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# MATHS expLOSION 2016



STEM

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The opinions expressed in this publication do not necessarily reflect the position or endorsement of the Editors or of the Mathematical Association of Victoria.

## FOREWARD

The theme of the 53<sup>rd</sup> Annual Conference Proceedings of the Mathematical Association of Victoria is “Maths e<sup>x</sup>plosion”. The explosion is evident in the diversity of topics and interests of the contributing authors and presenters for 2016. This compilation of papers includes a range of issues and innovative approaches to the teaching of mathematics, such as; social justice issues that build the capacity to teach learners with special needs, innovative professional learning opportunities, inspiring fireworks in the classroom, and, questioning who the winners are in high-stakes testing. The papers contribute to the growing body of knowledge in the field across all ages, from Foundation through to the secondary years, and onto pre-service and in-service teacher education.

We thank the authors for their outstanding contributions and their ongoing commitment to sharing their insights into the teaching of mathematics with the educational community. We were delighted to attract first-time contributors as well as our regular, seasoned professionals who continue to support the MAV. It has been a pleasure and privilege to work closely with such a dedicated team of authors.

The MAV conference organising committee work tirelessly to bring together an educationally rich conference that is instrumental in improving mathematics education for our students now and into the future. Their role in mathematics educators’ ongoing professional learning through this conference and proceedings cannot be over-emphasised. We thank them for their valuable contribution.

Finally, we thank our reviewers for their dedication, time and insightful comments. Their contribution to the continued high standard of the MAV conference proceedings is greatly appreciated.

We hope you enjoy this engaging and informative selection of papers.

Warmest regards,

*Dr. Wanty Widjaja, Dr. Esther Yook-Kin Loong, Dr. Leicha A. Bragg*  
*Editorial team - Deakin University, Melbourne*

## **The Review Process for the Mathematical Association of Victoria's 53<sup>rd</sup> Annual Conference Proceedings**

The Editors received 24 papers for reviewing. 12 for the Double blind review process, for which the identities of author and reviewer were concealed from each other. Details in the papers that identified the authors were removed to protect the review process from any potential bias, and the reviewers' reports were anonymous. Two reviewers reviewed each of the papers and if they had a differing outcome a third reviewer was required.

In addition we received 5 papers for the Peer review process and 7 Summary papers. These papers were reviewed by a combination of external reviewers and the editorial team.

In total 23 papers are published in the Mathematical Association of Victoria's 53<sup>rd</sup> Annual Conference Proceedings. A total of 21 reviewers assisted in the process, all of whom provided thoughtful feedback and were outstanding in responding quickly to requests.

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**Maths e<sup>x</sup>plosion**

**Blind Review**



# MORE THAN SUBSTITUTION: AN ONGOING JOURNEY OF THE 1:1 IPAD PROGRAM

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*The drive for the incorporation of digital technologies in the primary mathematics classroom is becoming more prevalent in our schools and curriculum documents. It is vital these technologies enrich the learning experience to make this integration worthwhile, and not simply be employed as a substitution for already satisfactory pedagogical approaches. The SAMR Model (Puentedura, 2006) incorporates four levels of technology integration in the classroom, from the simplicity of “Substitution” through to the transformative level of “Redefinition” (p. 7). In this paper, teachers from Ringwood North Primary School share how the SAMR Model impacted their teaching practice to support, enhance, and personalise student learning in mathematics through the 1:1 iPad Program. Tasks created with user-friendly and easily accessible digital resources such as Padlet, Kahoot! and Explain Everything are shared. Educators should consider the application and suitability of these digital technologies in conjunction with an appropriate level of integration prior to employment as a tool for enhancing mathematical understanding.*

## **Integrating Digital Technologies in the Primary Classroom**

Our global connectedness is increasingly facilitated and enhanced by digital technologies. More and more we see the proliferation of handheld devices for professional, personal and social connections. This digitally enhanced connectedness is evident in the process of collaborating on this paper which occurred over two hemispheres, via Skype for Business, Zoom, email, Google Docs, Adobe PDF, Microsoft word, Youtube, iPad applications [apps], and a multitude of internet sites. Reading this paper in ten years’ time, most or possibly all, versions of these software programs will be obsolete. Therefore, we as educators have a challenge to offer educational opportunities that build children’s understanding of mathematics within the present climate of this digitally enhanced world and prepare our students as life-long learners for what might come in the future.

Awareness of the importance of incorporating digital technologies in to students’ everyday experiences of school is evident in the Australian Curriculum: Mathematics through the requirement for teachers to develop in students “ICT capability when they investigate, create and communicate mathematical ideas and concepts using fast, automated, interactive and multimodal technologies” (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2015). There have been critics of the effectiveness of the implementation of the Australian Curriculum in relation to ICT. Philips (2015) claimed a general lack of teachers’ competencies, the outdated computing skills mandated in the learning outcomes, as well as the overwhelming choice of digital tools. These criticisms were refuted by ACARA (Lambert, 2015) who provided evidence to support the use of technology. This evidence included noting the currency of 21<sup>st</sup> century skills embedded in the Australian Curriculum, highlighting professional learning offered to teachers, and an acknowledgement of the wide range of digital tools on offer but

with a focus on developing key essential concepts, such as computational thinking that can be applied across these digital tools.

The integration of digital technologies in the curriculum is seen as one possibility to support deep learning. Deep learning is described as developing “the learning, creating and ‘doing’ dispositions that young people need to thrive now and in their futures” (Fullan & Langworthy, 2014, page i). This integration may prove challenging for educators who are skeptical about the effectiveness of implementing digital technologies, such as iPads, to develop deep learning. Novice implementation of digital devices can be employed in a superficial way (Romrell, Kidder & Wood, 2014), and there is evidence that significant mathematical learning was not achieved through short-term usage of iPads (Carr, 2012). However, the London Knowledge Lab’s report (Clark & Luckin, 2013) collated a range of research evidencing iPads moving beyond simple drill and practice apps, to supporting learning collaboratively (Henderson & Yeow, 2012), providing personalised learning experiences (Gasparini, 2011), augmenting and enhancing deep learning (Heinrich, 2012), and promoting connectedness (Clarke & Svanaes, 2012). Further, O'Malley, Lewis, Donehower and Stone (2014) found iPads were effective in building basic mathematical fluency, engagement and interest in middle-years children with special needs.

Whilst research shows positive outcomes in mathematical learning through digital technologies, an examination by Highfield and Goodwin (2013) of the pedagogical quality of popular educational apps offered for iPads showed 89% are “drill and practice” instructive pedagogical designed apps – where learners learn the content and complete set questions; with few apps (11%) promoting manipulation; and zero percentage of apps offered constructive pedagogical design – where learners can collaborate and learn together through feedback and sharing. If we aim is rightly to shift away from “drill and practice” apps then careful consideration of the selection of apps that promote higher-order thinking and engagement with mathematics in a meaningful way, as well as, professional learning support for teachers (Attard, 2013) is necessary for successful integration of digital technologies in the mathematics classroom.

The plethora of digital technology choices available to educators can be exciting and inspiring for some, while others may find the current situation overwhelming, time consuming, and frustrating. For those teachers who recognise themselves in the latter group, the Substitution, Augmentation, Modification, and Redefinition [SAMR] Model (Puentedura 2006) described below, may be an especially beneficial approach to employing digital technologies beyond the introductory level of substituting newer technologies for already successful teaching approaches. We describe the ongoing and rewarding journey one school has taken as a result of the implementation of the SAMR model into their 1:1 iPad Program.

## **The SAMR Model**

Ringwood North Primary School has employed the SAMR Model (Puentedura 2006) as a guide for planning and using technology meaningfully in the primary classroom. The SAMR Model (Figure 1) encompasses four levels of technology implementation and impact on student learning; Substitution, Augmentation, Modification, and Redefinition. Each level is described below.

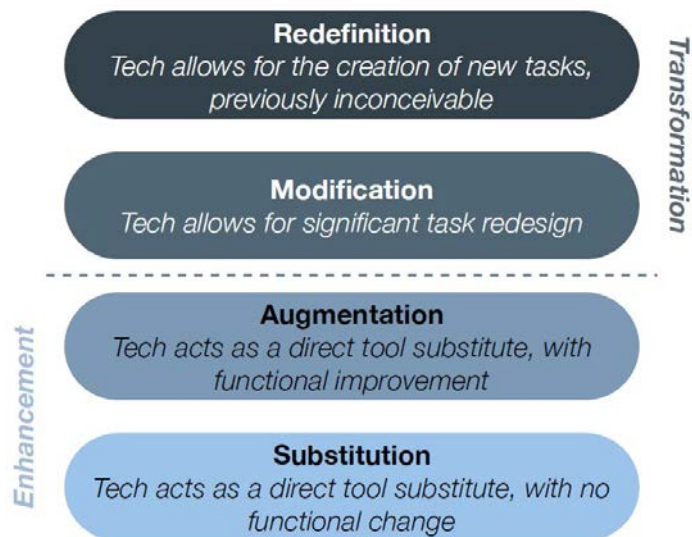


Figure 1. The SAMR Model (Puentedura, 2013).

The first stage, *Substitution*, is where technology is substituted without functional change. For example, a typical mathematics activity might be completing equations from a worksheet. Technology is incorporated when the Adobe PDF version of a mathematics worksheet is accessed and completed electronically via an iPad. This simplistic approach may be appropriate when teachers are starting to use iPads: it is easy to begin at this level whilst gaining skills for the integration of digital technology at higher levels. As Puentedura explained, “some substitutive uses are very good, very important, but you shouldn't expect to see great changes in student performance as a result of them” (2006, p.6). Although students may begin to make slight improvements in their performance, such as fluency through practice, it is not at the depth of the higher levels of the SAMR Model. Teachers must aim for higher levels of the SAMR Model to achieve higher student learning outcomes.

At the next level, *Augmentation*, there is a subtle change from the *Substitution* level; here technology is again substituted for other tasks but added features provide easier use or functional improvements to the task (Puentedura 2006). For example, When looking at the topic of angles, students can use the app Skitch (2016), to take photos around the school of different objects that reveal angles, e.g. playground equipment, basketball court lines. Students then measure their angles and annotate over the pictures with their findings.

The level of *Modification* is demonstrated through significant changes made to learning outcomes and improvements made to simple tasks (Puentedura 2006). For example, students can simultaneously record their voice and annotate concepts as they give an account of how they addressed a mathematical problem using Explain Everything (described below). Once completed, students can reflect on their problem-solving skills or contribute their ideas on an app such as Padlet (described below) where their peers have the opportunity to provide instant feedback. A task such as this exemplifies the *Modification* level of the SAMR Model as annotating and recording work samples provides greater insight into student understandings, whilst sharing ideas through the use of technology allows for instant feedback and promotes collaboration. By extending their audience, students can seek feedback on their work and share their knowledge with others in new, meaningful ways.

*Redefinition* is the highest level of the SAMR Model and where technology alters the learning tasks through creating learning experiences that were previously inconceivable or unable to be achieved without digital technology intervention (Puentedura 2006). For example, students may demonstrate their learning by collating their work into a multimodal iBooks using iBooks Author (2016) and publishing to global communities through the iBooks store. Through sharing their work with a wide audience, students can access new levels of feedback, collaboration, and reach higher order thinking by reflecting on this process. Prior to the introduction of digital technologies in the classroom, students would not have had access to publishing their work to such a wide audience.

All layers of the SAMR Model offer gains in students' learning (Puentedura 2006); redefining each task is not necessary. Technology used in the right context can be useful and valuable at all levels of SAMR (Puentedura 2006). Focussing on students' learning outcomes should drive mathematics planning. Before implementing digital technology educators should ask what educational value technology holds in our lessons. The SAMR Model is employed as a guide to assist in the planning of lessons and inform teaching. It 'provides a framework to support educators and instructional designers in creating optimal learning experiences using mobile devices in education' (Romrell, Kidder & Wood 2014, p.1). Next, we describe our 1:1 iPad Program and which was guided by the SAMR Model.

## **The Evolution of the 1:1 iPad Program**

Ringwood North Primary School is a coeducational school located in the eastern suburbs of Melbourne. There is a current enrolment of 523 students across 21 classes from Foundation to Grade 6. Ringwood North Primary School is renowned for its innovative and embedded use of technology and is an *Apple Distinguished School*, meaning it meets the criteria for innovation, leadership and educational excellence and reflects Apple's vision of technology in schools (Apple, 2016). This innovative use of technology is achieved through the 1:1 iPad program in Grades 4 - 6, and through a class set of travelling iPad Minis and classroom desktop computers from Foundation to Grade 3. The school places great emphasis on its community engagement and global connections whilst also focusing on the key learning areas of the current curriculum.

The school's 1:1 iPad Program journey began in mid-2010 when it was involved in Victoria's Department of Education and Training's *iPads for Learning - In Their Hands Trial*. This trial saw over 700 iPads distributed across ten educational settings. At Ringwood North Primary School, 138 iPads were allocated to Year 4 and 5 students to commence the trial, with these iPads following the students into Year 5 and 6 in 2011. By 2013, as the technology evolved, the school moved to a 'Bring Your Own Device' system, where Year 5 and 6 parents had the option to provide their child with an iPad to bring to school or lease one of the trial iPads. In 2016, the 1:1 iPad Program was extended to include Year 4 students. The teachers and students have witnessed the evolution of the iPad first-hand and recognised it is critical to establish a well-designed strategic plan for implementation to be successful (O'Malley, Jenkins, Wesley, Donehower, Rabuck & Lemis, 2013). With a need for our own strategic plan, the introduction of the SAMR Model has allowed teachers to further create, innovate, and personalise student learning. Selected digital technologies, such as Padlet, Kahoot!, and Explain Everything, implemented at Ringwood North Primary School employing the SAMR Model are described below.

### **Padlet**

Padlet (2016) is a powerful online-based, collaborative tool that allows students to interact and brainstorm in one creative space at the same time. However, Padlet is fundamentally different to a simple brainstorm exercise as it reaches higher levels of the SAMR Model due to students and teachers being able to collaborate simultaneously and read other people's ideas as they are written. Mallon and Bernsten (2015) outlined that the use of these collaborative learning spaces create an online community where powerful collaborative work can occur and these experiences promote critical thinking and reflection. Padlet takes this further as it allows users to add pictures, hyperlinks, and videos. This multimodal space is also a powerful feedback tool and a space for students to share their work. As Fuchs (2014) outlined 21st century learners should be asking questions, offering opinions and participating in discussions. Padlet allows students to question and comment in a safe, online space. As technology develops, teachers should be providing students with different ways to express their understandings.

Padlet is a sharing based platform that can be employed for assessment purposes through brainstorming, collaborative feedback or uploading of tasks completed on students' devices. We use Padlet to evaluate our children's prior knowledge of mathematical concepts before commencing a unit of work (see Figure 2) and thus are more informed when planning our teaching. By asking questions in class, their answers are collected in the one space and discussed as a teaching team as to the needs for broadening our students' understandings. Padlet is a space where questions can be asked, ideas shared and images / audio recordings posted. For example, we have asked our students to explain how to teach others to complete a multiplication equation and post the explanation on Padlet. Our students also set up personal

Padlets where their mathematics tasks are posted and allows them to seek feedback from their classmates in real time. Tasks such as these achieve the *Modification* and *Redefinition* levels of the SAMR Model as this technology is used to redesign tasks and allows students to collaborate and seek feedback in ways not seen in the past.

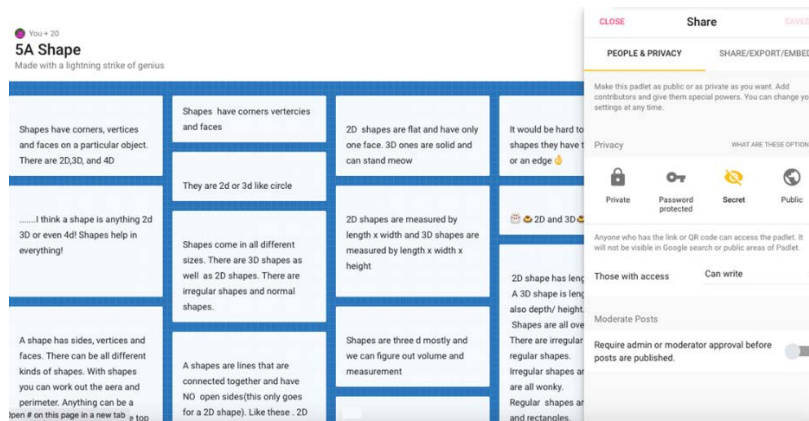


Figure 2. Screenshot of children sharing understandings of 2D shapes on Padlet.

