacher hands out the assignment and invites students to research polyhedra on the internet.</li> </ul>
Worked Example	The teacher (or student) writes out the steps involved in order to illustrate the type of solution expected to a problem or task with or without student involvement.	<ul> <li>For example:</li> <li>Teacher writes out the solution to a problem on the whiteboard, providing oral explanations and clarifications along the way.</li> <li>Non-example:</li> <li>Students recording their solutions at the board.</li> </ul>

Figure 2. A sample of definitions, examples and non-examples of the terms in the Australian lexicon.



#### LOCAL VALIDATION

The draft lexicon consisted of 69 terms and these terms and operational definitions were subjected to a local validation process to investigate the extent to which the local community of mathematics education researchers would endorse the terms as named as well as the descriptions, examples and non-examples.

We invited two groups of people to participate in an interview: i) Mathematics Education Researchers (Specialists) and ii) Education Researchers. We included the second group pf researchers to check on the extent to which the lexicon might appear to be cross-disciplinary.

#### Interview protocol

The researchers participated in an audio-recorded, one-on-one discussion, in which each interviewee was presented with ten terms for examination and discussion. The combination of terms and interviewees was organised such that no two people received the same set of terms and the entire set of terms was seen by at least one representative from each group (see Figure 3).



Figure 3. The allocation of terms to participants for the purpose of local validation.

Cards were prepared with the term on one side and the description with examples and non-examples on the reverse. The protocol for the interview involved the participant:

- sorting the ten terms (name only) into two piles: Familiar and Unfamiliar
- *matching* ten cards (name only) with ten other cards (description and examples) [Two sets of cards were prepared for this purpose]
- *suggesting improvements* to the operational definition (lexical bundle: term, description, example and non-example)

The participants suggested improvements to wording, to improve clarity, and to the examples, to improve helpfulness of these. They were also asked to reflect on whether there might be another term that teachers might use. In addition, participants were invited to take the remaining stack of cards and sort into Familiar and Unfamiliar.

This process of local validation confirmed our initial impression that the terms all identify general pedagogical practices and that no single term is unique to the mathematics classroom. For example, the practice of *Posing Problems* might be carried out quite differently in a Mathematics classroom from, say, a Science classroom, however, the term's description appears to be understood by both communities and would certainly be illustrated with different examples.



Following these interviews, the Australian research team met to consider whether to:

- accept or reject changes to expression/wording of the descriptions and examples of the Lexicon items;
- consider which current terms might need to be excluded from the lexicon given the feedback from the local validation interviews; and,
- consider the list of additional terms that were generated for possible inclusion.

The draft lexicon of sixty-nine terms was reduced to a lexicon of 63 terms that was offered up for national validation.

#### NATIONAL VALIDATION

The 63 terms in the Australian lexicon were validated with an online survey. 138 teachers across Australia responded to the survey and participated in questions in which they indicated familiarity with the term and its operational definition. Familiarity with each term was above the two-thirds threshold of inclusion in the lexicon. That is, if the term was found to be unfamiliar to more than a third of respondents it was considered not validated for inclusion in the lexicon. In fact, this did not occur for any of the terms.

#### THE AUSTRALIAN LEXICON

#### COMMUNICATING THE LEXICON

The Australian (middle school mathematics classroom) Lexicon consists of 63 terms that are considered familiar by teachers in the mathematics education community. In considering how best to communicate the lexicon, a university class of practising teachers was invited to group the items in the lexicon. Three categories identified across almost all the item clusters generated included: *Administration, Assessment* and *Classroom Management*. Two additional categories were proposed by the research team to capture the spirit of the teachers' suggestions: *Learning Strategies* and *Teaching Strategies*.

When organised into these five categories the 63 terms are distributed as follows: Administration (8 terms); Assessment (12 terms); Classroom Management (6 terms); Learning Strategies (27 terms) and Teaching Strategies (50 terms). Some of the terms appeared in more than one category; indeed 24 terms belong to both the Learning and Teaching Strategies categories (see Figure 4).

Administration	Assessment	Classroom Management	Learning Strategies	Teaching Strategies	Learning Strategies	Teaching Strategies
Collecting Work	Correcting	Disciplining	Answering Questions	Guiding	Board	d Work
Handing Out Equipment	Monitoring	Positive Reinforcement	On Task	Modelling	Clar	ifying
Marking the Roll	Test/Testing	Raising Hands	Practising	(use of a) Hook	Ques	tioning

Figure 4. A sample of terms present in each of the organisational categories.

#### FAMILIARITY AND USE OF THE LEXICON

One of the questions of the national survey invited teachers to indicate how familiar they were with each term and respond using a five-point Likert scale: *Extremely familiar; Very familiar; Somewhat familiar; Not so familiar;* and, *Not at all familiar.* With respect to the same terms respondents were also invited to indicate their use of the term by responding to the question, 'Do you use this term in conversation with your colleagues?' with: *Used extremely often; Used very often, Used moderately often; Used slightly often*; and, *Not at all used.* The charts in Figure 5 indicate the responses to a collection of ten terms from the lexicon.





Figure 5. Graphs representing Familiarity (left) and Use (right) of the same terms.

As you can see in the chart about familiarity (on the left) all of the terms met the two-thirds threshold to be validated as included in the lexicon. That is, the three responses Extremely familiar, Very familiar and Somewhat familiar added to more than 67%. The most familiar term in this group was the term *Encouraging (100%)* whilst *(use of a) Hook (77%)* and *Reciting (86%)* were slightly less familiar.

In responding to the question about using the terms with colleagues, the pattern of usage does not appear to match the pattern of familiarity. This may relate to the earlier assertion by Connell (2009) that the workplace opportunities to engage in discussion about the problems of practice are not always present. One hypothesis is that, as is the case with medical practitioners, a professional may be familiar with a range of technical terms relating to phenomena that occur only infrequently, but are nonetheless important. In such cases, the professional has an obligation to be familiar with a term, even though there may only seldom be occasion to make use of the term.

#### THE LEXICONS IN THE INTERNATIONAL PROJECT

English is used to describe the content and structure of all the lexicons. This reflects the underlying purpose and challenge of the Lexicon Project: to identify and make accessible to the international community the pedagogical principles and distinctions encrypted in different lexicons. Some non-English terms can be approximated in English (e.g., 'Teacher Feedback' adequately names 教师反馈 which is 'jiào shī fǎn kuì' in Chinese pinyin) but there are those that have no simple equivalent English term or phrase but can only be represented in pinyin and an extended English description (e.g., 课堂生成 which in pinyin is 'kè táng shēng chéng' and which refers to 'when the teacher makes full instructional use of an unexpected event beyond the intended plan for the lesson'). Similarly, the Czech term 'S cílem objevit' (literally, 'with the aim to discover') refers to the occasion when 'by solving the problem students discover something new.'

In combination, the nine international lexicons offer many forms of pedagogical insight:

(i) Terms that occur in many lexicons, suggesting that these are activities central to the professional practice of middle school mathematics teachers around the world (e.g., Teacher Feedback);

(ii) Terms absent from English, but which name practices common to middle school mathematics classrooms (e.g., the Japanese term, Kikan-Shido, which can be translated as 'Between Desks Instruction' and refers to the teacher activity of walking between desks while students work and observing or interacting with the students (see O'Keefe, Xu & Clarke, 2006);

(iii) Terms absent from English, naming practices that are seldom named or considered by many middle school mathematics teachers (e.g., 上课仪式, Shang ke yi shi, Ceremony of beginning class)

(iv) Terms that reflect a profound cultural situatedness (e.g., 画龙点睛 Hua long dian jing/qing, literally 'draw dragon, dot eye' which refers to a remark that succinctly summarises the main point of a discussion, possibly represented as 'Finishing Touch')


(v) Terms in transition – referring to practices that are becoming less frequent or more frequent in mathematics classrooms or indicating that one term is being replaced by another (e.g., 'catering for student abilities' has become 'differentiation').

We suggest that a variety of benefits will accrue from the sharing and comparison of the different lexicons.

#### **CONCLUDING REMARKS**

Each lexicon represents an important documentation of the professional vocabulary relating to middle school mathematics classrooms in that country. Documentation of this lexicon gives public recognition to professional vocabulary by which the community of mathematics teachers conceive, implement and reflect on their teaching. The structure of each lexicon offers insights into the way in which the teaching community conceptualizes and organizes its practice. A normalized technical vocabulary is one of the hallmarks of a profession. Such vocabularies are fundamentally dynamic and evolving. Reform initiatives must be undertaken as purposeful expansions of existing technical vocabularies.

Comparison with the lexicons of other countries offers insights into those classroom activities that have gone unnamed or which perhaps do not even occur in local classroom practice. Access to the lexicons of other countries offers possibilities for practice that may expand the professional repertoire of mathematics teachers around the world. Comparison with the lexicons of teachers elsewhere offers us the opportunity to reflect critically on both our professional vocabulary and the pedagogical practices named, invisible or absent from our mathematics classrooms. It is hoped that the publication of the lexicons of all nine participating countries will provide a major resource to mathematics education communities around the world.

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#### NOTES

Pinyin is a phonetic rendering of Chinese characters using the Latin alphabet employed in English and four basic tonal annotations.

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## The Volivoli Fish: investigating algebra within geometry

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Those of us who are teachers of school mathematics in Australia, compared with colleagues in many other countries, enjoy a high level of freedom regarding our ability to choose classroom activities for our students. Unencumbered by state- or nationwide overly prescriptive curriculum, we can address the perceived learning needs of our students using a range of resources and stimuli. This paper describes the development of a mathematically-rich task with multiple entry levels, inspired by the geometry and algebra embedded in a commercial logo

#### INTRODUCTION

Silver (1997) stated that the nature of mathematical thinking and the discipline of mathematics have, at their core, problem posing and solving. Thus, teachers whose lessons can reflect their own journey in discovering, formulating and solving problems of genuine interest are modelling the essence of being mathematicians for their students. The challenge as perceived by the author is for teachers to channel their own curiosity regarding problem exploration into meaningful and ability-appropriate tasks which allow students to pose the same or similar 'What if..?' questions as were first experienced by their teachers.

The author's previous experience in being part of the research team which developed the lesson materials for Marina's Fish Shop (Wander & Pierce, 2009), provided a springboard for curiosity when the logo at the centre of this investigation was first viewed in Fiji in 2015. In Marina's Fish Shop, the algebra emerging from a dynamic geometry representation of a fictitious fish-shaped sign led to analysis of the embedded quadratic function. The commercially developed Volivoli Fish logo, used for directional signage as well as brand recognition, was thought to have its own set of mathematical properties which invited further investigation. Figure 1 shows representations of each 'fish' and describes the underlying mathematical ideas for each.



(a) Marina's Fish Shop sign: Fish body and tail are formed by a square and right-angled triangle, respectively. Total horizontal width =10m, fish body width = b m; the total area is to be maximised as b varies. See HREF1



(b) Volivoli Fish logo: Three medium and one small square are removed from large square, forming the fins and the eye of the fish, respectively; *a*=2*b*+2*c*; the percentage of large square area remaining is to be investigated as the ratio *b*:*c* varies.

Figure 1. Fish-related geometric diagrams and the nature of the investigations.



#### **THE PROBLEM**

The author was on a holiday in Fiji where it was noticed that a resort was using a logo meant to represent a stereotypical tropical fish. This logo appeared in varying sizes with some appearing on promotional material, directional signage, furniture and walls. Others were seen submerged in the tiling of the swimming pool's base and steps. All representations, whether in or out of the water, showed the square tiling pattern which enabled an area calculation to be completed (ignoring the width of the grouting or drawn border lines) using simple arithmetic of counting tiles. See Figure 2.



Figure 2. : Representations of the Volivoli Fish logo, each showing b:c = 2:1

Problem 1 was posed and could be easily solved: What percentage of the original 'large square' remained after the four smaller squares were removed?

A more careful examination of a photograph of another below-water tiling pattern from the deep end of the pool revealed a different ratio was now being used. See Figure 3.



*Figure 3*. *b*:*c* = *3*:*1*.

In light of this discovery, it was decided to classify those Volivoli Fish with b:c = 2:1 as Type I, and those with b:c = 3:1 as being Type II. Problems 1 and 2 had identical wording regarding the percentage of original large square area remaining after the removal of the four squares for each of Types I and II.



When calculations unsurprisingly revealed different percentage values, and that Percentage (I) > Percentage (2) > 50%, the author looked at a suddenly more complex Problem 3: What ratio *b*:*c* is required for exactly 50% of the original large square to remain?

Pedagogical questions then surfaced as the author considered the artistic and mathematical benefits arising from students using materials to create their own fish, and using technology to model the likely complex equations arising from Problem 3.

#### **MULTIPLE PATHWAYS**

It was evident that, given a sufficient amount of time and access to the 'old' technology of the ruler, graph paper and scissors, students could design their own fish for any suitable (ie, easy to handle) square of side length a cm, provided the relationship a = 2b + 2c was maintained. Also, it would certainly be appropriate and useful for the teacher to provide same-sized plastic or wooden squares in large quantities for students to construct a fish in a manner similar to the swimming pool tiling pattern. Alternatively, e-technology such as that provided by Geogebra could assist a CAD approach. In either way, students' freedom to explore a ratio of their choice and make the necessary area calculations should ensure rich discussion of techniques and discoveries. These activities are ideal for late primary to middle secondary students who need application work to maintain engagement in standard curriculum topics such as area, measurement, ratio, similarity and arithmetic. However, senior students will also benefit from an initial hands-on approach before attending to the abstractions involved in exploring the algebraic formulations which emerge. Creative students who look at extreme ratio values may provoke interesting (and valid) aesthetic-based reactions from peers who may dispute the fish-like look of some of these creatures. Of course, the possibilities are entirely in the hands of the experienced teacher who designs the task to fit the needs and abilities of the students.

#### THE ROLE OF TECHNOLOGY

The author used TI-Nspire CAS technology to construct flexible Type I and Type II Volivoli fish as can be seen in Figure 4 below, where a Geometry application was used. These diagrams featured sliders to vary the value of a while at the same time preserving the proportionality of b:c required for each of Types I and II. Teachers can use these to point out the essential unifying feature of all Volivoli fish diagrams; that is, the fact that  $b + c = \frac{a}{2}$ .



Figure 4. Screenshots of dynamic geometry used in Nspire CAS to illustrate Types I&II properties.

Intended as a pedagogical device for teacher demonstration of similarity, the Nspire files developed for this activity have other pages (Calculator and Graphs applications) which the teacher can use to prepare students for Problem 3. Alternatively and hopefully, students can develop their own pages to complement or replace by-hand algebra for further analysis. This will of course be dependent on their confidence and competence in technology exploration. See Figures 5 and 6 for detailed explanations:

<Insert Figure 5>



Define areafish (a)= $a^2-3 \cdot b^2-c^2$ solve $\left(b+c=\frac{a}{2} \text{ and } b=2 \cdot c,c,b\right)$	Done $a$ $c = \frac{a}{6}$ and $b = \frac{a}{3}$	1 <sup>st</sup> : The function <u>areafish</u> is defined as showing the remaining area as a function of $a$ 2 <sup>nd</sup> : The relationship that variables $b$ and $c$ have to $a$ and the proportional relationship between $b$ and $c$ for Type I fish allow us to
$areafish(a) c=\frac{a}{6}$ and $b=\frac{a}{3}$	$\frac{23 \cdot a^2}{36}$	form simultaneous equations, enabling <i>b</i> and <i>c</i> to each be defined in terms of <i>a</i> . $3^{rd}$ : The function <i>areafish</i> is re-expressed
$\frac{23 \cdot a^2}{36}$	0.638889∙ <i>a</i> <sup>2</sup>	showing the exact fraction of the original square remaining in a Type I fish 4 <sup>th</sup> : The fraction is converted to a decimal, implying the solution to Problem 1 is ~64%.
Nspire Calculator application page	e (computer	Explanation of steps shown on this page

Nspire Calculator application page (computer view)

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FIGURE ) ACREENSHOLAND EX	nianaiion oi 4	proceaurai siens o	Ε Ο ΑΝ ΣΟΙΠΠΟΝΤΟ Ε	roniem i Livn	PII
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$areafish(a) _{c=\frac{a}{8}}$ and $b=\frac{3 \cdot a}{8}$ 0.5625 $\cdot a^{2}$	1 <sup>st</sup> : For Type II fish, the ratio 3:1 allows a similar calculation, showing the solution to
$solve\left(b+c=\frac{a}{2} \text{ and } b=k \cdot c, c, b\right) \qquad c=\frac{a}{2 \cdot (k+1)} \text{ and } b=\frac{a \cdot k}{2 \cdot (k+1)}$ $\implies areafish(a) c=\frac{a}{2 \cdot (k+1)} \text{ and } b=\frac{a \cdot k}{2 \cdot (k+1)} \qquad a^2 \cdot \left(\frac{3 \cdot k+1}{2 \cdot (k+1)^2 + 4}\right)$ $\implies solve\left(\frac{3 \cdot k+1}{2 \cdot (k+1)^2 + 4} = 0.5, k\right) \qquad k=0.236068 \text{ or } k=4.23607$ $\implies solve\left(\frac{3 \cdot k+1}{2 \cdot (k+1)^2 + 4} = 0.5, k\right) \qquad k=-(\sqrt{5}-2) \text{ or } k=\sqrt{5}+2$	Problem 2 is ~56%. $2^{nd}$ : The variable k is introduced as the unknown value of b:c, resulting in both b and c to be re-expressed in terms of a and k. $3^{rd}$ : The function <u>areafish</u> is re-expressed showing the exact fraction of the original square remaining in any fish where b:c = k. $4^{th}$ and $5^{th}$ : That fraction is set = 0.5, and k is found in both approximate and exact forms; thus the solution to Problem 3 is that a ratio $a = 6(\sqrt{r} + 2)$ .
Nspire Calculator application page (computer view)	Explanation of steps shown on this page

*Figure 6. Screenshot and explanation of 5 procedural steps of CAS solutions to Problems 2 and 3.* 

The solution to Problem 3 (that the ratio *b*:*c*, or *k* needed to be larger than 4:1) then leads to further questions: Is there an upper limit to the value of *k*? What happens when the restriction that b > c is relaxed so that any positive ratio is considered? At what point does a *k*-value alter the aesthetics of the logo so that its resemblance to a fish is lost? Technology allows us to explore the situation a bit further by replacing *k* with *x* and exploring the function  $f(x) = \frac{3x+1}{2(x+1)^2} + \frac{1}{4}$  in its graphical representation (see Figure 7 below).



Figure 7. Screenshots and interpretations of graphical representations of the function.



Thus, analysis of area remaining can occur on a number of levels, where the artistic implications of the logo as physically constructed can be expressed in mathematical terms. Extension work for students (again, upper-primary to senior secondary) could focus on specifics related to perimeter for this same shape.

#### CONCLUSION

For the author, one of the satisfying aspects of conducting this relatively simple mathematical exploration has been experiencing continual 'What if...?' questions being posed as initial results were obtained. This would appear to be consistent with the notion that creative activity is often generated by the posing and solving of problems (Silver, 1995). It is hoped that teachers who enable their students to explore the mathematical properties of artistic creations will be similarly inspired to promote an atmosphere of creative flexibility (as described in Griffiths, 2010) within their classroom. Mathematical ideas can start with real world experiences, and it is hoped that the examples discussed in this paper will generate similar rich learning activities for teachers and students alike.

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## **Problem solving in China**

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Problem solving promotes the idea that thinking is fun. 'The nine chapters of the mathematical art' is an anonymous mathematical work that originated in China at least 2000 years ago. The book consists of 246 problems organised into nine chapters. It is a tour de force in the history of mathematics in China. An English translation was produced by Kangshen, Crossley, and Lun in 1999. The aim of this paper is to present an overview of this book. What can we, in Australia, in 2017, learn about teaching mathematics through problem solving from this ancient work?

#### INTRODUCTION

Problem solving is a key part of contemporary mathematics education. The seminal publication Pólya (1990), which was first published in 1945, is often regarded as laying the foundation for modern problem solving in mathematics education. Schoenfeld (1985) is now a standard work that brings ideas from mathematics education to bear on mathematical problem solving. Stacey and Groves (1985) offers practical lesson plans on problem solving in the junior secondary classroom, and has been highly praised as a valuable Australian contribution to the literature. A special issue of *ZDM: The International Journal on Mathematics Education* (2007, vol. 39, nos. 5-6) was devoted to presenting an international perspective on mathematical problem solving. Smullyan (1992) and Poundstone (2012) contain many fascinating problems. The recent book Felmer, Pehkonen and Kilpatrick (2016) contains 22 research papers on problem solving in mathematics education.

Learning mathematics through problem solving has a long history. *The nine chapters of the mathematical art* is 'a text book of mathematical problems that originated more than 2000 years ago in ancient China' (Kangshen, Crossley and Lun 1999, p. vii). It is, approximately, as old as Euclid's Elements. This was an important and highly influential text which 'became the core foundation for the teaching of mathematics in the Tang Dynasty (Dauben 2007, p. 187). According to Stewart (2017), the work contained the 'first historical record of negative numbers' (p. 34).

The modern translation into English by Kangshen, Crossley and Lun (1999), is the focus of this paper. I will refer to this translated work simply as the *Nine Chapters*. The aim of this paper is to describe the contents of the *Nine Chapters*, and reflect on what we, as mathematics teachers in Australia in 2017, can learn from it.

For further information on mathematics in ancient China, Needham (1959, chapter 19, pp. 1-168), Kangshen, Crossley and Lun (1999), and Dauben (2007) are excellent sources.

#### STRUCTURE OF THE NINE CHAPTERS

The problems in the text are arranged under nine headings. Rectangular field (38 problems), millet and rice (46 problems), distribution by proportion (20 problems), short width (24 problems), construction consultations (28 problems), fair levies (28 problems), excess and deficit (20 problems), rectangular arrays (18 problems), and right-angled triangles (24 problems).

When discussing a problem, the author states the problem and then gives the answer without explanation; this may be followed by a few similar problems, each one introducing a new variation, and its answer. Finally the author will state a general rule for solving such problems. These days we call this teaching by variation (see Huang and Li, 2017).

The 246 problems in the *Nine Chapters* make up only a small proportion of the text of the English translation by Kangshen, Crossley and Lun (1999); it also contains a great deal of commentary by classical Chinese writers, especially Liu Hui who lived in the third century CE, as well as commentary by the translators themselves. The contributions by Liu were important in the recognition of the text as 'a classical mathematical textbook' Kangshen, Crossley and Lun (1999, p. 5).



#### HIGHLIGHTS

It is not possible in a short paper to present a comprehensive description of the *Nine Chapters*. So, let me present just a few highlights. But first, the titles of the nine sections might suggest that the *Nine Chapters* deals only with practical problems. This impression can be dispelled by problem 17 in chapter 1 which is stated as follows:

'Again given  $3\frac{1}{3}$  persons share  $6\frac{1}{3}$  and  $\frac{3}{4}$  coins. Tell: how much does each person get? Answer: Each gets  $2\frac{1}{8}$  coins.' (Kangshen, Crossley, and Lun 1999, p. 80).

#### Greatest common divisor

Finding the greatest common divisor arises in Problem 6 in chapter 1 which is stated as follows:

'Given another fraction  $\frac{49}{91}$ . Tell, reducing it, what is obtained?' (Kangshen, Crossley, and Lun 1999, p. 64).

To reduce a fraction to its lowest terms involves finding the greatest common divisor of the numerator and denominator. Let us denote the greatest common divisor of two positive integers m, n by (m, n). If m, n are positive integers, and m < n, then the set of common divisors of m, n is equal to the set of common divisors of m, n - m. Hence, (m, n) = (m, n - m). We can apply this result repeatedly, and obtain the following.

(49,91) = (49,91-49) = (49,42) = (7,42) = (7,35) = (7,28) = (7,21) = (7,14) = (7,7) = 7.

What we see here is, essentially, Euclid's algorithm for finding the greatest common divisor, using repeated subtraction rather than division. The first point to notice is that this was written in the *Nine Chapters* more than 2000 years ago, roughly around the same time that Euclid wrote *Elements*. Now it is a fair assumption that the authors of *Elements* and *Nine Chapters* did not even know of the other's existence. This causes me to marvel at the mathematics of ancient China just as much as I marvel at the mathematics of ancient Greece. The second point to notice is that the above method might be more readily understood by Australian students in primary or junior secondary classes than would Euclid's algorithm; I have used this method based on repeated subtraction with success in a Year 7 class. The third point is that Euclid's method would be more efficient than the above method (especially if the numbers involved were large) simply because division is more efficient than repeated subtraction. Finally, we note that the method in the *Nine Chapters* has the advantage that it would work for finding the greatest common divisor for more than two numbers. This is not a feature of Euclid's algorithm. The following example illustrates this point.

(385,390,715) = (385,5,330) = (380,5,325) = ... = (5,5,5) = 5.

#### Fractions

Many students, at all levels, have trouble with fractions. Chapter 1, problem 15 in the *Nine Chapters* is a lovely problem about fractions. It is stated as follows:

'Now, given fractions,  $\frac{1}{3}$ ,  $\frac{2}{3}$  and  $\frac{3}{4}$ . Tell: how much must be taken from the larger [fractions] and added to the smaller [fractions] to get the mean?' (Kangshen, Crossley, and Lun 1999, p. 77). Here 'larger' means 'larger than the mean' and 'smaller' means 'smaller than the mean'.

This problem would challenge students because it involves several aspects of fractions. The approach outlined in the *Nine Chapters* appears to work well when there are three fractions with two of them larger than the mean. The supplementary discussion offered by the translators show how the general approach can be extended to work with any number of fractions, but it suffices to say here that the method is non-trivial. This is an occasion that demonstrates the power of algebra in explaining an algorithm, and algebra was not available to the authors of the *Nine Chapters*.

#### Simultaneous linear equations

Simultaneous linear equations arise in Year 10 of the Australian Curriculum (ACARA, 2016). This topic is challenging for many students. Could the topic be approached through problem solving? What would be the result of simply



presenting Year 10 students with a problem, the solution of which will involve simultaneous equations, and let them loose on it? So I was keen to look at the problems in chapter 8 of the *Nine Chapters* that deal with solving systems of simultaneous linear equations.

Here is problem 1 in chapter 8; dou is a measure of volume or capacity.

'Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 1 bundle of low grade paddy, yield 34 *dou*. 1 bundle of top grade paddy, 2 bundles of medium grade paddy, [and] 3 bundles of low grade paddy, yield 26 *dou*. Tell: how much paddy does one bundle of each grade yield? Answer: Top grade paddy yield  $9\frac{1}{4}$  *dou* [per bundle]; medium grade paddy  $4\frac{1}{4}$  *dou*; [and] low grade paddy  $2\frac{3}{4}$  *dou*' (Kangshen, Crossley, and Lun 1999, p. 399).

There are some obvious notable points about this problem. It is interesting that this, the very first problem, involves three equations in three unknowns; one might have thought that a system involving two unknowns would be a better starting point. The answers are not particularly pretty, yet it would be easy to construct a problem with simpler answers.

On the other hand, it is more difficult to guess the solution to this system than a simpler system. The solver will have to find a method. In this sense, the problem conforms to contemporary views in mathematics education about the types of problems that are suitable for problem solving.

So, if I wanted to adopt a problem solving approach to simultaneous equations, I might start with a system of three equations where the solution is not easily guessed. Indeed, I might even start with a problem like this one that involves three unknowns rather than a problem where the system involves only two unknowns.

The outstanding point about this problem in the *Nine Chapters* is that the solution presented involves matrices ... and remember that this was written more than 2000 years ago. 'The solutions are correct, clear and appropriate and can be directly compared with modern mathematical methods' (Kangshen, Crossley, and Lun 1999, p. 388).

#### CONCLUSIONS

What can we, in Australia, in 2017, learn about teaching mathematics through problem solving from the Nine Chapters?

The Australian Curriculum aims to promote intercultural understanding as a general capability in Australian students. 'In the Australian Curriculum, students develop intercultural understanding as they learn to value their own cultures, languages and beliefs, and those of others' (ACARA, 2016). Obviously this can be done through subjects such as history, politics, languages, and art. Mathematics can also play a part in this national endeavour. Studying the mathematics from other cultures and civilisations is another way to increase the general interest of our students in those cultures. The study of mathematics, especially ancient mathematical cultures, provides an avenue to promoting understanding of those cultures. Frankly, it does not particularly matter whether Pythagoras' theorem was established in ancient China before or after it was established in ancient Greece. Rather, we should be impressed that this result flowered independently in these ancient cultures. This generates interest in, respect for, those cultures.

The *Nine Chapters* illustrates how problem solving was used to teach students in ancient China about applicable, mathematical methods. Contemporary ideas surrounding problem solving in mathematics have developed greatly since the times of Pólya, to the point where, now, there is a large body of literature surrounding this type of learning and teaching. In modern parlance, the approach of the authors in the *Nine Chapters* is teaching through variation. There is scope for further research in teaching mathematics through variation in Australia. A starting point would be Huang and Li (2017). Yoh and Ree (2010) have linked this ancient mathematical work with modern ideas in the theory of education.

Historical works such as the *Nine Chapters* are useful sources of mathematical problems that have a certain heritage or flavour. Zimmermann (2016) makes a similar point. Students learn some mathematics and some history at the same time. Let me close with a lovely problem from the *Nine Chapters*.

'Now a wild duck flies from the south sea to the North Sea in 7 days, and a wild goose flies from the North Sea to the south sea in 9 days. Assume that the birds start at the same moment. Tell: when will they meet?' (Kangshen, Crossley, and Lun 1999, p. 337).



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# Planning and assessing mathematical reasoning in the primary classroom

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Despite a growing emphasis in Australia on mathematical reasoning in curriculum documents and resources to support its understanding and teaching, reasoning still remains difficult to plan and assess for many teachers. In this paper, we share the planning of a number task intended to elicit reasoning in primary students, and the assessment of their reasoning through utilising a purposefully designed rubric to determine three reasoning actions; Analysing, Forming Conjectures and Generalising, and, Justifying and Logical Argument. These resources for planning and assessing reasoning not only offer teachers an avenue to examine their students' mathematical reasoning, but also as a springboard to future planning for reasoning opportunities in all mathematics lessons.

#### MATHEMATICAL REASONING

In the Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority, 2017), reasoning is one of the four key proficiencies expected to be embedded in every mathematics lesson. It states reasoning is the 'capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising' (p. 5), thus emphasising action from the learner (Herbert & Bragg, 2017). The term 'mathematical reasoning' describes various types of reasoning, such as, induction, deduction, abduction and adaptive reasoning (Kilpatrick, Swafford, & Findell, 2001). Reasoning is more than just thinking, it involves communicating that thinking to others. This communication can be verbal, written, diagrams, or gestures. However, evidence of 'mathematical reasoning is infrequent in many [Australian] mathematics classrooms' (Stacey & Vincent, 2009, p. 271).

Currently, in Australia, more resources are being dedicated to supporting teachers' noticing, understanding, and implementing reasoning (see Australian Association of Mathematics Teachers [AAMT] *Top Drawer* and *reSolve: Mathematics by inquiry* websites). However, raising a general awareness about mathematical reasoning is not enough (Bragg, Herbert, Loong, Vale, & Widjaja, 2016). Teachers need to understand the nature of mathematical reasoning, plan for reasoning, and assess reasoning to better equip their students to reason effectively. 'It takes time for teachers to become aware of the nuances of mathematical reasoning and be able to express their understanding of it', (Herbert, Vale, Bragg, Loong, & Widjaja, 2015, p. 36). Below we describe a teacher's planning of a task specifically intended to provide opportunities for students to develop their ability to reason mathematically and the teacher's assessment of reasoning through employing the *Assessing Mathematical Reasoning Rubric*.

#### **PLANNING FOR REASONING**

Planning is a key component of teaching practice whereby teachers' decision-making shapes their students' learning opportunities (Superfine, 2008). Teachers' mathematical knowledge for teaching informs planning judgements (Davidson, 2017) and the selection of the tasks they teach (Ball, Thames, & Phelps, 2008). As previous research has suggested, teacher knowledge of mathematical reasoning is crucial in assisting students' development of mathematical reasoning (Lannin, Ellis, & Elliott, 2011; Stylianides & Ball, 2008). It is the mathematical knowledge for teaching combined with an understanding of reasoning that ultimately influences planning of the pedagogical approach and tasks to stimulate students' reasoning actions.

So, how do we inject reasoning into our lessons? One approach is to plan for reasoning prompts or questions to be inserted into any lesson to elicit different reasoning actions. These prompts (Victorian Curriculum and Assessment Authority, 2002; AAMT, 2011) may include:

- What is the same and different about ...? (Analysing)
- Alter an aspect of something to see (such and such) effect. If we change this what will happen? (Analysing)
- Can you go through that step by step? (Justifying)
- What is the pattern here? (Generalising)
- Is that ... (pattern) always going to work? (Generalising)
- What happens in general? (Generalising)
- Is there a rule? (Generalising)



Such prompts have proved effective in drawing out the language of reasoning (Bragg, et al., 2016) through the logical conjunctions used in mathematical explanations, such as, so, thus, or, then, and because, as well as clauses, such as, but..., and, on the other hand (Clarkson, 2004). Of particular note is Clarkson's suggestion that the meaningful use of these reasoning words can only be learned in context through interactive lessons. Therefore, an effective approach to the planning of reasoning includes the consideration of interactive tasks which offer opportunities for developing reasoning language and thinking. The following Number Tower task (adapted from Noyce Foundation, 2012) is one such task.

#### Number Tower

The number towers task involves placing numbers 1, 2, 3, 4, and 5 in the bottom row of the tower and adding two adjacent number boxes together to make the total in the number box above it (see Figure 1). This task offers students practice in addition, however, it was adapted to facilitate students' development of mathematical reasoning by including the aim of building a tower with the highest possible number at the top of the tower. Students work in pairs to explore various number towers to uncover the way to build a tower with the highest number at the top. Each attempt is recorded in a blank number tower. The pairs are encouraged to form and test conjectures during their trials, and produce a generalisation to explain how the number tower with the highest number at the top is generated.



Figure 1. Blank and completed Number Tower.

This task provides students with the opportunity to: analyse a problem; develop and test conjectures; and, form generalisations through exploring the numerical structures of a number tower generated through addition. Students consider the trials (inductive reasoning) to develop, explain, and justify conjectures. They explore and notice relationships between numerical structures (analysing); compare and contrast examples to develop ideas (conjectures); generate and test conjectures (generalising); and use understanding of number properties to justify conjectures (deductive reasoning) through the use of the reasoning language 'If...then...'

Enabling prompts assist students to begin or progress in a task. For example, 'What totals can you make using the numbers 1-5 on the bottom row?' or offering a tower with a reduced number of levels or set of numbers. Extending prompts encourage deeper thinking and challenge students to explore numerical structures further, such as: 'How do you know you have all possible solutions?', 'Is this true for any set of numbers?', 'Convince another student you are right.' This task is designed for Year 3 students but may be extended for Years 5 and 6 through the complexity of reasoning encouraged and/or the use of a different set of five integers, decimals, or fractions.

#### **ASSESSING REASONING**

Understanding the complexities of reasoning, planning for these complexities within the classroom, and then assessing reasoning for learning is a challenge for many teachers (Bragg & Herbert, 2017).

Assessment for learning involves teachers using evidence about students 'knowledge, understanding, and skills to inform their teaching. Sometimes referred to as 'formative assessment', it usually occurs throughout the teaching and learning process to clarify student learning and understanding. (Board of Studies New South Wales, 2012)

Formative assessment of reasoning is undertaken to discover how students analyse, generalise and justify their reasoning. Teachers engage with students to assess how they communicate their findings and arguments - this is where the prompts presented earlier can successfully elicit reasoning. Reasoning can be assessed through students' oral and written communication, and this assessment impacts future planning of reasoning opportunities for students. To capture the multifaceted nature of reasoning and to assist teachers in the complex endeavour of assessing reasoning, the *Assessing Mathematical Reasoning Rubric* (see Figure 2) was developed by the authors and colleagues as part of the *reSolve: Mathematics by inquiry* collection of supportive resources (Australian Academy of Science and Australian Association of Mathematics Teachers, 2017).