In terms of privacy, Padlet provides a safe learning environment that can only be accessed by those who have the link. The program settings allow you to make Padlets private and hidden from the view of others. For your own records, you can export Padlets to your computer as images or a PDF, and share via email too. Padlet is a powerful tool as it is versatile, accessible across a range of devices, and participants are not required to create an account (Fuchs 2014). The use of technology in today’s classrooms should be planned and purposeful. Padlet is an engaging collaboration tool that offers students an outlet for creative ideas in a shared space (Mallon & Bernsten, 2015).

### Kahoot!

Kahoot! (2016) is a free online quiz platform that can be utilised in many classroom settings. Students of today have the disposition to create and do (Fullan & Langworthy, 2014) and creating online quizzes like Kahoot! allow them to showcase their knowledge through digital mediums. Kahoot! seamlessly integrates into any lesson for assessment purposes, is ideal for transitioning between lessons, or a simple way to invigorate children after the lunch break. The quiz is run through the teacher’s device, revealing the questions and multiple choice answers on the screen (see Figure 3). After the quiz has been started, four ‘buttons’ will appear on the students’ screen. They need to tap the ‘button’ that corresponds to the answer they think is correct. The quicker the students respond, the more points they will earn. There is also an option to place a time restraint on each question or leave it open so each student has a longer opportunity to answer, however the lack of a time limit can slow down the activity considerably.



Figure 3. Screenshot of Kahoot! ‘What are the odds!?’ quiz and students’ device

Kahoot! is a powerful resource teachers can incorporate into the everyday mathematics classroom. It can be used for assessment purposes as it offers teachers the option to export a spreadsheet with results after the quiz has been completed. To ensure the assessment is successful, it is prudent to inform students that random guessing should be avoided as it may have a negative impact on the outcome. We

utilised this option at the beginning of the school year when undertaking operations and place value to track our students' accuracy with automatic recall of basic number facts along with, and more importantly, gauging current understandings after the summer break. Kahoot! may also be used to refresh concepts from the previous lesson so teachers might evaluate if it is appropriate to move on with new content, or revisit current thinking.

Kahoot! provides students' with the opportunity to create their own quizzes both independently or in a collaborative group. The quiz can be shared with peers, allowing students to seek feedback and publish work to a wider audience. These attributes offer the possibilities of this task reaching the *Redefinition* level of the SAMR Model as this task is not achievable without the technology required to run the program. Access to Kahoot! offers students the opportunity to apply solutions, skills, and understandings to real-world problems with authentic audiences (Fullan & Langworthy, 2014) by giving students an online platform to share their work and collaborate in ways formally not possible this technology. By publishing their work to the wider world, students are able to seek feedback and use this information to improve their work.

### **Explain Everything**

Explain Everything (2016) is an iPad application that runs on the premise of student self-explanation of tasks and promotes reflection and critical thinking. Self-explanations that promote critical thinking, inferencing, and deeper understanding of concepts are powerful tools for learners (Chi, Bassok, Lewis, Reimann & Glaser, 1989). The strengths of self-explanations lie heavily with conceptually orientated tasks such as modeling problems or talking through difficult concepts (Nathan, Mertz & Ryan, 1994) and it is shown to facilitate deep learning (Weerasinghe & Mitrovic, 2006; Chi et. al., 1989).

Explain Everything is a powerful application that allows students to simultaneously record their voice and annotate concepts as they explain them. The app works with the use of simple pen and eraser tools, as well as, the use of the iPad's microphone. As students create their work and explain their understandings, they can pause and re-record parts of their presentation with the use of slides. These functions can also help students separate their work into steps so that they may explain their reasoning of each section in detail. In teaching mathematics, we have implemented this program through students recording their understandings of mathematical concepts (see Figure 4) and uploading their work to their digital portfolio. In this space, we have provided feedback to students and evaluated their insights and misconceptions of the concept for assessment purposes and future planning.

This year, we have commenced creating a digital archive of 'answers' to hands-on problem-solving boxes. To do this, we offered small groups of students (3 to 4 children) a physical problem-solving box that included worded problems with hands-on materials and invited them to find possible solutions. Once they solved the problem to their satisfaction, they planned and recorded a digital 'answer'. This answer is then stored digitally on the app SeeSaw (2016) and catalogued so that other groups may access these answers to check against their solutions. Students can provide feedback on others' work in this space. The use of collaborative work in the digital realm where feedback and published work distributed to a wide audience is a previously inconceivable task that exemplifies the *Redefinition* level of the SAMR Model.

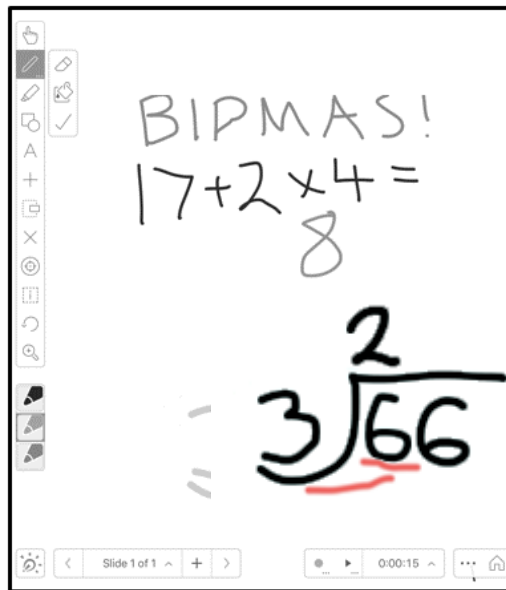


Figure 4. Screenshot of students' work sample from Explain Everything.

Explain Everything has a range of potential uses in the classroom; from explaining mathematical concepts, talking about and illustrating comprehension tasks, and annotating other peoples' work are just some possibilities. In teaching mathematics, we aim to improve student outcomes and promote deep levels of understanding. In order to do this, promoting self-explanations and critical thinking can help students develop their ideas in more detail. Studies show that students do not necessarily self-explain spontaneously, but rather, through prompting or guided activities (Bielaczyc, Pirolli & Brown, 1995; Chi, De Leeuw, Chiu & Lavancher, 1994; Weerasinghe & Mitrovic, 2006). As teachers, we should aim to provide well-planned activities with the use of digital technologies that scaffold our students' learning through mathematical concepts and allow them the chance to self-explain and share their understanding of these concepts.

## Conclusion

The SAMR Model (Puentedura 2006) provides a framework to encourage the use of digital technology in meaningful ways and ultimately enables deep learning of mathematical concepts. With the implementation of the 1:1 iPad program and the SAMR Model as a planning framework at Ringwood North Primary School we have found that technology offers a supportive collaborative environment that enriches the learning experience. By designing tasks in conjunction with the SAMR Model we are able to develop and implement well-planned programs that benefit students and create a new environment in the mathematics classroom for teachers and students that was previously inconceivable. While each level of the SAMR Model provides different opportunities for students' engagement in digital technologies, it is only through the quality of the learning task that deeper student learning will ultimately develop. By promoting mathematical deep learning (Fullan & Langworthy, 2014; Heinrich, 2012) and collaboration (Henderson & Yeow, 2012) through the use of technology, we are able to offer personalised learning in the mathematics classroom (Gasparini 2011). As O'Malley et al. (2013) advised 'in the 21<sup>st</sup> Century, teachers must know not only how to use technology, but when and why to use it.' (p.14). For the success of such programs in other schools, a systematic, whole school approach is needed (O'Malley et al. 2013) and the school community needs to be on board for change. While our understanding of the functionality of the digital technology has enhanced over the last six years through the 1:1 iPad project, it is the incorporation of the SAMR Model that keeps us mindful of our role as teachers in persistently considering the appropriateness and suitability of these digital technologies.

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# ARE THERE ANY WINNERS IN HIGH-STAKES MATHEMATICS TESTING? A QUALITATIVE CASE STUDY EXPLORING STUDENT, PARENT AND TEACHER ATTITUDES TOWARDS NAPLAN NUMERACY TESTS IN YEARS 3 AND 5

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*Through the annual implementation of National Assessment Program – Literacy and Numeracy (NAPLAN), testing of mathematical standards across Australia invokes questions about the impact that high-stakes testing has for the teaching and learning of mathematics. According to recent studies on high-stakes testing, the role of the teacher is instrumental in children’s achievement results. The purpose of this case study is to explore perspectives about NAPLAN from key participants at one Western Australian Primary School, namely: students, teachers, and parents. The paper will report on the extent to which instructional pedagogy at one school has been affected by the implementation of NAPLAN testing and subsequent publication of results. Consistent with a phenomenological perspective, the qualitative data for this investigation were collected through semi-structured interviews and field notes. These data offered particular insights into how key participants viewed the impact NAPLAN testing has had on the instructional pedagogy in Year 3 and Year 5 classrooms.*

## **Introduction**

Since the implementation of the National Assessment Program – Literacy and Numeracy (NAPLAN) in Australian schools in 2008, debate concerning the rationale of such testing has escalated steadily. The purpose, value, and results of NAPLAN have come under scrutiny from teachers, parents, ministers for education and politicians (Belcastro & Boon, 2012). The amount of criticism towards high-stakes testing of school-aged children continues to rise, with much debate focussing largely on the benefits of NAPLAN and the effects this procedure has on the well-being of all involved. Chiefly the debate centres around the question of ‘Are there really any winners in high-stakes testing?’ In answering this question, White and Anderson (2012, p. 61) doubted whether a high-stakes test could improve learning, “particularly when we consider it within the context of the time it takes to get back, the excessive time often taken to prepare for it, compounded as it is by the pressure many schools feel as a result of NAPLAN being published online via the *My School* website”. Furthermore, much has been written concerning the negative impact NAPLAN has had on the teaching and learning of mathematics. Some of the prominent issues include educators teaching to the test, a perceived narrowing of the curriculum, and the disempowerment of teachers (Bagnato & Yeh Ho, 2006; Klenowski & Wyatt-Smith, 2011). For instance, Thompson and Harbaugh (2013) highlighted that teachers face increased pressure for their

students to succeed due to publication of NAPLAN results on the *My School* website which can be viewed internationally. Perso (2009) underscored how mathematics teachers are concerned about getting the educative ‘balance’ right in terms of adequately preparing their students for NAPLAN and not altering their programs to teach to the test. Other writers have identified various negative effects NAPLAN testing has had on students’ mental health and well-being in general (Carter, 2012; Quinell & Carter, 2011; White & Anderson, 2012).

The Whitlam Report (ACARA, 2013) outlined the benefits of having a national a high-stakes testing procedure. The report stated that if used with other appropriate assessments, NAPLAN can “provide valuable data on student numeracy and literacy outcomes to a range of stakeholders as part of NAPLAN reporting” (ACARA, 2013, p. 7). At the same time, the Australian Curriculum Assessment and Reporting Authority (ACARA) cautions that NAPLAN results should be used as a snapshot of students’ achievement and should be viewed as only one of the high-quality assessments in the course of the year. In a similar manner to which the Programme for International Assessment (PISA) and the Trends in International Mathematics and Science Study (TIMSS) results are viewed, NAPLAN is the measure through which governments, education authorities, schools, and the community can determine the extent to which young Australians are meeting important educational outcomes (ACARA, 2014; Belcastro & Boon, 2012; Klenowski, 2010). Furthermore, Supovitz (2009) noted that high-stake test results have become the primary indicator of school and student performance within Australia, with monetary or non-monetary rewards, and a range of interventions offered for low-performing schools.

## **Purpose and Justification**

The purpose of this research is to investigate the impact of high-stakes testing on the teaching and learning of mathematics for students, classroom teachers, and parents. Au (2007) states that a test is considered ‘high stakes’ when its results are used to make important decisions that affect students, teachers, administrators, communities, schools, and districts. The research also explored the extent to which NAPLAN test conditions contributed to student performance and student self-perceptions in mathematics. To obtain these perspectives from key stakeholders, qualitative data were gathered through the exercise of semi-structured interviews and researcher-generated field notes. Using these methods, the researchers wished to give individual children, their parents, and their teachers, a voice in sharing their experiences about how NAPLAN has affected their relationships with mathematics. In doing so it hoped that educators can better understand student, teacher, and parent perceptions about NAPLAN testing, which has been identified as an area of need (Belcastro & Boon, 2012).

## **Key Research Question**

The key research question for this project is: *What is the impact of high-stakes testing on the teaching and learning of Mathematics in one Western Australian Catholic Primary School?*

## **Sub-questions**

Three sub-questions were developed from the key research question.

*What is the impact of high-stakes testing on mathematical teaching and learning for Year 3 and Year 5 students?*

*What is the impact of high-stakes testing on mathematical teaching and learning for Year 3 and Year 5 teachers?*

*What is the impact of high-stakes testing on the understanding of mathematical teaching and learning of the parents or guardians of Year 3 and Year 5 students?*

## **Methodology**

### **Case Study**

This research was conducted through an intrinsic case study (Stake, 1995) where all data were collected from one Western Australian Catholic primary school within a low socio-economic area as defined by the *My School* website. A case study approach was chosen because the researchers wished to carry out

a detailed investigation over a period of time within a particular context (Hartley, 2002). By involving students, parents and teachers from Year 3 and Year 5, the researchers were able to explore the extent to which high-stakes testing had an impact on the teaching and learning of mathematics for these participants. Specifically, the case study design enabled the researchers to discern similarities and differences through both Year 3 and Year 5 cohorts for students, teachers, and parents.

## Participants

The parents, children, and teachers involved with the Year 3 and Year 5 classrooms were the key participants in this research. Six children from each year were selected purposively by their classroom teacher based on their academic ability; two students achieving results at an A level or higher, two students achieving at the intended target for that year level, and two students achieving at a D level or lower. To ensure the holistic nature of the research, parents of participating students and the Year 3 and Year 5 classroom teachers were also interviewed.

Table 1 *Number of participants*

	Students	Parents	Teachers
Year 3	6	3	3
Year 5	6	3	3

The adult participants were chosen to discern the extent to which their experiences affected their understanding of NAPLAN, and their relationship with both the teacher and the school. To allow for an appropriate commentary, the parents were interviewed shortly after NAPLAN results were disseminated.

## Methods

The researchers used semi-structured, qualitative interviews and took field notes as data for this research. Individual interviews were conducted face-to-face with key participants soon after the NAPLAN test had been administered (teachers and students) and after the results had been disseminated (parents). Conducting interviews at this time allowed the researchers the best opportunity to ascertain a true account from all participants regarding their perceptions of the testing. The interviews were recorded so they could be transcribed and analysed at a later date. The researchers also took field notes during the interviews to note any salient observations or emerging thoughts arising during the interviews.

## Data Analysis

The researchers analysed qualitative data collected from the child and adult participants according to a framework offered by Miles and Huberman (1994) which comprises the stages: data collection, data reduction, data display, and conclusion drawing/verification. Following the data collection stage, and within the framework employed, the researchers used a content analysis process to interrogate the data. According to Berg (2007, p. 303), content analysis is “a careful, detailed systematic examination and interpretation of a particular body of material in an effort to identify patterns, themes, biases and meaning”. Using this analytical process the researchers were able to generate themes, and ultimately, key findings.

## Presentation of Findings

The results of the study are organised into three categories according to research participant grouping, namely: students, teachers and parents. The most prominent theme that emerged for each stakeholder group is presented in the table below.

Table 2. *Emergent themes from data analysis*

Stakeholder Group	Theme
Year 3 and Year 5 Students	Anxiety about sitting NAPLAN
Year 3 and Year 5 Teachers	Changes in Pedagogy in preparation for NAPLAN
Year 3 and Year 5 Parents	Discrepancy in results from school report and NAPLAN results

## Students

All Year 3 and Year 5 students reported feeling a heightened sense of anxiety when sitting the NAPLAN test. Compared with Year 5 students, Year 3 students described a higher level of anxiety as they had not sat the NAPLAN test before. To illustrate, one Year 3 student recalled: “[The teacher] explains things better when it’s not NAPLAN stuff. Because they don’t explain the test before you do it. We spread the desks out, so we could concentrate on the test”. However, even though Year 5 students had sat NAPLAN before, 80% of students expressed feeling nervous. For instance, a Year 5 student remembered “I get just a little anxious if I don’t get the question and I skip it and don’t get time to go back”. Overall the students’ reported anxiety was related to the test conditions they were placed under and the arrangement of the classroom, rather than the actual test itself. As one Year 3 participant noted: “NAPLAN day is different because I am not allowed to ask my teacher questions”. According to eight of twelve students interviewed, the classroom environment was altered from the regular classroom setting. The desks were placed in single file, which led to a feeling of isolation as one Year 5 student explained “I felt a bit nervous and insecure because I felt like it was only me getting tested and no one else”.

## Teachers

Teachers indicated that they felt some pressure from parents in regards to NAPLAN. Year 3 teachers were able to reassure parents who were anxious about NAPLAN, with one teacher noting: “We just try to calm the parents down so that it’s not a big deal. It’s only one test. Most parents are good with that explanation”. However, Year 5 teachers experienced greater pressure from parents as NAPLAN results are used as entrance criteria for secondary school. One Year 5 teacher relayed general concerns from parents, “I still think a lot of parents here put a lot of value on the results, because they are worried about the results for high school. What if they don’t perform well and they need the results to get into high school?”

Initially, all teachers declared that they did not alter their pedagogical approaches in preparation for NAPLAN. One teacher stated “Personally, it’s not a big deal for me, and I don’t like teaching to the test, so I’d never teach something just because they’re going to include it”. However, after probing during the interview, all teacher participants conceded that they did alter their mathematics lessons mainly through the arrangement of the classroom. To illustrate this another teacher stated

*It would definitely be fair to say that it is different from the normal way you teach, you move the desks, the week before or whatever are moved into their test conditions, we’re just sitting our test like this because of classroom conditions.*

Teachers also commented how they used NAPLAN practice tests within their classrooms to ensure that children were adequately prepared for the test under time constraints. One teacher commented

*We do more practice tests prior to NAPLAN; we have photocopied practice tests and get them to complete them under timed pressure so they get used to sitting for 40 minutes and working non-stop for 40 minutes.*

In addition, all teachers stated how the time they spent using practice tests resulted in a 'narrowing' of mathematics education, in that they were unable to teach the prescribed mathematics curriculum to students.

## Parents

NAPLAN results affected the relationship between the school and parents differently within the two cohorts. Year 3 parents were not concerned that the school performed under the national average in mathematics, as they felt the school report was more important than the NAPLAN results. One Year 3 parent commented, "No I wasn't concerned. I will wait for her school report. For me the school report is more important, the children spend a lot of time with those teachers". However, the relationship between the school and the Year 5 parents appeared somewhat damaged through expressed concerns about the NAPLAN results affecting their child's acceptance into secondary school. One Year 5 parent stated her concerns, "I don't want to hand this application into [school] knowing that it's not a true evaluation of my child....I've got the primary school that doesn't worry about it but then the high school that does". Four of six parents noted the difference between their children's achievements in school reports and in NAPLAN. In particular, parents stated that they were confused by these differences and recalled feeling unsure as to "who was telling the truth". One parent stated

*So what worries me is, is our academics maths class really an academic maths class, when compared with the rest of Australia? Maybe our whole level is not right? Maybe we shouldn't have an academic maths class if we're not up there with the academics.*

The parents articulated confusion and disappointment with the results and began to question the merit of the school. For instance, one parent remarked on the difference in the results between the school and the neighbouring school. This was evidenced by one parent who noted "I did go on the *My School* website to see what the other school got and I was thinking why I am paying money at this school when the other school has better results?" This real discrepancy in reported results was echoed by three other parents interviewed.

## Discussion and Conclusion

The aim of this research was to investigate the impact of NAPLAN on the teaching and learning of mathematics for students, teachers and parents at one school. Findings indicated that the anxiety students experienced was caused by alterations to the classroom environment and the time restrictions placed on them. These findings support the earlier work of Belcastro and Boon (2012) who suggested the effect an unfamiliar classroom environment can have on 8 year old children. Walking into the altered environment gave students a sense that they were about to engage in an unfamiliar activity. The test conditions also contributed to their heightened sense of anxiety, which is consistent with other commentators (Carter, 2012; Watson et al., 2002). For instance, Year 3 and 5 students at this school are usually allowed considerable time when completing regular classroom tests. However, NAPLAN time constraints resulted in students feeling nervous to complete the test as they did not have enough time to check their answers as would have been typical classroom practice. Teachers indicated changing their pedagogy mainly through alterations to the physical environment and the inclusion of sample tests in mathematics lessons. According to teacher testimony, the conditions prescribed by ACARA have inadvertently forced them to change their pedagogical approaches. Such practice 'narrowed' the mathematics curriculum by replacing lessons with sample NAPLAN tests, a finding consistent with Perso (2011). Parents commented that NAPLAN results had affected their relationship with the school, due to the concern and confusion arising from disparate NAPLAN results and school reports (Downey, 2000). On the whole, Year 5 parents found it more difficult than Year 3 parents to reconcile the differences due to their children preparing for secondary school.

Anxiety experienced by students, pressure on teachers to alter pedagogical practice and the confusion and disappointment experienced by parents concerning NAPLAN reporting, all indicate that there are no winners in high-stakes testing for the participants of this study.

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# ACCESSING MATHEMATICAL CONTENT THROUGH THE PROFICIENCY STRANDS

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*The Australian Curriculum: Mathematics differs from previous state versions of curricula by the inclusion of four Proficiency Strands alongside the three Content Strands. The Proficiency Strands are the power behind the curriculum as they shape the way in which the content should be accessed. Mathematically rich tasks that use a problem solving or inquiry approach allow students to develop deep understanding of mathematical concepts through reasoning and communication that will lead to greater fluency.*

## **The Proficiency Strands**

The Australian Curriculum: Mathematics (ACM) (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2016) is constructed around three Content Strands; Number and Algebra, Measurement and Geometry and Statistics and Probability. These strands provide teachers with guidelines about what to teach. Sullivan (2012) called these the ‘nouns’ of the curriculum. The ACM does more than inform teachers about the content areas that need to be addressed. Through the four Proficiency Strands; Understanding, Fluency, Problem Solving and Reasoning, the ACM shapes the way in which the mathematical content should be accessed, providing some direction on how this may be achieved. Askew (2012) explained that these Proficiency Strands should be enacted during the learning of mathematical content rather than just when applying it. Maintaining the grammar analogy, Sullivan (2012) referred to the proficiencies as the ‘verbs’ of the curriculum. The importance of these ‘verbs’ was endorsed by Burns (2012) who affirmed that the Standards for Mathematical Practice, the equivalent to the ACM Proficiency Strands in the US, should be front and centre of mathematics teaching and learning.

The Proficiency Strands in the ACM were adapted from Kilpatrick, Swafford and Findell (National Research Council, 2001) who conveyed five strands:

- Conceptual understanding
- Procedural fluency
- Strategic competence
- Adaptive reasoning
- Productive disposition (p. 5)

They saw the strands as being mutually dependent and intertwined like a rope, demonstrating that the individual strands should not and cannot be viewed in isolation from each other, or from the content they are acting upon. Kilpatrick et al. (2001) argued that problem solving is where all of the mathematical proficiencies come together, to provide a way for students to weave all of the strands together in such a way that allows teachers to assess student performance on all of the proficiency strands. Problem solving, after all, is what mathematicians do (Holton & Lovitt, 2013; Rigelman, 2013).

Lappan and Phillips (1998) in the Connected Mathematics Program developed criteria for a good mathematics problem:

- The problem must have important, useful mathematics embedded in it.



- Students must be able to approach the problem in multiple ways, using different solution strategies.
- The problem should allow various solution strategies or lead to alternative decisions that can be taken and defended.
- The problem should engage students and encourage classroom discourse.
- Solution of the problem should require higher-level thinking and problem solving.
- Investigation of the problem should contribute to students' conceptual development.
- The mathematical content of the problem should connect other important mathematical ideas.
- Work on the problem should promote skilful use of mathematics and opportunities to practise important skills.
- The problem should create opportunities for the teacher to assess what students are learning and where they are experiencing difficulty. (pp. 87-88)

## Task Selection

These criteria for good problems look very similar to the qualities of mathematically rich, investigative tasks (e.g. Day, 2012, Flewelling & Higginson, 2003; Lovitt & Clarke, 2011). Both have all of the Proficiency Strands embedded within them as well as stressing the importance of mathematical content. The choice of mathematically rich inquiry tasks is critical, although the tasks themselves need to be driven by quality pedagogy and teacher decision making (Aubusson, Burke, Schick, Kearney, & Frischknecht, 2014). As Boaler (2016) pointed out "Teachers are the most important resource for students." (p. 57). Sullivan and Davidson (2014) identified key considerations teachers need to make when selecting tasks as having multiple entry and exit points, encouraging deep and sustained thinking as well as argumentation. Hunter (2014) suggested that rich mathematical reasoning ensues when teachers play the vital role of demanding that students justify results and encourage generalisation.

Boaler (2016) identified six questions teachers should ask themselves when adapting or designing tasks for better mathematical learning:

1. Can you open the task to encourage multiple methods, pathways, and representations?
2. Can you make it an inquiry task?
3. Can you ask the problem before teaching the method?
4. Can you add a visual component?
5. Can you make it low floor and high ceiling?
6. Can you add the requirement to convince and reason? (pp. 77-86)

Lovitt (personal communication, 2012) would add: Can you make it kinaesthetic? When selecting tasks, teachers should consider whether these criteria have been addressed.

It is important when selecting, adapting or designing tasks that teachers consider both the mathematical content as well as the pedagogical considerations of how the mathematics is to be 'mined' from the task in such a way that develops deep conceptual understanding. Mason (2015) maintains that in order for mathematical thinking to take place a conjecturing atmosphere needs to be developed in classrooms. Carefully crafted tasks, in the hands of skilful teachers, "offer students more and deeper learning opportunities" (Boaler, 2016, p. 90).

## A Sample Task

'Greedy Pig' is a well-known dice game that is used in many classrooms (see *Figure 1*). There are several versions and adaptations of this game (e.g. [www.maths300.com](http://www.maths300.com), [www.nzmaths.co.nz/resource/greedy-pig-0](http://www.nzmaths.co.nz/resource/greedy-pig-0), <https://rich.maths.org/1258>, [http://www.curriculumsupport.education.nsw.gov.au/digital\\_rev/mathematics/assets/stage4/g\\_pig.pdf](http://www.curriculumsupport.education.nsw.gov.au/digital_rev/mathematics/assets/stage4/g_pig.pdf))

The class stands up, the teacher rolls a dice and students score whatever is rolled. This is repeated, with students aggregating their scores. They decide when to sit down and hence keep their score to that stage. However, if the number 2 is rolled while they are standing up their entire score for that round is zero. A game is made up of 5 rounds and the first 2 rolls of each round are free, so even a 2 can be scored. The investigative question becomes *When is the best time to sit down?*

*Figure 1.* How to play ‘Greedy Pig’ (Lesson 5, maths300)

When I first saw this activity, I thought it was a great game and I took it straight back to my classroom. The students loved playing ‘Greedy Pig’ and often asked if they could play the game. What I had neglected to consider was what mathematics I hoped the students would learn as a result of playing this game, or how I could engineer the task to ensure that there was rich mathematical learning occurring rather than the students just having fun. I have nothing against students having fun in mathematics lessons, as long as they are not just having fun, but are also learning important mathematical concepts and skills. Just playing the game does have some incidental learning associated with it, such as being too ‘greedy’ provides feedback about how likely it is that a two is rolled. Similarly by playing too safe students soon see that they never amass a large score. However there is so much more mathematics that can be ‘mined’ from this simple game and by employing the Proficiency Strands the quite sophisticated mathematical content can become accessible to the students.

The game setting of ‘Greedy Pig’ allows all students to enter the task in a non-threatening environment. By physically playing the game, the context for making sense of the mathematics is set and, as students tend to enjoy this game, they want to know more about how to determine a successful strategy for winning. When investigating the potential for this task and aligning it to the Australian Curriculum: Mathematics (ACARA, 2016) content descriptors were identified from Years 3-12 that could be accessed through this task. In Years 5-9 there were eleven possible Statistics and Probability content descriptors that could relate to this task, depending on the path the teacher chooses to take (see Appendix 1). This fulfils the criterion that good tasks are easy to begin and have multiple exit points as well as having rigorous mathematical content within it (Boaler, 2016; Lappan & Phillips, 1998; Sullivan & Davidson, 2014).

The collection of class data after playing and totalling the points from five rounds begins the students’ mathematical journey with this task. The use of student-generated data is much more relevant to students and this data is best represented with a stem and leaf plot. Students tend to find stem and leaf plots quite intuitive and, if they are new to this representation of data, I find that I only have to show a few students how the plot works and then the others demonstrate their understanding of the process by watching and learning from others. This to me is a much more powerful approach than running a formal lesson on how to generate stem and leaf plots. It is another example of incidental learning within this task. The data collection provides the springboard to using statistical inference to start thinking about when is the best time to sit down when playing Greedy Pig, which in turn leads to other aspects of understanding and fluency in statistics and probability.

The idea of a probability distribution is visually apparent within the stem and leaf graph, demonstrating the likelihood of scores falling within a particular range. The use of statistical measures such as range, median, quartiles and other comparative statistics can easily be addressed once a stem and leaf graph is constructed. By finding patterns within the graph and suggesting reasons for those patterns begins the process of statistical inference and allows students to start to generate hypotheses for when the best time is to sit down. It is also important that students recognise that the graph represents a limited number of trials which is an indicator of a general population, as this will lead to the big probability idea of long run frequency (The Law of Large Numbers).

The problem solving and reasoning aspects of this task are introduced by the question “When is the best time to sit down and how do you know?”. Once students have generated some hypotheses it is time to test those hypotheses. I always get students to work in pairs for this part of the task, as the communication and reasoning aspect is heightened when the students have someone other than me to convince. This is when the introduction of technology for students to gather data about the performance

of different strategies is useful. In doing this students are beginning to model what mathematicians do and the role technology plays in their investigations. By using the maths300 software students are able to play a hundred or more games and keep statistics in the form of stem and leaf graphs, mean, median, upper and lower quartiles as well as highest and lowest scores for each strategy. The software also introduces the visual box and whisker display which changes as each game is played, so students can see how it is formed and the effect of large numbers of trials of stabilising the statistics. When students have narrowed their preferred strategies down to two, they can use the software to compare the strategies. For older students this could entail an investigation of the levels of confidence needed to be convinced whether one strategy is superior to another.

Once students have decided on their preferred strategy, it is important to put their strategy to work by playing five more rounds. There is an opportunity for discussion about expected frequencies versus observed frequencies at this stage, as well as the introduction of back-to-back stem and leaf plots so that before-and-after strategy development comparisons can be made with the class data. Other challenges, such as investigating the effect of changing the ‘killer’ number can be introduced as can calculating the theoretical results. A full explanation of a method to calculate the theoretical probabilities may be found in Holton and Lovitt (2013). Finally, students can write a mathematical report about how they developed their strategy, including the reasoning they used when testing strategies, the content that they learnt and how they used statistical analysis to predict probability outcomes.

## Conclusion

This is just one example of a task in which employment of the Proficiency Strands allows teachers and students to access a range of interconnected and important content through a mathematically rich inquiry task. Rather than just playing the game and hoping that some mathematics will be incidentally learnt by students, carefully designed learning experiences based on problem solving, reasoning, fluency and the development of concepts that promote deep understanding mean teachers can help students to access important mathematical content in an interconnected way. My analogy for the Content Strands and the Proficiency Strands is a car. I see the Content Strands as the passengers in the car, or the ‘nouns’ as Sullivan (2012) referred to them. Passengers are important, we want to get our passengers from point A to point B safely and securely, albeit sometimes with some detours on the way. I see the Proficiency Strands as the engine of the car, or the ‘verbs’ as Sullivan described them, as they are the power behind the ACM. In order to move the passengers to their destination, I need the engine of the car to enable the car to transport the passengers safely and securely.