#### Assessing Mathematical Reasoning Rubric

	Analysing	Forming conjectures and generalising	Justifying and logical argument
lot evident	<ul> <li>Does not notice numerical or spatial structure of examples or cases.</li> <li>Attends to non-mathematical aspects of the examples or cases.</li> </ul>	<ul> <li>Does not communicate a common property or rule for pattern.</li> <li>Non-systematic recording of cases or pattern.</li> <li>Random facts about cases, relationships or patterns.</li> </ul>	<ul> <li>Does not justify.</li> <li>Appeals to teacher or others.</li> </ul>
Beginning	<ul> <li>Notices similarities across examples</li> <li>Recalls random known facts related to the examples.</li> <li>Recalls and repeats patterns displayed visually or through use of materials.</li> <li>Attempts to sort cases based on a common property.</li> </ul>	<ul> <li>Uses body language, drawing, counting and oral language to draw attention to and communicate:</li> <li>a single common property</li> <li>repeated components in patterns.</li> <li>Adds to patterns displayed verbally and/or visually using diagrams or through use of materials.</li> </ul>	<ul> <li>Describes what they did and why it may or may not be correct.</li> <li>Recognises what is correct or incorrect using materials, objects, or words.</li> <li>Makes judgements based on simple criteria such as known facts.</li> <li>The argument may not be coherent or include all steps in the reasoning process</li> </ul>
Developing	<ul> <li>Notices a common numerical or spatial property.</li> <li>Recalls, repeats and extends patterns using numerical structure or spatial structure.</li> <li>Sorts and classifies cases according to a common property.</li> <li>Orders cases to show what is the same or stays the same and what is different or changes.</li> <li>Describes the case or pattern by labelling the category or sequence.</li> </ul>	<ul> <li>Communicates a rule about a:</li> <li><i>property</i> using words, diagrams or number sentences.</li> <li><i>pattern</i> using words, diagrams to show recursion or number sentences to communicate the pattern as repeated addition.</li> <li>Explains the meaning of the rule using one example.</li> </ul>	<ul> <li>Verifies truth of statements by using a common property, rule or known facts that confirms each case. May also use materials and informal methods.</li> <li>Refutes a claim by using a counter example.</li> <li>Starting statements in a logical argument are correct and accepted by the classroom.</li> <li>Detecting and correcting errors and inconsistencies using materials, diagrams and informal written methods.</li> </ul>
Consolidating	<ul> <li>Notices more than one common property by systematically generating further cases and/or listing and considering a range of known facts or properties.</li> <li>Repeats and extends patterns using both the numerical and spatial structure.</li> <li>Makes a prediction about other cases:         <ul> <li>with the same property</li> <li>included in the pattern.</li> </ul> </li> </ul>	<ul> <li>Identifies the boundary or limits for the rule (generalisation) about a common property.</li> <li>Explains the rule for finding one term in the pattern using a number sentence</li> <li>Extends the number of cases or pattern using another example to explain how the rule works.</li> </ul>	<ul> <li>Uses a correct logical argument that has a complete chain of reasoning to it and uses words such as 'because', 'if then', 'therefore', 'and so', 'that leads to'</li> <li>Extends the generalisation using logical argument.</li> </ul>
Extending	<ul> <li>Notices and explores relationships between:</li> <li>common properties</li> <li>numerical structures of patterns.</li> <li>Generates examples:</li> <li>using tools, technology and modelling</li> <li>to form a conjecture.</li> </ul>	<ul> <li>Communicates the rule for any case using words or symbols, including algebraic symbols.</li> <li>Applies the rule to find further examples or cases.</li> <li>Generalises properties by forming a statement about the relationship between common properties.</li> <li>Compares different symbolic expressions used to define the same pattern.</li> </ul>	<ul> <li>Uses a watertight logical argument that is mathematically sound and leaves nothing unexplained.</li> <li>Verifies that the statement is true or the generalisation holds for <i>all</i> cases using logical argument.</li> </ul>
Ev	idence of student's reasoning (work sa	mple and orally).	

Figure 2. Assessing Mathematical Reasoning Rubric.



The Assessing Mathematical Reasoning Rubric describes the development of reasoning over three reasoning actions: Analysing; Forming Conjectures and Generalising; and, Justifying and Logical Argument. When Analysing, students explore the problem by considering or generating some examples. They notice variance and invariance in the examples. Forming Conjectures and Generalising involves conjectures generated from their noticing of variance and invariance in the examples considered and use other examples to test their conjecture. They generate and communicate statements about the problem yet to be tested, then identify commonalities across more than one example. Justifying and Logical Argument consists of verifying conjectures and generalisations, possibly using one or more logical steps of the form 'if..., then'. A watertight logical argument is a series of mathematically correct steps which clearly explains all aspects of the solution.

These three reasoning actions are represented in the columns of the rubric. The development of each reasoning action is described in five levels: Not evident; Beginning; Developing; Consolidating; and, Extending. So each cell of the table represents a description of a reasoning action at a particular level. The contents of each cell is a bulleted list of attributes displayed by students at that level for the particular reasoning action. A teacher may use the rubric by highlighting the bullet points to establish a student's level for each reasoning action. It is important to be aware that for any particular task, students may demonstrate different levels for each reasoning action. In the example below we illustrate how a student's work sample may provide evidence of this difference in levels for each reasoning action occurring. In addition, the levels are not aligned with particular year levels, although it is likely that older students would be assessed at higher levels, but this is dependent on the task and the frequency of opportunities students are given to reason. A space is provided at the bottom of the rubric to record any oral communication of reasoning the teacher noticed.

#### Assessing Number Tower work sample

When considering the assessment of a work sample, the teacher in our project highlighted statements within the rubric to determine the level for each reasoning action. Figure 3 (students' work duplicated electronically for clarity) and the quotes below comprise of a work sample collected from one pair of students who worked together on the Number Towers task. Here we demonstrate which dot points from the rubric are highlighted to identify reasoning levels.



Figure 3. Exploring Number Towers (electronically generated to clarify).

#### Analysing

In analysing this problem the pair have generated 8 Towers (see Figure 3). The Towers show the pair analysing the problem by exploring what happens when the numbers 1 to 5 are placed in the first row.

So the following dot points in rubric are shaded and identifies Analysing to be at the Developing level.



- Sorts and classifies cases according to a common property.
- Orders cases to show what is the same or stays the same and what is different or changes.
- Sorts and classifies cases according to a common property.

#### Justifying and Logical Argument

In reviewing the Towers in Figure 3, we notice the pair trying out different combinations of the numbers 1 to 5 on the bottom row of the tower. In Tower 4 the pair they have shown that using 5 at the end of the first row will not generate a number tower with a higher number at the top. So the following dot point in rubric is shaded.

• Refutes a claim by using a counter example.

Towers 7 and 8 show the pair checking their conjecture about the placement of the 5 in the middle box of the first row. This corresponds to the dot point.

• Verifies truth of statements by using a common property, rule or known facts that confirm each case.

The pair responded to the question 'What is the one thing you have learnt today?' by stating: '*I have learnt with the pyramids that you can't have the big numbers on the side if you want a big number [at the top]*'. This response demonstrates a statement which is the first correct step in a logical argument. So the following dot point in the rubric could be shaded.

• Starting statements in a logical argument are correct and accepted by the classroom.

#### Forming Conjectures and Generalising

The pair's response to the question 'What helped you find that out?' was '*Firstly we did some trial and error then put the 5 in the middle, the 3 and the 4 on the sides of the 5. Lastly we put the small numbers [1 and 2] on the ends*'. This response indicates that the pair were attempting to form a conjecture based on their exploration of the problem. So the following dot point in the rubric is shaded.

• Identifies the boundary or limits for the rule (generalisation) about a common property.

Taken together the dot points shaded in the rubric indicate this pair is classified as 'Developing' for the *Analysing* action, 'Consolidating' for the *Forming Conjectures and Generalising* action, and 'Developing' for the *Justifying and Logical Argument* action. Whilst we have employed a written work sample here to illustrate the application of the rubric, other forms of communication such as verbal, diagrams and gestures may also be used to assess students' reasoning.

#### CONCLUSION

Teachers place a priority and value on the results of their assessment tasks to determine planning (Sullivan, Clarke, Clarke, Farrell, & Gerrard, 2013). Therefore, being able to effectively assess reasoning has an impact on future planning for reasoning. The *Assessing Mathematical Reasoning Rubric* offers teachers an effective way to interrogate their students' reasoning and provides a clear pathway to future planning for more complex reasoning opportunities. Further support material to assess reasoning can be sourced on the AAMT ReSolve website http://www.resolve.edu.au/

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**Peer reviewed papers** 



## How to write excellent multiple-choice items

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Multiple-choice questions have a mixed reputation in education, particularly now with the emphasis on NAPLAN results. This new online test presents items in different interactive formats but essentially the nature of choosing from three to five options remains the same. Unfortunately, due to the high cost of marking and subjectivity of markers of open-ended assessment, the multiple-choice style of item will become an even more dominant form of assessment in the future. This paper will provide teachers with the tools to construct high quality multiple-choice items in the context of mathematics and the Australian Curriculum. Hess' Cognitive Rigor Matrix (2009) will be used to map items according to the Webb's Depth of Knowledge and Bloom's Taxonomy (Perkins, 2008). We will show how an ordinary item can be altered to engage students by setting it in a relevant context, choosing plausible distractors and adjusting the depth of knowledge and level of thinking required to solve it.

#### ASSESSMENT FOR LEARNING

Assessment for learning is a process by which students' responses in an assessment are used by teachers and students as a diagnostic tool. Their respective teaching and learning strategies can be adjusted based on misunderstandings and concepts not yet fully understood by the students. It can also be used by teachers to gather evidence of students' skills, knowledge and understanding of specific topics, and by students to guide their revision program.

#### PURPOSE OF MULTIPLE-CHOICE ITEMS

Teachers need efficient and reliable ways to monitor the progress of their students. The use of multiple-choice items as assessment for learning can help teachers to quickly identify and address common misconceptions (Rogers & Zoumboulis, 2015) and lack of prerequisite skills early in the learning process. Pre-tests, daily quizzes and post-tests can be constructed using this format as they provide immediate feedback to both the student and the teacher. The results of a pre-test can be used to adjust the program for the next unit of work so it is relevant to the students' needs and levels of understanding.

#### STYLES OF MULTIPLE-CHOICE ITEMS

Typically, multiple-choice items have 4 or 5 choices but they can also include true/false, drag and drop, multiple answers such as A and C, and other variants in which the test taker matches related pairs of objects.

The standard multiple-choice item called 'selected response' is the most common form of a style of test question. Essentially, any question where the test taker picks from a set of pre-chosen responses is a selected response item.

The true/false style of item is typically knowledge-based and requires only low-level thinking. These items, if used, are best placed at the start of an assessment.

Example 1: 27 is an even number. T/F.

Drag and drop items can vary in difficulty. It is an interactive way for students to demonstrate their understanding.

Example 2:Order these measurements from shortest to longest.6.45 cm5.5 m11 mm405 cm

Unlike constructed response (free response), multiple-choice items give the test-taker a clue to the solution and there is a chance of guessing by the process of elimination. The degree of difficulty can vary greatly depending on the choice of distractors. When thought out carefully, option distractors (wrong answers) can be used to diagnose misconceptions.

Example 3: Constructed response: 23 - 17 =

Example 4: Selected response: 23 - 17 = ?

(A) 6 (B) 7 (C) 14 (D) 40



Combining both selected-response items and qualitative explanation would give a more accurate assessment of student knowledge.

#### STEPS FOR CONSTRUCTING QUALITY MULTIPLE-CHOICE ITEMS

Much thought needs to be put into developing an excellent multiple-choice item so that it can be used as a diagnostic tool.

#### CHARACTERISTICS

#### A good item should

- Have a specific outcome, or set of outcomes, it is testing.
- Include clear diagrams and images that are relevant to the item.
- Use authentic quantities, measurements and information.
- Have one key (correct response) and non-trivial, plausible distractors that take into account common misconceptions and incorrect methods of solution.
- Be phrased positively where possible.
- Have all the information needed for the solution presented in the stimulus.
- Use short sentences to reduce the cognitive load with vocabulary only as complex as it needs to be.
- Use the active voice and avoid using 'if' as this could be ambiguous.
- Not be a trick such as 'Where were the survivors buried?'
- Not give clues about the key such as grammatical errors in the distractors, options of different lengths, other items in the test providing clues, only one sensible option and one option resembling the wording of the stem.

#### STRUCTURE

A test item is generally structured in three parts.

*Stimulus:* The text and/or images containing the relevant information needed to solve the problem. *Stem:* A specific question or instruction. *Response area:* Where or how the student answers the question.

Example 5: Stimulus: Mike and his brother, Tim, each threw a shot put. Mike's shot put landed 1.3 m ahead of Tim's. Stem: Which two distances could they have thrown their shot puts? Response area: \_\_\_\_\_ Options: (A) 5.1 m and 4.8 m (B) 5.1 m and 3.8 m (C) 4.6 m and 3.9 m (D) 4.1 m and 2.4 m

In this example, the distractors behind the options are:

- Option A: Trades 1 for 10 tenths but forgets to take the 1
- Option B: Key (correct answer)
- Option C: Subtracts 6 from 9, then 3 from 4
- Option D: Realises that decomposition is needed but still takes 1 from 4

Although this structure is neat, the distinction between each part isn't necessarily always clear nor does it need to be.

The choices are referred to as 'options', the correct choice is called the 'key' and the incorrect options are called 'distractors'.

#### **OPTION WRITING**

The major difference between a multiple-choice item and a constructed response item is the provision of options. Any reduction in guessing and any improvement in diagnostic information must come from the quality of the options.



One option is unique, the key, but the distractors are of equal importance. Each option can be thought of as an error trap. A student's encounter with a mathematics problem is not wholly unpredictable even if it is something governed as much by chance as by reason. The distractors we choose for an item to a great extent control what the item is testing and what information can be gained from it. They should cover a range of common but non-trivial errors and misconceptions.

#### DISTRACTORS

Distractor reasoning should mirror the reasoning behind the key and is important in the production of distractors and the key. It should indicate the faulty method that a student might follow to arrive at the distractor. Providing distractor reasoning forces the item writer to consider how the student may approach the question. This has a double benefit of sometimes identifying a flaw in the item – for example when a faulty method leads to the key. It also helps teachers to easily identify misconceptions and common errors made by students.

#### COMMON DISTRACTORS

Common errors and misconceptions: This is the most common form of distractor and the best for diagnostic purposes. It also helps students feel less stressed as their answer will usually match one of the options.

Partial solution: For complex problems, a common choice for a distractor is based on an incomplete method. That is, the student has chosen a response, which matches with a correct calculation at some stage of the problem rather than the final answer.

- Example 6: 120 students went on a trip. Two thirds travelled by bus. Each bus could carry 23 students. How many buses were needed?
  - (A) 3 (partial solution: divides 80 by 23 but does not round up)
  - (B) 4 (key)
  - (C) 5 (does not read that two-thirds travel by bus and has rounded down instead of up)
  - (D) 6 (partial solution: knows to round up but ignores the information about two-thirds travelling by bus)

For the purposes of assessment for learning, a distractor based on reasoning more sophisticated than the key can be used to find out what a student already knows. This is not recommended for a summative test as is may disadvantage these high achieving students.

#### PLAUSIBILITY

Writing distractors involves a little cognitive psychology and teaching experience, as three plausible ways of answering each problem incorrectly need to be found. This follows from their role in hiding the key. The key should not stand out or be obviously the correct answer without regard to the skill being tested. With mathematics items, keys that can be deduced logically are not as problematic as keys, which can be discovered because of grammatical or linguistic clues.

Some mathematics items (particularly ones with numerical answers) often appear to have distractors based on providing three numbers similar in size to the key. This is sometimes characterised as one-below, one-above and one-very close. This approach is inadequate for several reasons.

Firstly, it assumes that students hunt for an answer that matches the conditions given in the item. This assumption does accord with some multiple-choice items particularly in other subjects or tests of recall. However, a better assumption is that students engage with a mathematical task and arrive at a response. If the response is not present in the options then the student has effectively been given a second chance, which can affect the performance of the item.

Another issue is key-balance. Distractors chosen in this way lead to keys being either option-B or option-C when the options are listed in numerical order. As a consequence, sometimes the most plausible distractor may need to be swapped for a less plausible option so avoid giving away the key and to maintain a key balance.



#### **COMMON ERRORS**

A more sophisticated method of distractor choice is to consider the likely errors students will make. This not only helps to provide good distractors but also meaningful diagnostic information for the teacher.

Identifying the common errors can be made less subjective by examining the performance of similar items in previous tests. Knowledge of the skills and knowledge of the test population is also helpful.

One approach to identifying common errors is to look at the steps needed to find the key. For example, 28 - 2x = 36 requires the student to solve a simple equation. The options could be based on common errors made by students when solving two-step equations.

Example 7:	28 - 2x = 36
	(A) $-10$ (Rearranges the equation correctly to $2x = -8$ but subtracts 2 instead of dividing by 2)
	(B) $-4$ (key)
	(C) 4 (solves $2x = 36 - 28$ )
	(D) 10 (Halves 36 to get $28 - x = 18$ then solves the remaining equation correctly)

As we need three plausible distractors, the task in the item has to have enough complexity to generate three errors. More complex problems may generate so many plausible errors that selecting the best three can be difficult.

#### WHAT TO AVOID

Option driven items: An item that can't be answered without its options is called 'option driven'. Typically, the item gives a set of conditions, which apply to many objects in theory and ask which of the four options the condition applies to. In this case, the distractor reasoning should indicate why the student would mistakenly think the conditions were true for that option.

Negative items: Some items ask about a condition, which does NOT hold. In this case the three distractors are. In effect, true answers which do match the given condition. Such items should be used only rarely but when they are used the distractor reasoning should indicate why a student might plausibly regard the options as being FALSE.

#### **ASSESSMENT OF LEARNING**

Multiple-choice tests can also form part or all of a larger summative assessment provided this format suits their purpose. They can cover a wide range of curriculum outcomes and can be designed to test critical thinking skills.

A good test should

- Cover a range of skills, depths of knowledge and levels of thinking needed to solve the problem.
- Be unambiguous and free from bias.
- Reduce guessing by providing engaging and suitably challenging items.
- Be peer- reviewed to ensure suitability, coverage and accuracy.
- Contain uncomplicated yet non-trivial contexts relevant to both boys and girls.
- Be used as a diagnostic tool.
- Have a specific outcome, or set of outcomes, it is testing.

#### COGNITIVE RIGOR MATRIX

Using a design framework to plan a multiple-choice style assessment will help to achieve a balance between both the content and cognitive domains. Karen Hess' Cognitive Rigor Matrix (CRM, 2009) combines Bloom's Taxonomy and Webb's Depth of Knowledge. Table 1 is adapted from the Hess CRM for Mathematics. Each cell contains content-specific descriptors that can be used to categorise, plan and write good quality selected-response items. Tables 2 and 3 show how a selected response item can be written at different Depths of Knowledge (DOK) levels of Hess' Cognitive Rigor matrix.



Revised Bloom's Taxonomy	Webb's DOK Level 1 Recall & Reproduction	Webb's DOK Level 2 Skills & Concepts	Webb's DOK Level 3 Strategic Thinking/ Reasoning				
Remember Retrieve knowledge from long-term memory, recognize, recall, locate, identify.	<ul> <li>Recall, observe, &amp; recognize facts, principles, properties</li> <li>Recall/identify conversions among representations or numbers (e.g., customary and metric measures)</li> </ul>	Not included in matrix					
Understand Construct meaning, clarify, paraphrase, represent, translate, illustrate, give examples, classify, categorize, summarize, generalize, infer a logical conclusion (such as from examples given), predict, compare/contrast, match like ideas, explain, construct models.	<ul> <li>Evaluate an expression</li> <li>Locate points on a grid or number on number line</li> <li>Solve a one-step problem</li> <li>Represent math relationships in words, pictures, or symbols</li> <li>Read, write, compare decimals in scientific notation</li> </ul>	<ul> <li>Specify and explain relationships (e.g., non- examples/examples; cause- effect)</li> <li>Make and record observations</li> <li>Explain steps followed</li> <li>Summarize results or concepts</li> <li>Make basic inferences or logical predictions from data/ observations</li> <li>Use models /diagrams to represent or explain mathematical concepts</li> <li>Make and explain estimates</li> </ul>	<ul> <li>Use concepts to solve non-routine problems</li> <li>Explain, generalize, or connect ideas using supporting evidence</li> <li>Make and justify conjectures</li> <li>Explain thinking when more than one response is possible</li> <li>Explain phenomena in terms of concepts</li> </ul>				
Apply Carry out or use a procedure in a given situation; carry out (apply to a familiar task), or use (apply) to an unfamiliar task.	<ul> <li>Follow simple procedures (recipe-type directions)</li> <li>Calculate, measure, apply a rule (e.g., rounding)</li> <li>Apply algorithm or formula (e.g. area, perimeter)</li> <li>Solve linear equations</li> <li>Make conversions among representations or numbers, or within and between customary and metric measures</li> </ul>	<ul> <li>Select a procedure according to criteria and perform it</li> <li>Solve routine problem applying multiple concepts or decision points</li> <li>Retrieve information from a table, graph, or figure and use it solve a problem requiring multiple steps</li> <li>Translate between tables, graphs, words, and symbolic notations (e.g., graph data from a table)</li> <li>Construct models given criteria</li> </ul>	<ul> <li>Design investigation for a specific purpose or research question</li> <li>Conduct a designed investigation</li> <li>Use concepts to solve non- routine problems</li> <li>Use &amp; show reasoning, planning, and evidence</li> <li>Translate between problem &amp; symbolic notation when not a direct translation</li> </ul>				
Analyse Break into constituent parts, determine how parts relate, differentiate between relevant- irrelevant, distinguish, focus, select, organize, outline, find coherence, deconstruct	<ul> <li>Retrieve information from a table or graph to answer a question</li> <li>Identify whether specific information is contained in graphic representations (e.g., table, graph, T-chart, diagram)</li> <li>Identify a pattern/trend</li> </ul>	<ul> <li>Categorize, classify materials, data, figures based on characteristics</li> <li>Organize or order data</li> <li>Compare/ contrast figures or data</li> <li>Select appropriate graph and organize &amp; display data</li> <li>Interpret data from a simple graph</li> <li>Extend a pattern</li> </ul>	<ul> <li>Compare information within or across data sets or texts</li> <li>Analyse and draw conclusions from data, citing evidence</li> <li>Generalize a pattern</li> <li>Interpret data from complex graph</li> <li>Analyse similarities/ differences between procedures or solutions</li> </ul>				

Table 1: The Hess Cognitive Rigor Matrix for Mathematics

Retrieved from http://www.nciea.org/sites/default/files/publications/CRM\_math-sci\_KH11.pdf © 2009 Karin Hess permission to reproduce is given when authorship is fully cited khess@nciea.org



	Bloom's Taxonomy – Apply
DOK Level 1 Recall and Reproduce	1000 ÷ $= 500$ What is the value of $= ?$ (A) $\frac{3}{5}$ (divides 500 by 1000 and makes a place value error) (B) $\frac{2}{3}$ (divides 500 by 1000) (C) 2 (key) (D) 20 (divides by 50 or misplaces a zero)
DOK Level 2 Skills and Concepts	<ul> <li>Ann is given this equation.</li> <li>1000 ÷ ■ = ▲</li> <li>Anna knows that ■ is a positive whole number and ▲ is a 2-digit positive whole number.</li> <li>Which of these should Anna choose for ■ ?</li> <li>(A) 5 (this gives a 3-digit quotient)</li> <li>(B) 10 (this gives a 3-digit quotient)</li> <li>(C) 40 (key)</li> </ul>
DOK Level 3 Strategic Thinking	<ul> <li>(D) 125 (the quotient is a single-digit number)</li> <li>Ann is given this equation.</li> <li>1000 ÷ ■ = ▲</li> <li>Anna knows that ■ is a positive whole number and ▲ is a 2-digit positive whole number.</li> <li>How many combinations of numbers for ■ and ▲ are possible to solve the equation?</li> <li>(A) 3 (B) 4 (C) 5 (D) 6</li> <li>Analysis</li> <li>Students need to determine a strategy to use for this investigation and there is more than one answer to be considered.</li> <li>The strategy used by most students would be to make a list of the 2-digit factors of 1000 for the triangle and select the correct values for the square.</li> <li>Solution (triangle, square): (10, 100), (20, 50), (25, 40), (40, 25) and (50, 20)</li> </ul>

Table 2 - Year 6Australian Curriculum Codes ACMNA121 and ACMNA123 (All with the same key)Copyright UNSW Global Pty Limited



	Bloom's Taxonomy – Analyse							
Stimulus	The graph shows the number of people in a group playing different sports. Each ball in the graph represents one person that plays the sport.							
	Cricket							
	Basketball							
	Tennis							
	How many people play tennis?							
DOK Level 1	(A) 5 (number playing basketball)							
Recall and	(B) 6 (number playing cricket)							
Reproduce	(C) 9 (key)							
	(D) 20 (total number of people)							
	How many more people play tennis than cricket?							
DOK Level 2 Skills	(A) 1 (compares cricket and basketball)							
and Concepts	(B) 3 (key)							
	(C) 4 (compares tennis and basketball)							
	(D) 9 (number playing tennis)							
DOK Level 3	There are 14 people in the group. Each person plays at least one sport. Only one person plays all three sports.							
Strategic Thinking	How many people play exactly two sports?							
	(A) 4 (key: easy to solve by circling one group of three then 4 groups of 2)							
	(B) 5 (includes the person who plays all three sports)							
	(C) 6 (20 balls subtract 14 students leaves 6)							
	(D) 7 (20 balls subtract 14 students less the student who plays all three leaves 7)							

Table 3 - Year 3 Australian Curriculum Code ACMSP070Copyright UNSW Global Pty Limited



#### CONCLUSION

Multiple-choice tests are a useful teaching and learning tool. They provide teachers and students with rapid feedback and can be used regularly in the classroom to enhance learning and diagnose problems. This style of assessment is easy to implement and quite reliable in monitoring student progress. If coverage of the content and cognitive domains are planned, a balanced multiple-choice test will result that will challenge a large range of students. Also if the principles of item writing are adhered to and the distractors are plausible, the items are more likely to be a reliable measure of student understanding and aid instruction. To minimise the problems of guessing and making small errors which lead to incorrect choices, this style of test should be supplemented with other forms of assessment to further improve reliability (Clements & Ellerton, 1995).

Although this style of item is not suitable for open-ended problems, good multiple-choice assessments are still the most efficient and objective to mark despite being hard to set. When used as formative assessment, the results can help to guide teaching and learning. Teachers can also gain additional information if the distractors result from common errors and misconceptions and if students are encouraged to show their working. A balance of the Australian Curriculum codes and use of the CRM are important when designing a summative assessment. The assessment should have a clear purpose, be well planned and cover a range of levels of thinking and learning outcomes.

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## Fractions

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Fractions – a word that strikes fear into far too many people, including school students, preservice teachers, the public, and even some practicing teachers. In this paper, we provide an account of the framework we used to build confidence and discuss the work around unpacking concepts in ways of teaching fractions designed to benefit both teachers and students. Our aim is to provide an account of what was taken into consideration in order to build a professional development session designed by preservice teachers with academics for inservice teachers. We hope that in sharing our journey we are able to provide a broad overview for others, in terms of what needs to be considered, when building an understanding of how to teach fractions.

#### INTRODUCTION

This paper outlines an approach to teaching fractions constructed by La Trobe University pre-service teachers and academics in mathematics education. We share how our experiences, knowledge and expertise in conceptualising fractions was rebuilt into a package that provided us with the knowledge we needed to develop conceptual understanding and procedural fluency. We came together as a group of volunteers with the aim to provide in-service teachers with three two-hour professional development (herein PD) sessions in mathematics. 2017 is the third year of this PD work and this year the in-service teachers requested a focus on fractions. Our 2017 group consists of two academics, one (the first author) who conceived the work discussed here, as well as third and fourth year pre-service teachers, and again, it is important to note the valuable contributions of two of the fourth-year pre-service teachers in this group, both of whom share authorship of this paper. Each member of the overall group furthers their knowledge of fractions, further develops their pedagogy, and of course the pre-service teachers use their involvement as a means of setting themselves apart from the greater number of their peers, people with whom they will soon be competing for teaching positions.

#### PURPOSE

We offer the following account of our framework to others as an example of how we worked to target fraction arithmetic, one of the most problematic and anxiety-causing areas for students, for as discussed by Smith (2002, p.3), 'no area of school mathematics is as mathematically rich, cognitively complicated, and difficult to teach as fractions, ratio and proportion'. Our framework addresses:

- 1. Developing and promoting growth mindsets building happy, positive dispositions, valuing understanding, realising that everyone can get better at fractions;
- 2. Unpacking common misunderstandings and weaknesses;
- 3. Promoting best practice by relating vocabulary, models and algorithms, to build conceptual understanding; and
- 4. Tying pedagogy with practice.

This paper is not an account of our journey, rather it focuses on the approach we took toward teaching fractions. What we offer here is an outline of the things we took into consideration when building the fraction PD. Our aim is to assist others to work with confidence when teaching fractions. It is based on quality texts, primarily our prescribed text, Teaching Primary Mathematics by Booker, Bond, Sparrow, and Swan (2014) and uses the previously noted four points as headings to guide the conversation. Importantly, most of our learning came from members of the group continually discussing and focusing attention on this design framework, as it constantly required each member of the group to share, support and challenge thoughts and actions.

#### A GROWTH MINDSET

At the forefront of all our work was a challenge to become teachers that are constantly alert to Jo Boaler's work on building a growth mindset in mathematics education. From this work, we were alert to the idea that successful maths teachers need a good understanding of content, and a good knowledge of their students and how their students learn



(AITSL, 2017). Through engaging with research in this area and discussing outcomes, we agree that in being alert to how students view themselves, teachers are able to help students tackle poor belief structures and overcome negative beliefs. We understood the importance of immediately addressing statements such as, 'I don't have a maths brain' or 'I was never any good at maths'. Our classes are underpinned with Jo Boaler's growth mindset messages (Boaler, 2017). Her positive norms provided a baseline and reminded us that:

- 'Everyone can learn math to the highest level', with practise
- Mistakes are valuable in providing an instruction tool on how to improve
- Questions clarify understanding; resolve misunderstandings before moving forward
- Math is about
  - creativity and making sense not memorising rules and procedures
  - making connections and communicating
  - learning not performing, formative assessment promotes the whole understanding picture
- Depth is more important than speed

Having this growth mindset focus enabled us to develop an awareness of the benefits gained from providing encouraging environments and building positive attitudes toward mathematics, points which we admit highlighted numerous strengths and weaknesses in our initial practice.

#### KEY MISUNDERSTANDINGS/WEAKNESSES

Next, we explored key misunderstandings. We found that when learning about the concept of fractions it is important to have a good understanding of whole number and place value. We found that just as materials were used to build that knowledge we also needed to use models, such as fraction walls, number lines, number expanders, to build understandings of fractions. Again, using materials but this time using them to build an understanding of the concept of ones. When working with fractions we found it important to understand how they are represented and understood. Booker et al. (2014, pp. 162-186) & others (as noted) list the most likely problems with fractions:

a. Children lack understanding that equal parts are needed

- b. Circular regions may cause problems:
- 'equality among the parts is not intuitively obvious'
- While easy to see as a whole; it is difficult to partition' (Reys et al., 2012, p. 281)



d.  $\frac{3}{8}$  seen as larger than  $\frac{3}{5}$  as 8 is larger than 5 or  $\frac{2}{3}$  larger than  $\frac{7}{10}$  because there is 'only one difference between the numerator and denominator' (Goos, Stillman & Vale, 2007, p. 167)

e. Problems interpreting the symbolic representation of a fraction as a single number e.g. students asked to place  $\frac{1}{2}$  on a number line put a mark half way between 1 and 2 (Siemon et al., 2015).

f.  $2\frac{5}{6}$  calculated as  $\frac{16}{6}$  – the 'rule' (multiply then add) used incorrectly, rather than reasoning that 1 one has 6 sixths, so 2 has 12 sixths and 5 more sixths is 17 sixths.

- g. Not understanding 3 thirds is the same as a whole or 8 quarters the same as 2 wholes
- h. Finding an equivalent fraction for  $\frac{3}{4}$  and using the 'rule' incorrectly, multiplying only the denominator, hence  $\frac{3}{12}$
- i. Students confusing the addition algorithm with the process used for multiplying fractions
- j. Students incorrectly reasoning that multiplication always results in a larger number
- k.  $\frac{1}{2} \div \frac{1}{4} = \frac{2}{1} \times \frac{1}{4} = \frac{2}{4}$  misconceptions that arise with the 'invert and multiply' rule perform the 'rule' in the order read,



thus inverting the first term, then multiplying, even inverting both terms, then multiplying.

These examples demonstrate some of the problems caused by teaching rules without conceptual explanations. (Authors of key maths texts cited in this paper offer more detailed accounts). Siemon et al. (2015, p. 428) further note, 'a fraction is most commonly described as a part of a whole or as the result of dividing a collection or quantity into a given number of equal parts or shares'. Siemon et al. (2015) argues that while this only represents the part-whole representation of fractions it remains 'helpful to describe fractions in terms of their components; for example, the 'bottom' number (denominator) tells you how many parts the whole is divided into and the 'top' number (numerator) tells you how many parts you have' (p. 328). However, they add that 'students need to build an understanding of *all* of the different ways fractions can be interpreted and represented' (p. 428).

Having a proficiency in fractions concepts is an essential building block for the advancement of computative thinking in mathematics (Booker et al., 2014). Whole numbers are commonly understood and represented in everyday experiences, whereas fractions are described as artificial numbers - the extension of an abstraction idea tends to be difficult and confusing. Understanding fraction concepts is however a prerequisite for moving onto decimal fractions, common fractions and percentages. Therefore, conceptualising fractions is essential because it supports children's ability to generalise.

#### **PROMOTING BEST PRACTICE**

Next, we considered how to promote best practice. Authors such as Booker et al. (2014) and Siemon et al. (2015) emphasise the significant role of the language of instruction in the learning process. We cannot argue strongly enough of the value of this focus on the language of instruction. It is imperative that you take careful note of how the language directly connects to terms used in teaching and learning maths, in particular the consistent reinforcement of place value terminology. The texts cited here spend a lot of time demonstrating how to develop clear terminology. Good mathematical language, clear descriptions of actions, processes and outcomes, enables all learners to engage in the work as they see the purpose in each step along the way. This point is supported by and clearly described by Booker et al. (2014) when discussing the problems of a number having many names, i.e. *one half* can also be *2 quarters*, *2 fourths* and be referred to as *equivalent*. Furthermore, *1 and 3 fifths* is a *mixed* number that can be renamed as an *improper fraction*. A strong way to explore the concept of fractions is through the use of models such as rectangular region models and number lines, as explored below.

Another area to consider in regard to best practice is that based around learning standard algorithms. Researchers such as Clarke (2001) see the role and timing of the teaching of algorithms in primary mathematics as key in learner success. Cramer, Post and Del Mas (2002), also encourage educators to adopt less traditional approaches and argue for the inclusion of physical models as well as translations within and between other modes such as pictorial, verbal, real world and symbolic representations. Reys et al. note that the Australian Curriculum also emphasises the benefits of young to middle primary school 'students focusing on developing the meaning of fractions – as equal parts of a whole or equal parts of a set ... and that upper primary students should be freer to solve problems with fractions at a more personalised rate' (2012, p. 278). All those cited here agree on the importance of teaching fraction arithmetic, and finally common fraction arithmetic, processes and thinking. A strong way to encourage this understanding is the use of models (region models, number lines) and later algorithms to teach fraction concepts and an understanding of place value and renaming, all vital for the successful implementation of fraction arithmetic.

Region models, number lines and overlay grids (regions split into parts and printed onto transparency papers) are useful in illustrating how multiplication between two fractions can occur. Our first example of this work relates to multiplication and comes from Booker et al. (2014, p. 195):





I make  $\frac{3}{4}$ , take that to be the 'whole' for the moment, then find two fifths of that 'whole' by partitioning (as in the third diagram). You can then see that 2 fifths of 3 quarters is  $\frac{6}{20}$ .

The work encourages students to create their own rule based on the pattern.

#### $\frac{2}{5} \times \frac{3}{4} = \frac{6}{20}$

Both the numerators are multiplied to get the answer of 6 parts, and the denominators are multiplied together to get the answer of 20 parts in total.

Another example from Reys et al. (2012, p. 295), this time on division, asks students to start with a 6 metre roll of ribbon, and to cut this large ribbon into smaller sections that are each 3 quarters of a metre long. The question: How many of these pieces would I be able to cut?

We may write this as:  $6 \div \frac{3}{4}$ 

In presenting the question as a diagram, we would show the 6 metres:

Next, show quarters, and finally show 'how many groups of 3 quarters' exist.

											1

The above shows there are 8 groups of 3 quarters. In other words; we can cut eight *3 quarter* metre ribbons from 6 metres of ribbon.

In linking the process to the algorithm, the '6 metres' was multiplied by 4 to show how many quarters were present; 24 quarters. This was then divided by 3 because each group had to be of size 3 quarters. In summary, we multiplied by 4 and divided by 3, shown as:  $6 \div \frac{3}{4} = 6 \times \frac{4}{3}$ .

Using models to assist in developing understandings, students 'come to view algorithms as tools for solving problems rather than as the goal of mathematics' (Booker et al., 2014, p. 195).

Of course, there are many points that need to be considered on the processes and thinking that follow on directly from the teaching of whole number arithmetic, decimal fraction arithmetic, and finally common fraction arithmetic. Many of these points are further elaborated in the workshop provided.



#### **PEDAGOGY AND PRACTICE**

The work in building a framework to address how to work with fractions, as undertaken by the pre-service teachers and academics and outlined here, comes from a shared experience. The group from La Trobe built a PD program for primary school in-service teachers on how best to approach and teach fractions. In developing the PD sessions, the pre-service teachers were consistently challenged to consider pedagogical approaches, and worked to develop a deepened knowledge and expertise in how fractions work. We were aware that, to ensure any degree of mathematical proficiency, we needed to be teaching for understanding. We found that to teach effectively we needed to know *more* than the mathematics; we needed to be able to illustrate what was happening through a variety of means and we needed to encourage situations in which peers feel supported enough to share thoughts and ideas. The work undertaken by our group was considered no different to that undertaken in classrooms. In building the PD program we reflected on and used our own experiences as a means to create a logical progression. Maths opportunities were explored through fractions knowledge, and through the power of positive messaging.