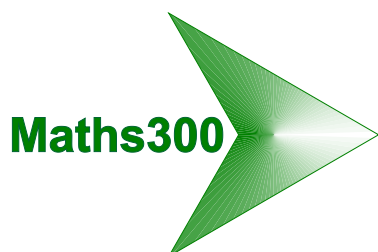
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## Appendix 1

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Connecting with the Australian Curriculum: Mathematics

### Greedy Pig

Year Levels: 3-12

**Proficiency Strands:** Problem Solving, Reasoning, Understanding, Fluency

**General Capabilities:** Literacy, Numeracy, Critical and Creative Thinking, ICT Capability

**Content Strands:** Number and Algebra, Statistics and Probability

#### Number and Algebra

Year 3: Recall addition facts for single-digit numbers and related subtraction facts to develop increasingly efficient mental strategies for computation ([ACMNA055](#))

#### Statistics and Probability

Year 3: Conduct chance experiments, identify and describe possible outcomes and recognise variation in results ([ACMSP067](#))

Year 4: Identify events where the chance of one will not be affected by the occurrence of the other ([ACMSP094](#))  
Select and trial methods for [data](#) collection, including survey questions and recording sheets ([ACMSP095](#))

Year 5: List outcomes of chance experiments involving [equally likely outcomes](#) and represent probabilities of those outcomes using fractions ([ACMSP116](#))

- Pose questions and collect categorical or [numerical data](#) by observation or survey ([ACMSP118](#))
- Year 6: Describe probabilities using fractions, decimals and percentages ([ACMSP144](#))  
 Conduct chance experiments with both small and large numbers of trials using appropriate digital technologies ([ACMSP145](#))  
 Compare observed [frequencies](#) across experiments with expected [frequencies](#) ([ACMSP146](#))
- Year 7: Assign probabilities to the outcomes of events and determine probabilities for events ([ACMSP168](#))  
 Construct and compare a range of [data](#) displays including stem-and-leaf plots and dot plots ([ACMSP170](#))  
 Calculate [mean](#), [median](#), [mode](#) and range for sets of [data](#). Interpret these statistics in the context of [data](#) ([ACMSP171](#))  
 Describe and interpret [data](#) displays using [median](#), [mean](#) and range ([ACMSP172](#))
- Year 8: Investigate techniques for collecting [data](#), including [census](#), sampling and observation ([ACMSP284](#))  
 Explore the variation of means and proportions of random samples drawn from the same [population](#) ([ACMSP293](#))
- Year 9: Construct back-to-back stem-and-leaf plots and histograms and describe [data](#), using terms including ‘skewed’, ‘symmetric’ and ‘bi modal’ ([ACMSP282](#))  
 Compare [data](#) displays using [mean](#), [median](#) and range to describe and interpret [numerical data](#) sets in terms of location (centre) and spread ([ACMSP283](#))
- Year 10: Determine quartiles and [interquartile range](#) ([ACMSP248](#))  
 Construct and interpret box plots and use them to compare [data](#) sets ([ACMSP249](#))
- Year 10A: Calculate and interpret the [mean](#) and [standard deviation](#) of [data](#) and use these to compare [data](#) sets ([ACMSP278](#))
- Years 11/12: Calculate measures of central tendency, the arithmetic [mean](#) and the [median](#) (ACMEM050)  
 Perform simulations of experiments using technology (ACMEM150)  
 Recognise that the repetition of chance events is likely to produce different results (ACMEM151)  
 Identify relative frequency as probability (ACMEM152)  
 Determine the probabilities associated with simple games (ACMEM157)  
 Use relative frequencies obtained from data as point estimates of probabilities. (ACMMM055)  
 Recognise the mean or [expected value](#) of a discrete [random variable](#) as a measurement of centre, and evaluate it in simple cases (ACMMM140)

# ALGORITHMIC ADVENTURES IN TEACHING “KEN YOU DO THE KENKEN?”

**Georgia Gouros**

*Distance Education Centre Victoria*

*The KenKen (KenKen Wikipedia,2016) puzzle that appears daily in many newspapers around the world is the trademarked name for a style of arithmetic and logic puzzle invented by Japanese math teacher Tetsuya Miyamoto. The puzzles require the integers 1 to n to be placed in each row and column of a (n × n) square grid or matrix. Similar to Sudoku (Sudoku Wikipedia,2016), the final solution to KenKen has the digits 1 to n appear only once in each row and once in each column. The point of difference of KenKen compared with Sudoku is that in the former there are adjacent grid cells grouped together into so called cages which may be linear, corner shaped, L-shaped or multi-row-column. Each cage has an arithmetic target number which must be met by the digits in these cells when applying the operation on the cage rule. In this document the solution to the KenKen puzzle is explored using Brute force methods and Backtracking patterns of algorithm design for implementing a coded solution. There are two aims that the author wishes to achieve, the first is to impart an understanding to students of how to abstract a problem or puzzle together with an exploration of possible patterns that can be identified and used to form a solution. Secondly students consolidate their learning by implementation of algorithms using a block based programming language such as Edgy (Bird et al,2016), which has been especially developed for the study of VCE Algorithmics HESS (Victorian Curriculum and Assessment Authority,2014) for High School students in Victoria, Australia. The Edgy language has evolved from the SNAP (University of Berkeley,2016) a block based programming language and incorporates basic abstract data types and graph/network structures together with associated methods and operations.*

## **Introduction**

Most teenagers engage with technology and algorithms everyday with little understanding of the underlying principles of the software that they use on their mobile phones and personal computer devices and tablets. The study of problems and the creation of algorithms in High School is an opportunity for the curious teenager to find out about how these algorithms are created and implemented. The popular puzzle KenKen can be understood and completed by the average high school student without any knowledge of advanced mathematics, requiring only simple logic and arithmetic skills. The puzzle is used by teachers in classrooms around the world to drill basic arithmetic skills and logic to find the solution. Teachers use many resources to support their classes including materials developed at Berkeley by Tom Davis (*Davis.T,2010*). The KenKen puzzle ranges in difficulty from very simple such as in the case of a (3 × 3) grid to the more difficult with puzzles typically of size (6 × 6) appearing regularly in newspapers, and to the extreme challenge of the (9 × 9) size grids that appear on the so called “official” KenKen site (*KenKen Puzzles that make you smarter,n.d.*).

In this activity students can be scaffolded into the analysis, design and implementation of algorithms. To initiate this activity, students should begin by solving KenKen puzzles by hand and exploring rules of thumb and heuristics that can be employed to quickly find a solution.

The aim is for students to have some “Fun with Algorithms” while solving a popular puzzle and learn some algorithmic concepts at the same time. Teachers are also able to use this resource as inspiration for their own classes.

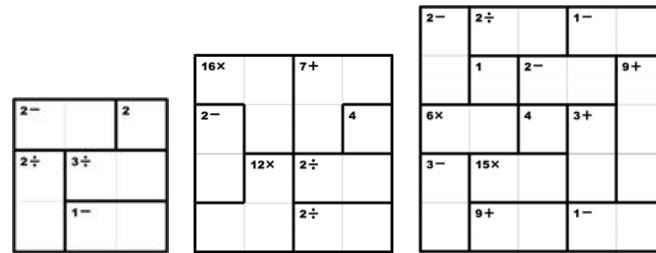


Figure 1. Examples of KenKen puzzles of dimensions  $(3 \times 3)$ ,  $(4 \times 4)$ ,  $(5 \times 5)$

In this activity students examine Brute force methods combined with some heuristics that are employed by experienced players to find the solutions. Following the exploration of the Brute Force methods students look at the Backtracking design pattern of algorithm design and create their own implementation in the block based language Edgy.

## Model

The KenKen puzzle is a square grid of dimension  $(n \times n)$  which contains  $k$  cages  $\{where\ n, k \in N\}$  each cage is a grouping of cells together with a target value and an arithmetic operation, application of the operator on the permissible integers enclosed in the cage achieves the target. Students can explore data structures that can be used to store the cages as well as enabling the visualisation of the evolving steps as it progresses to the final solution. There are many abstract data structure options available to the student to use in combination; selecting from lists, arrays, stacks, queues and graphs.

The initial state of the puzzle is set with the simplest cages of size  $(1 \times 1)$  are identified and filled with the specified integer. Many experienced players employ shortcuts and heuristics to determine other initial starting values such as identifying linear cages of length  $(n - 1)$ , in these cases if the operation is addition (+) or multiplication ( $\times$ ), the  $n$ th digit can be identified by simple arithmetic.



Figure 2. Example of finding the  $n$ th digit when presented with a linear cage of length  $(n - 1)$  and operations (+) or ( $\times$ ). The shaded cell is “4” since  $9+4=13$ , and for a  $(4 \times 4)$  KenKen each row and column will sum to 13.

## Brute Force

The first method presented to students in this study is Brute Force Model: Generating all possible arrangements or permutations. Single digit cell values and patterns of possible solutions are explored by the student to reduce the size of the solution space that needs to be checked against the cages to find the final solution.

## Backtracking

An alternate algorithm design pattern that students can explore for finding the solution is Backtracking. Cages can be ranked using a known (or invented by the student) heuristic in order from the least difficult to the most difficult to evaluate. Each cage is evaluated in sequence according to its ranking to eliminate as many wrong paths as possible and thereafter possible permutations of digits are explored and recorded as potential solutions. This method progressively works toward the final solution and explores

each permutation for its validity and integrity in comparison to possible values allowed in that row or column. Students learn how solution contenders are stored for later evaluation if needed. Permutations that do not meet the criteria are marked as dead ends and discarded and the algorithm backtracks to another possibility that is retrieved from a store of untried potential solutions.

## Exploring Some Algorithms for Solving KenKen

### Brute Force or Exhaustive Searching

This strategy involves generating all the possible permutations of digits 1 to  $n$  for a row or column that lead to a solution for the puzzle. The permutations can be generated in many ways. A recursive approach can be explored with students. An algorithm using recursion to generate all the linear permutations of  $1..n$  integer values.

```

Algorithm Gen(Input {ilist},n)
// Input: ilist a list of integers
// Input: n the size of the KenKen
if (length(ilist) > 0) then
  // remove ith item from list
  For i=1 to n do
    Report ithitem,Gen({ilist-ithitem},n-1)
  End do
End if
End Algorithm

```

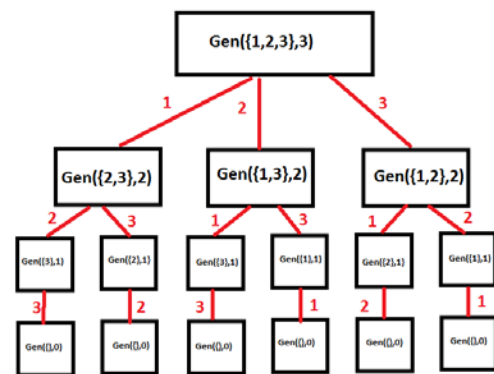


Figure 3. A call tree representing the recursive Algorithm. Traversal of edges gives all possible permutations in a row or column of a KenKen.

Students of VCE Algorithmics (*Victorian Curriculum and Assessment Authority,2014*) are introduced to time complexity analysis and should be able to deduce a recurrence relation describing the time complexity of the “Algorithm Gen” which generates all possible permutations for a row or a column. Students are also taught telescoping techniques to resolve the time complexity recurrence to a function. Since the “Algorithm Gen” progresses by trying a potential integer in a cell and then reducing the problem by that one integer and then recursively calling itself to solve the sub-problem, this results in work required described by a recurrence of  $T(n) = nT(n - 1) + c$ , where  $T(1) = 1$  as the base case of calls and “ $c$ ” indicates a constant amount of work is done outside the recursion.



Table 1. *Telescoping the recurrence relation*

The recurrence relation	Telescoping by rearranging recurrences so that constant value is on the right hand side.
$T(n) = nT(n-1) + c$ $T(n-1) = (n-1)T(n-2) + c$ $T(n-2) = (n-2)T(n-3) + c$ ..... ..... $T(2) = 2T(1) + c$	$T(n) - nT(n-1) = c$ $nT(n-1) - n(n-1)T(n-2) = nc$ $n(n-1)T(n-2) - n(n-1)(n-2)T(n-3) = n(n-1)c$ ..... ..... $n(n-1)(n-2) \dots (3)T(2) - n(n-1)(n-2) \dots (3)(2)T(1)$ $= n(n-1)(n-2) \dots (3)c$
	<hr/> <i>by elimination of like terms on either side of the "=" gives</i> $T(n) - n!T(1) = c + nc + n(n-1)c + n(n-1)(n-2)c + \dots + n!c$  $T(n) - n!T(1) = c + c(n + n(n-1) + n(n-1)(n-2) + \dots + n!)$ <i>reversing the terms in the bracket on the right hand side</i> $T(n) - n!T(1) = c + c(n! + \dots + n(n-1)(n-2) + n(n-1) + n)$ $T(n) = n!T(1) + c + cn! \left(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots + \frac{1}{(n-1)!}\right)$ <i>expressed as "big-Oh" notation showing the worst case time complexity of which is asymptotic behaviour as <math>n \rightarrow \infty</math></i> $T(n) = n!T(1) + c + \lim_{n \rightarrow \infty} (cn! \left(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots + \frac{1}{(n-1)!}\right))$ $T(n) = n!T(1) + c + cen!$ $T(n) = O(n!)$

Application of Stirlings (*Stirling's Formula Wikipedia,2016*) approximation  $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$  to the factorial gives  $T(n) = O(n^n)$ , showing intractability as  $n$  increases. The biggest KenKen seen by the author in the wild is a size  $(9 \times 9)$  which is beyond both the capabilities of the block based Edgy programming language and the patience of most teenagers.

Following the generation of possible linear permutations of  $1..n$  integer values further work follows for these permutations to be assembled into a  $(n \times n)$  grid to solve the problem. The number of possible solutions which are restricted to have each integer appearing only once in each row and column via Brute Force for a  $(3 \times 3)$  KenKen is  $3! 2! 1!$ . Generalising this method for an  $(n \times n)$  KenKen gives  $(n)! (n-1)! (n-2)! \dots (1)!$  arrangements to check for a final solution.

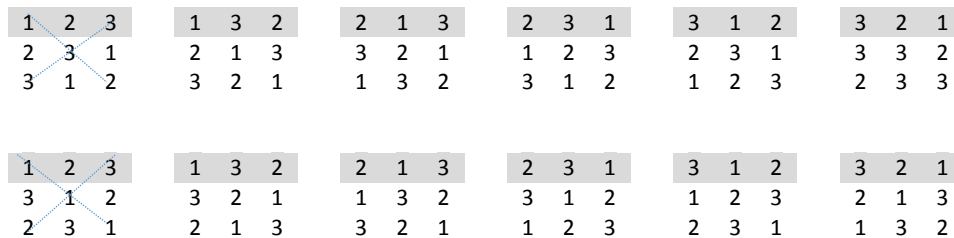


Figure 4. Example of the possibilities for any  $(3 \times 3)$  KenKen using a Brute Force method ignoring cage rules.

Students are encouraged to look for patterns in the possible solutions, for example if we look closely at the diagonals in the  $(3 \times 3)$  example, to avoid duplicating a value in a row or column, the said value

is propagated in either of the main diagonals and minor parallel diagonals, while the normal main diagonal has each permissible digit in it.

Further we consider in most of the instances of the  $(3 \times 3)$  KenKen puzzles there will exist single cell cages that hold a specified digit. There is usually one single integer in a  $(1 \times 1)$  cage given to begin the puzzle as shown in Fig. 1.

Students can realise that some work in generating the possible solutions can be reduced if we consider the restrictions imposed by single cell cage rules. This can be extended to include cases of linear cages of length  $(n - 1)$  and operations  $(+)$  or  $(\times)$  since the sum and product of digits 1 to  $n$  is easy to calculate.

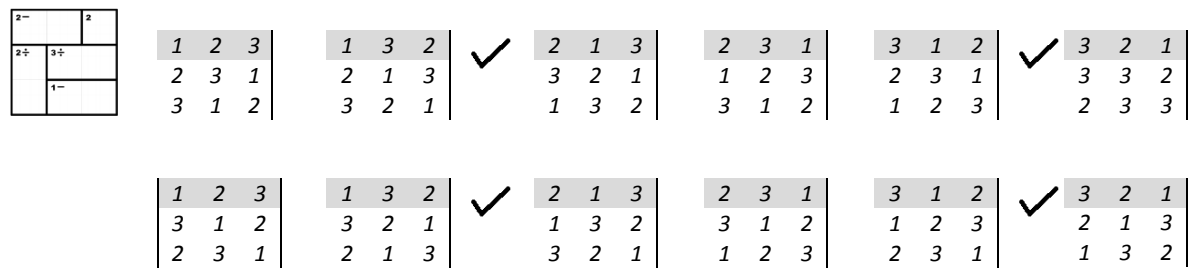


Figure 5. Example of reducing the possibilities for a given  $(3 \times 3)$  KenKen using single cell cages.

In small sized KenKen puzzles the work required to check each cage on the presented possibilities is trivial.

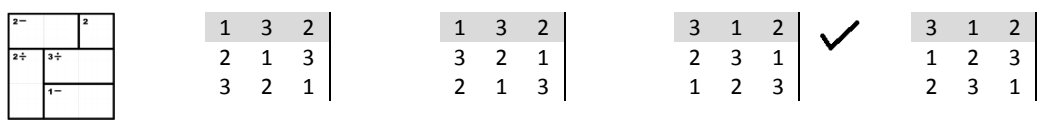


Figure 6. Example of applying cage rules to the possible solution candidates for an instance of a  $(3 \times 3)$  KenKen.

Further work can be done with students to look for patterns in KenKen  $(4 \times 4)$  puzzle solutions for many instances of the puzzle. Recognition of patterns gives an insight into the requirements for creation of new puzzles by students as well as possible ways to generate lots of potential solutions. We assume that each KenKen has a unique solution in all puzzles.

If we investigate in detail the  $(4 \times 4)$  example, using symbols to denote the digits and set theory we use patterns for generating the possible solutions to be checked by Brute Force give the following arrangements:

$$a = rand(\{1,2,3,4\}), \quad b = rand(\{1,2,3,4\} - a), \quad x = rand(\{1,2,3,4\} - a - b), \quad y = rand(\{1,2,3,4\} - x - b - a)$$

$x$	$a$	$y$	$b$	$x$	$b$	$y$	$a$	$a$	$y$	$b$	$x$	$x$	$b$	$y$	$a$	$a$	$y$	$b$	$x$
$y$	$x$	$b$	$a$	$a$	$x$	$b$	$y$	$y$	$b$	$x$	$a$	$b$	$x$	$a$	$y$	$y$	$a$	$x$	$b$
$a$	$b$	$x$	$y$	$y$	$a$	$x$	$b$	$b$	$x$	$a$	$y$	$a$	$y$	$x$	$b$	$b$	$x$	$y$	$a$
$b$	$y$	$a$	$x$	$b$	$y$	$a$	$x$	$x$	$a$	$y$	$b$	$y$	$a$	$b$	$x$	$x$	$b$	$a$	$y$

Figure 7. Examples of some patterns of solutions in a  $4 \times 4$  KenKen puzzle

Single cell cages can be used together with any recognised patterns to reduce the number of potential solutions that need to be checked against all the cages. Capable and keen students can extend this investigation and explore the pattern recognition activity for  $(n \times n)$  KenKen puzzles for  $n > 4$ .

Students are asked to explore how the time complexity of the Brute force method or Exhaustive search approach scales as the value of  $n$  increases.

Table 2. *Informal time complexity analysis of the Brute Force method for KenKens*

$(3 \times 3)$	If arrangements are generated without duplicating digits in row or column there are $3! 2! 1! = 12$ possible solutions
$(4 \times 4)$	$4! 3! 2! 1! =$ possible solutions
$(6 \times 6)$	$6! 5! 4! 3! 2! 1! =$ possible solutions
$(n \times n)$	$n! (n - 1)! (n - 2)! \dots (1)!$ NP-hard as $n$ increases problem becomes intractable, heuristics need to be employed to find the solution.

Finally students are required to develop and implement an algorithm that uses their own insights and known heuristics to reduce the number of arrangements that need be checked against the cages to find the solution.

**Algorithm KenKenBruteForce Overview**

```
// Input the dimensions (nxn) of the KenKen puzzle
// Input all the cage rules from the KenKen puzzle
// Generate all possible arrangements of digits 1..n (n)!(n-1)!
// reduce the number using heuristics and insights on possible solution
patterns
```

```
Arrangement:=1
While Arrangement <=count(Arrangements) do
  While (current arrangement is not checked) do
    // Check that this arrangement has digit 1..n appearing once
    // only in each column
    If (This arrangement does not follow the game rules) then
      // get the next arrangement
      Arrangement:=Arrangement+1
    End if
  End do
  // Check the cage rules are followed on this arrangement
  Cage:=1
  While cage <= count(cages) do
    If cage is not ok on arrangement then
      Arrangement:=Arrangement + 1
      Cage:=1
    Else
      // move to next cage
      Cage:=Cage+1
    End if
  End do
  If cage > count(cages) then
    // match found on arrangement
    Arrangement found
  Else
    // keep looking
    Arrangement:=Arrangement + 1
  End if
End do
End Algorithm
```

## Backtracking Algorithm

A very elegant solution can be found using recursion for the Backtracking design approach, usually the details of recursive Backtracking processes is hidden from view as the progress toward the solution is occurring using variables to store temporary solutions within the computer interface and the programming language. As part of learning about the Backtracking design pattern students are asked to make this process visual in its progress by creating their own structures that maintain temporary solutions and possible alternative solutions and displaying this at each stage.

Students can experiment with this design and can choose to represent their model using their own data structures, to continue the exploration in this paper we agree to use notation to represent the  $(n \times n)$  KenKen grid where each digit is stored in a string of length  $n^2$ . The visualisation of the solution as it progresses step by step could be achieved in Edgy by using a graph structure laid out in a grid, where the cage rules are *edges* connecting cage cells and *node* labels are raster images (*a raster image is an image represented by a matrix of coloured pixels*) of the digit that is being tried in that position.

<p>Cells are numbered sequentially starting from the top left corner to the bottom right corner as shown.</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr><td>[1]</td><td>[2]</td><td>[3]</td><td>[4]</td></tr> <tr><td>[5]</td><td>[6]</td><td>[7]</td><td>[8]</td></tr> <tr><td>[9]</td><td>[10]</td><td>[11]</td><td>[12]</td></tr> <tr><td>[13]</td><td>[14]</td><td>[15]</td><td>[16]</td></tr> </table>	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]	[15]	[16]	<table border="1" style="margin-left: auto; margin-right: auto;"> <tr><td></td><td></td><td></td><td>2</td></tr> <tr><td></td><td></td><td></td><td></td></tr> <tr><td></td><td></td><td>4</td><td></td></tr> <tr><td></td><td></td><td></td><td></td></tr> </table>				2							4						<p>The initial solution with the single cell cages filled represented by 0002000000400000 as a string.</p>
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4	3	2	1																															

Figure 9. Representing the solution as a string or word during the evaluation of the final solution.

A recursive solution using Backtracking is presented as a suggested solution after students have attempted the activity. This algorithm can be reviewed and improved by students either individually or collaboratively as part of the learning process.

```

Algorithm SolveKenKenRecursiveBackTracking(n, cageId, solution)
// This algorithm uses the backtracking design pattern
// with tail recursion to build up the solution that satisfies all
// the cage rules
// Input: n is the dimension of the KenKen puzzle
// Input: cageId is the cage identification, each cage is assigned
//         integer value from 1 to the maximum number of cages
// Input/Output: solution is a string that maintains the digits in
//                the KenKen solution so far
If all the cages have been processed then
    Return solution
Else
    For each possible digit permutation of the current cageId do
        If (this combination is valid so far) then
            SolveKenKenRecursiveBackTracking(cageId+1, solution)
        End if
    End do

End if
End Algorithm

```

As part of this exploration of algorithms for solving KenKen, students are encouraged to study the mechanisms involved using Backtracking. Students are required to create their own algorithm and

implement a highly visual solution so as they can see the Backtracking process as it is working toward the solution that satisfies all the cage rules. Rather than use recursion to find the solution which hides the variables and changing environment from the observer, students are encouraged to simulate the Backtracking process by setting up their own data structures for implementation in an iterative solution.

A basic algorithm for achieving an iterative solution is supplied to students as a suggested solution after they have attempted this activity.

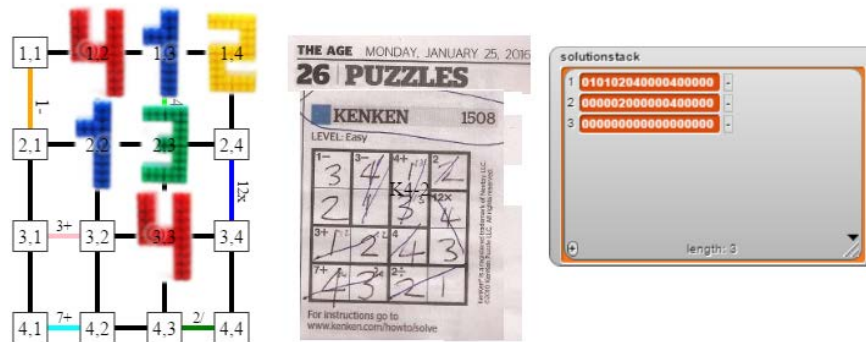


Figure 10. Visualising the solution using a graph laid out as a grid structure together with a stack of alternate permutations to try as it backtracks to and fro to final solution.

```

Algorithm SolveKenKenBackTracking(n, cageId, solution)
// This algorithm uses the backtracking design pattern
// together with a stack structure to keep track of the
// solution candidates
// Input: n is the dimension of the KenKen puzzle
// Input: cageId is the cage identification, each cage is assigned
//         integer value from 1 to the maximum number of cages
// Output: solution is a string that maintains the digits in
//         the KenKen solution so far

//Initialise the KenKen
Get the first cage
for each possible digit combination that satisfies this cage do
    push the cage id and solution onto the solution stack
end do

while (cageId < the count of cageIds) do
    pop a solution off the stack
    // explore the next cage
    Get the next cage
    For each possible permutation of this cage on current solution do
        Push the cage and solution onto the solution stack
        // Count the possibilities
        Count=Count+1
    End do

    If (Count is zero) then
        // this path is a dead end and we must backtrack
        // to a prior possibility
        Pop the solution stack back to the previous cage
        Get the previous cage
    Else
        // Keep going, this could be the one!
        Get the next cage
    End If
End do

```

## End Algorithm

Refer to the Figure 11.1 for the Edgy implementation of the SolveKenKenBackTracking subroutine.

All student solution algorithms need to check the solution for correctness against all the cage rules in sequence as a final pass.

## Learning Aims of This Activity

The KenKen puzzle is easily understood by students and can be evaluated by hand with a basic understanding of logic and arithmetic together with a few rules of thumb formally defined as heuristics. It can be used to introduce and consolidate many concepts taught in Computer Science such as modelling of the problem using data structures and describing the steps to finding a solution in terms of an algorithm.

To reinforce this learning the final algorithms are implemented in a programming language. The choice of programming language in this case is the block based Edgy, to maintain the accessibility of this activity for students with little programming experience. The simple block based programming language can be easily implemented from algorithms in natural language and or pseudocode that students have created. The Edgy language is powerful enough to achieve the solution with far less syntactical knowledge and without use of libraries for graphics and display manipulation in comparison with mainstream programming languages.

Students abstract the KenKen puzzle into appropriate data structures. A Brute Force algorithm is explored in the initial phase and its time complexity evaluated. The Backtracking solution is developed and studied as an alternative with the implementation left to the students with the proviso that the evolution of the solution must be able to be observed.

At the completion of this activity students have met many if not all of the learning objectives in cognitive domains according to Bloom's Taxonomy (*Bloom's Taxonomy Wikipedia, 2016*).

In the cognitive domain students (taking the liberty of modifying Bloom's Taxonomy (*Bloom's Taxonomy Wikipedia, 2016*)) in this activity have the opportunity to exhibit all learning objectives of:

- **Remembering** - knowledge of the universals and abstractions in a field - principles and generalisations, theories and structures. This is achieved through knowledge about available abstract data types and design patterns that can be used in algorithms.
- **Understanding** - demonstrate understanding of facts and ideas by organising, comparing, translating, interpreting, giving descriptions, and stating the main ideas. Students are able to state their intended approach to finding a solution.
- **Applying** - using acquired knowledge. Solve problems in new situations by applying acquired knowledge, facts, techniques and rules. Students apply their knowledge by implementing their algorithm in a programming language.
- **Analysing** - make inferences and find evidence to support generalisations. Students look for patterns and heuristics to enable a generalised pattern for solution.
- **Evaluating** - present and defend opinions by making judgments about information, validity of ideas or quality of work based on a set of criteria. Students are given the opportunity to analyse and compare different methods for finding a solution as well as evaluating solutions created by others.
- **Creating** - builds a structure or pattern from diverse elements. Students bring together all the elements to build their solution to this problem.

The "official" KenKen site (*KenKen Puzzles that make you smarter,n.d.*) states that "solving KenKens makes you smarter", if that is true then writing algorithms to solve KenKens surely moves you to a higher level of "smarter". Students are able to achieve a sense of achievement in producing an interactive solution that is highly visual, while gaining valuable understanding of processes that can be applied to many other problems.

## A Suggested Solution for Coding the Backtracking Algorithm in Edgy (Bird et al,2016)

Each student will of course generate their own unique implementation following the basic design principles. The following is a suggested solution provided at the conclusion of the activity and can be evaluated and critiqued by students who can suggest improvement and modifications as an individual and or a group activity. This particular implementation requires set up of lists and dictionaries to maintain visual information as well as cage rules and possible combinations that satisfy those rules. A few of the main blocks and subroutines are shown.

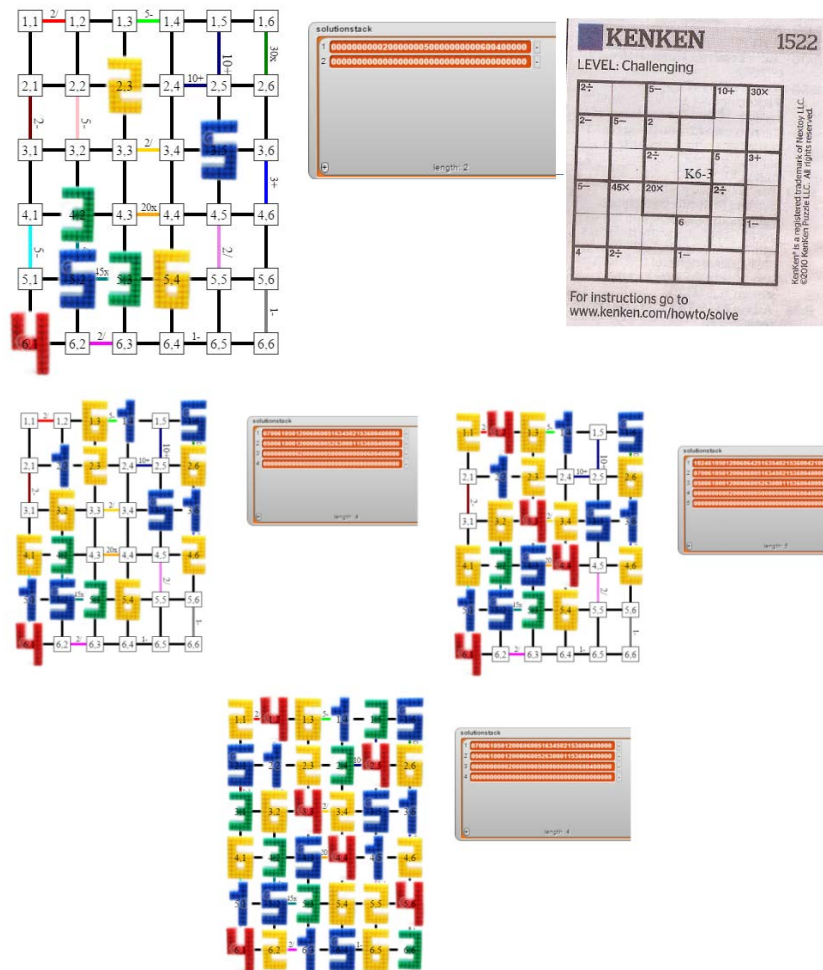


Figure 11. Visualisation of Backtracking algorithm for an instance of a KenKen (6 x 6) puzzle.

```

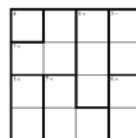
+SolveKenKenBacktracking+ n + pause +
script variables cageNo1 currSolution checkSolution
set cageNo to 1
SolveCage cageNo
repeat until cageNo > maxCages and cageNo > 0
  set curSolution to top of stack solutionstack
  set solution to currSolution
  pop from stack solutionstack
  UpdateGrid n currSolution
  say join C cageNo for 2 secs
  if pause = pause
    pause all
  SolveCage cageNo + 1
  if PushSolutionCount = 0 and cageNo < maxCages
    say DeadEndSolutionBacktrack for 1 secs
    set checkSolution to top of stack solutionstack
    say checkSolution for 2 secs
    repeat until
      (item 1 of split checkSolution by letter x 10 +
       item 2 of split checkSolution by letter) < cageNo + 1
      and
      (item 1 of split checkSolution by letter x 10 +
       item 2 of split checkSolution by letter) > 0
    pop from stack solutionstack
    set checkSolution to top of stack solutionstack
    set cageNo to
      (item 1 of split checkSolution by letter x 10 +
       item 2 of split checkSolution by letter)
  else
    change cageNo by 1

```

Figure 11.1. The main part of the Edgy program that solves the KenKen puzzle using an iterative implementation of Backtracking.