We hope that through sharing this work we assist others needing a framework for teaching fractions. Overall, we found that working together strengthened individual beliefs and abilities. As a demonstration of this claim, the paper concludes with direct quotes on the impact of the PD work from the pre-service teachers involved:

- The maths here has transformed my teaching experience beyond the classroom. The opportunity to collaborate with peers and present my pedagogical knowledge to students and fellow teachers was a whole different ballgame. The idea of me being a mathematician has been a change in belief and an understanding of what real success in mathematics looks like not just about getting the right answer, success in mathematics is comprehending the algorithm, students and teachers working collaboratively to solve problems. Success is having the confidence to have a go.
- Through peer mentoring, I have gained a valuable insight into the different ways that learners think; and as a result, have developed many different teaching strategies to cater for these diverse needs. This understanding of the needs of others when learning mathematics has led me to a conceptually based teaching approach. I like Duchesne, McMaugh, Bochner, and Krause's (2013) point, that if we enable students the ability to conceptualise the content, then they can apply this learning to other scenarios emphasising the interconnectedness of the different branches of mathematics.

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## The use of hands on materials in the teaching of mathematics

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Hands on materials can be a link for students to develop a deeper understanding of mathematical concepts, particularly when using a Concrete-Representational-Abstract model. The interactive role of the teacher as a facilitator, and the use of learning conversations around the materials are discussed in this article. Eight examples of hands on materials for classroom use are offered, with suggested concepts they apply to and some starting questions for discussions. There are examples of how two of these materials could be used as a learning task in a primary school setting.

#### INTRODUCTION

Using hands on materials to enhance the teaching of mathematics has long been an accepted practice, especially in the younger primary years (Ball, 1992; Howard & Perry, 1997; Marshall & Swan, 2008). Being able to handle and manipulate objects assists students to increase their understanding, and is supported by primary school teachers across a variety of years (Swan & Marshall, 2010). In the USA, the Common Core State Standards Initiative (2010) makes mention of using concrete materials to help visualize problems and construct arguments. There is a wide variety of concrete materials available for schools and in classrooms, and sometimes they are not used to the extent that they could be. This article will look at some examples of hands on materials and how they might be used in the classroom.

In the various literature, different terms: 'manipulatives', 'concrete materials', 'physical materials', are all used to describe the objects we use to enhance our understanding of mathematics. Swan and Marshall (2010) defined a mathematical manipulative as 'an object that can be handled by an individual in a sensory manner during which conscious and unconscious mathematical thinking will be fostered.' (p. 14). This author has chosen to use the term 'hands on materials', to emphasise the handling of the materials, as opposed to demonstrating or modelling using materials, by a third person.

#### CONCRETE - REPRESENTATIONAL -ABSTRACT MODEL

The Concrete – Representational –Abstract model [CRA model] (also known as Concrete – Pictorial – Abstract) has been well documented (Flores, 2010; Mudaly & Naidoo, 2015; Witzel, 2005). It allows students to experience mathematical concepts first through manipulation of hands on materials, then by representing their findings on paper through drawing, writing or creating diagrams. This is followed by a shift to an abstract understanding of the mathematics, using symbolic or numeric representations.

The first part of the model involves students exploring a mathematical concept using hands on materials. The use of 'hands on' materials provides the concrete example of a concept, which allows students to connect with a visual representation. (Shaw, 2002). The second phase of the CRA model encourages the student to move from the physical to representing their thinking and discovering in a written format. Just using materials is not enough to create understanding. Being able to transfer knowledge from one context to another is highly important. For learning to be consolidated, the knowledge gained from play needs to be able to be transferred to other contexts. In order to show depth of understanding students should be able to create multiple representations of the concept, both by using concrete materials and their own written representations. Multiple representations assist students to have a better understanding of the abstract idea (Kilpatrick, Swafford & Findell, 2001). The final part of the model moves students from their own representations of the concept to the abstract concept inherent in all mathematics. In this stage, it is expected that students will begin to use conventional mathematical symbols and algorithms to explain their understanding.

Building students' knowledge and understanding through a CRA model, and using materials to connect into mathematical concepts, allows for deeper understanding and less reliance on rules. (Shaw, 2002).

#### USING HANDS ON MATERIALS EFFECTIVELY

The role of the teacher is fundamental in the effective use of 'hands on' materials. Generally, it is the teacher who consciously selects which materials are most appropriate to be used, and scaffolds the learning with a particular



mathematical concept in mind. The teacher needs to have the depth of knowledge to be able to assist students to make connections from their explorations and discoveries. The aim is to move students to 'develop increasing sophisticated and mathematical representations.' (Clements, 1999, p. 47).

Using hands on materials is not a 'quick fix' or instant access into mathematical knowledge. 'They [hands on materials] require careful use over sufficient time to allow students to build meaning and make connections' (Kilpatrick, Swafford & Findell, 2001, p. 353). Students should be given time to 'play' with the materials first, before using them as a learning model. The purpose of this 'play' time is twofold. Firstly, this allows the students time to engage with and experience the material informally, so that when required to concentrate later on a task they may not be (as) distracted by the novelty of the material; and, secondly, while the students are 'playing' they discover properties of the materials that can develop into understandings as the teaching occurs. For example, while playing with pattern blocks the students may discover which side of the blocks fit together to make other shapes, leading into an exploration of compound shapes, or creation of tessellations, or modelling fraction parts of shapes. After the students have had a brief time to play, more guidance can be given to lead to learning, using the materials.

#### MATERIALS ARE NOT A GUARANTEE OF BUILDING MATHEMATICAL CONNECTIONS

For younger students the connection between the hands on material and mathematical concept may not be obvious, as their prior experience of mathematical concepts is limited (Ball, 1992; Clements, 1999; Kilpatrick, Swafford & Findell, 2001). This limitation in making the connections between the materials and the concepts is why teachers are important facilitators in helping students decide the best materials to use, and to enhance the possibility of making the links through good questioning. 'Students must link their informal knowledge and experience to mathematical abstractions. Manipulatives (physical objects used to represent mathematical ideas), when used well, can provide such links.' (Kilpatrick, Swafford & Findell, 2001, p. 9).

The use of meaningful conversations in conjunction with the hands on materials develops mathematical understanding. Marshall and Swan (2008) explain 'Language is the main tool in helping to make the bridge from the concrete to the abstract,' (p. 340). Hands on materials provide students with a visual prompt to help communicate their ideas. When students are using hands on materials they talk about what they are doing in more focused discussions than when they do not have models to work with. (Shaw, 2002). Teachers can listen to and see the students' ideas and refine them, using and modelling correct mathematical language. The use of hands on materials is an interactive process between the student and the materials, and the student and the teacher.

Hands on materials need to be appropriate for the topic to be explored. When counting, adding or subtracting, a collection of like items can be used, but when dividing or working with fractions, the items need to be those which can be realistically divided – 'cakes' or blocks or rods are a more practical material to employ. It also helps when the manipulatives are more easily transferrable to a representational model – blocks lined up, transfer to arrays, and hence to a grid and area model of multiplication. (Kilpatrick, Swafford & Findell, 2001).

#### EIGHT USEFUL 'HANDS ON' MATERIALS

Hands on materials can be used for exploring varied mathematical concepts – not just the original intended purpose of the materials. The following is a list of hands on materials preferred by the author. These materials have been selected due to their flexibility of use for exploring multiple mathematical concepts. Examples of questions or instructions which could be used as a starting point to promote mathematical thinking are provided with each material. Some of the questions are deliberately open-ended, e.g. 'Use matchsticks to measure your desk', to encourage thinking of different ways that students could measure their desk – length, perimeter, area, height, width. Open-ended questions encourage students to share different viewpoints, stimulate higher levels of thinking, and students and teachers learn from the students' engagement with the problem (Sullivan & Lilburn, 2004).

#### UNIFIX BLOCKS

Unifix (Figure 1), as single blocks, can be used for counting and sorting, and adding and subtracting materials. They can be made into 'tens sticks' for place value. Unifix can be used to model repeated and growing patterns. In groups, they can be used to model skip counting, arrays and multiplication and division, and to make a 'whole' to model fractions and



decimals. They can be used to calculate measurements (length, area and volume) informally and as to assist in creating measurement calculations and formulae.



Figure 1. Unifix used for multiplication.

#### **Questions:**

- Show me all the tens facts using unifix (number facts).
- Use your unifix to show me different ways you could make 40 by multiplying (number facts).
- Given two boxes, prove which box is bigger (volume).

#### TWO SIDED COUNTERS

Similarly to unifix, two sided counters can be used to model counting and the four processes. 'Shake and spill' involves taking a number of counters, shaking them in your hand or a cup and then 'spilling' them onto the table or floor. The random fall of the counters with two different colours facing up can be used to show number facts or to create fractions. (See below for 'Shake and Spill fraction ideas). Two sided counters can also be flipped like a coin for probability activities.

#### **Questions:**

- Make a skip counting pattern with the counters (patterns).
- Rearrange your pattern to make an array. What is the relationship between your pattern and the times tables? *(number facts)*.
- If you flip a counter ten times, which colour will come up most often? Why do you think this? (probability).

#### TANGRAMS

Tangrams (Figure 2) are a practical tool for problem solving and spatial reasoning. The traditional puzzle – making a square from the 7 pieces – is a good problem solving activity for students and leads them to explore the properties of the shapes e.g. which sides fit together (and which do not). The tangram shapes can also be used to make and model other shapes and compound shapes. (Figure 3) Rearrange the shapes to explore patterns, transformations, tessellations and symmetry. Compare the size of the shapes to investigate fractions and ratios.






Figure 2. The 7 Tangram pieces.

Figure 3. Tangram squares.

#### **Questions:**

- Can you make a square with one of the tangram shapes, with two of the tangram shapes, with three of the tangrams shapes, 4 shapes, 5 shapes, 6 shapes? *(shape)*.
- How many different shapes can you make with two or more of the tangram shapes? (shape).
- If the whole tangram square is equal to one whole, what is each part worth as a fraction? (fractions).

#### MAGNETIC CONSTRUCTION SHAPES (COMMERCIALLY KNOWN AS POLYDRON).

Use magnetic construction shapes, to model 3D shapes and unfold them to show the matching nets and investigate the properties of 3D shapes (Figure 4). The shapes can be used to make patterns, tessellations and show shape transformations. They can be used to measure area and surface area of shapes. Students can give directions to others to recreate shapes and patterns.



Figure 4. Cubes and nets.

#### **Questions:**

- How many different 3D shapes can you make just using triangles? Draw and name them all correctly. (shape).
- Make a compound shape and show translations, reflections and rotations using your shape (transformations).
- Create a shape and give directions to another student so they can recreate your shape without looking at it (shape).



#### MATCH STICKS

Matchsticks are another material that can be used for counting and bundling into tens to model place value. They can also be used to make 2D shapes and investigate sides and perimeters and patterns (Figure 5). Add blutak to create 3D shapes to highlight edges and vertices. The regular length of matchsticks can help model a number line, or be used to measure length. There are many matchstick problem solving puzzles which can lead to algorithmic and algebraic thinking.



Figure 5. Matchstick squares

#### **Questions:**

- Use matchsticks to measure your desk (measurement).
- Create different shapes that each have a perimeter of 20 match sticks (measurement).

#### COLOURED CONSTRUCTION PAPER/COVER PAPER (APPROX. 125GSM)

Paper can be used to make shapes, nets, 3D shapes, and to make your own playing cards, digit cards, flash cards and dominoes. It can be used to model fractions, (Figure 6) to make patterns and tessellations, explore symmetry, and create attribute cards. Use paper to create clocks and explore angles.



Figure 6. Showing quarters in many ways.



#### **Questions:**

- Use squares of paper to show quarters in different ways (fractions)
- Make a 'paper protractor' https://www.youtube.com/watch?v=zFNBEZ9-8t8 (angles).

#### BALL OF THIN CORD (APPROX.5MM THICK)

Cord is useful to measure length when a ruler is not long enough, and perimeter for shapes of all sizes, particularly irregular shapes. It can be used as a 'washing line' to model number lines for whole numbers or fractions (Figure 7). It can be used for teamwork activities to make shapes and number symbols.



Figure 7. 'Washing line' number line.

#### **Questions:**

- Have two students hold up a 'washing line' and mark points 0 and 1 with a peg or label. Ask another student (or students) to mark where they think  $\frac{1}{2}$  (or  $\frac{1}{4}$ , or....) is. Discuss 'How do you know you are right?' The number line can be extended beyond 1 to show mixed fractions *(fractions)*.
- An outside activity. Tie a long (approx. 30m) rope in a circle. Positions students around the circle of rope. Call a shape for them to make with their rope and listen to the discussion about how they work it out *(shape)*.

#### DRESSMAKER'S TAPE MEASURES

Dressmaker's tape can be used for measuring using millimetres, centimetres and metres, not only in straight lines, but also to measure curved lines which are difficult to measure with inflexible ruler. The dressmaker's tape measure can be used to model number lines, make shapes and measure perimeter, and to create a pie graph. (See Figure 9 below for making a pie graph with a tape measure).

#### **Questions:**

- Find something with a perimeter of 135cm? (measurement)
- How many tape measures would it take to measure the basketball court? (measurement).

Having a variety of materials available for use and giving the students opportunities, time and experience in using them, helps the students learn to make choices about the materials that enhance their learning. Using materials in class should be the norm and be expected, not just for lower level ability students, but for all students, at all levels.

#### TWO EXAMPLES OF USING HANDS ON MATERIALS

#### SHAKE AND SPILL FRACTIONS

Choose the value of the denominator of the fraction to be explored, e.g. eighths. In this case, eight is the number of two sided counters to use. Shake and spill all the counters. Count the number of counters of each colour out of the total number of counters and express these as fractions in numbers and in words. For example, if a student throws 8 counters, 3 may turn up blue and the other 5 turn up yellow. Possible responses might be; 3 out of eight are blue, three eighths are



blue,  $\frac{3}{8}$  are blue, 5 out of eight are yellow, five eighths are yellow,  $\frac{5}{8}$  are yellow. Beware students who try and make  $\frac{3}{5}$  out of 3 blue and 5 yellow – a common misconception. Moving students from the concrete to the representational by drawing a whole into eighths and colouring the fraction amounts that match the 'shake and spill' can help diminish the misconception. Multiple representations of the same concept help make connections and reinforce learning.



Figure 8. Two sided counters modelling fractions.

#### CREATING A PIE GRAPH WITH A DRESSMAKER'S TAPE MEASURE

Students often are familiar with pie graphs from seeing them in media and on tests before they have the knowledge of percentages and angles to be able to create them themselves. This task is an easier way for students to create a pie graph. Count the total number of items to be shown in the graph e.g. in a graph of pets there could be 3 dogs, 5 cats, 4 fish and 3 birds equals 15 items. Use the tape measure to create a circle with the same circumference (in centimetres) as the total of items in the graph. Trace around the circle. Mark the centre of the circle. Students segment the circle, matching the number of centimetres of the circumference with the number of each item in the graph 3 dogs = 3 cm of circumference, 5 cats = 5 cm of circumference, etc. This then opens conversations about the sectors of the graph actually representing amounts and comparing the different sectors when interpreting the graph.



Figure 9. Making a pie graph with a tape measure.



#### CONCLUSION

The use of hands on materials engages students in doing mathematics (Swan & Marshall, 2010). Mathematics should be about doing, exploring, inquiring and discovering. Mathematics should not be about writing answers to a predetermined set of questions as in a worksheet. Encouraging students of all capabilities, at all grade levels to use hands on materials as the norm in the classroom enhances their opportunity to learn. Learning happens when students make connections between objects, symbols and ideas (Kilpatrick, Swafford & Findell, 2001). The CRA model, using concrete 'hands on' materials to facilitate and focus discussion, and providing a scaffold for students to make their own representations, strengthens learning for all students. The teacher's role is vital in connection objects and understanding through facilitating discussions, and mathematical thinking and encouraging curiosity (Marshall & Swan, 2008).

All teachers have the opportunity to use a CRA model which promotes engagement and understanding for students and facilitates discussion and conversation about mathematical concepts and ideas. The use of hands on materials is an asset in all classrooms and a teaching tool which should be made more use of in Australian classrooms.

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# Using rich narratives to engage students in mathematics: a narrative-first approach

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Setting students problem-solving tasks that are simultaneously engaging and mathematically important is central to primary mathematics instruction. Often an attempt to develop engaging tasks involves first determining the meaningful mathematics to be learnt, and then creating a 'mini-narrative' as a vehicle for exploring these concepts. However, in our experience, the more familiar, enjoyable and deeply developed the narrative, the more engaging the task is for students. Consequently, we demonstrate how there might be value in inverting the process- that is, beginning with rich narratives, and mapping on the mathematics- through creating mathematical tasks embedded in examples of well-known children's literature. This is termed the Narrative-First Approach. We discuss one specific text – Fish Out of Water – and an associated mathematical investigation in some depth, including commenting on student work samples and student post-lesson reflections.

#### USING CHILDREN'S LITERATURE IN THE MATHEMATICS CLASSROOM

The use of children's literature as a tool for teaching mathematics is not a new phenomenon. It is contended that using texts, typically picture-story books, can support mathematical learning across a number of mathematical content areas including probability (Kinnear & Clark, 2014), functional thinking (Muir, Bragg & Livy, 2015) and equivalence (Russo, 2016). More generally, some benefits of using children's literature to teach mathematics may include:

- To help contextualise mathematical ideas;
- To support learning and promote mathematical reasoning;
- To engage students in their learning;
- Sometimes used to launch a unit of mathematical work (adapted from Schiro, 1997 and Muir et al., 2017).

#### CURRICULUM-FIRST APPROACH

A possible and, in our experience, perhaps typical, (at least amongst generalist primary school teachers) three-step process for developing a mathematics lesson with links to children's literature may be described as a *Curriculum-First Approach*. Step 1 involves educators identifying the relevant **curriculum focus**, that is, the targeted mathematical concepts of interest, linked to the curriculum. Step 2 has educators identifying appropriate **supporting resources** that can support the teaching of these concepts, including picture-story books. Step 3 requires educators to **develop activities** and tasks that link the relevant mathematical concepts to the supporting resource.

For example, a Year 1 teacher may be launching a unit of work on shapes. From the Victorian (Australian) curriculum, they identify the relevant content description, in this instance: 'Recognise and classify familiar two-dimensional shapes and three-dimensional objects using obvious features' (VCMMG098; VCAA, 2017). They may decide that to support the introduction of this topic, they will draw on the resource, *The Greedy Triangle*, by Marilyn Burns, as the text stimulates student thinking about the properties of shape. An associated activity the educator may develop to link the content description to the supporting resource could be:

- 1. Encourage students to identify triangles in the text.
- 2. Evaluate their properties (by contrasting with examples of non-triangles also in the text, e.g., quadrilaterals)
- 3. Ask students to find real world examples of triangles (and non-triangles) in the schoolyard.

Similarly, a Year 2 teacher may be planning a unit of work on division, and identify the relevant content description: 'Recognise and represent division as grouping into equal sets and solve simple problems using these representations' (VCMNA109; VCAA, 2017). They may identify *The Doorbell Rang* by Pat Hutchins as an appropriate supporting resource. After reading students the story, they may ask them to engage with a task such as: Show using diagrams how the 12 cookies were shared equally between the different combinations of children.



The Curriculum-First Approach clearly has value in helping to make Mathematics more engaging and contextualised. However, we concur with Muir at al. (2017) that 'books selected should provide an authentic reading experience in and of themselves, and it is essential to read and appreciate the story before focussing on the mathematical aspects of the lesson' (p. 9). In this way both the types of books chosen and the way they are used in classrooms should be subject to scrutiny. In our view, one way to realise the benefits of literature as a tool to promote meaningful mathematical learning experiences is to be led by the story, not by the curriculum. We have termed this process the *Narrative-First Approach*.

#### NARRATIVE-FIRST APPROACH

The Narrative-First Approach is a four-step process for integrating mathematics with children's literature.

**Step One** involves the educator **identifying rich narratives**, specifically picture-story books or novels that they enjoy, are inspired by and believe their students will find engaging.

**Step Two** requires the educator to **identify key components of the story**, such as the central complication (or its solution), a key theme or an aspect of characterisation.

**Step Three** invites educators to **develop rich problem solving tasks**, through making connections between potential mathematical ideas and the key components of the story.

**Step Four** asks educators to make **curriculum links** retrospectively; that is, make links from the problem solving task back to the curriculum. Our suggestion is that the Victorian (Australian) Curriculum is flexible, covering a range of interconnected skills and content knowledge, thereby allowing for this creativity.

For example, one of our favourite picture-story books is *The Sneetches* by Dr Seuss. Every year, we read this book with our respective grades for pure pleasure, although its reading inevitably leads to animated discussions about equity and discrimination (and the arbitrary nature of fashion!). There are many interesting themes and components to the Sneetches story. Two simple, obvious ideas are the fact that there are two types of Sneetches that live on beaches ('Plain Belly' and 'Star Belly'), and that the 'villain' Syvester McMonkey McBean makes a fortune from helping the Sneetches change from 'star' to 'plain' bellied and vice-versa. Rich problem solving tasks that draw on these components can be developed. Two such tasks are presented below. Both have been developed for upper primary (i.e., Year 5 and 6 students).

- *How many Sneetches are on the beaches?* Altogether, there are 101 Sneetches that lived on the beaches. At the beginning of the story, more than one-half of the sneetches were Star-Belly Sneetches and more than two-fifths were Plain-Belly Sneetches. How many of each type of Sneetch might there have been? There are exactly 10 solutions to this problem. Can you find them all?
- *Mr McBean's Fortune*. The Sneetches spent all of their money on going through Mr McBean's 'star-on' and 'staroff' machines, but they also wasted a lot of their time as well. It takes one minute to line up and go through the 'star off' machine and one minute to line up and go through the 'star on' machine. If the poorest Sneetch on the beach had \$780, how much time will it spend going through the machines? The richest Sneetch on the beach spends exactly one week going through the machines. How much money did it have to begin with? For an extra challenge: There were 99 other Sneetches on the beaches, who spent, on average, exactly one day going through the machines. How much money did Mr McBean leave town with? HINT: In the story, it costs \$3 to go through the 'Star-On' machine and \$10 to go through the 'Star-Off' machine.

There are many links between the above two problem-solving tasks and the Victorian (Australian) curriculums. For instance, in relation to the *How many Sneetches are on the beaches?* task, some relevant curriculum links include:

- Year 5: Use estimation and rounding to check the reasonableness of answers to calculations (VCMNA182);
- Year 6: Find a simple fraction of a quantity where the result is a whole number (VCMNA213);
- Year 6: Compare fractions with related denominators (VCMNA211; VCAA, 2017).



We believe the benefits of the Narrative-First Approach for supporting the integration of mathematics with children's literature are considerable. In addition to the aforementioned benefits which apply more generally to using children's literature to support teaching mathematics (e.g., contextualizing mathematical ideas, engaging students), we believe this Narrative-First Approach has some additional advantages, including that it:

- Emphasises connectivity of mathematical concepts;
- Provides a greater focus on the narrative 'hook';
- Allows for greater flexibility and creativity when planning maths activities (that can be energising and enjoyable for teachers to plan);
- Introduces rich tasks that allow for the development of the four proficiencies: understanding, fluency, problem solving and reasoning.

#### A FISH OUT OF WATER INVESTIGATION

In addition to the development of rich tasks, it is possible to build a mathematical investigation from the Narrative-First Approach. The first author has a particular affinity with the story *A Fish Out of Water* by Helen Palmer, having fed his goldfish an entire box of rice bubbles as a 5 year old because it 'looked hungry' (with the obvious disastrous consequences). Consequently, we decided to use the picture-story book *A Fish Out of Water* to explore the theme of exponential growth, given that a key component of the story is that Otto (the fish) is growing rapidly as a result of being overfed by his irresponsible owner. The investigation is appropriately differentiated as it possesses multiple entry and exit points, and is appropriate for students from Year 3 to secondary school. It covers a large range of mathematical concepts and involves students engaging with all four proficiencies (VCAA, 2017; see Figure 1).

	Level 2	Level 3	Level 4	Level 5	Level 6	Level 7
Patterns and Algebra	Apply repetition in arithmetic operations, including multiplication as repeated addition and division as repeated subtraction	Describe, continue, and create number patterns resulting from performing addition or subtraction	Solve word problems by using number sentences involving multiplication or division where there is no remainder Explore and describe number patterns resulting from performing multiplication	Follow a mathematical algorithm involving branching and repetition Describe, continue and create patterns with fractions, decimals and whole numbers resulting from addition and subtraction	Design algorithms involving branching and iteration to solve specific classes of mathematical problems Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence	Introduce the concept of variables as a way of representing numbers using letters Create algebraic expressions and evaluate them by substituting a given value for each variable
Number and Place Value	Recognise, model, represent and order numbers to at least 1000	Recognise, model, represent and order numbers to at least 10 000 Represent and solve problems involving multiplication using efficient mental and written strategies and appropriate digital technologies	Recognise, represent and order numbers to at least tens of thousands Represent and solve problems involving multiplication using efficient mental and written strategies and appropriate digital technologies	Recognise, represent and order numbers to at least hundreds of thousands Use efficient mental and written strategies and apply appropriate digital technologies to solve problems	Select and apply efficient mental and written strategies and appropriate digital technologies to solve problems involving all four operations	Recognise and solve problems involving simple ratios Investigate index notation
Measurement and Geometry		Tell time to the minute and investigate the relationship between units of time Measure, order and compare objects using familiar metric units of length, area, mass and capacity	Convert between units of time		Convert between common metric units of length, mass and capacity Connect decimal representations to the metric system Solve problems involving the comparison of lengths and areas using appropriate units	

Figure 1. Curriculum links for the A Fish Out of Water investigation.

#### THE POSING OF THE INVESTIGATION

The task that begins the investigation is stated as follows:

• At the beginning of the story, the little fish, Otto, was only 5cm long but by the end of the story, Otto was so big he was just over the length of a 50m Olympic size swimming pool! If Otto doubled in length every ten minutes, how long did it take Otto to grow to this size?



Students are then invited to deepen their investigation into exponential growth and the predicament of Otto the fish by engaging in the extending prompt:

• If Mr Carp hadn't of dived into the pool and saved the day, Otto would have kept on growing and growing. How long would it have taken Otto to grow so big, he wouldn't fit into Albert Park Lake? How long until he would have outgrown the length of the Yarra river? How long until he would have outgrown the Pacific Ocean? How long until he would have outgrown the earth? How long until Otto was so long, he would have stretched all the way to the moon? What about the sun? Pluto? Alpha Centauri? The known universe?

The second author taught a lesson built around this particular story book and the associated investigation to a grade of Year 5 and 6 students. He organised the lesson accorded to the launch-explore-discuss (and summarise) structure for cognitively demanding tasks posited by Stein, Engle, Smith and Hughes (2008).

#### STUDENT APPROACHES TO THE INVESTIGATION

The open-ended nature of the investigation allowed students with a range of mathematical competencies to approach the task in different ways. Students approached the problem using their existing mathematical skills and, if they reached a point where they required support, they requested guidance from their peers or teacher on a specific area of need.

Students were given the choice to work on the task independently or with a partner. Student A and Student B decided to work together and, without any prompting, chose to use a table to show the relative growth of Otto over time. They used mental computation skills to solve the first part of the problem, doubling Otto's length ten times to 5120 cm in 100 minutes. In their working out they demonstrated an understanding of conversion between metric units, showing that 5120 cm is larger than the size of an Olympic swimming pool, using the 'greater than' symbol. They also demonstrated an understanding of converting 100 minutes into 1 hour 40 minutes and stating 'It takes 1 hour 40 minutes to get bigger than an Olympic Swimming pool'. The students then moved on to the extension task; for example, researching the length of the Yarra River (242 km), converting this length to cm (24,200,000 cm) and then applying a multiplication algorithm to solve the required time for Otto to reach this length (see Figure 2).

50m = 5000 cm hour DU minutedentes It took Otto 30 minute to reach 40cm Time Length The Yourna is 242 km = 24200000 cm It takes not these fline to get ligger that the dynaptic environment part 20 20 30 40 It takes 230mins 40 80 50 160 80 1280 5120 > 50000 100 5120 ----South Qual 20480 00000 10000 1000 Extention 10 200 20480 m cm 1342177280 190 40960 41943040 250 1342177 280 26 84 354 560 15 Soch = 1550000000

Figure 2. Students A and B's approach to the task.



Student C worked on the problem independently. She decided to initially represent the comparative growth of Otto over time using a number line. The number line was the most common approach in the class for solving the initial problem. However, as the numbers got larger, the student decided to use a more systematic approach and generated a table comparing size with time. She continued to use mental computation skills to double Otto's size beyond the initial task, although a minor mistake when converting minutes to hours made it difficult to solve the extension problem (Figure 3).

$\frac{1 \text{ norest}}{5 \text{ cm}} = \frac{10 \text{ min}}{10 \text{ min}} = \frac{10 \text{ min}}{10 \text{ min}} = \frac{10 \text{ min}}{20 \text{ cm}} = \frac{10 \text{ min}}{20 \text{ cm}} = \frac{10 \text{ min}}{10 \text{ min}} = $	10 min ? How high the fish? 10 min ? How ward & Ward 10 min ? Ward ward ? Ward 10 min ? Ward 20 min
5cm 10/14   10cm 10/14   20cm 20   40cm 30   80cm 40   160cm/// 1m 60cm 50   320cm/// 5m 20cm 60/// 1 hour   640cm/// 6n 40cm 70// 1 hr 10mins   1880cm// 12m 80cm 80// 1 hr 20mins   2560cm// 25m 60cm 80// 1 hr 20mins   2560cm// 51m 20cm 100// 1 hr 40 mins	5/20cm//51m 20cm 100//1hr 50minc 10240cm//102m 40cm 110// the Ominc 2hrs 20480cm//102m 80cm 120// 2hrs 20mins 100400cm//409m 69cm 120// 2hrs 40mins 87920cm//409m 69cm 120// 2hrs 40mins 1045 40cm

Figure 3. Student C's approach to the task.

Student D also used a number line to represent the problem. Her final solution can be seen in Figure 4. This student decided she wanted to convert the centimetres into metres in order to simplify the problem, but initially converted 160 cm to 1.06 m. When she compared her progress to that of a peer, she recognised a discrepancy and checked in with the teacher. The teacher explained that her issue was with the conversion and prompted her to reconsider what she had done.

Teacher: 'How many centimetres in a metre?' Student: '100.' Teacher: 'What about two metres?' Students '200, of course.' Teacher: 'Well, how about 1.6 metres? Is it closer to one or two metres?' Student: 'Aha!'

Following this discussion, the student converted 1.6 metres to 160 centimetres and, with teacher prompting, drew on an existing mathematical skill of dividing by 100. Once the student was confident converting centimetres to metres she was able to double the decimal amounts and solve the initial problem.



Figure 4. Student D's approach to the task.



Student E initially found the problem challenging and was uncertain how to represent the comparative values of distance and time. With prompting from a peer, he initially tried to use a table to represent the data. However, he found it difficult to work through the problem in this way and so decided to use the class whiteboard to create a visual representation of the problem, with a series of lines showing 'the length of Otto' at different times (see Figure 5). This visual representation of the problem facilitated an understanding of the exponential growth phenomenon and then, using mental computation, the student was able to systematically double the size of Otto each ten minutes and reach the solution of 5,120 cm after 100 minutes.



Figure 5. Student E's approach to the task.

#### STUDENT REFLECTIONS ON THE INVESTIGATION

Students were invited to reflect on the lesson, and share their views in relation to both what they enjoyed about the activity and their own learning.

It was apparent that students responded positively to the investigation, and particularly valued its framing in terms of the picture-story book. Although most students reflected that they found the activity enjoyable, they provided different explanations for this positive evaluation. Some students made explicit reference to the use of the narrative as a 'hook' for engaging them in the associated investigation. For example:

I thought it was a really good idea starting a maths lesson with a story because it opened up my mind in a way and made me want to know what this had to do with the lesson. So I was feeling intrigued and sort of open. The lesson was really fun and interesting as well.

The story linked with the lesson so it was more fun that doing maths.

Other students instead emphasized how the picture-story context helped make the mathematics more meaningful and understandable. For example:



It was fun because if we read a book or something like that it puts a picture in my mind making the maths more fun.

It was fun to read the story at the start and then go into maths because it helped explain some things.

Because we read the story at the start which helped me understand better.

Finally, there was also some references to students engaging in and enjoying the mathematical process of doubling (and the depth of the investigation), without making explicit reference to the narrative-catalyst. For example:

I learnt about how doubling again and again can makes things big in such a short amount of time!

It was fun to double and get into such high numbers! It was also fun learning about how big some things are.

#### **CONCLUDING THOUGHTS**

We believe that the Narrative-First Approach has considerable potential to make planning, teaching and learning mathematics more engaging, enjoyable and contextualized. We would encourage teachers to experiment with developing tasks around well-loved picture-story books and novels, in the spirit of beginning with the narrative, rather than the curriculum. Our contention is not that mathematics should be contrived onto any and every storybook, but rather that many of the ideas, themes and characters that emerge from our favourite stories can be reconstructed through a mathematical lens. For other examples of our attempts to employ this approach, see Russo and Russo (2017a, 2017b, 2017c, 2017d).

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# Using challenging and consolidating tasks to improve mathematical fluency

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Following an overview of teaching with challenging tasks, we explore the nexus between using both challenging and consolidating tasks to simultaneously develop conceptual understanding and procedural fluency. In particular, we argue that it is critical that students are provided with parallel opportunities to work on consolidating tasks, in order to connect conceptual understanding to improved strategy efficiency. This discussion makes reference to the Big Ideas in (primary) mathematics (Charles & Carmel, 2005) and provides three examples of challenging and consolidating tasks, each of which support students in grappling with a different 'Big Idea'.

#### CHALLENGING TASKS, PROMPTS AND DIFFERENTIATION<sup>1</sup>

Challenging tasks are complex and absorbing mathematical problems with multiple solution pathways, where the whole class works on the same problem (Sullivan & Mornane, 2013). Challenging tasks can be viewed as an interpretation of cognitively demanding tasks that meet specific criteria that are outlined below (adapted from Sullivan et al., 2011).

The challenging task must:

- be solvable through multiple means (i.e., have multiple solution pathways) and may have multiple solutions;
- involve multiple mathematical steps (i.e., as opposed to a single insight facilitating completion of the problem);
- have at least one enabling prompt and one extending prompt developed prior to delivery of the lesson; The challenging task should:
- be initially perceived as challenging by the majority of students;
- engage students (i.e., students are motivated to solve the problem);
- involve students spending considerable time working on the task (although the exact length of time will vary substantially, depending on the nature of the task, the age group and the student in question, it is generally expected that students will spend at least 10 to 15 minutes engaged with the problem);

As indicated above, challenging tasks are differentiated through the use of enabling and extending prompts. Enabling prompts are designed to reduce the level of challenge through simplifying the problem, changing how the problem is represented, helping the student connect the problem to prior learning and/ or removing a step in the problem (Sullivan, Mousley, & Zevenbergen, 2006). When developing enabling prompts, it is critical that they do not undermine the primary learning objective of the lesson by 'giving too much away'. By contrast, extending prompts are designed for students who finish the main challenge and expose students to an additional task that is more challenging but requires them to use similar mathematical reasoning, conceptualisations and representations as the main task (Sullivan et al., 2006).

When engaged in a challenging task, students should be encouraged to access enabling prompts at their own initiative. Enabling prompts should be a student's first point of call if they feel they need some assistance to make progress with the problem (i.e., rather than immediately asking for support from the teacher/ a fellow student). Consequently, students need explicit support around how to use enabling prompts most effectively and when to use them, particularly if they have not previously been exposed to challenging tasks. It is important, therefore, that the teacher ensures that all students know where the enabling prompts are in the room, and that there is no stigma associated with accessing an enabling prompt (e.g., an overly competitive classroom climate, where it is assumed that 'good mathematicians don't need help').

In my classroom (first author), I call the enabling prompts the 'hint sheet', and print one prompt on each side of this sheet. During each challenging task, I include a pile of hint sheets up the front of the classroom on a chair, so students know exactly where they are. By contrast, I call the extending prompt the 'super challenge' and generally place the extending prompt on the flip-side of the challenging task.



When working on a challenging task, the expectation is that all students engage with the same primary learning objective; however, accessing enabling or extending prompts may modify, add or remove secondary learning objectives (Russo & Hopkins, 2017a). It is important that all students work on a similar core task with a focus on the same primary learning objective to lay the foundation for a highly-participatory, meaningful and inclusive discussion around the relevant mathematics.

#### STRUCTURING LESSONS INVOLVING CHALLENGING TASKS

It is often assumed that work on the challenging task should precede any teacher-facilitation discussion of the key mathematical ideas (Sullivan et al., 2014; although see Russo and Hopkins, 2017b, for an exception). Consequently, teaching with challenging tasks typically involves a three-stage process: the *launching* of the task by the teacher, the *exploration* of the task by the students, and the whole-class, student-centred *discussion* (and teacher-led *summary*) of student solutions and the underpinning mathematical concepts (Stein, Engle, Smith, & Hughes, 2008).

The teacher begins by *launching* the challenge, which involves presenting the problem, engaging students in the relevant mathematical mindset, and highlighting resources students have at their disposal (e.g., enabling prompts, concrete mathematical materials). After the challenge is launched, students *explore* the task, either individually or collaboratively, and the teacher encourages students to develop at least one potentially appropriate solution. The next stage of the lesson involves the teacher facilitating a whole-group *discussion*, which provides students with an opportunity to present their particular approach to solving the task.

As highlighted by Stein et al. (2008), this discussion component generally involves the teacher organising student responses in increasing order of mathematical sophistication. This sequential structure supports meaningful student participation and helps to build on the discussion of key mathematical concepts. However, the Stein et al. note that effectively coordinating this discussion can require considerable practice, skill and planning. Consequently, teachers beginning to experiment with challenging tasks in their classrooms should view it as a learning opportunity and not be overly self-critical or immediately discouraged if the discussion does not flow in the manner they expect. The teacher will usually close the lesson by offering a brief *summary*, reiterating the learning objective(s) and presenting a sample of student work that supports this objective.

More recently, an additional phase to lessons involving challenging tasks has been put forward. Specifically, there has been an emphasis on students engaging with supplementary tasks to consolidate their understanding following the discussion phase of the challenging task, and prior to the teacher summary (Russo & Hopkins, 2017c; Sullivan et al., 2014). Therefore, the three-stage lesson structure suggested by Stein et al., (2008) could be re-construed as a five-stage structure, as it now makes sense to separate out the teacher-led discussion and the teacher-directed summary stages. This revised five-stage structured for teaching with challenging tasks is presented below, with approximate periods for each phase indicated in parentheses (assuming a 60-minute block):

- 1. Launch (5 minutes)
- 2. Explore (20 minutes)
- 3. Discuss (15 minutes)
- 4. Consolidate (15 minutes)
- 5. Summarise (5 minutes)

Developing meaningful consolidating tasks that complement a given challenging task is an integral focus of the remainder of this paper.



#### CHALLENGING AND CONSOLIDATING TASKS, AND THE 'BIG IDEAS' IN MATHEMATICS

Challenging and consolidating tasks operate in unison to allow students to effectively grapple with the Big Ideas in mathematics. A Big Idea refers to 'an idea that is central to the learning of mathematics... that links numerous mathematical understandings into a coherent whole' (Charles & Carmel, 2005, p. 10). For example, a Big Idea in primary school mathematics is the notion that 'the base ten numeration system is a scheme for recording numbers using digits 0-9, groups of ten, and place value' (Charles & Carmel, 2005, p. 13).

In the first instance, the challenging task builds conceptual understanding through providing a context in which students can engage with the relevant Big Idea. It is a reflection of the deliberate design of such tasks (including, but not limited to, the high level of cognitive demand) that it is expected that several students will consider these Big Ideas simply through their engagement with the task itself, and independent of any teacher instruction or explanation. For other students, engagement with these Big Ideas is instead more likely to occur during the discussion phase of the lesson, when students have an opportunity to reflect on how they approached the task and become both more explicitly aware of the mathematics they employed and the relative efficiency of their particular strategy. The key is that, in either instance, students are not having the Big Idea explicitly unpacked and explained to them. Rather, the Big Idea appears to emerge organically out of the process of students undertaking the task, justifying their solution method, listening to other student approaches, and, importantly, listening to the teacher make links between student approaches and the associated mathematical concepts. The major role of the teacher during the discussion phase is therefore to contextualize, organise and shape student work, mapping it onto the underlying mathematical structure (i.e., Big Idea). This is what is meant by the phrase a 'student-centred discussion'.

Having primed student engagement with the relevant Big Idea, the consolidating task(s) offers students the opportunity for problem-based practice working on tasks which are similar in structure to the main task, however involve a lower level of cognitive demand. The goal of the consolidating task(s) is to build mathematical fluency in the context of student engagement in a given Big Idea. This in turn implies that there are multiple mathematical fluencies, each associated with a particular Big Idea.

We will now present three examples of challenging tasks, with the associated consolidating tasks included. Each task will relate to a different Big Idea in mathematics, as presented by Charles and Carmel (2005). The tasks are targeted at Year 1 and Year 2 students.

#### TASK 1: SKIP-COUNTING PATTERNS

Big Idea: *Patterns*. 'Relationships can be described and generalizations made for mathematical situations that have numbers or objects that repeat in predictable ways' (Charles & Carmel, p. 17).

Primary Learning Objective: Overlaying multiple skip counting sequences will result in some numbers being covered more than once (i.e., numbers with many factors) and some numbers not being covered at all (i.e., potential prime numbers).

Challenging Task: *Twos, threes, fours and fives; which numbers will survive?* Starting at 0, I skipped counted by 2's to 40, crossing off the numbers as I went. Then I did the same thing, but instead skip counted by 3's. Next, I did it by 4's. Finally, I skip counted again, but counted by 5's. Some numbers were crossed off more than once, but some numbers survived – they weren't crossed off at all. Can you guess which 10 numbers survived? Now check if you are right.

Enabling Prompt: Hint about skip-counting patterns (see Figure 1).



1	2	3	4	5	6	7	8	9	10		1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20		11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30		21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40		31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50		41	42	43	44	45	46	47	48	49	50
1	2	3	4	5	6	7	8	9	10		1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20		11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30		21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40		31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50		41	42	43	44	45	46	47	48	49	50
					1	2	3	4	5	6	7	8	9	1	0					
					11	12	13	14	15	16	17	18	3 19	2	D					
					21	22	23	24	25	26	27	28	3 29	> 30	0					
					31	32	33	34	35	36	37	7 38	3 39	7 40	D					
					41	42	43	44	45	46	47	7 48	3 49	7 50	D					

#### What counting patterns can you see in these charts?

Figure 1. A hint to students about the relevant skip-counting patterns.

Extending Prompt: What if I also skip counted by 6's, 7's, 8's, 9's and 10's? Would all 10 numbers still survive? How many more numbers would get crossed off?

Consolidating Tasks:

- Starting at 0, I skip counted by 2's to 20, crossing off the numbers as I went. Next, starting at 0, I skip counted by 3's to 20, crossing off the numbers as I went. Some numbers were crossed off more than once, but some numbers survived they weren't crossed off at all. Can you guess which 7 numbers survived? Now check if you are right.
- Starting at 0, I skip counted by 2's to 20, placing a counter on all the numbers I landed on. Next, I skip counted by 5's to 20, again placing a counter on all the numbers I landed on. Finally, I skip counted by 10's to 20, again placing a counter on all the numbers I landed on? What are the numbers with three counters on them the numbers I landed on three times?
- Starting at 0, I skip counted by 3's to 20, placing a counter on all the numbers I landed on. Next, I skip counted by 5's to 20, again placing a counter on all the numbers I landed on. What is the only number with two counters on it?

#### TASK 2: EQUIVALENCE

Big Idea: *Equivalence*. 'Any number, measure, numerical expression, algebraic expression, or equation can be represented in an infinite number of ways that have the same value.' (Charles & Carmel, p. 14).

Primary Learning Objective: Towers of a given height can be represented in a number of different ways.

Challenging Task: *Towers* (adapted from Sullivan, 2017). The class is presented with an image of Unifix towers (or actual Unifix towers), as per Figure 2. Students are given the instructions:



- 1. Create the towers like the ones in the picture.
- 2. Can you create two towers of equal height using *all* of these smaller towers? Record your solution.



Figure 2. Image of the smaller towers for Tower challenge.

Enabling prompt: The student is presented with an adapted image of Unifix towers (see Figure 3). Students are given the instructions:

- 1. Create the towers like the ones in the picture.
- 2. Can you create two towers of equal height using *all* of these smaller towers? Record your solution.



Figure 3. Enabling prompt for Tower challenge.

Extending prompt: Can you do it another way? How many solutions can you find?

Consolidating Task:

- 1. Create the towers like the ones in the picture (see Figure 4).
- 2. Can you create a tower that is exactly 10 blocks tall from these smaller towers? Record your solution.
- 3. There are exactly nine solutions. Can you find them all?



Figure 4. Consolidating task for Tower challenge.



#### TASK 3: PARTITIONING

Big Idea: *Basic Facts and Algorithms*. 'Basic facts and algorithms for operations with rational numbers use notions of equivalence to transform calculations into simpler ones.' (Charles & Carmel, p. 16).

Primary Learning Objective: When adding a sequence of numbers, partitioning and regrouping into lots of 10 can make addition easier.

Challenging Task: *Christmas Shopping*. Jeffrey did some Christmas shopping for his two sisters. He decided to get them both tickets to see Katie Perry in concert. The tickets cost \$99 each. He also got his dog, Sook, a plastic bone for \$2. How much money did he spend on his Christmas shopping?

Enabling prompt: Students are presented with a simpler problem (19 + 19 + 2 =), and a hint about the relevant partitioning (see Figure 5).

- 19 + 19 + 2 = ?
- If we break the 2 into 1 and 1, we can rewrite the number sentence as 19 + 19 + 1 + 1 = ?



Figure 5. Enabling prompt for Christmas Shopping challenge.

Extending prompt: Jeffrey forgot that his 4 cousins and their dog Fletcher were also going to be at his house on Christmas day. He decided to buy his cousins entrance to Luna Park, which cost \$49 each. For Fletcher, he bought a new collar for \$5. How much more money did poor Johnny have to spend?

Consolidating Tasks:

- 9+4=?
- 9+7=?
- 3+9=?
- 19 + 6 = ?
- 29 + 29 + 2 = ?
- 19+9+19+3=?
- 49 + 39 + 2 = ?
- 4+18+19+9=?
- 5+19+18+19=?

#### **CONCLUDING REMARKS**

Consistent with Sullivan et al. (2014), we have argued that consolidating tasks should be considered a critical aspect of teaching with challenging tasks and importantly, should be presented in the same lesson. Specifically, we have argued that such tasks are vital for building mathematical fluency, after exploration and discussion of the challenging task has



primed student engagement with the relevant Big Idea. The three examples of consolidating tasks that we have provided suggest that such tasks can come in a variety of forms. Task 1 provides a series of three consolidating tasks, each that has a similar structure to the original challenging task, however is less complex and cognitively demanding. By contrast, the consolidating task for Task 2 has the structure of an additional challenging task in itself. The key difference is that students are more likely to have success with this consolidating task than with the main challenge (i.e., to find at least one solution). However, the invitation for students to find all solutions associated with the consolidating task (and the implication that they need to be systematic in order to do so) ensures that the task still has an appropriately high ceiling. Finally, Task 3 provides nine consolidating tasks designed to encourage students to continue to experiment with partitioning strategies to support addition. The tasks are presented in order of increasing complexity, again to ensure that students are confronted with an appropriate level of challenge.

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<sup>1</sup> In response to reviewer feedback requesting more information about how to teach with challenging tasks, the first part of this paper draws heavily on an article that appeared in *Australian Primary Mathematics Classroom* (Russo, 2016).



### Understanding spatial reasoning in a Year 5 geometry lesson: tool mediation with the Mathomat template

John Lawton, Objective Learning Materials

This paper discusses the way in which the Mathomat template was used in a year 5 geometry classroom. It reveals students 'thinking and how that can be used in conjunction with an understanding of the van Hiele levels to help students with the difficult creative process that is involved in expanding their understanding of geometry. Teachers use of constructivist tools, such as the van Hiele levels and the Mathomat template are hindered by significant practical difficulties in classroom situations. Their usefulness is restricted by the fact that much of the learning associated with them involves visualisation processes that are difficult in practice for teachers to gain access to. This paper concludes with discussion of new research into areas related to the van Hiele levels that might help in overcoming these difficulties, thereby offering a more powerful approach to the use of Mathomat in geometry education.

#### UNDERSTANDING THE VAN HIELE LEVELS AS CONSTRUCTIONIST TEACHING TOOLS

Constructivist teaching involves the intrinsically powerful idea that students need to actively construct knowledge, developing deep understanding of the subject by participating in carefully constructed learning contexts. This approach involves learning in a social context. The term *constructivism* refers to the way in which *individuals* make sense of their world. Knowledge needs to be constructed in a social context and it is therefore more appropriate for the purpose of the classroom interactions with Mathomat in this paper to refer to it as what Crotty (1998, p. 42) calls *constructionism*, being 'the construction of knowledge in and out of interaction between human beings'.

Successfully constructing knowledge in geometry involves students in both geometric and spatial reasoning. Battista (2007, p. 843) argues that the former involves students in 'invention and use of formal conceptual systems to investigate shape and space' while spatial reasoning underlies this and involves 'the ability to 'see', inspect, and reflect on spatial objects, images, relationships and transformations'. While geometric reasoning is defined in depth, as a separate strand, within the Australian Curriculum Mathematics, spatial reasoning is only referred to in very general terms.

Geometry is a difficult subject for mathematics teachers and students to deal with because it involves students in making shifts between entirely different ways of reasoning as they construct knowledge of the topic. Research studies such as Senk (1989) found that in any particular class geometry students were scattered across several different thinking levels, making it impossible to successfully use any one teaching strategy for the whole class. The van Hiele levels are a useful tool in dealing with this problem. Van Hiele theory argues that geometry students need to ascend through five distinct and discontinuous levels of reasoning. Students at early secondary school are generally treated as being at van Hiele level 3, which means that they are expected to have made three distinct shifts; from visual reasoning about shapes as a whole to analytical reasoning about their properties (van Hiele level 0 to level 1), from analysis of shapes to non- formal deduction about their properties (van Hiele level 1 to level 2) and from non- formal to formal deduction (van Hiele level 2 to level 3). Each of these shifts involves students in the use of a different paradigm in the way that they are reasoning.

Traditional instruction in secondary school presents geometry in a form that assumes students are already at a high level of understanding in van Hiele terms. Vinner (1991) describes this traditional approach as definition based, presenting geometry as a 'skeleton of axioms, theorems and proofs'. The problem with this approach is that students need to acquire knowledge through fundamentally different reasoning processes. Students need to acquire concepts in geometry by developing a *concept image*, defined by Vinner (1991, p. 7) as the 'total cognitive structure that is associated with the concept which includes all mental pictures and associated properties and processes'. The *concept definitions* on which traditional geometry textbooks rely are important because geometry is a technical subject, however students need to interact with them while making the concept image their central focus as they learn. This can be seen as an underlying process upon which the transition between van Hiele levels relies, and it is summarised in Figure 1.



### Concept acquisition in geometry education; alternative pathways



*Figure 1. Concept acquisition in geometry education: alternative pathways. Source: Written by John Lawton, illustrated by Juliet Snape, 2017. Based on the original set of four diagrams in Vinner (1991)* 

In Figure 1, pathway number 1 is the traditional definition based approach in which students are assumed to reason entirely through the use of formal deduction. In reality students need to develop a concept image of the geometry task and to interact with formal definitions in order to acquire higher levels of van Hiele reasoning. This approach is shown in pathways 2 and 3 of figure 1. What is important in teaching geometry is to ensure that students do not take pathway 4 by relying on a natural tendency to make exclusive use of concept image without reference to the rigour of definition based thinking.

Tools are central to constructionist learning. This was first recognised by Vygotsky, when he identified the process of 'reverse action'. In reverse action, students begin by using a tool as an instrument (say by making a drawing using the Mathomat template) but they subsequently begin to inquire about the properties of the tool they are using, eventually treating it as what Nuhrenborger and Steinbring (2008, p. 163) describe as a 'field of action'. This process of reverse action means that tools begin to play a central role in the construction of knowledge.

#### THE LIMITATIONS OF CONSTRUCTIONIST TEACHING

Constructionism is often criticised for being impractical. For instance, Smith and Simon (1996, 1997 as cited in Groves, 1997) argue that 'attempts to reform school mathematics have undermined teacher's sense of efficacy by condemning the traditional expository model of teaching without replacing it with a clear new alternative' (p. 137).

This problem of impracticality can be seen in constructionist tools such as the van Hiele levels. As a broad description of student behaviour the van Hiele levels are compelling; they chart what Battista (2007, p. 856) describes as a 'natural progression from intuitive, everyday reasoning to formal scientific reasoning'. However, when individual student behaviour is closely observed the distinction between the van Hiele levels becomes blurred (Battista, 2007, p. 848).

Underlying these problems is the fact that only two of the five geometric objects that are present in a geometry lesson are



readily accessible in teaching; these two accessible geometric objects are *material* objects and *concept definitions*. The full spectrum of geometric objects present in a lesson are defined by Battista (2007, p. 844) as the *material objects* just mentioned (these are actual physical entities such as a drawing on paper), *sensory objects* (being what our senses such as sight and touch tell us is present), *perceptual objects* (how we interpret what is sensed), *conceptual objects* (being the meaning or way of thinking that is evoked by the senses) and the *concept definitions* previously referred to (which are the abstract mathematical specifications of the object). In the first part of the 20th Century behaviourist teaching practices formally ruled out any reference to the second, third and fourth of the objects in this spectrum because they involved purely mental operations. In recent decades constructionist teaching practice has opened up the theoretical possibility of referring directly to mental objects in geometry instruction. However, as Battista points out this area is in its infancy 'because researchers are investigating cognitive processes that cannot be observed' (2007, p. 858).

The problem with ignoring the mental objects involved in geometry is that it leads to the difficulty described by Laborde (2001, p. 115), that teaching 'confuses drawings and the theoretical geometrical objects that the former represent'. Presmeg (1997, as cited in Battista 2007, p.844) points out that this issue is 'the source of many difficulties in visualisation based mathematical reasoning'

#### THE DEVELOPMENT OF MORE EFFECTIVE CONSTRUCTIONIST TEACHING MATERIALS.

Despite the practical difficulty of dealing with inaccessible mental processes there has been steady progress by research studies in making them more explicit and understandable.

Battista (2007) elaborated the original van Hiele levels based on a steady progression of work by earlier researchers arguing that each one of those studies 'helps us to better understand the nature of the levels and student's geometric thinking' (p. 854).

A significant advance towards understanding student visualisation strategies was made in a 2007 study of student teacher's *figural concepts* by Fujita and Jones. A figural concept (Fischbein, 1993) is a fusion of real world and abstract mathematical properties; as such it can only exist in the mind. Figural concepts are complimentary to Vinner's (1991) theory of concept acquisition; they are a formally defined entity whose construction by students emerges through an underlying process of interaction between concept image and concept definition. The importance of the Fujita and Jones study was the development of the idea of *formal* and *personal* figural concepts. These two concepts were used by Erdogan and Dur (2014) to define the gap between personal and formal figural concepts employed by a large group of pre-service primary school teachers in Turkey.

An example of a figural concept, relevant to the location and transformation topic taught in the research study in this paper, are what Bronowski (1973, p. 122) referred to as the 'mental cross-wires in the visual field'. Battista (2007, p. 889) elaborates on this figural concept, in depth, in his discussion of how students conceptualise coordinate systems and locations.

#### MATHOMAT IN ACTION

The two figures below are photographs of the author's research study of Mathomat mediation of geometry learning (involving the Cartesian plane as new knowledge) in a grade five class at a P-9 public school in an outer Western suburb of Melbourne.

In figure 2, the student has been experimenting with inscribing a rhombus shape on her Mathomat template inside one of its circles. The teacher points to the rhombus on the student's Mathomat and asks her to name it. The student hesitates, confused by the difference between the rectangle and the rhombus which are close together on her Mathomat. The teacher asks the student about the difference between the properties of the rhombus she has been using and the rectangle on her Mathomat. To answer this question the student puts her Mathomat down on the table and uses her hands to demonstrate how to 'squish' a rectangle into a rhombus.

Concept image is central to the way in which the student in Figure 2 is thinking. She is operating on mental images created by her engagement with the physical Mathomat template. If the student in Figure 2 had chosen to work from a definition to answer the question she would have probably continued using the physical drawing in front of her.



Instead, there is interplay between concept image and concept definition in the student's thinking represented by pathway 2 in Figure 1.



Figure 2. A student operating on a mental image constructed through engagement with the physical Mathomat template

What makes this exchange powerful is that the teacher and student are at different van Hiele levels in their understanding of shape property. The student is being challenged to make a shift in her thinking from van Hiele level 0 (visual) to level 1 (analytical), as she attempts to mentally transform a rectangle into a rhombus, probably noticing that the right angles have to change in the process. Tool mediation by Mathomat is central to this learning, with both teacher and student able to operate on a shared construction of mental images of the physical template.

In Figure 3, the student has completed the formal assignment in the research lesson ahead of most of the class; he has been reflecting on a creative drawing made with his Mathomat template and then translated to a new location on his Cartesian axes drawing. The student asks the teacher 'is it (my drawing) the same?' He appears to be wrestling with the concept of translation, wondering if the internal structure of his drawing has been retained. The teacher extends this discussion by responding in an ambiguous way, saying 'is it exactly the same? No, because one's in a different position'.



Figure 3. Student and teacher discussing if the student's Mathomat drawing has 'stayed the same' after translation.

Figure 3 is an example of *reverse action*. The student has initially used the Mathomat as a drawing tool but he then begins to inquire about the properties of his drawing and, with teacher guidance, the student begins to explore an important concept underlying the lesson task. At the same time there is interaction between the student's concept image and concept definition of translation occurring. As with the example in Figure 2, the learning that is occurring in this case



is constructionist; formed by teacher and student discourse mediated by the Mathomat template.

The classroom teachers involved in this study found Mathomat to be an inspiring tool for students to use; one that enabled them to directly address students geometric and spatial thinking through discussion of their creative drawing work.

#### THE LIMITATIONS OF MATHOMAT AS AN ACTION TOOL

Like all constructionist tools, Mathomat is limited by the difficulty in teaching of accessing the visualistion strategies that students employ when using it. For example, the student in Figure 2, with the intervention of her teacher is engaged in a transition from holistic surface level reasoning about shape property to analytical reasoning, and this is compelling; but her underlying thought processes are obscure. As Battista (2007, p. 853) points out, we do not know if this student is engaging in a specific type of reasoning, if she is demonstrating a particular level of reasoning (involving a mixture of reasoning types), or if her thinking is best explained in terms of underlying psychological mechanisms.

An example of the drawing versus figure confusion discussed earlier can be seen in Figure 4. In this case the student (who is at the front of the classroom) has been asked to point to South. The student appears to be pointing to a mental image of South (a *conceptual object* in Battista's terms) by pointing his pencil back over his right shoulder.



Figure 4. Student pointing to conceptual South

He is not pointing to South as a *physical object* by pointing to it in the Cartesian plane drawing in his exercise book, or on the projector display in front of him, as expected.

In both of these cases there is a rich world of spatial reasoning at work in the students thinking, that intertwines with geometric reasoning. This world can only be hinted at if the theoretical objects of geometry described earlier are regarded as personal to students, and not directly addressed in teaching. In the case of Figure 4, as discussed earlier, the mental objects being employed by this student are central to his geometric reasoning, and they have actually been defined in formal and explicit terms.

#### CONCLUSION AND FUTURE RESEARCH

Mathomat can be a powerful classroom tool giving teachers access to a learning process that is directly aligned with the way in which students actually acquire geometry concepts in the early van Hiele stages.

Ongoing research into constructionist tools such as the van Hiele levels makes it possible to deal more directly with student's spatial reasoning strategies. In the case of our study, synthesising a formal description of the spatial reasoning involved in finding the location of objects with the idea of *formal* and *personal* figural concepts, it may become possible in future research to develop instruction for Mathomat that addresses the actual thinking of the students in cases such as in Figures 3 and 4 directly. This would involve the development of questions that required a full understanding of the complete spectrum of geometric objects involved in the lesson such as coordinate systems. Research could then focus on defining the gap between this formal figural concept and the personal figural concepts in use by students.



Not addressing the theoretical objects involved in geometry directly results in densely packed instructions, such as 'remember to estimate' when measuring. If these instructions can be unpacked in a practical manner in terms of actual classroom instruction, then the teaching power of constructionist based materials such as Mathomat can be fully realised.

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## **Critical and creative thinking in the maths classroom**

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The introduction of the general capability Creative and Critical Thinking in both the Australian Curriculum and Victorian Curriculum emphasises the importance of encouraging our students to become active learners. Teachers are no longer expected to only teach mathematical content but to also explicitly teach the knowledge and skills that will enable students to think creatively and critically, reflect and improve on their learning. The following article provides an overview of this general capability and offers suggestions on how teachers can implement it in the mathematics classroom.

#### **CRITICAL AND CREATIVE THINKING**

The Victorian Curriculum includes four capabilities in addition to the eight learning areas. These capabilities 'are a set of discrete knowledge and skills that can be and should be taught explicitly in and through the learning areas' (Victorian Curriculum and Assessment Authority, 2017) and are a part of the curriculum from Foundation to Year 10. Included in these capabilities is Critical and Creative Thinking which covers three strands: Questions and Possibilities, Reasoning, and Meta-Cognition. It is extremely important that teachers are aware that they are expected to explicitly teach these skills to students and to not assume that they will develop automatically through teaching the eight learning areas.

Learning should not be centred on content but should also include developing the skills that allow students to be lifelong learners. Being able to reflect is a valuable skill yet, in mathematics, the dense curriculum means we tend to move on quickly to the next topic without considering what the students have learnt. We must remember that just because it was taught and because students were exposed to it in the textbook it does not mean that students have actually learnt anything.

As teachers, we need to encourage our students to not only answer questions but to question their answers. The most basic of all is 'does your answer make sense?' Calculating the height of a person to be 150 metres tall is not a possible answer but time and time again we see students give similar impossible answers. We also need them to look for other possible solutions and also expose them to problems where there are other possible answers. Expose students to contexts that they face every day but also to unfamiliar contexts so that their thinking is challenged.

Unfortunately, textbooks do not know our students' strengths and weaknesses, their likes and dislikes, their learning styles and their learning needs. The majority of students will not learn mathematics from a textbook once they leave secondary schooling therefore we must expose them to the mathematics they will face in real-life. This last statement is supported by (Taplin, 1994) who argues:

Being able to apply a rote-learned algorithm to a question which is usually devoid of any real-life context does not equip students to deal with unfamiliar situations. To do this effectively students need to be able to build up their own algorithms for specific circumstances and to be aware of what is appropriate in a particular context (p.14).

Students need to be exposed to unfamiliar problems so that they develop their thinking skills, gain confidence in their ability to apply their skills and to ease any anxiety associated with application problems. Learning is a dynamic process that requires a dynamic environment. Mathematical understanding should not come from learning skills but from applying them and, as Stein, Grover, and Henningson (1996) state, 'students should not view mathematics as a static, bounded system of facts, concepts and procedures to be absorbed but, rather, as a dynamic process of gathering, discovering and creating knowledge in the course of some activity having purpose' (p.456).

Students need to understand that the strategies they use and the thinking involved in getting a solution are just as important as getting the correct answer. This is a difficult task because students are often rewarded only for correct answers and not for the thinking that took place to achieve the correct (or incorrect answer). Giving students an opportunity to use thinking strategies in class is important and offering them a variety of problem solving tasks allows them to use these strategies. Stein et al. (1996) emphasise the importance of mathematical problem solving tasks and acknowledge, '…students were more apt to be working from innovative materials and/or from teacher developed



materials than from a textbook' (p.482). They go further to state that tasks should be designed to allow for 'multiplesolution strategies, multiple representations, and require that [students] explain or justify how they arrived at their answers' (p.482).

It is also important to cater for the students who are at the high end as they can be overlooked in favour of supporting the weaker students. Tomlinson and McTighe (2006) identify that these highly able students 'are often left to fend for themselves' to find challenges and therefore 'would benefit from tasks designed to foster complex and creative thinking' (p.20). Therefore, the challenge for teachers is to be able to design lessons that are meeting the needs of all students so that every student has the opportunity to improve their learning.

#### **CLASSROOM APPLICATIONS**

The teaching of the standards within the Critical and Creative Thinking capability do not have to occur solely within mathematics, however, thinking about how these standards can be introduced allows teachers to be more creative with their mathematics lessons. It is easy for teachers, especially in secondary schools, to fall into the trap of allowing the textbook to guide the teaching. Mathematics textbooks do differentiate the content and slowly increase in difficulty to cater for different learning abilities but a textbook doesn't necessarily cater for different learning styles. A longitudinal study completed by Jo Boaler (1997) found that taking an applied learning approach (that is, placing mathematics in real-life context) in the middle secondary years can make a significant difference. Her main results concluded that taking an applied learning approach can help develop a positive attitude towards mathematics and decrease students' anxiety when completing tasks. She also found that students were able to transfer their skills from inside the classroom to outside the classroom and they retained their knowledge better than those students who worked from a textbook.

In their work, Sullivan, Clarke, and Clarke (2013) have defined six key principles of effective mathematics teaching (see Table 1). These principles focus on developing a student's understanding by making them an active member in their own learning. The principles strongly reinforce the importance of not only learning the mathematical content but also developing the personal and interpersonal skills that will prepare students for their future.

Principle 1	Identify big ideas that underpin the concepts you are seeking to teach, and communicate to students that these are the goals of the teaching, including explaining how you hope they will learn.
Principle 2	Build on what the students know, mathematically and experientially, including creating and connecting students with stories that both contextualise and establish a rationale for the learning.
Principle 3	Engage students by utilising a variety of rich and challenging tasks that allow students time and opportunities to make decisions and which use a variety of forms of representation.
Principle 4	Interact with students while they engage in the experiences; encourage students to interact with each other including asking and answering questions, and specifically planning to support students who need it; and challenge those who are ready.
Principle 5	Adopt pedagogies that foster communication and mutual responsibilities by encouraging students to work in small groups, and using reporting to the class by students as a learning opportunity.
Principle 6	Fluency is important, and it can be developed in two ways: by short everyday practice of mental calculation or number manipulation; and by practice, reinforcement and prompting transfer of learnt skills.

#### Table 1, Six Key Principles of Effective Mathematics Teaching (Sullivan et al., 2013)

These six principles not only emphasise the importance of teaching mathematical content but also the importance of other skills including goal setting, collaboration, reporting, asking questions, and considering other ideas. As teachers, we also need to think critically about our own teaching by reflecting on what is working well within our classrooms and our schools and also identify areas that require improvement.

The Trends in International Mathematics and Science Study (TIMSS) is a series of international assessments of the mathematics and science knowledge of students around the world. In their review of the 1999 TIMSS study, Lokan, McCrae, and Hollingsworth (2003) identified the following differences between Australia and Japan:



- Australia dedicated more lessons (28%) to review compared to Japan (5%);
- Australia introduced new material for 30% of the lesson time compared to Japan at 60% of the lesson time;
- 26% of Australian lesson time was spent on working on the same problem compared to 64% in Japan;
- In Australia homework was assigned in 62% of the lessons compared to 36% in Japan (however, the authors noted that students often attend private lessons outside of school in Japan);
- 45% of problems per Australian lesson were classified as 'applications' (applying procedures learnt in one context to solve problems in another context) compared with 74% in Japan;
- The average time spent on one problem in Australia was 3 minutes compared to 15 minutes in Japan.

Other significant statistics relating to Australia include:

- More than three quarters of the problems used by teachers were rated as low in procedural complexity;
- The curriculum level of Year 8 Maths was lower than most other countries in the study;
- Seventy-six per cent of all problems were repetitions of one or more problems students had done earlier in the lesson;
- Just over one quarter of problems were set up with the use of real-life connections;
- More than 90% of problems were presented to students as having only one solution.

From these results, it is quite evident that the lessons in Japan focus on a particular mathematical understanding and the students are given the time to work through a problem in order to develop this understanding on their own. In contrast, the statistics indicate that Australian students learn through completing short questions with very low procedural complexity and with only one answer.

Stevenson and Stigler (1992) noted that Japanese teachers aren't necessarily focused on drilling and testing their students but on stimulating real understanding. These types of lessons use structured problem solving whose purpose is 'to create interest in mathematics and to stimulate creative mathematical activity in the classroom during the collaborative work of students' (Takahashi, 2006, p. 39). The structured problem solving approach (see Figure 1) requires students to work on a problem by themselves before sharing their solution with other students and then with the entire class. The teacher leads students in a whole class discussion to allow students the opportunity to share and learn from each other.



Figure 1. Structured problem-solving (Takahashi, 2006).



As the lesson usually focuses on solving one problem or completing one activity it is important that the teacher ensures the lesson is planned so that students achieve the expected learning outcomes and also so the teacher can be prepared for any questions or misconceptions the students may have. Sullivan et al. (2015) also support the concept of exposing students to problems that they haven't previously been shown how to solve and therefore challenging them to find their own solution. Sullivan et al. also acknowledge that 'actions that result in a decline of the cognitive demand include routinizing approaches to tasks, emphasis on completion rather than comprehension, inadequate time on tasks, inappropriate choice of tasks and expectations for high level performance not being communicated' (2015, p. 125).

The structured problem-solving model encourages students to think about how to solve the problem and highlights that there is more than solution. An example of a lesson could be presenting students with a range of containers and asking them which one holds the most liquid. This problem can be solved in a variety of ways and allows students to make the connection between maths and the real world. Also, through discussion, students are learning from each other and can discover a different (and perhaps easier or more suitable) way to solve the problem.

#### **A SAMPLE LESSON**

The task, 'Shopping for shoes' (Sawatzki, 2014), is an example of a task where the thinking and reasoning are more important than the final answer. As long as students have a basic understanding of financial literacy the task can be used or easily modified to suit students in both primary and secondary school. The purpose of this task is to link social and mathematical reasoning and, as Sawatzki *(ibid)* emphasises, asking for more than one response is 'critical for creating an awareness of alternative possibilities' (p. 132). The task as it was presented to students is shown in Figure 2.

#### Task 1.1:

Jenny and Carly go shopping for shoes. Jenny chooses one pair for \$110 and another for \$100. Carly chooses a pair that costs \$160.

When they go to pay, the assistant says that there is a sale on, and they get 3 pairs of shoes for the price of 2 pairs.

- a. Give two options for how much Jenny and Carly should each pay?
- b. Explain which of these options is fairer.

#### Task 1.2:

Jenny and Carly go shopping for shoes. Jenny chooses one pair for \$110 and another for \$100. Carly chooses a pair that costs \$60. When they go to pay, the assistant says that there is a sale on, and they get 3 pairs of shoes for the price of 2 pairs (the cheapest pair becomes free).

- a. Give two options for how much Jenny and Carly should each pay?
- b. Explain which of these options fairer.
- c. Explain in what ways the fairer solution depends on the cost of Carly's shoes.

The students in the study completed Task 1.1 first and then shared their responses with the class. From the 38 mathematical calculations recorded, the most common (n = 22) was to share the cost equally so that both Jenny and Carly pay \$135. The teacher asked for students to present their solutions to the class and she led a discussion on their mathematical reasoning as to what was deemed 'fair'. The emphasis was on the students' reasoning and not their answer and by having no 'correct' answer meant that students could see the validity in the multiple responses.

Task 1.2 is very similar to Task 1.1 however students were now able to approach it differently after hearing the strategies and reasoning used by other students. Students now also had to consider the change in cost of Carly's shoes (now \$60



Figure 2. Shopping for shoes (Sawatzki, 2014).

from \$160) and how this would impact their reasoning. From the 30 calculations recorded, 10 stated that Jenny and Carly should share the cost, a lower cost than the same response in Task 1.1. Another interesting finding was that six students stated Carly should pay \$70 (dividing the total cost by the number of shoes) and eight stated Carly should pay \$60 (only Jenny should benefit from the discount).

You can imagine the depth of discussion that can follow after sharing all the responses with the class. Asking students why they changed their idea of 'fairness' allows them to share their thinking. They are able to make comparisons, explain why some responses were better than others, make links to the real-world, and support their reasoning using mathematics. In addition to being an engaging and effective task for covering the three strands of Creative and Critical Thinking, it also allows every student to participate and learn.

The important thing to remember is that in the mathematics classroom, quantity is not better than quality. A page full of answered questions may seem like a huge achievement because students can actually see their work but often it is work they already know how to do. Completing tasks like 'Shopping for shoes' do require a lot more thinking and, because thoughts can't be seen, the discussion that takes place before, during and after the task is extremely important in assessing what the students are learning and what they have learnt. Both teachers and students need to understand that quality learning can be achieved by just responding to one task and it is perfectly acceptable to spend time completing it.

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### The integration of mathematics and everyday knowledge: Grade 10 mathematics and mathematical literacy learners' solution strategies

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This paper is about the learning of mathematics through integration with contexts that draw on learners' everyday experiences for Mathematics and Mathematical Literacy learners. The study is located within a qualitative approach, adopting a case study which focuses on one secondary school in South Africa. In this school, two classes – Mathematics and Mathematical Literacy, with approximately 30 learners in each, were chosen for responding to a contextualized mathematics task. three learners in each class (Mathematics and Mathematical Literacy) were selected for interviews based on what they have written for the task. Using Bernstein codes of recognition and realization rules, the study shows that Mathematical Literacy learners use their everyday knowledge without reference to mathematics knowledge.

#### INTRODUCTION

It is of grave concern that even when students have procedural fluency of specific concepts in mathematics, they often struggle with solutions to problems that are posed in a contextualized form. Researchers (Cooper & Dunne, 1998, 2000; Lubieski, 2000; Nyabanyaba, 1999, 1998; Sethole, 2004) highlight the difficulties of learning mathematics through every day experiences. This study investigates how the learning of mathematics by grade 10 learners is influenced by integration with contexts that draw on learners' everyday experiences, both in Mathematics and Mathematical Literacy classes. The study is guided by the following key critical question: How is the learning of mathematics by grade 10 learners by grade 10 learners influenced by integration with contexts that draw on learners' everyday experiences? The research sub question driving the study was as follows: How does the learning of mathematics through integration with learners' everyday experiences affect the recognition and realisation rules within the classroom? To answer the above questions, the study looked at how other researchers nationally and internationally draw on the Bernstein's (1996) constructs of recognition and realisation rules in the context of learning mathematics through the integration with learners' everyday experiences.