Puzzle from "The Age" newspaper Melbourne.



[6] Puzzle from **KenKen For Teachers** Tom Davis March 4, 2010





Figure 11.2. Setting up instances of KenKen puzzles in Edgy that can come from many sources



Figure 11.3. Implementation of permutations for a sum as a dictionary for each digit set. That can be added to as new rules encountered.

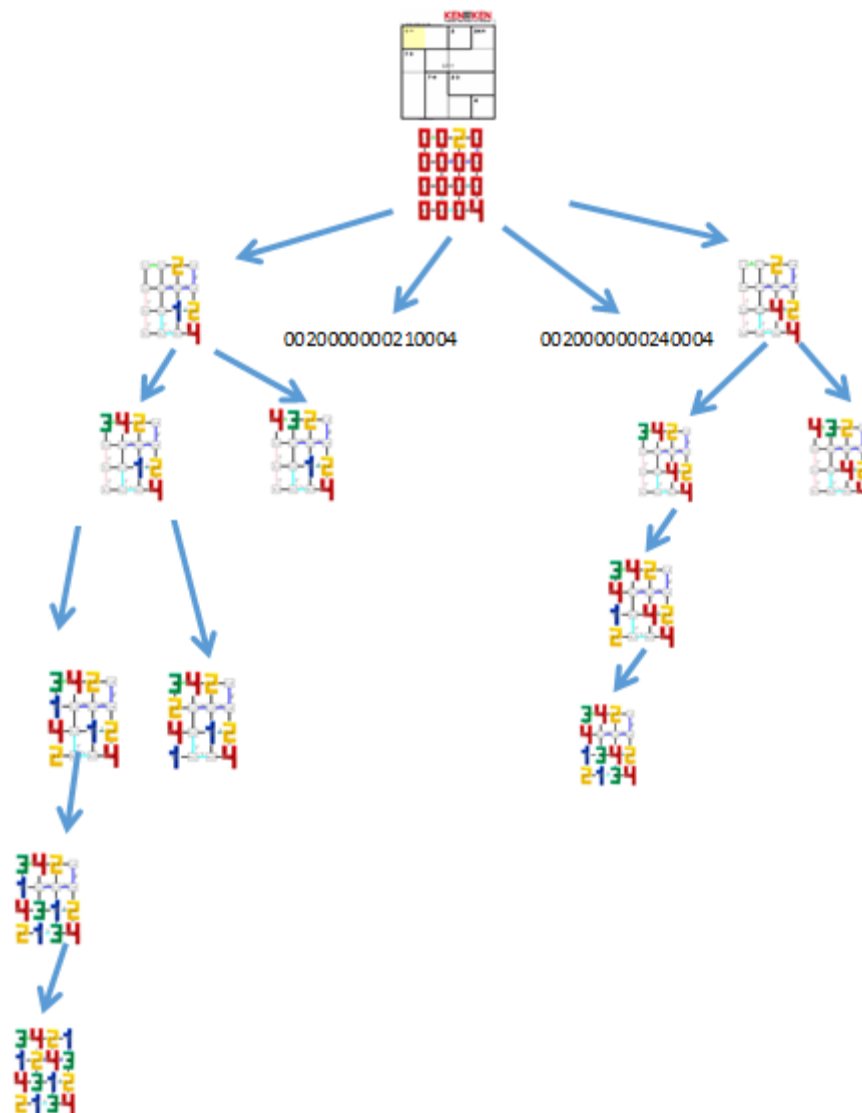


Figure 11.4. Solving an instance of a KenKen by Backtracking.

The full Edgy implementation of the suggested solution which can be downloaded and imported into the Edgy platform can be found at:

<https://drive.google.com/open?id=0B5gbhm6XKwJyaNCR2hHUGxINzQ>

A narrated recording of the Backtracking solution solving two KenKen puzzles running in Edgy exists on Youtube at this URL: <https://youtu.be/6MoqzcgZ66k>

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Davis,T.(2010). *Kenken For Teachers*. Retrieved October 2016 from [http://mathcircle.berkeley.edu/archivedocs/2009\\_2010/lectures/0910lecturespdf/KenKen\\_tom\\_davis.pdf](http://mathcircle.berkeley.edu/archivedocs/2009_2010/lectures/0910lecturespdf/KenKen_tom_davis.pdf)

*KenKen Puzzles that make you smarter*. (n.d). Retrieved October 2016 from <http://www.KenKenpuzzle.com/>

*Blooms Taxonomy Wikipedia 2016*. Retrieved October 2016 from [https://en.wikipedia.org/wiki/Bloom%27s\\_taxonomy](https://en.wikipedia.org/wiki/Bloom%27s_taxonomy)

*Stirling's formula Wikipedia 2016*. Retrieved October 2016 from [https://en.wikipedia.org/wiki/Stirling%27s\\_approximation](https://en.wikipedia.org/wiki/Stirling%27s_approximation)

Stirling's approximation (or Stirling's formula) is an approximation for factorials.

# MATHEMATICS CLASSROOMS - WHY ARE SOME FIREWORKS AND OTHERS JUST SPARKLERS?

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*University of Notre Dame Australia*

*The name of this paper and the presentation which accompanies it is “Mathematics classrooms – why are some fireworks and others just sparklers? Why is it that some classrooms are vibrant places of lively debate, reasoning and mathematical inquiry and others are places of anxiety and disengagement? The following article will address research that underpins some of the affective elements that need to be considered in order to tackle the content of mathematics in a way that makes students feel engaged in the mathematics learning process.*

## **Introduction**

Can I ask you to please recall a time when you really loved something -- a movie, an album, a song or a book -- and you recommended it wholeheartedly to someone you also really liked, and you anticipated that reaction, you waited for it, and it came back, and the person hated it? So, by way of introduction, that is the exact same state in which I spent every working day of the last six years. I teach high school math. I sell a product to a market that doesn't want it, but is forced by law to buy it. I mean, it's just a losing proposition. Meyer, D. (2010) <https://www.youtube.com/watch?v=qocAoN4jNwc>

Above, is an extract from a “TED Talks” clip by an American mathematics educator, Dan Meyer, Dan is expressing a feeling and often a frustration, that is well-known to many people who teach mathematics. That feeling, that much of the general population does not seem to see the importance, or the relevance, let alone the elegance, of mathematics.

I would like to start by making a statement and then posing a question. The STEM (Science, Technology, Engineering and Mathematics) subjects are vital to Australia and the world. In 2013 The Chief Scientist highlighted the five most significant societal challenges as being: living in a changing environment; promoting population health and wellbeing; managing our food and water assets; securing Australia's place in a changing world; and lifting productivity and economic growth. He stated that the way to address these challenges was through high quality STEM enterprises which called for building the quality of the disciplines that are the foundation of STEM. I actually do not know of anyone who disagrees with STEM being a priority, not even those journalists renowned for opposing anything and everything, can find the will, to be dissenting voices. So why is it then that STEM subjects, and in particular Mathematics, has trouble in attracting participation?

This paper will look at just some of the myriad factors which may lead to engagement or disengagement in mathematics from our students, focusing particularly on some elements of the affective domain; the domain, which it can be argued, is the domain where teachers have the greatest opportunity to make choices.

## **Beliefs and Attitudes, Part of the Affective Domain**

If you have been around students it is not hard to argue that the affective domain, someone's beliefs and attitudes, plays a vital role in the learning of mathematics. Research is rich in studies showing that

the affective domain has a great deal of impact on teaching practices and student learning (Attard, Ingram, Forgasz, Leder & Grootenboer, 2016). It tells us that teachers who entertain positive attitudes and beliefs about the teaching of mathematics tend to: embrace innovations (Gresham, 2008); use a variety of instructional strategies (Swackhamer, Koellner, Basile & Kimbrough, 2009); be highly motivated (Bandura, 1993); encounter less stress (Tschannen-Moran, Woolfolk-Hoy & Hoy, 1998) and; have higher expectations and goals for their students (Zambo & Zambo, 2008).

If teachers with positive attitudes and beliefs are, as the research indicates, more likely, to be highly motivated, take on a variety of instructional strategies and embrace innovations; there should be a flow-on effect to the students. It does not seem unreasonable to think that these positive attitudes and beliefs will be transmitted to the students and in turn make them feel more positively disposed towards mathematics. This assumption is supported by research such as that completed by Calder and Campbell (2015) who found that reluctant students could be encouraged to become involved in mathematics through the use of engaging pedagogy. This then opens the question about what pedagogies might engage students, something which has been the topic of much research (Marshman & Brown, 2014) and much teacher debate. I will pursue this later in this paper.

## **Beliefs**

According to Tschannen-Moran and Woolfolk-Hoy (2007) there has been an accumulation of compelling evidence to support the link between teachers' beliefs about their ability to enhance student motivation, and student achievement. Beliefs are usually defined as being understandings, premises or propositions that are held true (Goldin, Rosken, & Torner, 2009), and beliefs, unlike knowledge, maybe held with various degrees of conviction (Philipp, 2007). Although much has been written about the inconsistency between teachers' beliefs and teacher practice (Beswick, 2006), teacher beliefs should not be discounted as being unauthentic but rather as representations of intended practice (Liljadhar, 2009). What this means is that a person might espouse a firm belief without necessarily enacting it, but this in no way negates or dilutes the perceived 'truth' of that belief. For instance, I can firmly believe that an inquiry-based mathematics class is highly beneficial to students but might not feel comfortable, for a whole host of reasons, about presently running my class this way.

Beliefs and action develop together and influence each other, and so can be logically argued to be related (Grootenboer, 2008). Furinghetti and Moselli (2009) wrote that "...beliefs are behind reasons for teachers' decisions and in this role, relies the importance of beliefs in relation to practice" (p. 61). Raymond (1997) proposed a model (Figure 1) to represent the relationship between teachers' mathematical beliefs and their teaching practice.

It can be seen through this model that the relationship is quite complex and I would suggest that this model only hints at the relationship's gestalt. It is clear that beliefs are complex and are subject to many mitigating and strengthening influences, for instance the many ways that the dynamic of the immediate classroom situation can influence the decisions that are made. Any experienced teacher will tell you that no two classes of students are the same, and what works brilliantly well with one group can prove an abject failure with another. For example, using manipulative materials with one cohort may promote deep, positive, reflective learning, whilst with another it provides the opportunity to play without purpose.

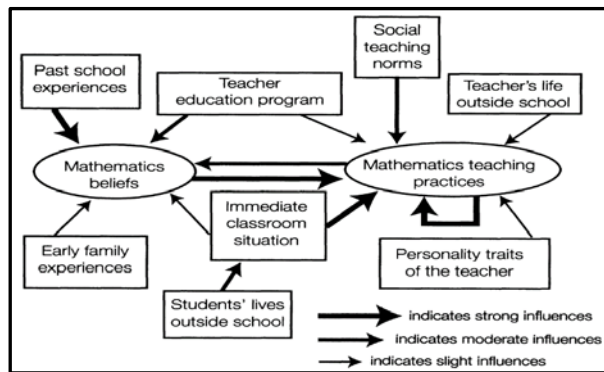


Figure 1. Raymond's (1997) revised model of the relationships between mathematical beliefs and practice

The model also illustrates the influences on teachers of their past school experiences and the teacher education programs with which they were involved. For instance, Goulding et al. (2002) wrote that beliefs are inextricably tied to subject matter knowledge as they determine how a teacher may approach mathematics teaching. For instance, if due to their own school and teacher training experiences, a teacher has a strong belief in the rote learning of routines, then this will probably have an impact on their teaching. This impact on teaching will then have a direct influence on students' beliefs. That is, the student are likely to develop a sympathetically strong view that rote learning of routines is important and therefore, this becomes their paradigm as to what mathematics is and what you require to be a mathematician.

If a teacher decides to adopt new beliefs about the teaching of mathematics they need to be offered credible, well-researched pedagogical practices that are powerful and effective. Practices which enable them to blend or shed previously held beliefs and that further enhance their students' capacity to engage in productive classroom mathematical practice.

## Attitudes

Attitude is an ambiguous concept (Hannula, 2002) and there is a diversity of definitions and constructs regarding attitudes. These constructs are often seen to conceptualise attitudes as having emotional elements that places them nearer the affective than the cognitive end of the spectrum; as having an impact on intention and hence behaviour (Ajzen & Fishbein, 1980); and as being dependent upon experience (McLeod, 1992) and beliefs (Ajzen & Fishbein, 1980). Cognition and emotion are seen as being two complementary aspects of mind and the interaction between the two is so entwined and intense that neither can be separated from each other (Hannula, 2002). That is, we ignore the affective at the peril of the cognitive. In search of the definition of the mathematics attitudes, Di Martino and Zan (2010) proposed a three dimensional model for attitude (Figure 2) that shows interconnections between emotional dispositions towards mathematics, vision of mathematics and perceived competence in mathematics.

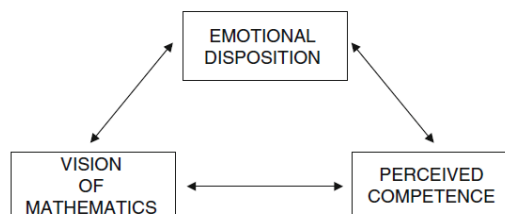


Figure 2. The TMA model for attitude (Di Martino and Zan, 2010)

Di Martino and Zan (2010) described mathematical activity as the interaction between cognitive and emotional aspects, and recognised that teachers are aware that mathematics is a subject which triggers

strong negative emotions amongst students, which can eventually lead to disengagement and block thinking processes.

Attitudes are learned and they predispose a person to some degree of consistency and can be judged to be either positive or negative (Hannula, 2002). They are linked to beliefs, as each person has a corresponding attitude to each belief. This attitude shows itself in the performance of a behaviour or task and the manner in which it was performed. The positive relationship which exists between attitudes and achievement is widely documented (Wilkins, 2008). It is no secret to teachers that a student's positive attitude can go a long way to increasing their level of performance.

## **The Effect of Teachers' Beliefs and Attitudes**

There have been a number of studies completed on teacher confidence and attitude (Bobis & Cusworth, 1994) which paint a fairly unsettling picture. Research indicates that many school teachers, particularly primary school teachers, are fearful, anxious, pessimistic and resentful of the subject of mathematics and hold beliefs and attitudes about mathematics which can be narrow and debilitating (Szydlik, Szydlik & Benson, 2003). These beliefs have been found to be firm, tenacious and resistant to change (Swars, Hart, Smith, Smith & Tolar, 2007). Goldin et al. (2009) asserted that beliefs are unlikely to be replaced unless they are challenged and proven to be unsatisfactory. These tenacious and unchallenged beliefs often prevent teachers from teaching mathematics in ways that empower students (Schuck & Grootenboer, 2004) and can affect the level of cognitive complexity in instruction (Charalambous, 2010). Aguirre (2009) argued that the beliefs that teachers hold, ultimately make the difference between failure and success in classroom reform. If, as Ajzen and Fishbein (1980) report, beliefs and attitudes influence behaviour, and as Wilkins (2008) asserted, knowledge, beliefs and attitudes all influence instructional practice, then negative attitudes and beliefs about mathematics will probably manifest in actions contrary to increased mathematical learning taking place. This could be demonstrated through decreased instruction time, over-reliance upon text books, lack of subject content knowledge, an inability to provide conceptual explanations, a reluctance to adopt alternative pedagogies or to allow for a spirit of inquiry and argumentation to take place.

If we want to challenge the unproductive and debilitating attitudes that teachers might have, then we need to better inform them of alternative visions of mathematics (Di Martino & Zan, 2010). It is pointless to deconstruct someone's set of negative attitudes without providing them with demonstrations of practices which allows them to replace the negative attitudes with a more positive disposition towards effective teaching and learning.

## **Beliefs and Attitudes About Gender**

There is a long history of writing concerning gender inequity issues in mathematics (e.g. Fennema & Sherman, 1977) which in recent times has morphed into concerns regarding the lack of female participation in STEM subjects (Office of the Chief Scientist, 2013) and further commentary on the poor participation rate of females in Year 12 mathematics subjects across Australia (Barrington & Evans, 2014). The sad fact of the matter is that females are underrepresented and their lack of participation is injurious to the welfare of society. This is despite the evidence that psychological research suggests there are no gender differences in children's cognitive abilities and therefore no difference, on average, in the potential for females and males to achieve in mathematics (Spelke, 2005). Despite these findings, gender-stereotyped views are still prevalent (Forgasz, Leder, & Tan, 2013).

We have to ask ourselves, why it is that females are less likely to be attracted to mathematics than males. Is part of the reason that mathematics is associated as being a masculine pursuit (Mendick, 2006)? Who are the role models that females can aspire to emulate? It is true that across both genders not many students (or teachers for that matter) would be able to recognise a picture of Andrew Wiles or Terrence Tao, but it is far sadder that many people are absolutely unaware of the contribution to mathematics and therefore the world that women have made (for example; Hypatia of Alexandria, Maria Gaetana Agnesi, Marie-Sophie Germain, Amalie Emmy Noether (considered by Einstein to be most important woman in the history of mathematics), the 2014 Field's Medal winner Maryam Mirzakhani, Australia's own Cheryl Praeger).

Just consider, there are significantly less females (25%) engaged in Year 12 intermediate or advanced mathematics than males (34%), and yet 70% of all teachers in Australia are female (Figure 3 is as high as 81% across primary schools) (ACER, 2016). Therefore, in a profession that employs predominantly females, we are drawing from a pool of mathematics students which is under-represented. If we want our teachers, both primary and secondary, to be the best possible candidates then this is not a situation that should be allowed to continue.

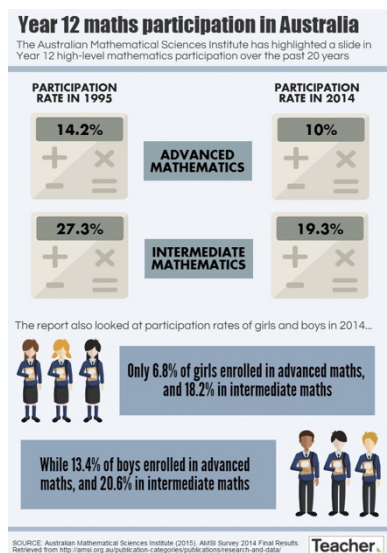


Figure 3. Infographic: Declining maths participation (ACER, 2016)

## Pedagogy and Engagement

If we are trying to engage students in mathematics learning, then we must take specific actions to do so. As much emphasis needs to be placed on the social aspects of the classroom, the mathematical interactions between the students, as on the mathematics itself (Leach, Hunter & Hunter, 2014). There is a need to encourage the communicative processes for mathematics students through judicious modelling and scaffolding of argumentation; processes such as explaining, justifying, and responses to challenges (Marshman & Brown, 2014).

Research conducted in Australia has shown that mathematics teachers are dependent on a variety of commercially produced materials, particularly the student workbook (Watt, 2004). International research by Johansson (2006) suggested that up to 90% of mathematics lessons employ a textbook to form content, sequencing and instructional activities and ideas for lessons. Other international studies have shown that textbooks influence what teachers teach by delineating what topics are covered and how these topics are presented (Stein, Remillard, & Smith, 2007), how they teach mathematics, and what homework or activities they assign to their students (Alajmi, 2009). The research by Jamieson-Proctor and Byrne (2008) further confirmed that contemporary schooling has a heavy dependence on textbooks in spite of the fact that there is a lack of conclusive evidence to support the efficacy of them (Shield & Dole, 2013). Such unquestioned reliance on textbooks could be quite problematic as it could lead to what Haberman (2010) calls the “pedagogy of poverty” (p. 45) where teachers are overly didactic, dole out information and assign problems. It should however be noted that Hill, Ball and Schilling (2008) offered a countervailing argument to the objection to using text books:

Teachers are flooded with messages *not* to use their textbooks, starting with scholarly work (Ben-Perez, 1990) and continuing on to the materials thrust upon them in professional development and ending with district curriculum documents that piece together units from disparate resources. This may have been appropriate in an era when most textbooks were similar in their mathematical drabness; however, the quality of available materials has sharply improved, yet this ethos persists. And we argue that solid mathematical tasks and representations that come from a drab textbook are preferable to teacher-created math lessons in the hands of teachers with little mathematical knowledge



for teaching. Without the ballast of mathematical knowledge, teachers' implementation of supplementary materials is chancy at best. (Ball & Schilling, 2008, p. 499)

However much Hill, Ball and Schilling's (2008) statement may seem to be at odds with prevailing opinion and research, it is a position which would resonate with many teachers. Surely then it is not necessarily a question of *if* we use a textbook, but rather *how* we use a textbook? Do we use the textbook as a default curriculum or is it just one tool of many to provide a rich and varied teaching and learning environment? This is a question each of us must answer, both as individuals, and as part of a teaching community.

## A Concluding Comment

Few teachers would truly believe that the study of mathematics is a purely cerebral one. In fact throughout the ages deep-thinkers both within the field of mathematics and outside of it have rhapsodised over the affective elements of this subject. If our classrooms are going to be places that are about fireworks rather than just sparklers then we as teachers need to understand that a mathematics lesson is more than just the important content that is being presented. The mathematics classroom is a place where participants, both students and teachers bring their whole being, it is a place where their attitudes, beliefs and understandings are exposed and tested. It is a place where current paradigms of what constitutes a mathematics classroom and mathematics learning, needs to be examined. It may be that in the classroom we can feel restrained by the syllabus we are required to work within, but in the space of being able to set the agenda regarding affect, we seem to have more affordances than constraints. Once we analyse our capacity to influence affect, then perhaps students will be better disposed to engage with mathematics. The difference between engagement and disengagement may be as simple (or as complex) as the teacher understanding that, "If you always do what you've always done, you'll always get what you've always got" - Henry Ford.

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# TARGETED TASKS FOR TAMING MULTIPLICATIVE THINKING

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*Multiplicative Thinking is a vital part of students' mathematical development and has proven to be a stumbling block for many students. This paper focuses on two tasks that have been developed to promote the development of a conceptual understanding of 'the multiplicative situation'. Both tasks are predicated on the need for the students to be active participants in the learning and for the proficiency strands of understanding and reasoning to be in sharp focus.*

## **Introduction**

The concept of 'big ideas' in mathematics is not a new one, but it is not necessarily one that has been rigorously pursued by researchers or curriculum developers. It was implied in the seminal works of Shulman (1986) who spoke of the 'syntactic structures within mathematics' and Ma (1999) who discussed the idea of 'knowledge packages'. Although Clarke, Clarke and Sullivan (2012) noted that a universal agreement as to what the big ideas of mathematics are is unlikely, Charles (2005) thought that the 21 ideas he identified were a good place to start a discussion. In constructing the Australian Curriculum one of the major goals was to make the curriculum deep rather than wide (ACARA, 2009), to focus on the important ideas for mathematical development. In this article we will not be arguing if the Australian Curriculum achieves this goal. Rather, we will be concentrating on one element of mathematics that is widely acknowledged as being a big idea (Hurst & Hurrell, 2015; Siemon, Blackley & Neal, 2012), that is, Multiplicative Thinking. In particular, we present some tasks which can support the development of some of the underlying understandings that need to be developed.

## **Multiplicative Thinking**

Multiplicative thinking underpins important mathematical concepts such as an understanding of fractions, proportional reasoning, and algebraic thinking. As such, its development is vital and needs to be considered carefully but the data suggest that as many as 40% of Year 7 and 8 students performed below curriculum expectations in multiplicative thinking (Siemon, Breed, Dole, Izard, & Virgona, 2006).

Multiplicative thinking is characterised by understandings such as the multiplicative relationship between places in the number system, basic and extended number facts, and properties of operations and associated relationships (Hurst & Hurrell, 2015). According to Siemon, Breed et al. (2006), multiplicative thinking is:

- a capacity to work flexibly and efficiently with an extended range of numbers (and the relationships between them);
- an ability to recognise and solve a range of problems involving multiplication and/or division including direct and indirect proportion; and

- the means to communicate this effectively in a variety of ways (e.g., words, diagrams, symbolic expressions, and written algorithms) (p.28)

## Tasks

There is no decision that teachers' make that has a greater impact on students' opportunities to learn and on their perceptions about what mathematics is than the selection or creation of the tasks with which the teacher engages students in studying mathematics.

Lappan and Briars (1995)

Many commentators have argued that tasks, and the decisions teachers make when choosing tasks are critical (Brousseau 1997; Christiansen and Walther, 1986; Hiebert and Wearne, 1997, Ruthven, Laborde, Leach and Tiberghien, 2009). Sullivan (2011) argued that without tasks, the mathematical actions described by Kilpatrick, Swafford and Findell (2001), which were adopted for the Australian Curriculum Mathematics (ACARA, 2016) proficiency strands, would not be possible. That is, "it is not possible to foster adaptive reasoning and strategic competence in students without providing them with tasks that are designed to foster those actions" (p. 32).

There has been a good deal of research regarding what makes for an effective task. The need for tasks to have multiple entry and exit points is paramount (Sullivan & Davidson, 2014), that is, that everyone can start but the problem's potential allows students to exploit it until they run out of enthusiasm for the problem, rather than mathematical possibilities. The task should also cause "sustained thinking" (Sullivan & Davidson, 2014). An effective task offers opportunity for students to engage in argumentation, to reason, explain and justify (Cheeseman et al., 2013; Fielding-Wells, Dole, & Makar, 2014) and to develop persistence (Sullivan & Mornane, 2014).

## Working with Teachers and Students

Over the last three years we have been working with teachers and students in the important mathematical area of Multiplicative Thinking. We have worked with eight primary schools situated in a range of socio-economic profiles, with ICSEA ratings ranging from 1000 to 1170 and representing all three school systems and sectors. In those schools, data has been collected through the administration to over 1000 students of a written quiz and a semi-structured interview in which 48 students participated.

Once the data were collected a process was followed to refine the main elements of Multiplicative Thinking. We consider that there are six main elements which can be considered separately, but are strongly interrelated.

- Understanding the 'multiplicative situation', i.e., the relationship between multiplication and division, the use of the multiplicative array, the language of factors and multiples, and the links with fraction, ratio, and proportion. A range of problem types and writing stories to describe number sentences.
- The notion of a number being '...times bigger' or '...times smaller' than another number. This is distinct from the additive notion that a number is '...more' than another number (e.g., 40 is 36 more than 4).
- Multiplication and division by powers of ten and, what happens when a number is multiplied or divided by another number.
- Use of a variety of unstructured materials such as bundling sticks, and structured materials such as MABs to develop an understanding of the multiplication and division algorithms.
- Properties of multiplication and division and the relationships between them. The commutative property, distributive property, inverse relationship and extension of number facts.
- Extension of multiplication and division beyond 2 digit by 1 digit or  $2 \text{ digit} \div 1 \text{ digit}$ , and including the use of algorithms based on multiplication properties (e.g., distributive and extended facts).

This refinement then allowed us to embark on the undertaking of collecting and writing tasks that support students in the learning, and teachers in the teaching, of these elements. In the interest of space the concentration in this article will be on two activities. The first activity is called “Lots of stories” and is an activity which requires the students to tell lots of (many) stories and stories of “lots of” (the multiplicative situations).

## Lots of stories

A boy has \$6 to spend on sweets costing 20c each. How many can be bought?	Twenty dollars is shared between six people. How much will each person receive?	Share 4 apples between 12 people.	If it costs \$4 for 12 pens, how much would each pen cost?
Mini chocolate bars come in packs that contain 20. How many would 6 packets contain?	Pencils cost 6 cents each. How many could a school buy for \$20?	There are 4 bags, each containing 12 apples. How many apples altogether?	Pencils cost 6 cents each. How many can I buy for \$2?
Notebooks are priced at 20 for \$6. How much does each notebook cost?	C.D.'s cost \$6 each. How many can you buy for \$20?	There are 6 apples to share between two people. How many does each person receive?	I have 2 pizzas to be shared amongst 6 people. How much does each person receive?
Share six large pizzas among 20 people. How much does each person receive?	Six people each give 20 dollars to a collection. How much is collected altogether?	Pencils cost 20 cents each. How much will 6 pencils cost?	A bill for coffee is \$20. If it is shared among 6 people, how much will each person pay?

Figure 1. Story cards

Possibly the variation of the activity that might be first employed is where the students select a story card (Figure 1) and then construct a number sentence, using the numbers and operations cards (Figure 2), which matches the story card. In order to make sure that all students may be involved the teacher may want to be judicious about the cards which are available in the initial stages. For instance a card such as “Share 4 apples between 12 people” is one that is likely to be syntactically familiar to the students, as is “There are 4 bags, each containing 12 apples. How many apples altogether?”. One aspect to note is the absence of the equals sign, or of a card which gives an answer. The benefit of not having these cards lies in the fact that the students are not required to be looking at the question as a piece of arithmetic, where finding an answer is the aim. The focus is hopefully on the students looking at the problem to find the process.

6	20	4
12	2	3
÷	×	)
60	600	200
400	40	6000
2000		

Figure 2. Number and operations cards

At all times the students are asked to articulate their choices. This articulation is not a corollary, or an incidental issue, it is a focal part of the process. It is here that the students need to show their understanding through reasoning and argumentation to convince their working partners of the validity of their solution. This articulation is a cornerstone in developing and clarifying their thoughts, giving them an opportunity to display a conceptual rather than just a procedural understanding of multiplication and division.

The reverse of this process should also take place where the students create a number sentence using the number and the digits card and then attach the story card or cards to that numbers sentence. If in the event that the numbers sentence they construct, does not have a matching story card, then it is expected that they will write an appropriate one. A further activity, is where the students are asked take a story card and retell the story with its inverse operation.

The assumption being made here is that there is a large element of formative assessment happening with this activity. As previously stated, as the students are engaging with the cards they are expected to be articulating their understanding. This articulation of understanding will become apparent though not only through the manner in which they physically manipulate the cards, but also through their conversations. Any areas of concern, or when good understandings are shown are noted, this gives the opportunity for focussed, at point of need, teaching and learning.

Once the students have been taken though a needs-lead process with the cards, they can then be used as a “concentration” activity to maintain understanding and reasoning and to increase their fluency in being able to recognise multiplicative situations. The students can then be invited to create their own games using numbers and situations of their choosing.

The richness of this activity is that it exercises a number of the elements of Multiplicative Thinking (understanding the ‘multiplicative situation’; the notion of a number being times bigger or smaller; multiplication and division by powers of ten and; properties of multiplication and division) and provides a reason to articulate these elements in language, first developed by the students with their peers, then refined through consultation with the teacher. The cards also facilitate multiple attempts at arriving at a solution, without the need to initially commit pen to paper, hopefully encouraging even reluctant students to have-a-go.

## Pick Up Sticks

The genesis of this task came from one of the questions in the semi-structured interview which asked students to give an answer for  $6 \times 17$ . They were asked to see if they could calculate it using a mental computation strategy and if they could, then asked how they arrived at their answer. For those who chose not to employ mental computation, they were then given the opportunity to create a written algorithm. Regardless of how they found a solution, the students were provided with a quantity of bundling sticks already arranged in groups of ten and secured with elastic bands, and a quantity of single sticks and asked to represent  $6 \times 17$ . Quite a number of the students represented the algorithm by constructing the addends from the sticks as shown in Figure 3 rather than showing six lots of seventeen (Figure 4).