#### RECOGNITION AND REALISATION RULES IN THE CONTEXT OF MATHEMATICS AND EVERYDAY CONTEXT

Many researchers (Cooper 1998; Cooper & Dunne, 2000; Nyabanyaba, 1998; Verschaffel, Adendorff, Cooper, Kasana, Le Roux, Smith & Williams, 2000) used the construct of recognition and realization rules within mathematics. Verschaffel et al. (2000) showed that children, in the context of a mathematics test, typically do not pay attention to realistic considerations when constructing responses to short word problems that embedded in everyday context. For example, one of their test items asked: '450 soldiers must be bussed to their training site. Each army bus can hold 36 soldiers. How many buses are needed?' There are at least two competing meanings that might be given to a 'realistic' response to such items. Many learners gave their response as 12.5. Children responding in this way might be viewed as responding or behaving 'realistically' or at least rationally - given that school mathematics problems conventionally paid little attention to the realistic consideration that might arise outside of school. The context in this case would be used as a vehicle to access the mathematics (Recognition rule). On the other hand, few learners gave a 'realistic' answer as 13 buses rather than 12.5. Verschaffel et al. (2000) argued that when learners faced with stereotyped and short word problems, children tend to act without apparent concern for what would be realistically meaningful outside of the classroom. This is because of the way mathematics is traditionally taught to schoolchildren. Their research showed that children typically remain apparently unwilling or unable to apply realistic consideration when solving supposedly realistic word problems, though research has also shown that children's behaviour in this domain does vary depending on the nature of the item, its context and the children's social background.

In contrast, Cooper and Dunne (2000), Cooper (1998), and Nyabanyaba (1998) showed that many children import extra school 'realistic' consideration into their solutions for this type of problems even though the designers – judging from the marking scheme – do not intend for them to do so, having employed everyday context merely as a vehicle for introducing some mathematical task.

Cooper and Dunne's (2000) study in the UK illustrated differential access to 'realistic' school mathematics items across social classes. They provided an epistemic explanation for the poor performance of working class children in the UK that crosses boundaries between the everyday and the school concepts. Their study illuminated how working class children



faced with a realistic question, remain tied to their everyday knowledge and thus failed to negotiate the boundaries between the everyday and the school context.

Furthermore, Cooper and Dunne (1998) illustrated with reference to Bernstein's (1996) recognition rules and realization rules the difficulties that students experience with realistic mathematics items. Cooper (1998) defines recognition rules as the means an acquirer employs to recognize the speciality of the context they are in and the realization rules as a means of allowing for the production of a legitimate text. Cooper (1998) uses Bernstein's (1996) framework to argue that even if children may have the recognition rules, those from the working class may not possess the realization rules necessary to perform well in realistic mathematics assessments. Mike, a child from the working class, was given a task to sort a set of objects. In his responses, Mike remained very much tied to the material base of the question. The question required children to sort the rubbish from the sports field into one of the circles drawn. The rubbish to be sorted included newspaper, cold-drink cans, a bottle of mustard, a pen, and a carton of milk. The marking scheme indicated that it was acceptable for children to sort by the shape of a container, by being edible or drinkable and so on (Cooper and Dunne, 2000: 50). Mike's sorting reflected as being very much related to the types of materials, either paper or metal, as he wrote 'meatle'. Another student, Diane, from the middle class environment responded by illustrating the recognition of both material and metacognitive sorting. Diane's sorting prioritised 2-dimentional over 3-dimentional diagrams; although she also illustrated awareness that sorting can be done at material level. Thus the investigation reveals that Mike possesses the recognition rule, but not the realization rule that Diane demonstrated. Thus in the context of this study, one was expected to find out how Mathematics and Mathematical Literacy learners respond to questions with realistic context embedded in it, would they (mathematics and mathematical literacy learners) respond the same or differently?

An interesting variation on the theme is the study carried out by SalJo and Wyndhamn (1993) focusing on the relation between the concrete condition for solving problems and the resulting nature of children responses. They set a task as follows: What would it cost to send a letter that weighs 120 grams in Sweden? The children were provided with a table that indicates various maximum weight grams of letters per rate. The task was administered to Mathematics learners and Social Science learners. They found that Social Science students responded to the task in a realistic manner expected in a post office, whereas Mathematics learners solved the problem in a mathematical way. This suggests a clear effect of context on response. In Bernstein's (1996) terms, students seem to have recognized the task as a different one when it was presented in the context of one school subject rather than the other. This application by students, within the context of classified school subjects, is what Bernstein (1996) termed a recognition rule, the production of responses that are more or less appropriate within each subject. This study investigated how the learning of mathematics by grade 10 learners is enabled or hindered through integration with contexts that draw on learners' everyday experiences within the context of Bernstein constructs of classification, framing, and recognition and realization rules.

#### THEORETICAL PERSPECTIVE

According to Bernstein (1996), the fact that mathematics is a content-oriented subject makes it strongly classified and framed, which resulted in its recognition and realisation rules being clearer. Bernstein (1996, p. 59) refers to classification as:

the nature of differentiation between contents. Where classification is strong, contents are well insulated from each other by strong boundaries. Where classification is weak, there is a reduced insulation between contents, for the boundaries between contents are weak and blurred

Framing is defined as a 'form of the context in which knowledge is transmitted and received and refers to the specific pedagogical relationship between the teacher and the taught' (Bernstein, 1996, p. 59). The concepts of classification and framing, yield to concepts of recognition and realisation rules. Recognition rules are criteria (special relationships) for making distinctions, for distinguishing the speciality of a thing, a practice, a specialisation or a context that makes it what it is (Bernstein, 1996). They are principles for recognising the 'legitimate text' (p. 50), the voice to be acquired, and are determined by the classification principle at work (relations between different knowledge discourses and practices). Conversely, realisation rules are the means for creating and producing the special relationship internal to what is recognised as the 'legitimate text', i.e., the means for reproducing/creating the speciality in practice. For Bernstein, this implies that the mathematics context can easily be recognised than mathematical literacy due to its strong classification of a knowledge structure, unique identity, unique voice and internal rules. The acquirer (the learner or the teacher) is



able to recognise mathematics pedagogical text (e.g. textbooks) in the classroom. This means that teachers or learners, in the context of contextualised tasks (those that are connected to learners' everyday experiences), are able to identify themselves in the context of mathematics than mathematical literacy. They can ably recognise 'the speciality of the context they are in' (Bernstein 1996, p. 31).

#### **RESEARCH DESIGN AND METHODOLOGY**

This study focused on one township secondary school in South Africa. The study was located within a qualitative, case study approach. Students in this school are predominantly from poor or less affluent background, where parents are typically of low-income earners. The student participants were provided with a task (Telephone option) to solve.

In the task, learners were asked to explore and critically analyse the context of phone charges - cell phones against a fixed phone, using mathematics to determine the best value for money telephone option. Two tables of tariffs for the different cell phone options as well as telecommunications provider, TELKOM rates for using a fixed phone were included. Learners were to investigate the financial implications of using a fixed phone versus using a cell phone. Learners had to think of all factors that influence the decision, for example, the cost of calls at different times or over different distances. They also had to decide which information is not necessary when solving the problems. As these activities are modeling tasks, learners had to make use of assumptions, investigations, justifications, generalizations, and calculating the cost of the calls made during peak time and off peak time. Their investigation and decisions must be based on mathematics and personal preference. The project had 3 activities. The first activity was about finding the formulae when given the two tables of tariffs of Voice Data Communication (VODACOM) and Mobile Telephone Network (MTN). The second activity was about finding the formulae when given TELKOM tariffs over time and distance. The third activity, reported in this paper, was about the comparison and decision making of the three telephone providers based on the calculation from first and second activities. Activity 3 was structured as follow:

Which telephone option is the best value for money based on your maths analysis in activity 1 and 2 your individual needs?

Consider your own need for a telephone. Take into account your current use and your available budget for a telephone. If you don't have a phone, plan and budget as if you want to get one. Decide if you should get a VODACOM phone, an MTN phone or a TELKOM phone.

- State your assumptions. That is, give some evidence of when you usually need to make calls, how many calls you make during a month, where you call to and so on. This information can be sourced from old accounts or by careful estimation.
- Make a decision for your situation and justify your decision mathematically.

After the written work, three learners from Mathematics and three from Mathematical Literacy class were chosen for an interview. The six learners were selected based on what they have written in their scripts and their responses were coded according to the class they attend. Thus, learners from the Mathematics class were coded as ML1, ML2, ML3 whereas the Mathematics Literacy learners were coded as MLL1, MLL2, MLL 3 respectively.

#### **FINDINGS AND DISCUSSION**

Two issues emerged in this study in relation to the learning of mathematics connected to learners' everyday experiences. The data shows that there is a dialectical relationship between mathematics and learners' everyday context. While Mathematical Literacy learners are tied up to the context - mathematics get backgrounded when solving tasks that are embedded to learners' everyday context.

#### MATHEMATICAL LITERACY LEARNERS' MATHEMATICAL KNOWLEDGE

When asked about the best network option between VODACOM, TELKOM and MTN, MLL1 and MLL2 said the following:

VODACOM, because is the leading network and they charge their services at low rates than other networking



providers. Like when you want to make a call on off peak time it charges R0. 68 and weekend and holiday (MLL1).

I think I will go for TELKOM, because I know that is better than VODACOM and MTN, we use TELKOM at home (MLL2).

In the above extract, MLL1 and MLL2 (Mathematical Literacy learners) did not use the mathematics as instructed in the task; instead they used their personal preferences. It is evident that these two learners remain tied to their everyday knowledge and thus failed to negotiate the boundaries between the everyday and the school context. Cooper (1998) uses Bernstein's framework to argue that, even if they may have the recognition rules, children from the working class may not possess the realization rules necessary to perform well in realistic mathematics examinations items. Whereas, the middle-class children could move from context to mathematics.

Similarly, MLL3 also became tied to the everyday experiences instead of using mathematical calculations to take decision, and justify their arguments when it comes to choosing the best phone options. When asked about the best network between MTN, VODACOM and TELKOM, MLL3 said:

TELKOM rates are cheaper during off peak time. I usually make calls during off- peak time and I make 2 to 4 calls daily and calling people here in Soshanguve and some in Mpumalanga so in a month. I probably make about plus minus 120 calls. During the day on a peak time a sms is 10c or R1.50 depending on the time of day. I sms more than calling because it is lesser and it allows you to say more. I send 2 to 5 sms daily and I usually speak for more than 3 minutes on the phone.

In addition, MLL3 said:

VODACOM is a network I choose because first of all when coming to advertisement is a network that is leading even if you don't like to look at advertisement but VODACOM will make you to enjoy, not only in advertisement, but also in free air time. I never heard that MTN is giving free cell phone or call discount. But with VODACOM you can get free cell phone and again is a South African leading network in terms of supporters. It also sponsors African soccer challenge.

MLL3 chose TELKOM and VODACOM because TELKOM rates are cheaper during off peak time, and she usually make calls during off-peak time. MLL3's choice is based on her experience with using TELKOM, not based on her mathematical calculations. Similarly, her choice for VODACOM network was based on the advertisement she saw on TV, not on her mathematical justification. In his study, Nyabanyaba (1999) found that learners answered Premier Soccer League football log contextualized task from their everyday life experiences, without concentrating on the mathematics embedded on the task. For example, when asked which team will win the league and what will be the score based on the comparison of two league tables, learners responded: 'Moroka swallows won 3-1, I know because I was there when it played'. This is one of the tensions that emerge when learners are dealing with contextualized tasks. Moreover, Nyabanyaba revealed that gender differences affect how student respond to the task could be created, since not all students can be in the position of understanding the key concepts such as the one used in football log tables. Thus, Nyabanyaba maintains that teachers must be careful to select tasks that would not perpetuate inequalities. He also argues that teachers should intervene when learners are dealing with such contextualized tasks.

## DIALECTICAL RELATIONSHIP BETWEEN MATHEMATICS AND EVERYDAY CONTEXT FROM MATHEMATICS LEARNERS

Conversely, Mathematics learners understood the usage of mathematics in context. They understood that to come up with the correct answers to the questions, they must do mathematical calculations. For example, ML1 indicated that she 'would prefer MTN network because you only pay R95.26 for connection fee and that is less than the R190.00 you pay for the VODACOM connection fee'. Clearly, there are some mathematical calculations done before a choice could be made, as ML1 explained:

I would prefer MTN network because you only pay R95.26 for connection fee and that is less than the R190.00 you pay for the VODACOM connection fee. And it is less than the VODACOM connection fee. In MTN you also get free goodies such as T-shirt, bags, caps, juice bottle and lanyard. And when you make call during off peak time your



monthly account will be monthly subscription +connection fee+ the amount charged for off peak time. But I think MTN is better excluding the free goods you get, MTN is also affordable.

Similarly, ML2 also could connect mathematics to everyday context properly without backgrounding the context:

According to the project that we have done and our different findings on contract phones with all the tariffs, that is, monthly subscription, connection fee, call rates. I happen to find TELKOM as the network provider so cheap but depending on the different distances, for instance TELKOM charges 16c for 0-50 km per minute (standard time). With TELKOM, I will make as many calls a possible. However, of course depending on my distance and on a call more time and the monthly rental of 57. 26. The disadvantage of this is with the landline phone you will have to wait for the call.

In addition, ML3 used the graphs she drew to decide the best network option:

I prefer VODACOM because in our system of axes whereby we compare MTN and VODACOM 's monthly costs option for peak time calls to another mobile we happen to find that the VODACOM costs are lower than the MTN according to our sequence of numbers.

Adler (2001) uses the concept of transparency (in Lave & Wenger, 1991) to explain the leap from the context to the school mathematics. Both (Adler, 2001; Lave and Wenger, 1991) argue that the context could be used as a resource or tool to access mathematics. The context should be seen to be visible and invisible. It needs to be visible so that it could be noticed and used, while simultaneously invisible so that attention is focused on mathematics not the context. Adler seems to suggest the context needs to be in the background with the focus more on the mathematics. Le Roux and Adler (2016) show that contextualized problems require students to work flexibly with movement within and across texts.

#### CONCLUSION

This study shows that Mathematics learners could use the connection between mathematics and everyday experiences without compromising either mathematics or every context. On the contrary, Mathematical Literacy learners were unable to move from context to mathematics. Although they could recognize the context they are in, Mathematical Literacy learners did not possess the recognition rules. The fact that Mathematics learners could possess the recognition and realization rules and Mathematical Literacy learners could not, could be attributed to the nature of mathematics and mathematical literacy. The attributes, internal rules and the structure of mathematics knowledge, for example, gives mathematics a special voice and unique identity in the social terrain because of the way mathematics knowledge is classified. The mathematical context is easily recognised due to its strong classification of knowledge structure, unique identity, unique voice and its internal rules. The acquirer (learner) is able to recognise singulars pedagogical text (e.g. textbooks) and classroom. The recognition rule of regions (Mathematical Literacy) is blurred. The acquirer (learner) can confuse or face difficulty in recognising the context as Mathematical Literacy.

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# What does excellence in MATHS look like? It is all about relationships and communication

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Mathematics is the study of relationships. Teaching and learning have as their foundation the relationship between teacher and student. So learning mathematics well relies doubly on relationships. To teach mathematics well also requires excellent communication skills. The primary indicator for this is how well the students understand what we say. This paper is a reflection on thirty years of involvement with teaching and learning mathematics in the classroom, individual tutoring and most recently, managing AMSI's Schools Outreach Team. It includes a combination of advice and insights garnered from many sources. Key among these is the Australian Association of Mathematics Teachers (AAMT) Standards for Excellence in Teaching Mathematics in Australian Schools (2006).

# THE FIRST DAY

On my first day as a teacher the deputy principal gave me two pieces of advice. The second of these was 'It doesn't matter how good a lesson you have prepared, if you don't bring the students with you, it is a waste of time'. Having reams of transparencies, or nowadays Powerpoints, full of content does not make a good teacher. Nor does an array of flashing lights and interactive screens. Like anything, these resources can be useful, but without the fundamental underpinning of good relationships they may in fact be more of a hindrance than a help.

As the thirtieth anniversary of my entering the teaching profession approaches, the theme for the 2017 conference allowed me to pause and examine what has been written about teaching excellence over the same time period. I began with the MAV's own conference proceedings. A seminal paper appeared in 1998 (Bishop & Clarke) entitled 'Are you an excellent teacher of Mathematics?'. This paper acted in many ways as a catalyst for conversation in the mathematics teaching community over the next few years. This conversation eventually saw the AAMT, of which Mathematics Association of Victoria (MAV) is a state affiliate, adopts the *Standards for Excellence in Mathematics Teaching* (the Standards) in 2002, which were further updated in 2006. Also in 2006, Bishop, Clarke and Morony authored the *Professional Learning Using the Mathematics Standards* report, which proposed a model for the continual deepening of teacher knowledge and skills against 'the Standards'.

The AAMT Standards pre-date, by several years, those promulgated by Australian Institute for Teaching and School Leadership (AITSL), and in many ways informed them. The advantage of the AAMT Standards is that they refer directly to mathematics and mathematics teaching. There is a total of ten standards spread over three domains, as opposed to AITSL, which has seven domains. The three AAMT domains (2002) are:

- Professional Knowledge
- Professional Attributes and
- Professional Practice.

Within each domain there are a further series of subsections, listed as 1.1, 1.2 and so forth. These subsections are referred to in this current paper by this numbering convention. With this in mind, it is recommended that the full Standards for Excellence document be placed beside this current paper while reading it.

# RELATIONSHIPS

In a conversation between Daniels Goleman and Siegel on Goleman's Leadership Masterclass series (Goleman, 2012), Siegel discusses the quality of seeing the mind of others that he observed in some surgeons when delivering difficult diagnoses to patients. The significant difference between these doctors and their equally intelligent and trained peers was their ability to relate empathically with their patients. In a similar way, the best teachers have a relational quality that goes beyond content and pedagogical practice. This accords well with Goleman's own work on Emotional and Social Intelligences over the last few decades (see Goleman, 2011). It is also at the heart of the advice my deputy principal provided me on my first day of teaching.



## **TEACHER-STUDENT RELATIONSHIPS**

Gervasoni (1994) and Kaur (1997) report on student impressions of what makes a good mathematics teacher in two very disparate contexts. In both cases it is enlightening to note the number of affective or relational characteristics that are cited. Students value teachers who care, who are patient, humorous, nurturing and who provide regular encouragement. These are linked to the following AAMT Standards (2006):

- 1.1 Knowledge of Students
- 1.3 Knowledge of Students' Learning of Mathematics
- 2.1 Personal Attributes
- 3.1 The Learning Environment

The common thread connecting these descriptors is other centredness. This is evinced by references to students' social and cultural contexts, effective role modelling, and the care and respect of students. This in turn leads to addressing psychological, emotional and physical needs by the teacher. Students are seen as persons in their entirety, not just empty vessels to be filled or blank minds to be written on.

## **TEACHER-TEACHER RELATIONSHIPS**

Teachers are often seen, and see themselves, as isolated within the walls of their classroom. On the one hand this can lead to a sense of infallibility, or a need to be seen as infallible. On the other it can dissuade honest searching and dialogue that could enable improved student understanding and results. Neither outcome is desirable. The Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment (PISA) data do not report on a school by school basis but country by country. In order to address our national decline in rankings, it is imperative that the profession works collaboratively. This is at the heart of Standard 2.2: Personal Professional Development (AAMT, 2006).

According to this standard, excellent teachers are described as those who 'are committed to continual improvement' and 'undertake sustained, purposeful professional growth' (AAMT, 2006). The key to achieving this is dialogue with other teachers. Teachers must first be learners, and enthusiastic learners at that. A major challenge in achieving this however, is that while teachers, like Winston Churchill, appear 'always ready to learn' we do not always 'like being taught' (Churchill, 1952). This in turn raises the question of how well we relate to ourselves.

#### **RELATING WITH OURSELVES**

For the first few years of my career I was a stickler for a quiet classroom. This stemmed from my own days as a student, where I associated a quite classroom with 'good' teaching. Then one day, I found myself becoming increasingly agitated by the level of noise in my classroom, but unable to pinpoint the source of my agitation. This does not mean I did not know where the noise was originating from. The school was undergoing renovations and on that particular day there was a jackhammer tearing up concrete within two metres of the classroom door. I was well aware of the origin of the noise, and conscious of having to cope with it. What I discovered about myself however, was that I was unable to tune the background noise out of my awareness. Looking around at the students in my class however, they continued to work without distraction. They were on task and focused, seemingly oblivious to the cacophony outside. This gave me pause to reflect on myself. Over the following months I discovered a congenital auditory processing disorder, which makes it difficult for me to. While there are techniques that help me to concentrate on individual voices, I have had to adapt my teaching style and classroom management approach to find a balance between productive buzz and potentially debilitating background noise. This taught me that in order to continually improve as teachers, it is necessary that we are honest and vulnerable. This is the essence of reflection and learning from experience, as stated in Standard 2.2: Personal Professional Development (AAMT, 2006).

#### MATHEMATICAL RELATIONSHIPS

Far too many in the wider community, and even many teachers, see mathematics as a series of rules to be followed like recipes. They see good mathematicians as those who can recall and use the recipes on demand. This reveals a lack of understanding of the rich interweaving of the true mathematical tapestry. The more we study mathematics the more



we find there is to know (see Standard 1.2: Knowledge of Mathematics, AAMT, 2006). I wish to illustrate this with an example involving the multiplication algorithm.

It is often lamented that students do not know how to multiply and cannot use the written algorithm. My experience in working with teachers in many schools over recent years is that what is valued is the written form of the result (see Figure 1) over any clear and meaningful approach to working out the product of two numbers. The 'algorithm' is what students are to learn and it is to conform strictly to the style shown in Figure 1.



Figure 1. Written vertical multiplication.

This in fact is not an algorithm. An algorithm is a set of steps, a recipe, to follow. In this example the steps would be:

- Multiply the tens digits of both numbers together
- Multiply the tens digit of the second number by the units digit of the first number
- Add the products together and write down the result
- Multiply the units digit of the second number by the tens digit of the first number
- Multiply the units digit of the second number by the units digit of the first number
- Add the products together and write down the result
- Add the two partial products together and write down the total

There are multiple opportunities for error in this process, little indication of where these errors might occur and the process can be completed by students with very little multiplicative thinking. For these reasons would it not be better for students to use the area model for recording the individual steps and then writing the neat vertical summary next to it?



Figure 2. Vertical multiplication with area model.

There is clearly much more to multiplicative thinking than just the algorithm. However, it is included here to illustrate the need for teachers to have a solid working relationship with the concepts and skills they are teaching and the absolute need for clarity in the way they communicate these ideas.

#### COMMUNICATION

I mentioned at the beginning that my first deputy principal gave me two pieces of advice. What was the first? Here it is: 'Teaching is acting.' Actors tell us stories. Not just with the words they say but with their gestures, body language, use



of light and materials and in so many other ways. Excellent teachers do the same. Communication is not just verbal. It is sight and movement, colour and emotion all employed together to impart knowledge and meaning. This provides us with a second lens through which to view our profession and our performance. The AAMT (2006) Standards of Knowledge of Mathematics (1.2), Planning for Learning (3.2), Teaching in Action (3.3), and Assessment (3.4) all require teachers to exhibit exemplary communication skills. For me, these involve, questioning, listening, feedback, and structuring experiences. I will explain each in further detail below.

### **QUESTIONING AND LISTENING**

Curiosity drives learning. Curiosity is articulated in questions. Sullivan and Clarke (1991) and Zager (2017) highlight the central role played by good questions. Of equal importance is the provision of opportunities for students, and teachers, to explore questions, grapple with them and argue about them rather than just answer them. A useful exercise for examining the strength of questioning in classrooms can be found on the NRICH Enriching Mathematics website (University of Canberra, 2013). This sets out a structured set of eight aspects for examining questioning in the classroom along with prompts for improving each aspect.

As important as good questioning is the ability to listen. How do we respond when we hear a student say 'I don't understand'? An unfortunate reality is that many times the response is simply to repeat the original instruction or explanation without variation. What the student is really seeking is another perspective, method or approach. To more adequately ensure student understanding, planning is essential. Wiggins and McTighe (2005) stress that the capacity for a teacher to respond flexibly to student enquiry requires both deep knowledge of the connections inherent in the subject and prior consideration of the possible paths that might present themselves during a lesson or topic. A student acknowledging that they do not understand the explanation given by the teacher is asking for the concept to be described in a different way. For a teacher to respond adequately he or she needs to be familiar with alternate approaches and to be able to call them to mind when needed in the moment. This is nicely encapsulated in a quote attributed to Marvin Minsky (Herold, 2005), 'You don't understand anything until you learn it more than one way.'

#### FEEDBACK

Feedback is often seen as what teachers provide to students by way of comments on an assignment or test or to parents during parent teacher interviews. As Hattie (2012) argues however, those same tests and assignments can also be a very powerful means of feedback to teachers. Teachers can glean information about which students are moving towards the success criteria and which are not. This is turn provides the teacher with a clear indication of what they have, or have not, taught well. Teachers who reflect on this can then identify both the strengths and weaknesses of their own teaching practice and make changes accordingly.

#### STRUCTURING STORIES AND EXPERIENCES

Booker Bond, Sparrow, and Swan (2014), Sousa (2015) and Mason, Burton and Stacey (1996) all stress in different ways a three-fold approach to the growth of understanding of mathematical concepts. The first stage is through a concrete, sensible interaction or manipulation of an object. The second phase is the development of a story, symbolism or sense of meaning about the object. The final phase is the articulation of the knowledge gained and meaning developed. This requires the acquisition of abstract forms, be they words or algebraic notation, with which to convey the meaning to others.

In my experience, the second stage is often glossed over in mathematics classrooms. Humans are a social species and it is through story that we develop and pass on what is important and valued from one generation to the next. Pimm (1987), in his preface states that: 'The teacher conventionally acts ... as an intermediary and mediator between the pupil and mathematics ... pupils are learning from the teacher the range of accepted ways in which mathematics is to be communicated and discussed' (p xiii).

The teacher, in this context, is the keeper of story. As teachers we need to have to work diligently at finding or constructing stories that will have interest and meaning for our students. Consider algebra as a case in point. Many students quickly become lost by the order of operations needed to solve either a numerical or a symbolic equation. Sometime ago I observed a teacher address these difficulties for her students. She cleverly related the unpacking of an



equation to the party game 'Pass the Parcel'. She described how each layer of wrapping needed to be stripped away by undoing or reversing what had been used to wrap it in the first place. By using concrete materials and the symbolic layering of increasingly more complicated equations, she was able to successfully convey the concept to her students. This teacher helped her students relate to abstract concept through her story telling. This is exemplary mathematics teaching at its best.

#### CONCLUSION

An excellent teacher of mathematics is one who can, and consistently does, improve the learning outcomes for all of their students. We have to take our students with us on a journey that they can relate to, and communicate the richness of the landscape to them in ways they can understand. Thirty years on, those two pieces of advice my deputy principal gave me, still, perhaps more so now than ever, ring true.

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# **Mathematics and the Commonwealth Games**

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The theme of 'Mindsets' includes the desire to promote that maths is fun, everyone can do it and its value in society. Embedding school mathematics in relevant and engaging contexts is one way of achieving this. The Commonwealth Games provides an ideal opportunity for students to experience the use of mathematics content that they are learning to an event beyond the classroom in sporting and other contexts, as well as making connections across the curriculum. This paper outlines aspects of mathematics and the proficiencies that can be included as part of a program focusing on the Commonwealth Games.

## WHY USE THE COMMONWEALTH GAMES IN YOUR PROGRAM?

Embedding school mathematics in relevant and engaging contexts for students is advocated in the Australian Curriculum - Mathematics (Australian Curriculum, Assessment and Reporting Authority, 2013). What could be more relevant and engaging for students in Australia, which is such a sporting nation, than an international sporting event of such diversity as the Commonwealth Games, especially with the Games being held on the Gold Coast in 2018.

The Commonwealth is a voluntary association of 70 sovereign nations and territories. About 30% of the world's population (2.1 billion people) lives in Commonwealth countries. The countries range from some of the world's largest populations, such as India with a population of 1.35 billion, to some of the world's smallest populations such as St Helena, a 308 square kilometre island in the Atlantic, with a population of 7000.

- Africa: Botswana, Cameroon, Ghana, Kenya, Lesotho, Malawi, Mauritius, Mozambique, Namibia, Nigeria, Seychelles, Sierra Leone, South Africa, Swaziland, Tanzania, Zambia
- Americas: Bahamas, Belize, Bermuda, Canada, Falkland Islands, Guyana, St. Helena
- Asia: Bangladesh, Brunei, Darussalam, India, Malaysia, Maldives, Pakistan, Singapore, Sri Lanka
- **Caribbean:** Anguilla, Antigua & Barbuda, Barbados, British Virgin Islands, Cayman Islands, Dominica, Grenada, Jamaica, Montserrat, St. Kitts & Nevis, St. Lucia, St. Vincent & The Grenadines, Trinidad & Tobago, Turks & Caicos Islands
- Europe: Cyprus, England, Gibraltar, Guernsey, Isle of Man, Jersey, Malta, Northern Ireland, Scotland, Wales
- **Oceania:** Australia, Cook Islands, Fiji, Kiribati, Nauru, New Zealand, Niue, Norfolk Island, Papua New Guinea, Samoa, Solomon Islands, Tonga, Tuvalu, Vanuatu

# SPORTS AT THE COMMONWEALTH GAMES

Six sports made up the Program of the first Commonwealth Games in Hamilton in 1930; Athletics, Aquatics (Swimming & Diving), Boxing, Lawn Bowls, Rowing, and Wrestling.

Now each candidate city looking to host a Commonwealth Games must include a minimum of 10 core sports on their Program: Aquatics (Swimming), Athletics, Badminton, Boxing (Men) with an option to add Women's events, Hockey (Men & Women), Lawn Bowls, Netball (Women), Rugby Sevens (Men), with an option to add Women's Sevens, Squash and Weightlifting.

They can then include up to an additional seven from a list of optional sports/disciplines: Archery, Basketball (Men & Women), Beach Volleyball (Men & Women), Canoeing, Cycling (Road and/or Mountain Bike and/or Track), Diving (as part of Aquatics), Gymnastics (Artistic and/or Rhythmic), Judo, Open Water Swimming (as part of Aquatics), Rowing, Sailing, Shooting (Clay Target and/or Fullbore and/or Pistol & Small Bore), Softball (Men & Women), Synchronised Swimming (as part of Aquatics), Table Tennis, Tennis, Tennin Bowling, Taekwondo, Triathlon and Wrestling.



2018 CGF Sports				
Archery	Athletics	Badminton		
Basketball	Beach Volleyball	Boxing		
Cricket	Cycling - Mountain Bike	Cycling - Road		
Cycling - Track	Diving	Fencing		
Gymnastics - Artistic	Gymnastics - Rhythmic	Hockey		
Lawn Bowls	Netball	Para powerlifting		
Rugby Sevens	Shooting	Squash		
Swimming	Table Tennis	Triathlon		
Weightlifting	Wrestling			

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The sport and mathematics connection provides a wonderful opportunity for students to experience the use of the content that they are learning at school to the society beyond the classroom. This event is also a rich context for the application of mathematics topics not necessarily related to sport, for example the exploration of the countries competing in terms of their populations [Australia has the 12th largest population of Commonwealth countries; India (1.34 billion), Pakistan (180 million), Bangladesh (168 million), Nigeria (155 million) and England (55 million) are the five largest nations], area, team size in relation to population etc. Virtually every aspect of mathematics could be included as part of a unit focussing on the Commonwealth Games, as well as the potential to integrate this with all learning areas across the curriculum. One possibility would be for students to 'adopt' a country and investigate where it is located, the distance the athletes need to travel to the Games, the particular sporting strengths of the athletes and the size of the team. Students could then track the results and consider factors that may affect the success or otherwise of the country (number of athletes, population, geography, standard of living ...).

#### MATHEMATICS IN THE COMMONWEALTH GAMES

All mathematics domains and topics can be linked to the Commonwealth Games. Here are some suggestions.

#### **Statistics and Probability**

- Predict whether there are families from Commonwealth countries in the school community. Survey and graph nationalities of students in the school/year level/class.
- Students develop questions to collect data on favourite sports/games; sports played by students; favourite event in the Commonwealth Games. Explore ways of representing data.
- Prior to the start of the games, students predict the number of medals certain countries will win (students choose countries) and justify their prediction. Keep a medal tally over the Games. Compare at the end and discuss.
- Students collect personal data during sport (running, jumping, swimming, goals shot/shots on goal, ...) and represent graphically comparing these with elite athletes.
- Investigate the schedule of events (available on GC2018 website).

#### Measurement and Geometry

- Estimating, then measuring lengths (100m, 200m, 400m, 1500m ..., long/high jump, shot put, javelin records)
- Time/measure themselves 'having a go' and comparing this to the actual results in 2018;
- Younger students can start with 20/50m and conduct races, time student races and compare with Games' records;
- Students estimate the height/length of the Games record jumps;



- Make sets of weights (appropriate to age/size of students), conduct weight lifting events and compare with records what is the difference?
- Dimensions and volume of the Games swimming and diving pools;
- Shapes of different arenas and the reasons for these shapes;
- Shapes of track and field event areas;
- Design of equipment shapes involved;
- Designing logos and flags;
- Location of Gold Coast and each country on a world map (include longitude and latitude for older students).
- Distance travelled, time taken for athletes to travel to Brisbane and the Gold Coast (include time zones).

#### Number and Algebra

- Ordinal number can be explored when young students are performing in a sports day, or having races with toys;
- Tallying medals; combinations of gold, silver, bronze if there are 'x' medals;
- Order numbers of athletes in each country's team;
- Compare/order populations of the Commonwealth countries;
- Cost of tickets;
- Explore decimals by comparing results of competitors in an event (timing, height/length/mass); comparing winners with past events; comparing with students' results;
- Investigate scoring of different events;
- Numbers of volunteers/officials needed in the various categories;
- Explore catering quantities by using local data (canteen/catering outlet ...);
- Explore waste management by investigating this at school level first;
- Speed of athletes/swimmers/cyclists how can this be determined?
- Number of laps needed to complete swimming, athletics and cycling races;
- Investigate fitness measures (pulse rate, heart rate at rest and when exercising ...);
- Investigate gearing on cycles.

Links to the Australian Curriculum – Proficiency strands can also be made using this context.

**Understanding:** build robust knowledge of adaptable and transferable mathematical concepts, connect related concepts and develop the confidence to use the familiar to develop new ideas, and the 'why' as well as the 'how' of mathematics.

**Fluency:** develop skills in choosing appropriate procedures, carrying out procedures flexibly, accurately, efficiently and appropriately, and recalling factual knowledge and concepts readily.

**Problem solving:** develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively.

**Reasoning:** develop increasingly sophisticated capacity for logical thought and action - analysing, evaluating, explaining, inferring, proving, justifying, generalising.



Reading through this list will trigger other ideas, and if you work together with your Professional Learning Team, the list will expand even more. There is also the possibility of developing integrated units, such as 'Plan a day at the Commonwealth Games' OR 'Plan a trip to the Gold Coast to attend the Games', which could include many of the above (and more), with students being personally involved in the decision making as an individual, or planning as a group. This would require investigating the schedule of events (available on the Australian Commonwealth Games website), the duration and location of desired events, distance between these and travelling time. Cost for the day would include tickets, travelling, meals and any souvenirs to purchase. This unit would involve many aspects of time (schedules, duration), money, mapping and distance. If this was expanded to travelling to the Gold Coast, accommodation and travel costs would need to be included as well as possibly taking in other attractions, even the weather and clothing to be packed!

Importantly, students should be participating in sporting activities at the school as part of the Commonwealth Games scheduling and seeing for themselves the relevance of mathematics to so many aspects of their lives. Enjoy!

#### **AVAILABLE RESOURCES**

There are many resources that could be used to support the development of mathematics and integrated units based on the Games.

- *Prime Number*, Volume 29 No.3, June 2014, Mathematical Association of Victoria. This journal includes an article on using the Glasgow Commonwealth Games for planning rich mathematics sessions and cross-curricula units.
- *Prime Number*, Volume 15 No.2, July 2000, Mathematical Association of Victoria. This was a special Olympics edition of this journal with many ideas and resources that could be used or adapted for the Commonwealth Games.
- *Prime Number*, Volume 19 No.2, June 2004, Mathematical Association of Victoria. This journal included more ideas to be added to those in the earlier publication.
- *MCTP Professional Development Package. Activity Bank Volume 1*, Curriculum Corporation, 1992. This invaluable resource includes the use of Olympic events such as 100m and 200 m running focusing on pace length and running pace, men's and women's marathon, long jump, high jump, swimming and more.
- *Maths300*, if your school subscribes to this web-based resource, includes activities that can be related to the Games: 'Potato Olympics'; 'Pulse Rates'; 'Country Maps'...
- Children's literature, a great context for maths investigations, includes titles that can be linked, for example *Koala Lou*, Mem Fox; *The Possum Creek Olympics*, Dan Vallely ...
- Media local and metropolitan newspapers, magazines, television and radio are reporting on the build-up to the Games. Collecting these from now on could provide useful material to be used both before and during the Games.
- Australian Commonwealth Games: https://www.gc2018.com/games
- Commonwealth Games Federation: http://www.thecgf.com/

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# The timeless geometry of sundials

**Tim Byrne** 

Sundials provide a natural opportunity to explore geometric relations from the macro to the micro. The tangible benefit of creating a working model appeals to the child in all of us. The step by step instructions combine two pieces; a plywood square and a bamboo skewer. The author takes some mathematical inspiration from some well preserved lumps of clay used by Babylonians to supposedly assist students with trigonometric calculations.

## FINDING MATHEMATICAL INSPIRATION FROM ANCIENT BABYLON

Recent research at UNSW involving a Babylonian clay tablet, Plimpton 322, have revealed that ancient mathematicians had worked out solutions to trigonometry using ratios rather than through the Pythagorean geometry of circles and angles. While searching for more information on Babylonian mathematics, I came across another small tablet from the same Babylonian period. It is formally called YBC 7289 from the Yale Babylonian Collection and measures about the 150 mm square (see Figure 1). This tablet seems to have been used as a teaching aid. My aim in drawing the attention of the reader to these wonderfully preserved items of mathematical antiquity is to make the point that deep mathematics is on display on the surface of a lump of clay. This paper aims to take the reader on a simple mathematical journey using a similar sized block of plywood.



# Figure 1. Clay tablet YBC 7289 (Lynch, 2016).

File is licensed under the Creative Commons Attribution- Share Alike 3.0 Unported. Bill Cassellman original holder of photograph (mailto:cass@math.ubc.ca)

To begin this exercise, there needs to be a mathematical context, which I have identified as geometric measurement and geometric reasoning strands from year 5 to year 7 using ACARA as the guide. Sundials are readily recognizable as objects, which rely on the user's understanding of time zones, latitude angles, triangle properties and the use of simple tools such as a protractor.

To supply a class with the best material for the job, I suggest buying a small panel of 12 mm BC grade plywood, which is cut into any square size to suit, I prefer 150 mm squares as the finished sundials are easy to prepare or read. The squares of plywood can be given to students to smooth up the roughened edges and to draw the intersecting diagonals to find the centre. The drill bench ensures that the centre hole is perpendicular to the surfaces, which is critical to making the sundial accurate. In Figure 2, some of the important geometry of sundials is explained. The gnomon needs to point directly to the South Celestial Pole (SCP).





Figure 2. Geometry of sundials.

Students use the centre point of the square and a set square to draw a perpendicular line, which becomes the noon or midday mark from which all the hour lines are calculated. Sundials have a peculiar specificity. They are designed to work in one place, using the local longitude and latitude. In Victoria, towns close to the 150 degree longitude line will notice that their noon line is the same as their 12:00 pm line. In Melbourne, which seems to be smeared over the 145 degree line, the noon mark will be 20 minutes later than the 12 o'clock hour line because the earth turns one degree every four minutes. Students need to take into account that the earth is a very large sphere with a different sunrise / sunset and noon mark for every place.



Figure 3. Materials needed for sundial.

Students can now begin to use the protractor to mark out all the hour lines using the local noon mark as the central datum point. I have included a table of major town locations across the state including both the longitude, latitude and longitude adjustment for 12:00pm. The rule of thumb is to adjust the noon mark four minutes for every one degree of



longitude west of the 150 longitude line. This means that Portland on 141.6 degrees of longitude will have a local noon approximately 8 x 4 or 32 minutes later than Mallacoota. All the hour lines are 15 degrees apart. Working in pencil will ensure that mistakes are temporary.



Figure 4. Use drill bench for 4 mm centre hole.

The sundial can be drawn up inside a single lesson. Students may be surprised to discover that the sundial needs to have a similar set of numbers on the reverse side to incorporate the winter/summer divide. Each side of an equatorial sundial only works for six months of the year. In fact, on two days of the year, at the equinoxes, the sundial will not throw a shadow as the sun is precisely in line with the dial plate. Readers will be aware that the sun seems to drift between the two tropics and the equinoxes occur as the sun passes over the equator, usually, on March 21 and September 23 each year. One side will reflect the daylight saving hours which requires that the noon mark is an hour later.



Figure 5. Use set square to draw noon mark.

The next step brings the sundial to life. A long skewer is inserted to act as a gnomon. Care should be taken with the point; some teachers may decide to remove the pointy end for safety reasons. The skewer is adjusted to form an angle with the ground equal to the local latitude. In Melbourne, the correct angle of the skewer is 37-38 degrees, depending on how far north or south is your workplace. A small right angled triangle made from cardboard will fit under the gnomon/skewer.



The skewer is gently pushed to and fro until the latitude angle is perfect. Then the next step is to take the sundial outside and line it up on the north south line, perhaps using a Smartphone's compass. The tip of the skewer points up to the South Celestial Pole (SCP), which, incidentally, is the point that the earth's axis points to. A photograph using a long exposure will show the SCP as the centre of all the star circles.



Figure 6. Using noon mark to find 12 pm.

Troubleshooting for sundials can involve a short checklist.

Check the following features of the sundial.

- Is the gnomon perpendicular?
- Are the morning lines on the west side of the sundial?
- Does the skewer point to the SCP and line up north / south?
- Are the summer hour lines on the south facing side of the dial plate?
- Does the angle the skewer makes to the flat surface reflect the latitude (37 degrees for Melbourne)?



Figure 7. Hour lines for winter.





Figure 8. Correct alignment for sundial.

Finally, take a moment to reflect on your achievements. Using a simple plywood square, the student has made a time keeper that will be accurate for thousands of years without any need for adjustment. The photograph in Figure 9 shows a group of Nepalese student monks aligning their wooden sundials to reflect their northern latitude (28.8 degrees N) and unique time zone (UTC + 5:45). A close examination of the photograph shows that the students were working on the sundials during the northern winter. A small or subtle correction to sundial time to bring it in line with civil time requires application of the Equation of Time (EOT). This graphical information takes into account all the speed bumps of the earth's tilt and its elliptical orbit of the sun. The US Naval Observatory (n.d.) has an excellent discussion of matters relating to the EOT. It also provides a comprehensive site map of astronomical applications that would impress ancient or modern Babylonians alike.



Figure 9. Nepalese student monks aligning their wooden sundials.

Teachers looking for further reading support can find all the theory and formulae for sundial construction in Folkard and Ward (1996). This book has the added benefit of an Australian perspective. The authors have been commissioned to construct many well known sundials. A student friendly text with actual 3D models and pop ups of sundials was created by Mitsumasa Anno (1986), a famous Japanese illustrator. Anno taught maths for ten years before embarking on his career as an illustrator. Anno's text provides a sundial model for every 10 degrees of latitude from the poles to the equator.



Location	Longitude	Latitude	Noon mark AEST
Mallacoota	149.75	37.55	12:00
Sale	147	38.1	12:12
Melbourne	145	37.5	12:20
Ballarat	143.85	37.56	12:25
Portland	141.6	38.36	12:33
Mildura	142.1	34.2	12:32

*Table 1. The longitude and latitude of major Victorian towns showing the adjusted noon mark for each location.* 

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**Summary papers** 

# **Targeting student understanding**

Sarah Ferguson, Clairvaux Catholic School and Australian Catholic University



As teachers of mathematics we are aiming for the centre of our target which is student understanding of mathematics. To have learnt mathematics well is to have built understanding of the connections, the concepts and the relationships in mathematics that allow us to solve problems. The middle ring is the teaching actions or pedagogy that we can draw on to create conditions for students to build understanding of key concepts. The outer ring is concerned with mathematics content – what are the big ideas and concepts that students need to build understanding in. Teaching is the mediating factor between the content and the students' understanding.

# WHAT IS UNDERSTANDING OF MATHEMATICS?

Richard Skemp first described two meanings of 'understanding' in mathematics in 1976. 'Instrumental understanding' which is when the learner builds a 'rules without reason' knowledge of mathematics such as applying a procedure or process to a mathematical question to achieve a correct result but not knowing why they apply such a process. 'Relational understanding' is when the learner knows both 'what to do and why' (p. 20).

We can look at these two types of understanding with the example of area. A learner can know the formula for area is length multiplied by width, an instrumental understanding. They can apply this formula to most area questions posed in school and achieve a correct answer. However with an instrumental understanding only, they cannot explain why the formula is length times width. This becomes a problem if the question they are given does not follow the normal format or if the formula is forgotten or misapplied. For example if it is confused with the formula for perimeter and the length and width are added, a learner with instrumental understanding will not be able to recognise that their answer does not make sense or how to correct it. A learner who has a relational understanding of area will know that length multiplied by width is the formula because area is multiplicative and can be thought of as equal rows of squares covering a surface. We can total these squares by finding out how many in a row (length) and how many rows (width) then multiply these. This is a more robust understanding of area and means that adding the length and width (perimeter formula) will not make sense to this learner as they understand area is covering the whole surface not just the edges. Open ended tasks or complex problems can be solved either by using the formula for area backed up by relational understanding or informal or intuitive strategies that have been developed as a result of this understanding of area.

Skemp (1976) asked the question which of these types of understanding is better. His immediate answer is 'relational understanding' but he goes on to explain that recognizing that instrumental understanding and relational understanding represented in education is an important consideration for educators. Skemp proposes that teaching mathematics in these two distinct ways is really two different types of mathematics. If there is a mis-match between mathematics taught instrumentally and mathematics taught relationally problems can arise. Students might want to learn instrumentally ('just tell me which process to follow') when teachers want them to understand.



Or students might want to learn relationally ('but why do we use this process?') and the teacher wants to teach instrumentally ('just follow the rule/ process, you won't understand why and don't need to know')

Educators, students, parents and schools can have instrumental understanding or relational understanding at the centre of their target for mathematics teaching and learning. Sometimes the centre of our target may change due to circumstances such as student pressures, parent expectations, school requirements or school system requirements. The centre of the target for our mathematics teaching, the bullseye that we are aiming for is connected to our beliefs about mathematics teaching and learning. This conference will explore ways to 'create and embed deeper understanding' – relational understanding. Examining how we keep this as the 'bullseye' of our mathematics teaching will be a key focus for delegates. Reflecting on our own beliefs regarding the target of our teaching and hypothesizing about the beliefs of our students, and their parents is a good start for a conversation about understanding and what it means in mathematics.

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# **Desmos animations**

Roger Walter, Education consultant, Editor of Vinculum (MAV)

Desmos is a new, free, highly versatile and very user friendly on-line graphics calculator. After introducing Desmos, this session will look at some of the possibilities of its ability to animate points and graphs, through varying the value of constants.

#### INTRODUCTION

For those unfamiliar with Desmos, I will firstly take a quick look at Desmos and what it can do in general, then focus on its capacity for animation of points and graphs.

Desmos is fundamentally an on-line graphing calculator, with added features. The main points are as follows.

- 1. It is free for schools to use. Having lived within subject faculty budgets, this is not unimportant. Some things are free, and worth even less, but I think you will find Desmos a useful educational tool.
- 2. It is developed by educators whose objectives (I believe) are to provide useful tools which facilitate learning.
- 3. It is very user friendly. This is important when navigating menus and finding the exact keywords for a task on a CAS or graphing calculator can be more difficult than doing the mathematics.
- 4. It is not CAS. The focus is graphing, with most functions available and a liberal dose of statistics thrown in.
- 5. It can be used on a computer, tablet or a smartphone. For a computer, go to www.desmos.com/calculator; for a tablet or smartphone, download the app.

#### **BASIC SKILLS**

To draw a graph, just type in its equation, either as

 $y = \dots$  or  $f(x) = \dots$ . You can use y for as many functions as you like. If you want to draw  $y = x^2$  for  $-3 \le x \le 3$ , enter the following in a formula box on the left. To get  $x^2$ , type 'x^2'. The power is automatically raised. Easy. You can restrict the values of x or y or both. You can turn a graph on or off by clicking the coloured circle on the left. You can change the graph's colour or line type by holding on this circle. For tablets and smartphones, you can change the size of the coordinate axes by using pinch and zoom. Otherwise use the tools on the top left of the screen.

 $y = x^2 \{-3 \le x \le 3\}$ 

Constants are entered as c = 2. Without asking you, Desmos will automatically add a slider, with default values ranging from -10 to 10, changing in steps of 0.1. This can be changed by clicking one of the numbers in the range. If you didn't want a slider, don't worry. Just don't use it.

After entering c = 2, enter  $y = (x + c)^2$  (in the box below). You can drag the point on the slider and see the effect changing the value of *c* has on the graph. An obvious activity is to enter constants *a*, *b* and *c* and observe how these impact  $y = a(x + b)^2 + c$ , or transformations of any function. But don't stop there. How does changing *a*, *b* and *c* impact  $y = ax^2 + bx + c$ ? Get your students to do it and then explain their answers. The effect of *c* should be obvious, but what about *a* and *b*?

#### ANIMATIONS

On the left of every box with a slider is something which looks like a play button. If you click it the slider will move backwards and forwards. To make it move forwards only, click the two arrows, which will then both point the same way. Clicking the arrows again makes it move in both directions. You cannot make it move backwards only. You can speed it up or slow it down by clicking the arrowheads.



The next section will look at three possible activities.

#### **ACTIVITY 1: PROJECTILE MOTION**

Desmos can also use parametric specification to draw graphs. This is useful in projectile motion, where velocity and displacement are resolved into vertical and horizontal components.

If a projectile is launched at an angle  $\alpha$  with an initial speed V, then, if  $g = 9.8 \text{ m/s}^2$ , we have

 $x = Vt \cos \alpha$ 

 $y = Vt \sin \alpha - 4.9t^2$ 

Before we use Desmos we need to know the limits for t.

 $Vt \sin \alpha - 4.9t^2 = 0$  $t = 0 \text{ or } t = \frac{V \sin \alpha}{4.9}$ 

When V = 20 and  $\alpha = 30$ ,  $\frac{V \sin \alpha}{4.9} \approx 3.53$  $Vt \sin \alpha - 4.9t^2 \ge 0$  when  $0 \le t \le 2.04$ 

You can get Desmos to calculate this automatically by entering  $\frac{V \sin \alpha}{4.9}$  into a Desmos formula box (after giving values to *V* and  $\alpha$ ). Use the forward slash for division.

Now we open Desmos and make sure it is set to degrees. Define formulae in boxes as below:

 $\alpha = 60$  (or any angle) Type 'alpha' to get  $\alpha$ .

V = 20 (the initial speed)

 $\frac{V\sin\alpha}{4.9}$  (optional)

 $(Vt\cos\alpha, Vt\sin\alpha - 4.9t^2)$ 

```
0 \le t \le 3.53
```

 $s = 0, 0 \le s \le 3.53$  step 0.01.

If you don't like using *s* for time, use  $t_1$  (type 't1').

 $(Vs \cos \alpha, Vs \sin \alpha - 4.9s^2)$ 

 $(Vs\cos\alpha,0)$ 

 $(0, Vs\sin\alpha - 4.9s^2)$ 

Set the animation to forwards only as before, and slow it to a suitable speed. You can clearly see the vertical component rise and fall while the horizontal component moves at a constant speed. I'm sure you will agree it is a good way for students to understand how projectile motion works, particularly those who are visually oriented.



*Figure 1. Demonstration of projectile motion using Desmos.* 



# ACTIVITY 2: CIRCLES, ELLIPSES AND LISSAJOUS FIGURES

This part was born when John Gough wrote an article for *Vinculum* about the ABC logo, an example of a Lissajous figure.

It is probably a little too complex to start with the ABC logo, so we can start our students with a circle. They should be able to progress from there, if the right questions are asked.

The parametric equations for the circle  $x^2 + y^2 = a^2$  are simple.

 $x = a \cos t$ 

 $y = a \sin t$ 

If we open Desmos (www.desmos.com/calculator), we can define a value for a in the top left box, say a = 3. Before going further we need to check we are in radian mode, by clicking the spanner (top right) and making sure 'Radians' is in green. If you decide to work in degrees, change  $2\pi$  to 360 in the appropriate places.

Underneath this, enter a point in parametric from shown below using *t*, the default parameter for Desmos.

 $(a\cos t, a\sin t)$ 

The default range of values for t is  $0 \le t \le 1$ . Change this to  $0 \le t \le 2\pi$ . by clicking on '1' and typing '2pi'. (Typing 'pi' produces  $\pi$ .) This will produce the full circle.

To animate the circle, define a new variable *s* below your parametric specification. It can equal any positive number less than  $2\pi$ , say s = 2 and change its limits to 0 to  $2\pi$  step 0.01 (or any small amount).

Under this, enter three points, each underneath each other.

On the circle:  $(a \cos s, a \sin s)$ 

On the *x*-axis:  $(a \cos s, 0)$ 

On the y-axis:  $(0, a \sin s)$ 

This should produce the circle as shown in Figure 3. The colours may be different and can be changed by long holding on the colour on the left of the relevant point or line, then selecting the colour you want. Desmos does not treat *s* as a parametric variable because it has already been defined to have a value.

Now the fun begins. On the left of the box defining *s* is what looks like a play button for a video. Clicking this will begin the animation. See Figure 4.



*Figure 4. Clicking the circle on the left will begin the animation.* 





Figure 2. The ABC logo. The two purple points represent the varying x and y co-ordinates as the green point traces out the shape.



Figure 3. Producing a circle using Desmos.

You may notice that the 'play' button is now a 'pause' button. See Figure 5. Clicking this will pause the animation.

# *Figure 5. The 'control panel' for animation of the variable s.*

You may feel the animation is too fast, and it is probably not ideal that it goes back and forth. On the top of figure 5, towards the left, are two arrows pointing in two directions, indicating that the values of *s* are both rising to the maximum, then falling to the minimum, and so on. If you click this, *s* will increase to the maximum, then re-start at 0 and increase again. The arrows show this by both pointing forwards. On the right are ' (slower),  $1 \times$  (the speed) and ' (faster). Click ' to slow down and ' to speed up the animation.

If the speed is right, students should be able to visualise what is happening with the circle. The first question to ask is either 'How can we make an ellipse?' or 'What would happen if the coefficients of cos t and sin t were different?' They should first make a prediction, and then check it by introducing a new variable b. They simply change every 'a sin' to 'b sin' and should come up with the standard ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

## Lissajous figures

If we make *a* approximately twice as long as *b*, we can ask, 'Since *x* has twice as far as *y* to go, what if we made *y* do 2 repeats for every cycle of *x*. What would the shape look like?'

An effective conceptual way to approach this would be to break up the cycle into quarters, corresponding to quadrants. If the point starts on the far right, *x* will move from a to 0 (one quadrant). *y* will start at 0 and either move up to *b* and back down to 0, or down to -b and back up to 0. Some students will realise that *x* and *y* will move faster when they're nearer the origin. This should also be conceptual, since near the extremes you would expect them to slow down before changing direction. Students therefore should predict something like Figure 6 for this part of the motion.





Figure 6. How a student may predict part of a Lissajous figure.



The rest of the shape may be predicted using the same method. See Figure 7.

The prediction can be tested. First set a = 4 and b = 2. We can make y move twice as fast as x by changing every  $b \sin t$  to  $b \sin 2t$  and every  $b \sin s$  to  $b \sin 2s$ . The boxes, from top to bottom, should now read a = 4; ( $a \cos t$ ,  $b \sin 2t$ ); s = ... 0 to  $2\pi$  step 0.01;

 $(a \cos s, b \sin 2s); (a \cos s, 0); (0, b \sin 2s); b = 2.$ 

As the animation runs, students should enjoy seeing their predictions work. *y* clearly repeats two cycles for every cycle of *x*, and the point moves in the way predicted, producing 'infinite' delight.

From here it is not too hard to work out what would happen if *y* moved 3 times as fast as *x*. While *x* moves from the right extremity to the origin (one quadrant), *y* will move three quadrants, from the origin up to *b*, back to the origin and down to -b. the curve will not go through the origin, and it should now be relatively easy to predict the shape of the curve (the ABC logo).



This can be confirmed by changing every 2t and 2s to 3t and 3s and running the animation.

This can gradually be extended to more complex scenarios, for example when y completes three quadrants as x completes two. And the interesting thing is that this is exactly how the point moves in the oscilloscope, with the x and y values corresponding to the horizontal and vertical voltage inputs.

Desmos can be used to illustrate other curves defined by parametric equations, but I will only look at these two.

#### ACTIVITY 3: PULLING A GRAPH THROUGH INFINITY

This idea fascinated me when I was at school and after leaving school, I drew a large number of these graphs, intending to animate them using my movie camera. Then I discovered I could do the same thing as follows.

Enter a constant a = 10. We will animate this constant.

Enter the graph  $y = \frac{10}{x^2 + a}$ .

A fraction is automatically created by typing '/'. If you want an expression for the numerator, highlight the expression before typing '/'.

Slowly drag a backwards, until you get to about 0.5. A hump starts to rise. As a approaches zero, the hump, a local maximum approaches positive infinity. What do you say when x = 0? I considered there would be an infinitesimally 'flat' portion of the graph at  $(0, \infty)$ .

What happens as *a* continues to decrease, becoming negative? You could say that the local maximum continues to rise, having passed through infinity, a bit like infinite modular arithmetic. This may or may not be valid, but it should stimulate the imagination. For negative values of *a*, you could also imagine the curve approaching infinity, and continuing to increase 'through' infinity, then rising through the negative values to reach a 'maximum'. There is also a continuity of the graph as a approaches infinity (giving y = 0), then rises from  $-\infty$  through the negative numbers. (Alternatively, a can approach  $-\infty$  and rise from  $+\infty$  through the positive numbers.)

#### CONCLUSION

There are no doubt many more and interesting applications of Desmos sliders, but I hope these activities will give you a taste for more. The great thing is that they are so easy to produce.

Any questions?

#### REFERENCES

Gough, J. (2017). The changeable ABC logo. Vinculum 54(3), 8-9.

McCarty, C., & Hannah, S. (2017). How Desmos trumps CAS calculators. Vinculum 54(2), 7-9.

Walter, R. (2017). Parametric animations. Vinculum 54(2), 10-13.

#### **Desmos URLs**

Start using Desmos: www.desmos.com/calculator

You can view the exercises discussed in this article at www.desmos.com/calculator/rhuakfaxum (ABC logo), www.desmos.com/calculator/rzadept0ot (infinity), www.desmos.com/calculator/guy7yexu9u (projectile motion).



# **Problem performing to problem solving and the effect on teachers' teaching**

# Charlotte Wilkinson, NCWilkinsons Ltd (The Wilkie Way) with Awapuni School Gisborne NZ

'In a range of meaningful contexts students will solve problems and model situations that require them to .....' This statement heads all achievement objectives in the Mathematics & Statistics learning area of the New Zealand curriculum document.

I became involved with the school to ensure the full implementation of the intent of the curriculum document. To use the document and further research evidence to move the teachers from doing word problems requiring the application of a particular strategy to actual problem solving using rich learning tasks.

#### A SCHOOL'S PROFESSIONAL LEARNING JOURNEY

Background: All staff at the school had been involved at some point with the Numeracy Project – some as a professional learning but most as a mathematics programme. The lead teacher had undertaken MST (Maths Specialist Teacher) university papers over the previous two years and part of the journey was to support her in her role as a lead teacher in the school.

#### INITIAL BARRIERS TO PROFESSIONAL LEARNING

- Teachers' confidence with mathematics
- Teachers' conceptual content knowledge of the mathematics
- Students' (and teachers') perception that the answer is the main goal. This leads to the belief that speed and efficiency are more important than reasoning and justifying- (often unconsciously on the part of the teacher).
- Student tendency to act as problem performers which prevented them from considering problem solving strategies such as draw a picture or act it out that would enable them to explore a problem.

#### **GETTING STARTED**

Initial modelling by myself and by the in school lead teacher enabled teachers to recognise that their students were capable of thinking with support at much higher levels and tackling much harder problems than they had been giving their students.

The feedback from teachers noted the need for much longer wait times before jumping in to assist students. The asking of questions to keep the students thinking was recognized as necessary but most teachers felt they did not have the conceptual knowledge to ask a question to develop a conceptual understanding. Most questions asked by teachers focused the thinking on a particular strategy for calculating the answer rather than exploring the problem and developing mathematical reasoning and justification

It was obvious that building teacher conceptual knowledge was going to be essential to be able to use rich learning tasks to develop conceptual understanding in the students.

# USING STUDENT EVIDENCE

For teachers to use evidence to improve teaching and learning in their classrooms they need information about what their students know and can do. At this point we hit an unexpected problem. The students could not provide us with recorded evidence of their thinking and many students oral evidence of their thinking was the recital of a rehearsed procedure to get an answer to an arithmetic word problem. Presented with a more complex problem or rich learning task they had no idea where to start.

'I don't think I will be able to do what you ask as my thinking is messy.' (Jasher aged 10)



A term was given over to the explicit teaching of recording thinking appropriate to a student's development. The class modelling book began to reflect using written recording to assist thinking – from drawing a picture to photographs of materials used to model the problem and using number lines and bar models to represent relationships between numbers in a problem. Students were expected to keep a record of their thinking using whiteboards and paper and pen.

The time invested in focusing on reading and writing mathematics has been exceedingly worthwhile. The difference between students who have come through the school, and imports from other schools shows in their attitude to mathematics as being another form of literacy. They have another language in which to think and communicate mathematically. The students have been empowered to use mathematics, their original perceptions of mathematics being about getting an answer (at speed) has been altered.

#### **RAISING STUDENT ACHIEVEMENT**

Using student evidence to inform teachers about their own practice and its impact on student learning required a mindset change from the teachers. Teachers are used to looking at student work as a summative assessment, to inform them of where a student is performing against a specific measure (Curriculum). The student work should also give the teacher evidence of where the student needs to go next using the same measure.

Using evidence for improvement in practice requires the mindset to extend to examining how the classroom practice and teacher knowledge is influencing the trends evident in the students' work.

The interpretation and use of evidence about student learning for guiding and directing teaching requires a mindset shift towards professional learning from evidence and a new set of skills.

To enable this process, teachers need to ask, with the help of relevant experts, what knowledge and skills they require in order to address students' identified needs, through some more detailed questions. How have we contributed to existing student outcomes? What do we already know that we can use to promote improved outcomes for students? What do we need to learn to do to promote these outcomes more effectively? What sources of evidence or knowledge can we utilize? (Timperley 2015)

Achieving the mindset shift in the teachers is a process that occurs over time. For some teachers this has proved to be straight forward but for others we have still not achieved the shift. The significant trait we have noted in the non-movers is in teachers who are highly competitive or are judged to be 'excellent' teachers and continually seek (and receive) praise from school leaders. This reflects Carol Dweck's work on growth mindsets.

#### ACHIEVING THE MINDSET SHIFT

Each term the school has created a rich learning task to fit with the mathematical content of the long term coverage plan. The tasks are designed to cover curriculum levels one to three with extension into level four. The tasks were initially to be used as assessment moderation tasks with teachers bringing annotated samples of student work to a moderation meeting to ensure teacher interpretations of curriculum levels were consistent. This would ensure that overall teacher judgments for National Standards would be more valid within the school.

In the first moderation meetings after using the student work to reflect the ability of the students, teachers were guided to relook at the student work to search for any trends. For example: in a fractions task, it was noted that students struggled with the recording of fractions greater than one and not a single student had recorded as a mixed number. In discussion no teacher had modelled or even discussed a mixed number and some teachers were not aware themselves that a fraction can have the numerator larger than the denominator. The implication from this small piece of evidence has led to the need for some in school teacher knowledge workshops on fractions.

The initial teacher discussions about student work focused on surface features but over time and by repeating the moderation process with different learning tasks the quality and depth of teacher discussion has changed significantly.

It is only by looking back and reading the records of the discussion that I really notice the significant change. Discussion of the mathematics is now at a conceptual level. Teachers are critical of their own conceptual knowledge.



'You don't know what you don't know '(a teacher's comment at one meeting)

The teacher inquiry process is now in progress and gaining momentum with teachers much more focused on the teaching-learning relationship and recognizing when their own lack of content knowledge puts ceilings on students learning. They actively seek learning opportunities for themselves. They are engaging in professional reading to build personal conceptual content knowledge and applying new learning with critical analysis in their classrooms.

This has placed a higher demand on the lead teacher of mathematics in the school who needs to be responsive to a higher level and greater quantity of teacher questions and requests. School leadership will need to consider the role of the lead teacher in maintaining and developing all teachers on their professional journey.

#### REFERENCES

Timperley H (2015) Using Evidence in the Classroom for Professional Learning, *Paper Presented to the Ontario Education Research Symposium* 



# Assessing young children's mathematical strengths

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# YOUNG CHILDREN'S MATHEMATICAL COMPETENCIES

Children begin developing mathematical skills from a very young age. As Geist (2009, p. 12) says, we should 'think of children as competent mathematicians'. International research has shown that young children demonstrate competence in regards to a range of mathematical concepts and processes, including number and counting, geometry, dimensions and proportions, location, and problem solving (Björklund, 2008; Reikerås, Løge, & Knivsberg, 2012). Both Australian and international research has shown that young children explore a range of mathematical concepts and processes prior to starting school (for example, Gervasoni & Perry, 2015; Sarama & Clements, 2015). The seminal Australian study, the *Early Numeracy Research Project* (see, for example, Clarke, Clarke, & Cheeseman, 2006) investigated the mathematical knowledge of over 1400 children in their first year of primary school. An important finding from the study was that much of the content which formed the mathematics curriculum for the first year of school was already understood clearly by many children on arrival at primary school (Clarke et al., 2006), a finding echoed in several other studies, both in Australia (for example, Gervasoni & Perry, 2015; MacDonald, 2010) and internationally (for example, Aubrey, 1993; Wright, 1994).

Of course, there will be substantial variance in the mathematical competencies children develop prior to school (Peter-Koop & Kollhoff, 2015), and both standardised tests and experimental tasks reveal marked individual differences in children's mathematical knowledge by the time children enter preschool (Levine et al., 2010). Research has emphasised the importance of early mathematical learning, with links being drawn between early mathematics and later achievement (MacDonald & Carmichael, 2016; Watts, Duncan, Siegler, & Davis-Kean, 2014). This research notes, in particular, the predictive power of mathematical knowledge at school entry for later mathematical achievement (Duncan et al., 2007). It has been found that children who enter primary school with high levels of mathematical knowledge maintain these high levels of mathematical skill throughout, at least, their primary school education (Baroody, 2000; Klibanoff, 2006). As such, it is important to consider the mathematical competencies of children in the early years in order to understand the foundation upon which subsequent mathematics education should build.

#### A STRENGTHS APPROACH TO ASSESSMENT

In order to make children's competencies visible, we need to utilise assessment strategies that attend to young children's capacities for sharing their unique mathematical strengths. 'Strengths' can be defined as people's intellectual, physical and interpersonal skills, capacities, interests and motivations (Mallucio, 1981, cited in McCashen, 2005). A person's strengths can also include the resources in their environment, such as family, friends, neighbours, colleagues, material resources, and so on (McCashen, 2005). Strengths-based assessment emphasises starting from what children *can* do, rather than what they *cannot*. A deficit view of children's mathematical ability can have self-reinforcing negative effects, whereby negative expectations of children result in negative experiences in mathematics education. This can be described as a 'deficit cycle' – an inadvertent process of disempowerment (McCashen, 2005). In contrast, a focus on strengths emphasises children's competencies and resources, which can be utilised to bring about positive change in their mathematics education experiences. This can be described as a 'competency cycle' – a process of creating positive expectations and opening the way for the development of new competencies (McCashen, 2005).

Assessment in mathematics education can be considered as 'the process of gathering evidence about a student's knowledge of, ability to use, and disposition toward, mathematics and of making inferences from that evidence for a variety of purposes' (National Council of Teachers of Mathematics, 1995, p. 3). Put another way, assessment is about 'observing, recording, and otherwise documenting the work that children do and how they do it' (Copley, 1999 p. 183). Assessment is essentially about making children's learning visible, and there are a number of strengths-based ways in which this might occur. Some example strategies are outlined below.

#### CHILDREN'S DRAWINGS

A child's drawing can be a useful source of information about their mathematics learning. For example, the 'draw a clock' task is an open-ended drawing task I have used many times with very young children both as part of my teaching



and as part of my research (see, for example, MacDonald, 2013; Smith & MacDonald, 2009). This drawing task provides a useful method of assessing children's understandings about time, and the measurement and representation of time. Children are given a blank piece of paper and are simply asked to 'draw a clock' – no further direction is given. This drawing activity gives children the opportunity to share what they know about clocks and the representation of time, including things like sequencing numbers and unit iteration. Some children also take the opportunity to record different 'times' that they know on the clock. Children may choose to draw a range of different clocks, including both digital and analogue, watches, cuckoo clocks, alarms clocks, and so forth. Many children also include an accompanying 'narrative' which reveals their different experiences with clocks in a range of contexts, and gives insight as to how their understandings have developed. Indeed, drawings become more powerful forms of assessment when the child's comments are added to the drawing and the drawing and comments are considered



Figure 1. Ethan's clock.

as a whole. For example, 5-year-old Ethan described his clock drawing (Figure 1) as follows: 'It is 6 o'clock because one hand is on the twelve and one hand is on the six. I only know times with a twelve in them. I have a little Shrek clock in my bedroom and it looks like that. [My brother] helped me learn some times.'

#### CHILDREN'S PHOTOGRAPHS

Selective photographs which capture children as they are engaged in mathematical learning can be valuable records (Aitken et al., 2012). Children can participate in taking the photographs and choosing what to photograph. Furthermore, the process of 'photo elicitation' (Prosser & Burke, 2008), whereby photographs are used as stimulus for further discussion, can reveal children's own perspectives of their mathematical activity and how it has contributed to their developing mathematical understandings. Children can also use cameras and tablets to record their own mathematical learning and create their own mathematical learning stories (Arthur et al., 2015).

#### LEARNING STORIES

Learning stories, pioneered by Margaret Carr in New Zealand for use with the *Te Whariki* early childhood curriculum, are qualitative snapshots, recorded as structured written narratives, often with accompanying photographs, that document and communicate the context and complexity of children's learning (Carr, 2001). Learning stories focus on children's strengths, and are positioned within a social and cultural context (Arthur et al., 2015). Learning stories may differ in length, the amount of detail included, whether they focus on one child or a group, and their structure (Hunting, Mousley, & Perry, 2012). In general, the construction of learning stories involves *noticing*, *reflecting*, and *responding* (Aitken et al., 2012). A mathematics-focused learning story is an opportunity to attend closely to the ways in which mathematical concepts being developed.

#### CONVERSATIONS

A conversation with a child about their mathematical explorations can provide in-depth evidence of learning (Aitken et al., 2012). Open-ended questioning during conversations invites the child to share their ideas and understandings. When recorded, transcripts of conversations can provide many insights into children's mathematical thinking and uses of mathematical language (Arthur et al., 2015). Conversations can also be key to understanding the development of alternate conceptions in mathematics, as they provide opportunities to consult with individual students about their thought processes and encourage children to explain their mathematical thinking (MacDonald, 2008).

#### **FINAL THOUGHTS**

Strengths approaches to mathematics assessment help to give value to the diverse mathematical experiences and understandings of young children, and assist teachers to see the different ways in which children construct and demonstrate their mathematical competencies. Given the compelling evidence of the relationship between mathematics at the time of school entry and later school achievement, it is important for teachers to use strengths-based assessment strategies to identify the mathematical competencies of children in order to understand the directions in which children's mathematics education should proceed; i.e., *knowing where children are going to by knowing where they are coming from.* 



# **The Bisection Method and Newton's Method**

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In the current VCE Mathematics Study Design (2016 - 2018) for the subject Mathematical Methods, additional algebra material has been added as follows:

Numerical approximation of roots of simple polynomial functions using the bisection method and Numerical approximation of roots of cubic polynomial functions using Newton's method.

The purpose of this article is to discuss these two methods and to illustrate their application with examples. In addition, exercises are provided together with answers.

#### THE BISECTION METHOD

The bisection method is a root-finding method that repeatedly bisects an interval and then selects a sub-interval in which the root must lie. Consider the function y = f(x) which is continuous on the closed interval [a, b]. If f(a) f(b) < 0, the function changes sign on the interval (a, b) and, therefore, has a root in the interval. The bisection method uses this idea in the following way.

If f(a) f(b) < 0, then we compute  $c = \frac{1}{2}(a + b)$  and test whether f(a) f(c) < 0.

If this is so, then f(x) has a root in [a, c]. So *c* is now reassigned as *b* and we start again with the new interval [a, b] which is now half as large as the original interval. If, on the other hand, f(a) f(c) > 0, then f(c) f(b) < 0 and *c* is now reassigned as *a*. In either case a new interval trapping the root has been found. The process can then be repeated until the required level of accuracy has been attained. Figures 1 and 2 illustrate the two cases discussed assuming f(a) > 0 and f(b) < 0. The bisection method is sometimes referred to as the method of interval halving.



Figure 1. The bisection method selects the left subinterval. The root is to the left of x = c and f(a) f(c) < 0.

#### EXAMPLE 1

Consider the continuous function  $f(x) = x^3 + x - 1$ .

- (a) Evaluate f(0).
- (b) Evaluate f(1).
- (c) Determine the sign of f(0) f(1).
- (d) What conclusion can you draw from (c)?

(e) Use the bisection method to obtain the root of y = f(x) to four decimal places.





Figure 2. The bisection method selects the right subinterval. The root is to the right of x = c and f(c) f(b) < 0.

а	b	<u>a+b</u> 2	f(a)	f(b)	$f\left(\frac{a+b}{2}\right)$
0	1	0.5	-1	1	-0.375
0.5	1	0.75	-0.375	1	0.1719
0.5	0.75	0.625	-0.375	0.1719	-0.1309
0.625	0.75	0.6875	-0.1309	0.1719	0.0125
0.625	0.6875	0.6563	-0.1309	0.0125	-0.0610
0.6563	0.6875	0.6719	-0.0610	0.0125	-0.0248
0.6719	0.6875	0.6797	-0.0248	0.0125	-0.0063
0.6797	0.6875	0.6836	-0.0063	0.0125	0.0031
0.6797	0.6836	0.6817	-0.0063	0.0031	-0.0015
0.6817	0.6836	0.6827	-0.0015	0.0031	0.0009
0.6817	0.6827	0.6822	-0.0015	0.0009	-0.0003
0.6822	0.6827	0.6825	-0.0003	0.0009	0.0004
0.6822	0.6825	0.6824	-0.0003	0.0004	0.0002
0.6822	0.6824	0.6823	-0.0003	0.0002	-0.0001
0.6823	0.6824	0.68335	-0.0001	0.0002	0.0001
0.6823	0.68335	0.68233	-0.0001	0.0001	0.00001
0.6823	0.68233	0.68232	-0.0001	0.00001	-0.00002
0.68233	0.68232	0.68233	0.00001	-0.00002	0.00001
0.68232	0.68233	0.68233	-0.00002	0.00001	0.00001

Table 1. After 17 iterations, we can safely take  $x_{root}$  to be 0.6823 to four decimal places.

## Solution to example 1

(a)	f(0)	=	-1
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(b) f(1) = 1

(c)  $f(0)f(1) = (-1) \times (1) = -1 < 0$ , i.e., the sign of f(0)f(1) is negative.

(d)  $0 < x_{root} < 1$ 

(e) See Table 1.

#### EXAMPLE 2

Consider the function  $f(x) = x^3 + 10x^2 + 8x - 50$ .

- (a) Evaluate f(1).
- (b) Evaluate f(2).

(c) Evaluate f(1)f(2).

(d) What is the sign of f(1)f(2)?

(e) What conclusion can you draw concerning your answer to part (d)?

(f) Use the bisection method to obtain the positive root of

y = f(x) to four decimal places.

а	b	<u>a+b</u> 2	f(a)	f(b)	$f\left(\frac{a+b}{2}\right)$
1	2	1.5	-31	14	-12.125
1.5	2	1.75	-12.125	14	-0.0156
1.75	2	1.875	-0.0156	14	6.7480
1.75	1.875	1.8125	-0.0156	6.7480	3.3059
1.75	1.8125	1.7813	-0.0156	3.3059	1.6328
1.75	1.7813	1.7657	-0.0156	1.6328	0.8075
1.75	1.7657	1.7579	-0.0156	0.8074	0.3976
1.75	1.7579	1.7540	-0.0156	0.3976	0.1934
1.75	1.7540	1.7520	-0.0156	0.1934	0.0888
1.75	1.7520	1.7510	-0.0156	0.0888	0.0366
1.75	1.7510	1.7505	-0.0156	0.0366	0.0105
1.75	1.7505	1.75025	-0.0156	0.0105	-0.0026
1.75025	1.7505	1.75038	-0.0026	0.0105	0.0042
1.75025	1.75038	1.75032	-0.0026	0.0042	0.0012
1.75025	1.75032	1.75029	-0.0026	0.0011	-0.0005
1.75029	1.75032	1.75031	-0.0005	0.0011	0.0006
1.75029	1.75031	1.75030	-0.0005	0.0006	0.00003
1.75029	1.75030	1.750295	-0.0005	0.0003	-0.0002
1.750295	1.750295	1.750295	-0.0002	-0.0002	-0.0002

Table 2. After 17 iterations we can confidently deduce that  $x_{root}$  is 1.7503 to four decimal places.



# Solution to example 2

(a) f(1) = -31(b) f(2) = 14(c)  $f(1) f(2) = -31 \times 14 = -434 < 0$ (d) The sign of f(1) f(2) is negative. (e)  $1 < x_{root} < 2$ (f) See Table 2. THE BISECTION METHOD EXERCISES

## Question 1

Use the bisection method to find the root of f(x) = 5x - 9 to one decimal place. Take a = 1 and b = 3.

[ANSWER:  $x_{root} = 1.8$ ]

#### Question 2

Using the bisection method, obtain the greater positive root of  $f(x) = x^2 - 6x + 7$  to four decimal places.

[ANSWER:  $x_{root} = 4.4142$ ]

#### Question 3

Obtain the greatest positive root to four decimal places of  $f(x) = -x^3 + 9x^2 - 20x + 6$  using the bisection method over the interval [5,6].

[ANSWER:  $x_{root} = 5.6458$ ]

#### Question 4

Use the bisection method to obtain the co-ordinates of the point of intersection, to an accuracy of four decimal places, of  $y = -x^3 + 4x^2 - 3x + 2$  and y = 2x - 7 over the interval [2,4].

[ANSWER: (*x*,*y*) = (2.8637,2.7274)]

#### Question 5

Obtain, to an accuracy of four decimal places, the co-ordinates of the point of intersection of the cubic functions  $y = 2(x-1)^3$  and  $y = -3(x-2)^3$  using the bisection method over the interval [1,2].

[ANSWER: (*x*,*y*) = (1.5337,0.3041)]

#### **NEWTON'S METHOD**

Newton's method, also referred to as the Newton-Raphson Iteration Technique, involves less iterations than the bisection method since its convergence is quadratic rather than linear.

The basic idea is that if  $x_0$  is an approximation to the root,  $x_{root}$ , of the equation f(x) = 0, then a closer approximation will be given by  $x_1$  where the tangent to the graph at  $x = x_0$  cuts the *x*-axis at  $x = x_1$  as shown in Figure 3.

Using the definition of derivative at  $x = x_0$ 



Figure 3. Newton's method for finding roots.



$$f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$
  

$$\therefore (x_0 - x_1) f'(x_0) = f(x_0)$$
  

$$\therefore (x_0 - x_1) = \frac{f(x_0)}{f'(x_0)}$$
  

$$\therefore x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$
  

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \text{ Eqn 1}$$

More generally, we may write

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
 Eqn 2  
where  $n = 0, 1, 2, 3....$ 

This equation is known formally as the Newton-Raphson Iteration Procedure for obtaining an approximation to the root of f(x) = 0.

#### EXAMPLE 1

Use Newton's method to find the positive root of  $f(x) = x^3 - 6x^2 - 9x - 1$  to an accuracy of four decimal places. Take the initial guess to be  $x_0 = 6$ .

## Solution to example 1

 $f(x) = x^3 - 6x^2 - 9x - 1$ 

$$f'(x) = 3x^2 - 12x - 9 = 3(x^2 - 4x - 3)$$

We shall take the initial guess to be  $x_0 = 6$ .

In accord with Equation (1), we define

$$k(x) = x - \frac{f(x)}{\frac{d}{dx}f(x)}$$

Initial guess is  $x_0 = 6$ .

We now carry out the following procedure using the Casio ClassPad II CAS calculator.

Define  $f(x) = x^3 - 6x^2 - 9x - 1$ 

Define

$$k(x) = x - \frac{f(x)}{\frac{d}{dx}f(x)}$$

6	EXE	
k(ans)	EXE	8.0370
	EXE	7.3777
	EXE	7.2623
	EXE	7.2588
	EXE	7.2588



After four iterations we obtain the required root correct to four decimal places as follows:  $x_{root} = 7.2588$ 

Clearly, Newton's Method is more efficient and substantially faster than the Bisection Method.

#### EXAMPLE 2

Find the positive root of  $f(x) = 4x^3 + 12x^2 - 32x - 29$  to four decimal places using Newton's Method. Take the initial guess  $x_0$  to be 1.5.

#### Solution to example 2

 $f(x) = 4x^3 + 12x^2 - 32x - 29$ 

 $f'(x) = 12x^2 + 24x - 32 = 4(3x^2 + 6x - 8)$ 

Initial guess is  $x_0 = 1.5$ 

Again using the Casio ClassPad II CAS calculator, we carry out the following procedure:

Define  $f(x) = 4x^3 + 12x^2 - 32x - 29$ 

Define  $k(x) = x - \frac{f(x)}{\frac{d}{dx}f(x)}$ 

1.5	EXE	
k(ans)	EXE	2.6774
	EXE	2.2706
	EXE	2.1872
	EXE	2.1837
	EXE	2.1837

After four iterations we obtain the positive root to this cubic function to four decimal places as follows:  $x_{root} = 2.1837$ .

#### **NEWTON'S METHOD EXERCISES**

#### Question 1

Use Newton's Method to find the greatest root of  $f(x) = x^3 - 4x^2 - 2x + 4$  to four decimal places. Take the initial guess to be  $x_0 = 4.5$ .

[ANSWER:  $x_{root} = 4.2491$ ]

#### Question 2

Find the root of  $f(x) = 2x^3 - 4x^2 + 5x - 7$  to four decimal places using Newton's Method. Take the initial guess to be  $x_0 = 1$ .

[ANSWER:  $x_{root} = 1.7263$ ]

## Question 3

The function  $f(x) = x^3 - 7x + 7$  has two roots on the interval [1,2]. Find these two roots to four decimal places.

[ANSWER:  $x_{root1} = 1.3569, x_{root2} = 1.6920$ ]

#### Question 4(a)

Find the square root of 17 to four decimal places using Newton's Method.

Hint: Let  $f(x) = x^2 - 17$ . Then f(x) = 2x. The iteration formula becomes



$$x_{1} = x_{0} - \frac{f(x_{0})}{f'(x_{0})}$$
$$= x_{0} - \frac{(x_{0}^{2} - 17)}{2x_{0}}$$
$$= x_{0} - \frac{x_{0}}{2} + \frac{17}{2x_{0}}$$
$$= \frac{x_{0}}{2} + \frac{17}{2x_{0}}$$
$$= \frac{1}{2} \left( x_{0} + \frac{17}{x_{0}} \right)$$

Take  $x_0$  to be 4 and start the iteration process.

[ANSWER:  $\sqrt{17} \approx 4.1231$  after only two iterations]

# Question 4(b)

Find the cube root of 28 to four decimal places using Newton's Method.

[ANSWER:  $\sqrt[3]{\sqrt{28}} \approx 3.0366$ ]

# Question 5

Use Newton's Method to find all three roots to four decimal places of the function  $f(x) = -5x^3 + 5x^2 + 18x - 8$ . Take the initial guesses to be -2, 0.5 and 2. See Figure 4.



*Figure 4.*  $f(x) = -5x^3 + 5x^2 + 18x - 8$ 

[ANSWER:  $x_{root} = -1.6901, 0.4163 \text{ and } 2.2738$ ]



# **Engagement and mathematics:** what is it and what does it look like?

## Catherine Attard, Western Sydney University

In education we talk about student engagement every day, but what do we mean? What does real student engagement look like in a contemporary mathematics classroom? The term engagement relates to students 'levels of interaction with classroom activities and content. However, educators often don't have a deep understanding of the elements required to ensure students experience true, sustained engagement, promoting lifelong learning and enhancing learning outcomes. The challenge to engage students of any age or ability and within any field of study is becoming increasingly complex. The nature of students and their learning needs continue to evolve as their social lives and studies intertwine due to the ubiquitous nature of technology. So how can mathematics educators engage these students? How should we effectively use technology in ways that enhance learning? How can we make mathematics learning more relevant and enjoyable for our students? I'll explore the construct of engagement and will discuss practical, hands-on ideas to enhance the teaching and learning of mathematics and increase student engagement.

#### **ENGAGING STUDENTS**

Try and recall one of your mathematics lessons that had the 'wow' factor. What was it about that lesson that had your students turned on to mathematics? Chances are, that great lesson included three factors that are critical for 'true' student engagement to occur: affective, operative, and cognitive elements of engagement, all working together. Perhaps your students were playing a challenging mathematics game, using technology in a meaningful way, or participating in a real-life mathematical investigation. Maybe they were collaborating in groups, or were given an element of choice in the task and were provided with constructive feedback on their work.

Deep engagement with mathematics occurs when students are procedurally engaged and interacting with the task and with each other, when there is an element of cognitive challenge within the task, and when they understand that learning mathematics is worthwhile, valuable and useful both within and beyond the classroom. It is easy to fall into the trap of thinking students are engaged when they appear to be busy working or 'on task'. True engagement is much deeper than on task behaviour – true engagement is 'in task' behaviour where all three elements; cognitive, operative, and affective, come together. This leads to students valuing and enjoying school mathematics and seeing connections between the mathematics they do at school and the mathematics they use in their lives outside of school.

The Framework for Engagement with Mathematics (FEM) (Attard, 2014) was derived from a three-year longitudinal study on the influences of student engagement. It provides insight for mathematics educators into the foundations necessary for students to engage with mathematics. The FEM highlights that it is not simply the pedagogical repertoires or the resources and activities that teachers use that influence student engagement, it is the deeper level of pedagogical relationships that develop between students and teachers that are a necessary foundation for engagement to occur. That is, the connections between teachers and students in relation to learning. If these connections do not exist, then it is unlikely that engagement will develop, regardless of how interesting the activities or tasks are. See figure 1.

It is not suggested that teachers strive to achieve every aspect of the framework in every mathematics lesson. Rather it is about the development of a classroom philosophy and approach. However, the framework may serve as a useful reminder and a starting point from which to plan mathematics teaching and learning experiences that represent student voice and promote affective, operative, and cognitive engagement.


#### FRAMEWORK FOR ENGAGEMENT WITH MATHEMATICS

#### In an engaging mathematics classroom, positive pedagogical relationships exist where:

- students' **backgrounds** and pre-existing knowledge are acknowledged and contribute to the learning of others
- the teacher is **aware** of each student's mathematical abilities and learning needs
- interaction amongst students and between teacher and students is continuous
- the teacher models **enthusiasm** and an enjoyment of mathematics and has a strong **pedagogical content knowledge**
- feedback to students is constructive, purposeful and timely

#### In an engaging mathematics classroom, engaging pedagogical repertoires mean:

- there is substantive conversation about mathematical concepts and their applications to life
- tasks are positive, provide opportunity for all students to achieve a level of success and are challenging for all
- students are provided an element of **choice**
- technology is embedded and used to enhance mathematical understanding through a **student-centred** approach to learning
- the **relevance** of the mathematics curriculum is explicitly linked to students' lives outside the classroom and empowers students with the capacity to transform and reform their lives
- mathematics lessons regularly include a **variety** of tasks that cater to the diverse needs of learners

#### Students are engaged with mathematics when:

- mathematics is a subject they enjoy learning
- they value mathematics learning and see its relevance in their current and future lives
- they see connections between the mathematics learnt at school and the mathematics used beyond the classroom.

Figure 1: Framework for Engagement with Mathematics (Attard, 2014)



### **Mindfulness in a secondary school:** investigating student and teacher experiences of a school-based mindfulness program

Elizabeth Shepherd, Canterbury Girls' Secondary College

#### MINDFULNESS

Mindfulness is more than a passing fad. When students are truly present they are better able to learn and achieve their potential. Students who are anxious, depressed or struggle to engage in the classroom may benefit from learning to practice mindfulness. This paper describes M Ed research into the use of mindfulness in an independent secondary school. The case study investigates experiences of the school leadership, staff, and students-while teaching, learning and using mindfulness.

#### THE STUDY

In order to gain some understanding of the effects and acceptability of teaching mindfulness in a secondary school setting, the mindfulness program at a non- government, girls' school was investigated (over the period November 2016 to June 2017). To obtain the student voice, 54 Year 7 students were surveyed. The questions mostly required selecting responses using a Likert Scale with some asking for individual answers. Also two focus group interviews were held, one of students who had completed the Year 7 survey and another where the participants were current Year 12 students. Various staff were interviewed individually to gain the perspective from school administration, Home Group teachers (the main teachers of mindfulness), and staff involved in student wellbeing. While the data is still being analysed some valuable insights have already occurred and are described below.

#### THE SCHOOL PERSPECTIVE

Mindfulness was introduced to students at this school following the training of a large group of interested staff in about 2010. Staff were trained over 8 weekly sessions by Dr. Craig Hassed, an Australian expert in the field based at Monash University. These staff have then supported other staff in using mindfulness at school. Each Monday morning at the weekly briefing, staff participate in a mindfulness activity, usually lead by the Head of Wellbeing (who was one of the original core group of trained staff).

At this school it is expected that the girls in every class, at each year level (7-12) will practice mindfulness at least three times each week in the Home Group (HG) time (10 minutes) before the first class of the day. Each class elects a student Wellbeing Leader who works with the HG teacher to plan mindfulness activities for the class. Responsibility for the content and implementation of mindfulness activities is left with the individual HG teacher.

#### STUDENT SURVEY HIGHLIGHTS

- Mindfulness activities were highly acceptable to students, only a few (7%) explicitly criticized learning mindfulness.
- Mindfulness coloring was the most popular activity (32%).
- 22% of students mentioned mindfulness meditation activities as something they found enjoyable or interesting.
- 20% said mindfulness enabled them to 'focus their attention'.
- 19% said mindfulness enabled them to 'manage stress/anxiety'.
- 43% of the Year 7 students surveyed reported using mindfulness since learning it at school.
- 15% of students used mindfulness outside school to manage stress/anxiety and/or to focus on their school/ homework.
- Not all the students reported actually taking any mindfulness activities.
- The mindfulness activities undertaken varied between classes.
- Many students (74%) found the mindfulness activities boring.



#### YEAR 7 FOCUS GROUP HIGHLIGHTS

- The girls displayed a clear understanding of mindfulness: 'Well for me Mindfulness is just a state of mind where you feel relaxed and calm' and 'I'd say it was being aware of your surroundings, cuddling yourself and relaxing yourself' (two focus group participants).
- They articulated the benefits they had experienced from practicing mindfulness, especially: Managing stress and anxiety, to focus on homework or study, to help them go to sleep.
- Most of the group used some form of mindfulness at home as well as at school.
- Several relied on using mindfulness to cope, 'just sort of being able to be free of the stress can also make you look at like all the tasks and things you have to do more clearly, and like less 'Oh my gosh we have so much homework' but 'Okay we will do this and this'. Yeah' (a focus group participant).
- These students were adamant that learning and practicing mindfulness at school is beneficial and should be continued.

#### YEAR 12 FOCUS GROUP HIGHLIGHTS

- Learning mindfulness was highly useful and has been of particular benefit during VCE.
- When in Year 7 they didn't realise how useful mindfulness actually was and tended to see it as boring.
- Learning to practice mindfulness is one skill they have learned at school which they will take with them into the post school world and which will be useful for the rest of their lives.

#### DIFFICULTIES AND SUGGESTIONS FOR IMPROVEMENT

The main difficulty students had gaining the most benefit from the mindfulness activities related to noise from within or outside the classroom (including talking/giggling from students).

The program could be improved by:

- Having mindfulness activities consistently throughout the year, rather than a lot some weeks and none at others.
- More variety of activities and more flexibility (allowing students to choose their own posture for meditation, or have background music or sometimes having guided rather than silent meditation).
- Going outside and focusing on the sounds of nature (rather than man made sounds).
- Timing of sessions so they weren't rushed, but not in lunchtime.

#### STAFF INTERVIEWS HIGHLIGHTS

- Staff were generally in favour of teaching mindfulness to students: 'The rationale for the introduction of mindfulness, I believe, is that it's another tool, it's another attribute that can be used in the whole arsenal of well-being,...that is a way of achieving success, that it's a way of focus....that opportunity just to calm the mind' (A Year 7 HG teacher).
- They indicated examples of their observations of the positive effects of mindfulness on some particularly anxious students.
- They found it difficult to routinely fit mindfulness activities into the 10 minute HG time so didn't often do it then.
- The level of comfort with teaching mindfulness varied greatly as did the amount of training teachers had received.
- Some staff were very comfortable leading mindfulness meditation while others didn't do this at all with their class and focused instead on mindfulness coloring.
- Staff would welcome training in mindfulness for new staff, refresher training for current staff, and a bank of mindfulness resources (including meditation scripts and other activities) as well as the time and opportunity discuss, ask questions, and share their experiences teaching mindfulness with other staff.



#### CONCLUSION AND FUTURE IMPLICATIONS

- At this secondary school, learning mindfulness has been beneficial for many students and damaging for none.
- For a sizeable fraction of those studied, learning and using mindfulness has been crucial in them managing anxiety, stress, their thoughts or emotions, focusing on school or homework, and going to sleep.
- Student experience of mindfulness was greatly influenced by their HG teacher.
- The level of interest, expertise, training and confidence (especially in leading mindfulness meditation) varied greatly between HG teachers.
- Both students and staff felt time pressures impacted negatively on their experience of learning and using mindfulness.
- The reflections of Year 12 students confirmed the positive and ongoing effects of them learning mindfulness at school.
- Staff teaching mindfulness should be well trained (including regular updates) and supported with resources.
- Students should be exposed to a variety of mindfulness practices to foster engagement and learn which suit them.
- Sufficient, regular, time must be allocated so that students develop their skill enough to use mindfulness as a tool, whenever needed, throughout their lives.



# Numeracy picture books and the lessons that bring their mathematics to life!

#### Anna Kapnoullas, Top Ten Resources

Where literacy and mathematics meet: Numeracy Libraries. Our team of leading teachers has selectively chosen the best 75 picture story books to introduce and hook students into excellent mathematics lessons. Multiple lesson plans for each picture story book include teacher modelling, photographs of lessons in action and work samples, extending and enabling prompts, as well as higher-order questioning specific to the pages of each story. Our catalogue organises each mathematical picture story with specifically recommended grade levels and best-matched concepts. For our 200 member schools, this has maximised the most effective use of the stories in the classroom to introduce and contextualise real-life mathematics throughout the year. Our tried-and-tested linked lesson plans (two or more for each book) use the context of the storylines to deliver hands-on, materials-based sessions that make sense of the maths in each story and literally bring it to life!

#### USING NUMERACY PICTURE BOOKS IN THE MATHEMATICS CLASSROOM

For years now, academics and mathematics curriculum leaders have advocated for the use of picture books in the classroom to launch and engage students in excellent mathematics lessons with rich contexts. A team of ten numeracy leaders has recently completed the process of selecting, setting up and successfully implementing this first-class pedagogy for curriculum leaders. In the classroom, the Numeracy Libraries with linked lessons capture the best picture books available for mathematics and transform them into multiple hands-on, rich-context lessons with instructive photographs and detailed enabling and extending prompts pre-planned for teachers. Please peruse these two short examples of how picture books can be transformed into hands-on lessons in the mathematics classroom.

#### MAKING SOCK SENSE OF CHALLENGING CONCEPTS: MISSING MITTENS

One of the least-understood concepts we come across as numeracy leaders working in schools is the seemingly simple concept of even and odd numbers. The mathematical picture story *Missing Mittens* provides the perfect context for a lesson created to develop a depth of conceptual understanding for students using a real-life link to socks and the necessity of having a pair (even number) as opposed to possessing an odd sock without a matching partner.

Our lessons occasionally encourage students to bring items from home (a show-andtell version of maths but for the items to serve as the manipulatives for the lesson). By connecting mathematics to objects beyond the conventional counter or dice, we are aiming to bring those genuine real-life links directly into the classroom not only for but also through the students. Our lessons provide templates for parent reminder notes in the early years and all materials involve no cost; these are items in every household such as pairs of socks. We do find that once students are in a routine of bringing things from home to make maths lessons real, the use of notes is mostly unnecessary because children want to use these materials and are genuinely excited about maths!



Back to the story: *Missing Mittens* by Stuart Murphy involves a farmer checking his animals have socks for the winter ahead and finding that a few are missing one. It is all quite odd because the animals are one sock short of an even number for their chilly legs. Teachers then transform this story into a mathematics lesson by asking students to bring in a few pairs of clean socks. First, students line the socks up in their matching pairs to practise skip-counting by two. Students then use the provided 0-120 chart to colour-code different numbers of socks as even (colour it toasty warm orange) or odd (colour it chillingly cold blue). The reasoning students use is whether that number would create a complete pair of socks (warm even feet) or whether the number would result in the wearer having only one sock (quite odd).

The extending prompt involves extension students multiplying larger numbers by two to calculate the total number of socks in each grade, each year level and then the entire school using our scaffolded template. Extension students can also be supported to use alternative strategies for large multiplications (including the very popular among students lattice



method or grid/box method) by watching supporting video clips available through our family maths website www. nextmaths.blogspot.com (use the multiplication side tab).

For the enabling prompt, support students are guided along the line to practise counting by two and to physically place their feet on top of each number to figure out whether it would be even or odd, focusing on mastering the numbers 1 to 20 before moving onto the later parts of the line. Students in need of significant support focus on counting the socks in the line (one-by-one) and mark this on a small number line or 1-20 chart to check their count.

As the exit ticket for the session, students are asked whether the number 17 is odd or even and to explain why using drawings and words. Prior to this lesson, many students simply drew an arrow to the digit '7' and wrote 'odd.' After this more in-depth concept work, most students drew a picture of a line of socks, pointing out that the final sock had no match and thus could not make a pair – a solid real-life link as the foundation for understanding odd/even rather than a superificial, rule-based view of maths! Other students drew 8 children with socks and one child with only one sock. Students' written descriptions included phrases such as, 'One leg will be left out/missing a sock,' and some students even showed the beginnings of understanding the odd/even link to divisibility by two, '17 cannot be shared between people with two legs without a leftover sock.'

To view the lesson plan with instructive photographs in an easy-to-read and copy-and-paste-able table format, including all templates, please visit www.toptenresources.com for the full free download.

#### ONE MATHEMATICAL PICTURE STORY: MULTIPLE HANDS-ON LESSONS ACROSS ALL DEVELOPMENTAL STEPS

Most picture books are focused on one main concept, for example Two of Everything is perfectly suited to efficient

addition strategies. Specifically it is a brilliant story to introduce the doubles facts for grades one and two in the context of a magical doubling pot, which an elderly couple stumble across on their farm. However, all the selected picture books in the Numeracy Libraries lend well to a second or third concept for another year level, for example *Two of Everything* is also fantastic for introducing input and output tables to grade five students. This is in the context of students creating their own mystery codes or magical pots in



a two-column format (what goes in as the input and what came out as the output). Students then walk around the room with their maths books solving the codes of like-ability partners in a gallery walk-style lesson. Extension students often create two-step rules involving different operations, for example  $x^2 - 30$ .

#### THE IMPORTANCE OF AN EASY-TO-NAVIGATE CATALOGUE TO ASSIST ALL TEACHERS

What if there was a perfect picture book to introduce the lesson you were about to deliver but you had to look through an entire library to find it? You wouldn't because as teachers we all know we are time-poor. Accordingly, putting great literature with matching hands-on lessons at our fingertips is the best way to ensure we use it at the ideal time within our scope and sequence. The Numeracy Library's included catalogue allows teachers to simply scroll to their specific grade level and specific maths concept, or even a precise developmental step where their class cohort is showing a point-ofneed (for example, skip-counting) to be shown a summary of the most relevant mathematical picture books. Teachers simply click the cover of their chosen picture book for the file to automatically open the electronic lesson plan.

To reduce the workload of curriculum leaders, we organise the picture books into key concepts with displays, making it easy to implement the use of the Numeracy Library the day after you receive it. The main catalogues are place value, addition, subtraction, multiplication, division, fractions and the minor units; within each of these, there are specific developmental steps ranging from before Foundation level to slightly beyond year six into some introductory secondary content. All picture books are provided in their regular format (not electronically) as we believe it is critical to still enjoy the engaging read-aloud and literacy time with students during the tuning-in/hook element of the lesson structure.

Due to the quality of the selected literature, many of the picture books have up to five tried-and-tested lessons linked to their storylines. Many of the picture books in our Numeracy Libraries are unavailable from other publishers as we have



had no boundaries in sourcing these, including requesting reprints of classic tales and importing stories from the UK and US wherever the quality of the mathematical picture story demanded its presence in Australian classrooms.

#### CLASSROOM MODELLING FINDINGS

Our classroom modelling with member schools has shown that students who prefer reading or writing but show less confidence in mathematics often thrive during sessions based on a storyline. More than that, for all students, the pedagogy and excitement of using a story for mathematics transforms the setting for lessons into the realm of captivating fiction and real-life materials at the same time! These picture books are not solely exceptional tools to tune-in students, but are also powerful platforms for explicit modelling and student practice of the more difficult mathematical concepts within the linked hands-on lessons. To view a video outlining the work of the ten numeracy leaders in selecting the picture books and creating the 250 hands-on lessons in schools please visit www.toptenresources.com.



# From closed to open: differentiated investigations allowing students to become proficient mathematical learners

Brendan Hodge, Lang Lang Primary School; Karyn Thomas, Wonthaggi Primary School; Kristy Miller, Woodlands Primary School;

Imagine walking into a classroom where every student believes with all their heart that they are an efficient and creative mathematician. They enjoy taking on new challenges and do not doubt their ability. Where teachers see themselves as the facilitator of exciting explorations in learning. That is what we are about; equipping and developing teachers to enable such a learning environment. This presentation will explore the idea of opening up closed activities to not only explicitly teach skills but to cater for problem solving, thinking, reasoning, and communication. Rich open investigations that all students can access.

#### WHY OPEN INVESTIGATIONS?

Open investigations typically:

- Engage all students in mathematics learning
- Have a range of appropriate responses
- Enable students to participate more actively in lessons and express ideas more frequently
- Enable teachers the opportunity to rove and probe students mathematical thinking and reasoning
- Assess a range of knowledge and skills
- Provide information about problem-solving strategies and thinking
- Provide opportunity for students to demonstrate higher levels of understanding.



#### FROM CLOSED TO OPEN





## Rich Open Tasks in the Classroom



There is a great difference between knowing and understanding, you can know a lot about something and not really understand it.

- Charles Kettering



# Harry Houdini and maths

Chris Ireson, Melbourne High School, Texas Instruments T<sup>3</sup> National Trainer

Learn how to perform a simple magic trick to inspire your students and then build a project based on the outcome of the trick and more. Students need to learn how to use the best technique to solve a mathematical problem with and without the use of technology depending on the type of assessment task they are attempting and the nature of the mathematical problem. The TI-Nspire CAS CX Calculator will be used to do some simple coding by writing User Defined Functions and the Notes Page will be used to make a unique project for each student.

#### DESIGNING PICTURES ON THE TI-NSPIRE CAS CALCULATOR

#### Part 1: Graphing equations with restrictions

A User Defined Function (UDF) allows repetitive calculations to be performed quickly on the CAS Calculator, thereby saving time pressing keys and scrolling through CAS Calculator menus. UDFs are predefined on the CAS Calculator by the User and are useful in saving precious time in Technology Active Tasks. The type of UDF defined in this project only works in the current open document and current open problem within a document. A UDF can be used in a Calculator Page or a Notes Page.

#### The five-pointed star

The five-pointed star below was created by using two UDFs, finding the equation of a straight line, and finding the intersection of two straight lines. The domain of the straight line, the set of *x*-values, was then restricted to form a straight line segment representing a side of the star. The process was then repeated several times to form the complete star.

The two UDFs used to design the five-pointed star are:

(1) Finding the equation of a straight line in the form y = mx + c that passes through two points  $(x_1, y_1)$  and  $(x_2, y_2)$ .

line2p
$$(x1, y1, x2, y2)$$
 := solve $\left(y - y1 = \left(\frac{y2 - y1}{x2 - x1}\right)(x - x1), y\right)$ 

(2) Finding the coordinates of the intersection of two straight lines.

intn2lines(eqn1, eqn2) := solve 
$$\begin{pmatrix} eqn1 \\ eqn2 \end{pmatrix}$$

Note: The assign variable symbol, ':=' was used in the UDFs rather than using the Define command to define the UDFs.

Using the coordinates of the five-pointed star shown below, find the following:

**Question 1.** Without using the CAS Calculator find the equation of the line passing through AC in the form y = mx + c.

**Question 2.** Type into a Calculator Page the line2p() UDF and check your answer to Question 1. UDFs can be accessed by pressing the 'var' button.

**Question 3.** Using the line2p() UDF find the equation of the line passing through *BD* in the form y = mx + c.

Question 4. Without using the CAS Calculator find the coordinates of the point *G*, the intersection of lines *AC* and *BD*.







Question 5. Type into a Calculator Page the intn2lines() UDF and check your answer to Question 4.

**Question 6.** To draw the line segment *GC* on a Graphs Page, open a Graphs Page and on the Graph Equation Entry Line (ctrl G) type:

 $f1(x) = mx + c \mid x$ -value of  $G \le x \le 6$ 

replacing 'mx + c' with the values of m and c found in Question 1 and the 'x-value of G' with the value found in Question 4.

E.g., Type in  $f_1(x) = -2x + 7 | 3 \le x \le 6$ , but with your calculated straight line equation and domain.

**Question 7.** Using the two UDF's finish drawing the five-pointed star on the Graphs Page. The completed five-pointed star must have the coordinates of the points *A* to *J* labelled on the Graphs Page.

Question 8 (challenge). Using Geometry Tools, shade in the five-pointed star in your School's Sport House Colour.

#### Part 2: Designing your own picture

Design a picture by using at least 5 different relations or functions from the mathematics course and at least 15 different equations for the picture. Transformations of equations must be used in the design.

(1) Before you begin, decide where the origin will be on a sheet of graph paper.

(2) Design your picture on the sheet of graph paper. To make it easier, try to make the major points on your design have integer values for coordinates such as (7, 0). Use transformations of equations – dilations, reflections, and translations.

Looking at the design of the spaceship below, the body of the spaceship has been formed by reflecting a square root graph in the *x* and *y* axes.

(3) Be creative.

(4) Once your design is complete, reproduce your creation on your CAS Calculator on a Graphs Page using what you have learnt from Part 1.

#### Submission of Part 1 and Part 2

(1) Part 1 Question 1 and Question 4 can be submitted on a sheet of paper or scanned and emailed to your class teacher with your class code and name in the file name. i.e., 10A\_Surname Initial Pt1 nonCAS.

(2) Please email Part 1 and Part 2 in one tns file to your class teacher with your class code and name in the file name. i.e., 10A\_Surname\_Initial\_Pt1\_Pt2\_CAS.

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# **Playful and joyful maths**

David Butler, University of Adelaide

I love maths. I derive great joy in finding the maths around me, talking about it, and solving problems I've never seen before. Yet, I know not everyone experiences this kind of mathematical joy. Some find the concept of joy in maths a little alien. Some like maths but struggle to find joy in the maths they face daily. Some do have the joy but wonder how to help their friends, students, and colleagues find it. I have come to realise that my own joy in maths has been created by a continual and concerted effort to play. I've organised more time to play with maths in my life, but also infused a playful attitude into the daily maths I do with students.

Here I will discuss how to take a playful approach to your maths in order to find more joy. I'll talk about what both joy and play feel like and how they make a difference to growth mindset. I'll share my experiences running puzzle sessions and in providing maths learning support to thousands of students. You will see how you might engage in playful maths activities, and find the play in maths that at first looks like hard work or like the 'same old thing'. I hope that you will learn some strategies to be more playful and hence find more mathematical joy.

#### JOY, PLAY, AND GROWTH MINDSET

Here is my definition of joy:

Joy is happiness that you know can't be taken away... ... because it belongs completely to you but at the same time is bigger than you.

Joy is the laughter at a really clever joke, the satisfaction of finishing a well-written book, the awe at a glorious sunset. In maths, it often happens when you notice that ideas fit together in unexpected ways, and yet you feel it couldn't have been any other way – the feeling that makes you say 'Oh!' and then 'Well, of course!'.

Here is my definition of play:

Play is trying out ideas inspired by your curiosity... ... and being absorbed in the experience.

Play is all about choosing something you want to try and trying it out, about choosing your own goals and how much you want to fulfil them, about putting things together in new ways to see how they work. In a state of play, you aren't worried about what others think of you or about whether what you are doing fits anyone else's goal. In maths it usually only happens in situations rich with possibilities where you have a chance to decide your own goal.

Growth mindset is one of the latest buzz-phrases around education. We spend effort encouraging students to believe that their abilities can improve and that trying and failing are part of improving. Yet it is hardly surprising to be afraid of failure if everything is so serious and goal-oriented all the time. Perhaps if we spent more time in a state of play, where we are pushed forward by our own curiosity and absorbed in the experience, we might be able to forget looking foolish and forget not being good enough and allow ourselves to grow.

#### **DESIGNATED TIME FOR PLAY**

Ten years ago I started a puzzle and games gathering at my university which has come to be called 'One Hundred Factorial', after the first puzzle we ever did together. At these regular sessions, staff and students join together to solve puzzles, play games, and discuss maths and life. Some students have attended for years, and talk fondly about the importance that the weekly play sessions have had in their experience of university life and the rest of mathematics. They love having a block of time each week where they don't have to worry about being assessed or judged, and where they can choose to engage at their own level. They say the joy they experience at One Hundred Factorial bleeds into the rest of their week. For myself, it is my weekly inspiration in a busy work life. I believe that this designated time for play makes a big difference to the likelihood of experiencing mathematical joy.



#### **EVERYDAY PLAY**

Play is not just reserved for designated play time, but can be infused in the daily work of maths learning. I run a Maths Learning Centre (MLC) where students from all disciplines visit to talk about their mathematical learning. In the MLC, I encourage moments of play by wearing mathematical t-shirts I am ready to discuss at a moment's notice, by having our latest mathematical art project available to build, and by having puzzles on display. Students often love to take a break from their usual work to play with mathematical ideas.

However, it's not just these moments between work that are playful. My students and staff talk highly of my playful attitude to everyday teaching. I am always ready to pull out play dough or blocks to help illustrate a concept, and have a bodyscale floor graph that I jump (literally) at the chance to use; I am always drawing illustrations for concepts, usually in coloured textas; and I regularly shout 'That's so COOL!' when a student shows me a new idea. This attitude that the business of learning doesn't have to always look so serious is one of the key ways that I encourage students who are seeking support to approach their maths with confidence. In short, a playful attitude helps student growth.

#### **PLAYLIKE WORK**

So far I have mentioned taking time to play in large designated blocks and also in moments across the day. However, you can actually infuse joyful play into the daily work of learning and doing maths. My definition of play involved trying out ideas inspired by your curiosity, and I noted also that joy often comes when things fit together beautifully. Mathematics is nothing if not a collection of curious ideas to try that fit together beautifully. The work of professional mathematicians is finding, exploring, and describing those ideas, and so the work of a mathematician can be seen as a form of play. We can make the daily work of maths more authentically like the work of mathematicians and simultaneously more like play by allowing more opportunities to be curious and seek connections.

From watching how we work at One Hundred Factorial, and also my own work as a mathematician, I have learned two principles that can be applied to make mathematical work more play-like:

#### The goal is not the goal. The end is not the end.

These principles mean that just because a puzzle or problem has a stated goal, it does not mean that this is what you are really aiming for by doing it. Instead, you might be aiming to find something beautiful to learn, or to make connections to other ideas, or to understand the thinking of others. The problem or puzzle is the thing that inspired your curiosity but it is not the goal itself. Also, getting the intended answer is not the end. After the answer is reached, there are always more questions to ask yourself and more connections to make. My most successful students never have the stated goal as their goal and the stated end is never the end – they are always chasing their curiosity. In short, they're always playing, and because of it, they understand the maths much better than if they had been satisfied with the answer.

We can give students the opportunity to make their work more playful and more mathematical by asking them questions that change the goals and ends. The questions 'What would happen if...?' and 'How is this connected to...?' are great to ask students while they are doing textbook problems. It's important not to just ask them at the end of a set of problems, because then only the fast students get the chance to play, whereas everyone deserves to have this opportunity.

#### **FINDING JOY**

I can't think why anyone would keep doing maths or teaching maths without finding joy in the experience. I believe that making time for play and making work more play-like will provide both you and your students with the opportunity to find this joy.



### What we have learned, and what we still wonder: Two oldish codgers reflect on what excellence means in maths education, with a focus on the critical role off assessment.

#### **Dave Tout and Ross Turner**

Between them, Dave and Ross have about 90 years' experience in mathematics education and assessment across a range of sectors and roles. This presentation will be an opportunity for two aging men to reminisce about the lessons they have learned over that period, and reflect on what they see as achieving excellence in maths education. Using assessment as a major focus, the presentation will start from their experiences and the resulting lessons, or at least stories, in relation to all five of the conference themes:

#### MINDSETS

You can teach almost anybody almost anything. There are not 'maths people' and 'others'. Why do we believe that? Does typical teaching practice reflect these statements? Do our assessment practices reflect these statements?

We were two of the lucky ones, successful enough at maths at school to progress, and have managed the transition from student to teacher. We review some of the influences that helped make that transition and that have stayed with us as our professional careers developed: particular colleagues and collaborators, professional learning experiences, and theories and ideas that we encountered along our journeys.

#### ASSESSING

Assessment is often seen as a dirty word. We know too well, our hands are filthy: STC, TOP, PISA, PIAAC, VCE, VCAL and more. Why do we see assessment and the collection of evidence as critical? What have we learnt from our experiences in the assessment and the collection of evidence about mathematics learning for children, youth, and adults?

#### TARGETING

Effective targeting of instruction depends on good information about where students are on a well-described learning path. Why are good 'learning progressions' important and why do all teachers need to know about them? And what role does assessment play?

#### HANDS-ON

The importance of hands-on learning and associated structured activities was something we learnt on our journeys from successful maths students to beginning secondary school maths teachers to where we are now. So we would like these ideas to be more than buzz-phrases: real-world problem solving, modelling with mathematics, cooperative team work, mathematical communication, critical thinking. How on earth can we also do that with **all** our students? We share some views developed through our research and professional work based on these ideas.

#### SHIFTING THE NARRATIVE

Why, at our age, are we still passionate about mathematics education and assessment? Our journeys and narratives will reflect how our own mindsets changed significantly over our careers, and how we now believe that excellence in maths education requires changed mindsets, effective assessments, properly targeted teaching and learning, and hands-on learning. And combined, these can build the engaged and successful mathematics students of the future.

Which brings us to wondering about what the future holds. What are the mechanisms we can get behind to promote mathematical competencies, problem solving and modelling, more effective mathematical communication, team work, and the so called 21st Century skills, for all, not just the few? What can we do to ameliorate mathematics anxiety, and to build levels of enjoyment and confidence in individuals to make use of the mathematical knowledge they hold? What needs to change to take account of recent research evidence that confirms the idea that worthwhile mathematical progress can be achieved by all students, not just the few?

How do we best open ourselves to the opportunities that new technologies provide, in promoting mathematics learning? Are the changes to the structure of schooling we see on the horizon (and much closer in some cases) a threat, or do they provide opportunities to deal with some of our challenges in a new way?



# STEM: Women are (still) all over it

Katherine Seaton, Department of Mathematics and Statistics, La Trobe University

50 women who have made significant contributions to mathematics and other sciences across continents and centuries were featured in 2015 on a shirt (STEM: Woman are all over it). Meet some of the mathematicians featured, some others who could have been, and the inspirational women of Hidden Figures.

#### THAT OTHER SHIRT

#ThatOtherShirt was the brainchild of Elly Zupko, who tweeted a photo-shopped image of a shirt that one might wish that the project scientist on the Rosetta mission had worn to his international press conference in 2014. His shirt that day featured scantily clad women holding weapons; Zupko's tweet 'There, I fixed it!' showed photos of women scientists such as Rosalind Franklin, Marie Curie, and Rachel Carson instead (HREF1).

Not only was this image retweeted thousands of times, people wanted this virtual shirt to become a reality that they could own and wear! Zupko crowd-sourced the funding to make this a reality; the enthusiasm for the project can perhaps be measured by the fact that the project was over 600% funded. Nominations for notable women in science, technology, engineering and mathematics (STEM) to include were also crowd-sourced. From a pool of hundreds, 50 were chosen. They represent different cultures and times in history, as well as different fields of science. Two have Australian connections: Ruby Payne-Scott (radio astronomer) and Helen Quinn (particle physicist) (HREF2).

#### MATHEMATICIANS ON THE SHIRT

The mathematicians featured on the STEM: Women are all over it shirt are:

- Hypatia (c. 360-415): a Greek mathematician, located in Alexandria (Egypt)
- Émilie du Châtelat (1706-1749): famous for her French translation of, and commentary upon, Newton's Principia, and a successful gambler
- Mary Somerville (1780-1872): Scottish mathematician, astronomer and supporter of women's suffrage
- Ada Lovelace (1815-1852): an English Countess, credited with the first algorithm
- Sofja Kowalewskaja (1850-1891): first woman mathematics professor in Northern Europe, though not in her native Russia, and first woman editor of a mathematical journal
- Emmy Noether (1882-1935): German algebraist, described in her day as the most important woman in the history of mathematics

Succinct information about their mathematical interests and achievements is to be found in a booklet produced alongside the shirt, but which is separately available (HREF2). Far more detail of these women's lives and achievements, and those of many others, can be found at the *MacTutor History of Mathematics archive* (HREF3) or at the site *Biographies of Women Mathematicians* hosted by Agnes Scott College (HREF4).

#### HIDDEN FIGURES AND THE GLASS UNIVERSE

Two recent books *Hidden Figures* (Shetterly, 2016) and *The Glass Universe* (Sobel, 2016), and the movie based on the former, have thrown light upon two groups of women scientists who made fundamental and lasting advances, built on underlying mathematical principles, in astronomy and space travel.

In *Hidden Figures* (the wonderful movie), Katherine Johnson, Dorothy Vaughan and Marjorie Jackson are portrayed as the main protagonists; for a more complete, chronologically accurate and nuanced perspective, Shetterly's book is a great resource and very readable. Some of NASA's female human computers are featured on #ThatOtherShirt: Melba Roy (1929-1990), who became a leading computer programmer, and Christine Darden (1942-), who openly



questioned the sexism that delayed her career, and who was promoted to aerospace engineer. A famous photo of MIT mathematician-turned-software engineer (the latter a term she invented) Margaret Hamilton (1936-) is featured. This photo shows Hamilton beside the enormous pile of paper comprising the navigation software for the Apollo missions.

Some fifty or so years before the era of Hidden Figures, another astonishing group of women was deployed (and underpaid) at the Harvard Observatory, categorizing the stars captured on long-exposure glass-plate photographs of the night sky (Sobel, 2016). Two of these women also feature on #ThatOtherShirt. Annie Jump Cannon (1863-1921) is co-credited with creating the stellar classification system (based on temperature, in turn based on spectral characteristics). Henrietta Swan Leavitt (1868-1921) made observations about variable stars that allowed astronomers to determine the distance from earth to other galaxies, and hence to the realization that the universe is expanding. Her observation was called the period-luminosity relationship until, one hundred years later, it was re-labelled as Leavitt's Law. Had she not died of cancer, she would have been nominated for the Nobel Prize.

*The Glass Universe* is again a very readable book by a well-known author, about some remarkable women and their times and achievements.

#### MATHEMATICIANS NOT ON THE SHIRT

Who else might one want to include on such a shirt?

Perhaps Florence Nightingale (1820-1910). Her graphical representations of data related to death from unsanitary hospital conditions (rather than in battle) saved many lives and advanced applied statistics; she was made a Fellow of the Royal Statistical Society, a fact not as well-known as it could be (HREF4).

What about Hanna Neumann (1914-1971), who was the first woman professor of mathematics in Australia? Like Noether, she left Germany during the rise of Hitler. Hanna is well-known for her leadership in Australian mathematics and is honoured with a building named after her at the ANU. And one might like to include as well her doctoral supervisor, Olga Taussky-Todd (1906-1995), who worked as an applied mathematician in Britain's World War 2 defence after leaving Austria, an important figure in matrix theory, and who finally settled in the US (HREF3).

Katherine Johnson (1918-), whose mathematical contributions to the American space program have been recognized in recent years, not only in Hidden Figures, but also in the naming of a building (the *Katherine G. Johnson Computational Research Facility* at NASA Langley) and by the Presidential Medal of Freedom which she received from Barack Obama – she belongs (HREF5).

And certainly include Iranian-American Maryam Mirzakhani, who sadly lost her battle with breast cancer this year at the age of 40. She was the first woman to be awarded the highest prize in mathematics, the Fields Medal, in 2014. She has been described as a tenacious explorer of abstract surfaces (HREF4).

One final nice thing about the *STEM: Women are all over it* shirt is that there is a blank silhouette on it. It represents the nameless and unrecognized women of STEM, but also the unknown figures of the future, whose contributions are still to come.

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## **Success in tertiary education requires mathematics**

Nicole Merlich and Trish Jelbart, Victoria University

#### INTRODUCTION

Since the 1990s mathematics entry requirements for many universities have been relaxed due to falling student interest in courses such as Engineering and Science (King & Cattlin, 2015). In 2015 only 14% of universities had intermediate mathematics as a prerequisite for Science (Prince et al., 2015). Firm pre-requisites have been replaced with 'assumed knowledge', the scope, definition and description of which has varied greatly between tertiary institutions. This has resulted in the widespread problem of students applying for courses whilst not possessing the required mathematical knowledge to succeed. The problem is not restricted to Science and Engineering, but includes courses such as Nursing and Paramedics, Education, Business, Trades and Sports and Exercise Science. Also, in order to compete for and attract more students, universities have tended to lower the suggested level of required mathematics for entry into these courses. The VTAC entry requirement for a Bachelor of Engineering at one university at the time of writing was 'a study score of at least 20 in any Mathematics', whereas an equivalent ability of at least 25 in Mathematical Methods and Specialist Mathematics is needed to be successful (Victorian Tertiary Admissions Centre, n.d.). In addition, many students are advised that to study Business or Sports and Exercise Science Year 11 or 12 mathematics is not required.

An associated problem is that students select Year 12 mathematics subjects in order to achieve a good study score, but that may not prepare them for the mathematical content in their tertiary course. For instance, students may complete Further Mathematics at Year 12 for Engineering but not select Mathematical Methods. From our experience, they then report that the more difficult content in Engineering Mathematics is Differential and Integral Calculus, not the statistics and matrices, which they have already studied in Year 12. Another issue arises when students study lower levels of mathematics to maximise their study score, and rely on their calculator instead of consolidating good basic skills which are required for many tertiary courses.

The collision of these issues has created several problems for students, teaching staff and universities. It is not uncommon for us, as Adult Educators at a university, to see failure rates in the order of 40-50% or higher in some first year subjects.

#### THE CURRENT SITUATION

In our experience, too many students enter courses with an insufficient mathematical foundation, attempt to complete first year units, experience difficulty and possibly fail. They then have to repeat subjects or drop out which affects their goals and dreams, mental health and financial situation. The lack of pre-requisites and co-requisites not only affects students in the first year, but all throughout a degree program. For example, if a student fails a first year Business Statistics unit - and they have a strong aversion to mathematics in the first place – it is possible that the student may not repeat the subject until their final semester regardless of the fact that second year Finance and Accounting subjects assume concepts learned in that unit.

Some institutions and courses have attempted to pre-empt students failing by requiring students to undertake initial mathematics assessment and recommending pathways and bridging courses to students. However, pathways may increase the time taken to complete the qualification and although bridging courses may be credited towards the qualification, they are often intensive and can demand that the student learn a lot of content in a very short time hence putting them under extra pressure. The student may then feel that perhaps they are not able to pursue university study, lose confidence and discontinue their course.

Teaching staff are also under pressure to pass as many students as possible (King & Cattlin, 2015) whilst ensuring course standards are maintained. If they do not achieve this, then the quality of their teaching is called into question. Teaching staff are finding that the lack of firm pre-requisites has resulted in a broader spread of knowledge and ability levels within the cohort (King & Cattlin, 2015) and there are many gaps in knowledge which is hindering student success. The amount of resources which are available to assist students varies across institutions, so the strategies used to address the



problem are also varied. Some examples are: giving the students very explicit direction regarding which materials to study in the learning management system and when they should be studying them; mastery learning (repeated testing and reinforcing concepts until the students attain a solid understanding of fundamental content); re-structuring classes to have smaller groups with a greater number of contact hours per week and individual academic support.

#### OUR PERSPECTIVE AS ACADEMIC SUPPORT LECTURERS

Although the mathematical knowledge required to successfully complete a course of study varies from qualification to qualification, it is very common for students to seek assistance with concepts as fundamental as Order of Operations in the first few weeks of a program of study. Although it is not practical to list here all the required mathematical concepts for each course, if we take Nursing and Paramedics as an example, the required knowledge (with and without a calculator) is:

- Multiply and divide decimals by 10, 100 & 1000 to convert units: kg, g, mg, mcg, L, mL
- Add, subtract, multiply and divide decimals
- Use formulae
- Multiply and divide fractions
- Convert fractions to decimals
- Rates, ml/hour, mg/min, beats/min
- Rounding off and estimating
- Ratio
- Times tables

#### STRATEGIES TO COMBAT THE PROBLEM

We, as Academic Support Lecturers, work with specific colleges to support students and improve their mathematical understanding. We offer:

- embedded support where we use our close partnership with subject co-ordinators to include scaffolding materials in the course, or run skill specific workshops inside the timetabled sessions; and
- complementary programs such as general or course-specific drop-in sessions, individual appointments, support tutorials, pre-semester and pre-exam workshops, and online support sessions which are all optional for students.

We also have Student Peer Mentors who run various mathematics sessions to assist students. At times, the students who need the most assistance find it difficult to access and use these resources for a range of reasons. For example, lack of confidence to approach support services, or lack of time due to work commitments.

In 2018, our multi-sector institution will be significantly changing the delivery of first year higher education units by introducing block teaching where units (8-10 during an academic year) will be taught intensively over a 4-week period but students only study one unit at a time and in small groups of no more than 30. Students will attend scheduled classes either three mornings, afternoons or evenings per week, leaving other times to undertake private study, complementary programs, undertake paid employment or other activities. It is hoped that this will provide a smooth and positive transition for our higher education students, a high proportion of which are from non-English speaking backgrounds.

#### CONCLUSION

A few universities are indicating that firm pre-requisites will be re-introduced in the next year or two, but the majority have no firm plans to follow as universities and TAFE colleges continue to compete in order to attract as many students as possible in the current tertiary education environment. Hence, students should be encouraged to fully research the mathematical content of any course that they are considering undertaking and ensure they have the required knowledge. The focus should always be on ensuring that students have a comprehensive understanding of mathematical fundamentals and encouraging them to study Year 12 units that will increase their chances of success into the future, not



those in which they can achieve an easy pass.

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## Getting creative with mathematics and growth mindsets

**Rebecca Harris, Dandenong Primary School** 

Children are born into the world with an innate sense of number, curiosity to explore patterns and problem solve, and the instinct to persist with challenges. Yet we as educators see this dwindle away as students learn to accept they are not good at mathematics. What implications does this have for our teaching practice? Explore research behind growth mindsets, and develop ideas and strategies to connect students to mathematics through art and creativity. Reignite the fire with colleagues, who themselves, see mathematics as a challenge. Foster the skills to build dialogue and reasoning strategies amongst students through visual artistic ideas.

#### INTRODUCTION

Hearts are pounding. Palms are sweating. The bell rings and the teacher opens the door. The anxiety builds as the children gather together on the floor. The maths lesson is about to begin. Anxiety fills the minds and sits in the pit of the stomach for too many of the students we teach. Especially during the maths lesson. This reality is not just true for students but also teachers. So how do we alleviate these negative feelings and attitudes towards mathematics? Recently the use of art as a means to engage students and teach increasingly complex and abstract concepts has been introduced into classrooms, although the connection between mathematics and the arts has been explored for centuries. This extension of STEM to STEAM opens up another avenue in which students can explore, imagine, and delve into the world of mathematics, bringing with them their creativity and appreciation of beauty and pattern. Bailey (2015) describes art as 'the discipline that most celebrates, encourages and embraces the original and the creative'. With this in mind, teachers can find a way to connect mathematical learning in meaningful ways and build a growth mindset in, not only the students in their classroom, but also themselves.

#### WHY ART? MAKING THE WHIMSY AND BEAUTY OF MATHEMATICS VISIBLE

How does art make us feel? What do we think when looking at artwork created by some of history's greatest? Leonardo da Vinci is renowned for his works using the golden ratio and Fibonacci sequence to create the aesthetically balanced Mona Lisa. M.C. Escher manipulates and plays with perspective in his mindbending work 'Relativity' (Figure 1). Brezovnik (2015) discusses how Escher's work demonstrates that 'art is an efficient transfer that brings mathematics and creative thinking closer to students.' Each artist embraces the mathematical aspects incorporated into their work that intrigues the viewer. In contrast, Booker (2014) explores how ancient mathematicians such as Pythagoras used geometry to explain music, art and science. So while a new idea in the classroom, there has been a natural connection embedded into our culture and history.



Figure 1: M.C. Escher's Relativity

If nothing else art makes the beauty of mathematics visible to us. Within the classroom, art offers the opportunity for students to explore mathematical concepts within a subject they find less stressful. The use of rich, open, creative tasks for students creates purpose for the concepts they are learning, and with mathematical concepts increasing with complexity and becoming more abstract the need for purpose and application of mathematical skills and knowledge also increases.

#### **GROWTH MINDSET FOR STUDENTS AND TEACHERS**

The notion of maths being something you're either good or bad at has become so engrained in our thinking that there is a social acceptance to finding maths challenging. Many of us could think of a student we've taught, or a colleague, who without hesitation makes comments about their own poor performance in maths. A growth mindset develops when students are encouraged to persist with challenges regardless of their perceived ability. Without this, students demonstrate a fixed mindset making difficult tasks an overwhelming experience. How can we change the feeling of dread students may feel during a maths lesson? The need to motivate and show real connections to the world is paramount.



Incorporating art and being creative while covering or exploring maths leads to students making positive connections with their maths learning. 'Students who feel the beauty, playfulness, challenge and utility of mathematics from the early school grades onwards would be more prepared to work harder on mathematics in the later grades, when the content becomes less visibly imaginable and more symbolic,' (Brezovnik, 2015, p.30).

Sousa (2008) explores the frustration and anxiety students feel about the mathematical competency, the impact this can have on their understanding and ultimately their mathematical achievement. In order to alleviate these feelings, Shields (2005) identifies 5 key areas; Teacher attitude, Curriculum, Instructional strategies, Classroom culture, and Assessment. We are the greatest influence on our students' learning. A teacher's attitude can be difficult to adjust if they had, or continue to, struggle with mathematics. We as teachers need to find mathematical learning meaningful and authentic to shift our own mindsets. Educators need to recognise the value of mathematics and the connections to other disciplines and society, promote student confidence and curiosity with interesting and relevant tasks, and create opportunities for success. Students experience enjoyment and relevance of what they are learning in a supportive and positive environment build a positive mindset ready for challenges and struggle. The use of art techniques and exploring art from around the world supports the building of a growth mindset.

#### MATHS + ART + CLASSROOM

Art and mathematics are both areas of curriculum that are covered in schools but the combining of the two can lead to wonderful learning and engagement of students. A simple search online unveils a plethora of activities and tasks covering nearly all areas of primary year mathematics. From the use of mosaics to explore fractions, shapes,

tessellations, and symmetry, or creating an illustration of stained glass windows to explore lines and angles, to using pattern blocks to create pictures and then counting the blocks, there is almost something for everyone.

The incorporation of art in a maths lesson could also include students using ombre colours to create arrays to demonstrate multiplicative thinking or investigating the Fibonacci sequence using Mondrian 'Composition II in Red, Blue, and Yellow', Below is an example completed by students in Foundation covering place value. Further application and differentiation of these tasks is open to wherever inspiration takes you. The activity titled 'Place Value Art' may be extended to use hundreds in a 2D model or use cubes to create a 3D sculpture. It could also be used to explore decimals with tens representing tenths and the minis representing hundredths.

#### CONCLUSION

The connection between art and mathematics as a catalyst for engaging students in meaningful mathematics is clearly established. Providing students with the opportunities to build positive connections with mathematical learning is a priority for all classroom teachers. While the potential to create activities that use a myriad of colour, shape and pattern, maybe a touch of glitter, is always likely to lead to students enjoying the lesson, it is important



Figure 2: Place Value Art-Foundation. Students used cut outs of tens and ones to create an image.

to remember why we are adapting our approach when teaching mathematics. We are working towards creating opportunities for success for our students, to break the social agreement that being bad at maths is ok, and to build flexible mindsets when faced with challenges. The mindset of teachers as maths learners plays a major role in the learning outcomes for our students. If we can shift our mindsets to view the beauty of mathematics all around us then we can shift the mindsets of our students as well.



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# **One task for all:** an inclusive teaching approach that encourages growth mindsets

Russell McCartney, Sean Webb and Chris Terlich. Wonthaggi North Primary School and Wonthaggi Primary School

No ability groups, just respectful engaging tasks that all students can access. The session is a variety of low floor-high ceiling tasks that provide multiple levels of challenge. The approach builds students' understanding and reasoning around mathematical concepts.

#### INTRODUCTION

Over the past two years, we have been involved in the Primary Maths/Science Specialist Program with 100 other teachers from throughout Victoria. During our training, we have had access to some of the most highly respected Maths Educators from around the country, including Doug Clarke, Charles Lovitt, and Dianne Siemon. Over this period, we have developed a philosophy of teaching based around providing all students with rich, respectful tasks. This means that all students have the opportunity to learn mathematical concepts collaboratively while completing low floor-high ceiling assignments. The key to this is allowing all students access to the tasks from the beginning of the lesson. The task is then extended so that every child experiences a challenge and is exposed to new learning.

#### MULTIPLE ENTRY AND EXIT POINTS (LOW FLOOR-HIGH CEILING)



#### TASK PREDICTS PERFORMANCE

In rich, respectful tasks, all students complete the same task, although these have many levels. It is absolutely necessary that all students can engage from the very beginning of the assignment. This may be achieved through a hook or a hands-on element to the lesson. The lesson will then progress in stages, with all students reaching a level where they are challenged. Teacher questioning and student reasoning are very important throughout each stage of the lesson.

#### DEEPER CONCEPTUAL UNDERSTANDING

The purpose of these rich tasks is for students to build a deeper conceptual understanding of mathematics. We believe this is achieved through building understandings through symbols, language, and materials. Children need to be able to see it, say it, and use symbols to record their thinking mathematically to build a strong understanding of any mathematical concept.





#### WORKING LIKE A MATHEMATICIAN - BUILDING A GROWTH MINDSET

For students to develop their mathematical thinking, they need easy access to strategies that will help them to solve problems in a number of ways and develop a growth mindset. A Mathematicians Strategy Board that is on display in the classroom is a strategy that we have implemented with great success. It not only gives the teacher ideas of where the students are working, but it gives ideas to extend their thinking and fluency skills.





### **Strategy choice is key for building number fluency:** new approachs for teaching the basic facts to children who are falling behind

#### Sarah Hopkins, Monash University

#### A NEW RESEARCH METHOD IS NEEDED TO UNDERSTAND CHILDREN'S DIFFICULTIES LEARNING MATHEMATICS

In the research literature, children who exhibit substantial underachievement in mathematics are often referred to as displaying a mathematics learning disability (MLD); children who exhibit mild but persistent underachievement are referred to as low-achieving, and children who exhibit expected achievement as average or typically achieving. Despite decades of research comparing groups of children categorized according to achievement, clear explanations of why some children have difficulty learning mathematics have not been forthcoming. There are no obvious cognitive antecedents of a MLD (e.g, language, spatial or numerical skills) that underpin or predict a MLD and so there are no obvious skills that can be targeted to help improve learning. Alternative research methods are needed to advance new knowledge in the field so to better understand and address children's difficulties learning mathematics.

A microgenetic approach is a research strategy that undertakes a microscopic look at the development of cognitive change and growth as it occurs. It is not new but was pioneered by Siegler and Crowley in 1991 to investigate strategy variability among typically achieving children. It involves studying individual performance on at task on many occasions, over a concentrated period of time, and collecting detailed information about how the task was executed on each occasion and the strategies used. In this keynote address, I present findings from a series of microgenetic studies investigating the role practice plays in building fluency with basic facts to help explain why some children have difficulties learning these facts and to investigate what can be done about it. While the application of microgenetic approaches to the study of learning difficulties is new, it shows much potential for finding better ways of teaching basic facts to children who are at risk of falling behind.

#### FLUENCY WITH BASIC FACTS: WHAT IS IT AND WHY TARGET IT

Fluency with basic facts encompass the ability to use basic facts (single digit addition and multiplication facts) to accurately, efficiently and flexibly solve single-digit and multi-digit computational problems. How children come to know basic facts is a critical determinant of how they use these facts. It is commonly acknowledged that fluency builds on a foundation of conceptual understanding and strategic reasoning, as well as problem solving experience (NCTM, n.d.).

Knowing the basic facts is defined here as being able to directly retrieve correct answers to single-digit addition and multiplication problems from memory. The educational goal is not for children to know (directly retrieve or recall) <u>all</u> single-digit facts but rather for them use a combination of direct retrieval and decomposition strategies (strategies that make use of known facts) to solve single-digit problems. Curriculum documents indicate that children will predominantly use retrieval-based strategies to solve single-digit addition problems by at least Year 3 and single-digit multiplication problems by Year 4.

Children who do not use retrieval-based strategies (retrieval and decomposition) at a time when they are expected to do so will be hindered in their ability to attend to important features of numbers during instruction and through experience. Consequently, poor fluency with basic facts makes learning mathematics increasingly difficult and is arguably the most influential single factor contributing to Australia's 'long tail' of underachievement in mathematics. While significant investments have been made in reform-orientated teaching approaches to improve children's understanding, problemsolving and reasoning abilities, the benefits of these will not be fully realised unless achieving fluency with basic facts is targeted at the same time. Research is needed to better understand the role of practice in building fluency and delineating the type(s) of practice children are most likely to benefit from. The focus of this summary paper is on single-digit addition.

#### THE ROLE OF PRACTICE IN LEARNING RETRIEVAL-BASED STRATEGIES

There are at least three types of practice that children have opportunities to engage with in the classroom. Children can be exposed to *problem-based practice*; that is, practice solving single-digit addition problems using strategies of choice.



Children can also benefit from being explicitly taught more efficient backup strategies like min-counting (also known as the count-from-larger strategy) and decomposition strategies. If children are required to practice a specific strategy like min-counting after explicit instruction, this is referred to here as *strategy-based practice*. Alternatively, children may be given opportunities for problem-based practice after they have been explicitly taught or exposed to a new strategy (for example, during whole class discussion). The difference between strategy-based practice and problem-based practice is the level of choice afforded during practice. As it is not possible to explicitly teach a direct retrieval strategy, a core component of interventions designed to increase the use of retrieval for single-digit addition has been *fact-based practice*. With this type of practice, children are repeatedly exposed to and rehearse each problem with the correct answer.

Problem-based practice can lead to the discovery and strengthening of more efficient back-up strategies: countingall strategies are replaced by counting-on strategies (including min-counting), decomposition strategies emerge, and retrieval eventually comes to dominate performance. But some children do not benefit from problem-based practice in this way; they do not construct more efficient strategies but stay reliant on inefficient strategies. It has been common wisdom to expose these children to alternative types of practice, including strategy-based practice involving mincounting and fact-based practice. While these approaches may help children to use a more efficient strategy, they are unlikely to build fluency. If teachers are to help all children learn basic facts in a way that promotes fluency, it is imperative to better understand the benefits of problem-based practice, why it doesn't always produce changes in strategy use, and how the efficiency of problem-based practice can be improved. Microgenetic methods are most suitable for investigating these issues.

#### A SUMMARY OF FINDINGS TO DATE

Research in Australia indicates that around 50% of students are still counting to solve single-digit addition when they are expected to use retrieval-based strategies (Hopkins, 2016; Hopkins & Bayliss, 2017). In a series of studies employing microgenetic methods (de Villiers & Hopkins, 2013; Hopkins & de Villiers, 2016; Hopkins & Russo, 2017), the following findings have come to light.

- Problem-based practice stimulates a constructive/transformative process of change in strategy use but much practice is needed before retrieval-based strategies dominate performance.
- The persistent use of inefficient counting strategies are easily resolved by exposing children to more efficient counting strategies and providing opportunities for problem-based practice.
- Errors occur for a variety of reasons (e.g., losing track of the count, faulty decomposition strategies, incorrect double facts, digit confusion, guessing) and can impede the benefits of problem-based practice depending on their consistency, but responses to errors need to be carefully considered.
- Tie facts (doubles) are harder for children to learn than is commonly anticipated and alternative backup strategies are needed.
- Teaching approaches emphasizing subitising and partitioning can improve the efficiency of problem-based practice but still much practice is needed before children come to know basic addition facts.
- Hypothesis: strategy choice is key for developing fluency and is the cognitive mechanism underpinning confidence.

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## Maths in the mulch: real-life learning through a kitchen garden program

Stephanie Davies, Stephanie Alexander Kitchen Garden Foundation

#### INTRODUCTION - DEVELOPING MATHEMATICAL MINDSETS IN THE KITCHEN GARDEN

Kitchen garden schools across Australia are using real-world inspired kitchen and garden spaces for students from Foundation to Year 10 to learn all kinds of mathematics. They do this through growing, harvesting, preparing and sharing their own fresh, seasonal, *delicious* food, and they're loving every minute of it. Excellence in mathematics teaching can be supported and achieved through a hands-on approach informed by physical, tangible kitchen and garden contexts as the lens for learning. When engaging directly with these sensory spaces, learning about maths implicitly and explicitly, students grow a healthy mathematical mindset to learning through real-life examples and practice, using a unique approach that underpins the curriculum and offers students – as well as educators – diverse educational opportunities and experiences.

#### THE KITCHEN GARDEN PROGRAM AS THE LENS THROUGH WHICH TO LEARN MATHS

Maths is colourful, dirty, textural and tasty! Students learn this is especially so when they've spent time outside plotting beds, planting out, and nurturing the plants over time, and following up with harvesting crops, cooking and sharing their produce. Learning in the kitchen and garden is inclusive – students identify and get creative with something they feel passionate about doing, and get involved when working with their peers and teachers. Through learning by doing, and without realising it, they explore mathematical principles at their fingertips. This gives rise to learning opportunities for students of all capabilities. Teachers are actively engaged in the modelling of mathematical concepts and understanding, to further deepen and enhance their student's knowledge and attainment of maths-based skills and abilities. This helps make textbook maths teaching less confrontational and embeds the mindset that lets students make active and authentic links. This makes a kitchen garden program an invaluable resource when used to extend on the four walls of the maths classroom.

#### CASE STUDIES OF MATHEMATICAL LEARNING TASKS

The following examples illustrate three mathematical learning tasks that can be taught to students in both primary and secondary settings, across Years 3 to 8. While mapped to learning outcomes in the Australian Curriculum, they can be adapted and differentiated to suit the learning needs of a range of students. Kitchen garden schools have actively participated in the teaching of these tasks with successful learning outcomes for their students.

#### SHAPE MAKERS - PRIMARY YEARS 3/4

In this learning topic students have an opportunity to explore the role that shape plays when shown in the production of different foods. This also includes the presence of learning about geometry when learning about foods that can be folded, such as pasta, pastry and dough. This task can be translated from the classroom into the kitchen, when learning in the kitchen there is a focus on these foods. Food is actively used as the teaching tool and resource.

As a topic that can be taught over a range of different lessons, students begin by exploring common shapes known to them including, but not limited to; triangles, circles, squares, rectangles, parallelograms, trapeziums. This knowledge is then used when exploring the types of pasta shapes available when either purchasing ready-made pasta from the supermarket, or as 'shape inspiration' when making it from scratch. Students begin to see the world of shapes that are all around them! Pasta shapes can include; agnolotti seen as a half circle, penne as a parallelogram, ravioli as a square, lasagna as a rectangle and fazzoletti as a triangle.

Learning continues from this lesson, into an in-depth study of triangles and their angles. In doing so, the food item of samosas are used as the learning stimulus. Students begin using paper in learning how to fold what would be a samosa when in the kitchen, while learning about the 30 and 60 degree angles related to the equilateral triangles being utilised for this task. In a similar nature, when rolling out pastry dough students investigate circles, domes and crescents when using pastry cutters and estimate how many shapes can be cut from *x* sized pastry sheets. These are tangible examples of the relationship between maths learning and the kitchen.



#### PIZZA PARTY - PRIMARY YEARS 5/6

In this learning task, over the duration of a series of six lessons, students are introduced to a range of mathematics outcomes including but not limited to; problem solving, data collection, budgets, financial plans, addition, subtraction, fractions and cost of food by weight. Working in small teams, over a set period of time students plan to hold a 'pizza party' using either their very own cooked pizzas or store bought pizzas – whichever may be more financially viable, after some intensive planning and consideration for this event!

Students begin by considering their own preferences for the pizzas for the event, and inquire into which type of pizza is 'better' and why, home-made or bought? Factors and consideration may include students own personal preferences for flavour, taste, cost and economy, any environmental consideration and the 'fun factor' when it comes to making their own pizza. Students collect data based on these factors and variables and beginning planning for their event from here. Once a comparison begins to be drawn between home-made and bought pizzas, students get down to thinking about pizzas by flavour and type based on ingredients (i.e. margherita), and the class devises a menu for which pizzas will be offered. From here price comparisons can begin to be researched for what types of pizzas cost what, and how this translates to cooking their own using the ingredients that are the stars of different types of pizzas.

Store bought pizzas costs are established, and students then focus on the maths behind recipe calculations and a pizza dough base master costing grid. Students therefore undertake a thorough investigation into the costs of ingredients, and divide these by weight based on how much of an ingredient would be used during the cooking and production process. Budgets are devised for the event, and order forms created from which guests can order their pizza from. Learning from this task comes from an in-depth inquiry into economy, costs, food production and time. When students as a class share their findings an amazing event to showcase their learning can be held.

#### MONEY AND FINANCIAL LITERACY - SECONDARY YEARS 7/8

When introducing the concept of economical shopping and 'best buys' through financial mathematics, students undertake an investigative task which takes them through a kitchen or household shopping list when purchasing food products through a supermarket. Real-world learning through either means of an excursion to a local supermarket, or through utilising an online shopping website, gives students an opportunity to develop an understanding of the costs of both packaged and fresh seasonal foods. This includes the factors that influence costs such as; seasonality, food miles, branding/packing along with weight and the cost per litres or grams.

Once students have established an understanding of economical shopping, and have a bench mark by which to consider the costs of food products during Year 7, this learning can be translated into developing a concept for holding a 'farmers market' stall when in Year 8.

Produce grown in the school garden can offer a fantastic opportunity for students to sell what they have grown through the real-world context of a farmers' garden market stall. Students can establish price points and set costs to sell produce on a per singular item basis, for bunches, by weights or through fractions when dividing whole food items into thirds, fifths, tenths and so on. In doing so learning about costs using the decimal system is enhanced, and their active involvement in the running of a stall allows them the opportunity to directly interact with the concepts of profit, loss and discounting. A farmers market stall can be held as a seasonal or annual event to demonstrate and support student learning in financial literacy and mathematics.