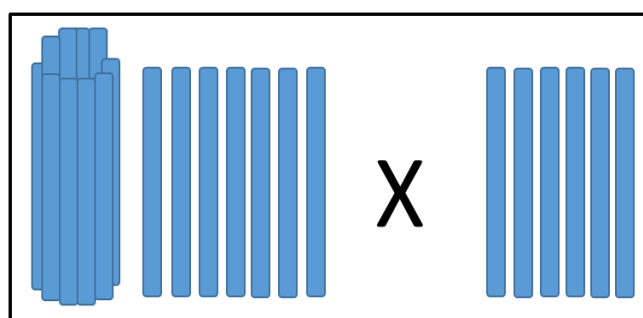


Figure 3.  $6 \times 17$  represented as addends

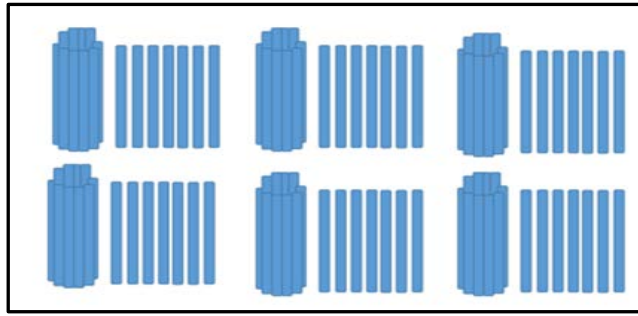


Figure 4.  $6 \times 17$  represented as six lots of seventeen

It is worth noting that the majority of students that displayed this lack of understanding were the students who relied on a written algorithm and in many cases were able to compute the correct answer using the algorithm.

As with the previous task (and fairly well all tasks) it is recommended that this activity be completed in groups of two or three to promote conversation and argumentation and therefore understanding and reasoning. Make available sets of bundling sticks, these can be toothpicks, pop-sticks or similar and have some of them pre-bundled into tens. Allow the students to create a couple of bundles of ten for themselves to internalise the understanding that one-ten (a bundle) is equal to ten-ones. Although this may seem self-evident to most people, experience has shown that it is not evident to all children.

Set a task such as:  $7 \times 13 = \square$  and ask the students to record the number sentence using a template (Figure 5).

Number of groups	×	Number in each group	=	Total
7	×	13	=	

Figure 5. Recording template

The students then need to model the number sentence with the bundling materials. After they have attempted this they are required to create a story about the number sentence and tell it to their partner. For example, “I have 7 paddocks of sheep and there are 13 sheep in each paddock. How many sheep are there altogether?” It is important that in their retelling of the story the students relate the numbers used in their story back to the template.

The students are then asked to use place value partitioning and partition the 13 to make the following number sentence:  $(7 \times 10) + (7 \times 3) = \square$ . Again the students are asked to model this number sentence using the bundling sticks and describe and defend what they have done. It is meaningful to pause at this point, to highlight and explore that, for many students, the partitioning of the number into tens and ones allows them to find a solution using mental computation. When the number sentence  $(7 \times 10) + (7 \times 3) = 70 + 21$  has been constructed it is prudent to observe how the students deal with the 21 sticks.

Even if this activity is terminated at this point there is a good deal to recommend it for developing student capacity with multiplicative thinking, however there has now been an opportunity created to explore the written algorithm to develop a conceptual understanding of this procedural tool. Through carefully developing vertical algorithms (Figure 6) from the horizontal algorithm used earlier ( $7 \times 13 =$ ), and matching each part of the algorithm to the actions taken with the bundling sticks, the students are given the opportunity to understand what actually happens with the numbers at each step of the algorithmic procedure.

Please note that in this instance the construction of the algorithm follows the syntax of the question “seven lots of 13” which is written as  $7 \times 13$  and therefore slightly more accessible to relate to the story, alternatively there is no issue if the double digit number is placed on the top line of the algorithm (as in Figure 7) if this is preferred. Likewise, the use of the term “carry” is commonly used, but is not preferred over other terms such as regroup or rename, and the use of the superscript 20 has been employed and it is acknowledged that others employ a superscript 2 with an implied zero.



$$\begin{array}{r}
 7 \\
 \times 13 \\
 \hline
 21 \quad 3 \times 7 \\
 70 \quad 10 \times 7 \\
 \hline
 91
 \end{array}
 \qquad
 \begin{array}{r}
 20 \quad 7 \\
 \times 13 \\
 \hline
 91 \quad 3 \times 7 \text{ (carry 20)} \\
 \hline
 10 \times 7 + 20
 \end{array}
 \qquad
 \begin{array}{r}
 20 \quad 7 \\
 \times 13 \\
 \hline
 1 \quad 3 \times 7 \text{ (carry 20)} \\
 90 \quad 10 \times 7 + 20 \\
 \hline
 91
 \end{array}$$

Figure 6. Vertical algorithms to represent  $7 \times 13$

$$\begin{array}{r}
 20 \\
 \times 13 \\
 \hline
 91
 \end{array}$$

Figure 7. Alternate vertical algorithms to represent  $7 \times 13$

This activity has particular links to the Multiplicative Thinking elements of: the use of a variety of unstructured materials and; the extension of multiplication and division beyond 2 digit by 1 digit calculations. As previously stated, the tactile nature of the activity and the explicit making and unmaking of the bundles of 10 creates a conceptual understanding which the teacher can use to underpin increasingly complex recording of algorithms.

## Conclusion

Both of the activities presented in this paper rely on two key elements; the thoughtful and knowledgeable instruction and guidance of the teacher, and the students taking an active role in their learning. Both activities are suitable for developing the vital skills and understandings of multiplicative thinking, and both are greatly enhanced when the students are made an overt part of the learning, and are required to articulate their understanding and their reasoning at every stage.

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# CAPTAIN ZERO . . . HERO OR VILLAIN?

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*Our current research into children's multiplicative thinking has shown that children have the capacity to think multiplicatively and that some aspects of multiplicative thinking are more thoroughly understood than others. We look at a data set obtained over three classes in the same year level and explore the considerable variation in responses to several key questions on a multiplicative thinking quiz. The questions relate to the 'times bigger' notion in comparing numbers, the ability to use standard place value partitioning when operating, the ability to articulate what happens when numbers are multiplied and divided by powers of ten, and the role of zero. It could reasonably be implied that the variation in understanding across the three classes may be due to pedagogical emphases.*

## **Introduction**

'Big idea' thinking in mathematics has been in evidence for some time, perhaps initiated by the connectedness of the work of Ma (1999) and the seminal paper by Charles (2005), and followed more recently by the work of Clarke, Clarke & Sullivan (2012). Multiplicative thinking has been identified as a 'big idea' (Hurst & Hurrell, 2015; Siemon, Bleckley & Neal, 2012) which underpins much of the mathematical learning that occurs in late primary years and beyond. Amongst other things, it provides important foundational understanding for fraction concepts, decimals and percentages, ratio and proportion, and algebraic reasoning (Siemon, Bleckley & Neal, 2012). Unfortunately, many students are not strong multiplicative thinkers and as many of 40% of them in Years 7 & 8 perform below expectations (Siemon, Breed, Dole, Izard, & Virgona, 2006). Over the past three years, we have conducted research into children's multiplicative thinking for several reasons. Firstly, we wanted to understand the specific mathematics that constituted multiplicative thinking. Secondly, we wanted to identify the aspects with which students and teachers experienced the most difficulty. Thirdly, we were keen to develop some tasks and pedagogies that would be of benefit to teachers and students.

## **What Constitutes Multiplicative Thinking?**

Two instruments were used to gather data – a written quiz and a semi-structured interview. The quiz has been administered to over 1000 students with about fifty being interviewed. Our initial ideas about what comprised the component parts of multiplicative thinking were refined as the data were analyzed and six themes were established as follows:

1. The 'multiplicative situation', or the relationship between multiplication and division, the use of the multiplicative array, the language of factors and multiples, and the links with fraction, ratio, and proportion, with all of these points being expressed and described in a range of problem types, stories, and number sentences.
2. The notion of a number being '...times bigger' or '...times smaller' than another number. This is distinct from the additive notion that a number is '...more' than another number (e.g., 40 is 4 more than 36).
3. Multiplication and division by powers of ten and, what happens when a number is multiplied or divided by another number.

4. Use of a variety of materials such as bundling sticks, and MABs to develop an understanding of the multiplication and division algorithms.
5. Properties of multiplication and division and the relationships between them including the commutative property, distributive property, inverse relationship and extension of number facts.
6. Extension of multiplication and division beyond 2 digit by 1 digit or 2 digit  $\div$  1 digit, and including the use of algorithms based on multiplication properties (e.g., distributive and extended facts).
  - This paper reports on the use of aspects of the quiz with three unstreamed, heterogeneously grouped Year 4 classes at the same school. Interesting observations can be made about aspects of Themes 1, 2, 3, and to some extent, Theme 5. When taken on their own, the responses to each section of the quiz could be seen as ‘unremarkable’ but when considered together, they suggest some clear pedagogical differences across the three classes.

## Results and Discussion

### Theme 1 – Numbers of Equal Groups; Representation With Arrays

Students were asked the answer to the number fact  $8 \times 7$ , to explain what the numbers in the fact told them about, and to write a story about it. They were then asked to represent the number fact with a drawing. Responses are summarised in Table 1 and given as a percentage of the class total. Class 1 had 30 students, Class 2 had 23 students and Class 3 had 28 students.

Table 1. *Responses of students in three classes to Theme 1 questions*

Criterion	Class 1	Class 2	Class 3
Knows about group size and number of groups, and/or writes appropriate story about given number fact	43	32	11
Represents a number facts as a number of separate groups	37	36	18
Represents a number fact as a multiplicative array	43	25	42

Students in Class 3 were generally unable to articulate about number and size of equal groups in the multiplicative situation yet nearly half of them drew a multiplicative array, considered to be a powerful representation of it. In Class 1 a similar percentage of students drew an array and a greater proportion of them articulated about groups than students in Class 3. Of interest is the fact that twice as many students in Class 1 also depicted the situation with a drawing showing equal groups than did students in Class 3. Also less students in Class 2 drew an array than did students in Class 3 but more of them were able to talk about numbers of equal groups. What might this indicate?

It is problematical to draw a clear conclusion from the data in Table 1 other than to suggest that it might be due to pedagogical influences. Perhaps the teacher of Class 3 had explicitly taught the use of arrays but had not explicitly made the connection with numbers of equal groups. Perhaps the teacher of Class 1 had explicitly taught the concept of numbers of equal groups and also linked it to representing with drawings, both of arrays and separate groups.

### Theme 2 – The Role of ‘Captain Zero’, and the Notion of ‘Times Bigger’

Data related to this section is presented in two tables and some figures. Students were asked to explain what happened to a number when it was multiplied by ten and then were asked to say ‘how many times bigger’ a number was than another number. These numbers can be seen in Table 2.

Table 2. Summary of explanations for multiplying a number by ten

Criterion	Class	Class	Class
	1	2	3
Explanation based on 'adding a zero'	77	72	46
Explanation based on other ideas	17	14	11
No response given	6	7	0
Explanation based on digits moving a place, and/or moving to the left	0	7	43

There are some stark differences between the three classes with these data. The great majority of students in Classes 1 and 2 explained the result of multiplication by ten in terms of 'adding a zero', while only two students (7%) in Class 2 explained it conceptually in terms of the movement of digits to a higher value place. However, almost half the students in Class 3 explained it in that conceptual way. The difference is even more noteworthy when one considers the particular way in which students in Class 3 explained their thinking. Figure 1 contains five examples of their work.

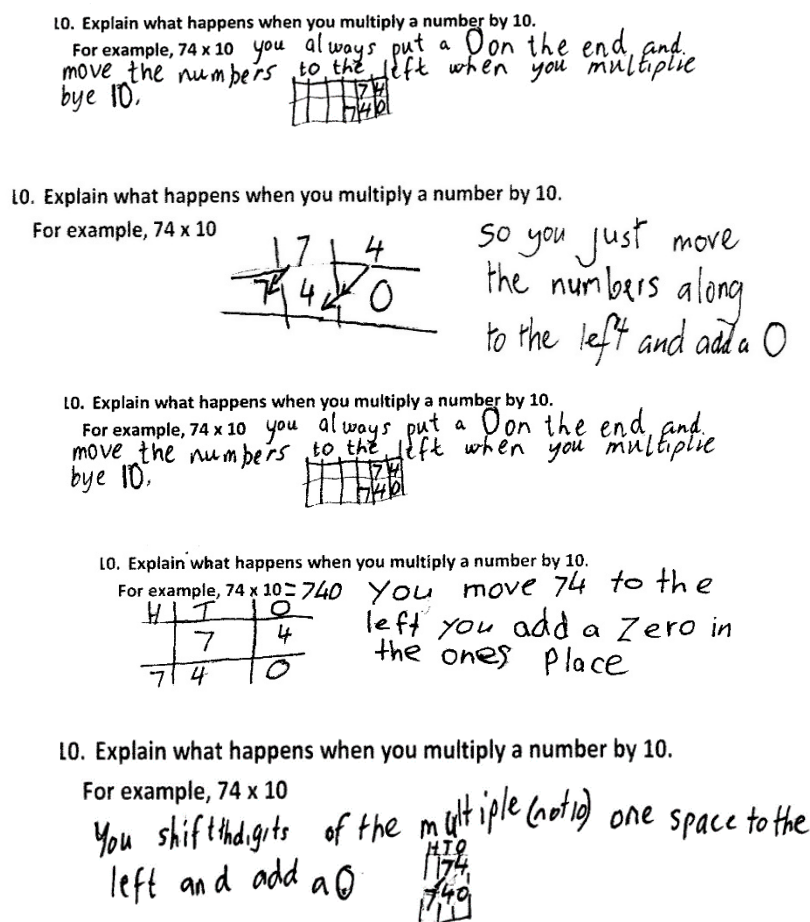


Figure 1. Examples of conceptual explanations of multiplication by ten

It seems likely, given the level of explanation in the Figure 1 samples that, in Class 3, there has been some explicit teaching of what happens when numbers are multiplied by ten. Even though a similar percentage of students explained it in terms of the 'Captain Zero' phenomenon, the proportion of students demonstrating conceptual understanding is considerably higher in Class 3 than in Classes 1 or 2. The 'adding a zero' explanation is considered to be procedural and not likely to be underpinned by conceptual understanding. The difference is that, while students in Class 3 did talk about adding a zero, they stated the vital aspect about the digits moving to a place of higher value. The situation becomes even more intriguing when data from the 'how many times bigger than . . .' questions are considered.

Table 3. *Summary of responses to the ‘how many times bigger’ questions*

Criterion	Class	Class	Class
	1	2	3
Identifies 40 as 10 times bigger than 4	70	71	32
Identifies 400 as 10 times bigger than 40	43	18	25
Identifies 4000 as 10 times bigger than 400	33	21	25
Identifies 400 as 100 times bigger than 4	53	43	32

The data in Table 3 are interesting in themselves as they show some clear differences between some of the classes and the responses of students. For instance, and taken at face value, students in Class 3 do not seem to have responded as well to these questions about ‘times bigger’ as they did to the previous question about multiplying by ten, yet the underpinning ideas in both are at least very similar. To clarify this situation, we need to consider the range of responses given by students to the four questions. These responses are contained in Table 4.

With some exceptions, students responded in one of three ways. Response Set 1 contains four correct responses demonstrating an understanding of the ‘times bigger’ notion. Response Set 2 indicates that students have likely considered only the size of the first number in each question with no consideration given to the ‘times bigger’ notion. They were able to give a correct response for the first and fourth questions, but likely for the wrong reason. Response Set 3 is purely an ‘additive’ response obtained by subtracting one number from the other, with no understanding of ‘times bigger’.

Table 4. *Summary of sets of responses given to the ‘times bigger’ questions*

Criterion	Set 1	Set 2	Set 3
Identifies 40 as 10 times bigger than 4	10	10	36
Identifies 400 as 10 times bigger than 40	10	100	360
Identifies 4000 as 10 times bigger than 400	10	1000	3600
Identifies 400 as 100 times bigger than 4	100	100	396

We can now consider the results shown in Table 3 in a somewhat different light and present Table 5 showing only the results for the second and third questions (where understanding the ‘times bigger’ notion is essential in arriving at the correct answer), and including the percentage of students who gave the ‘additive’ responses.

Table 5. *Summary of sets of responses given to the ‘times bigger’ questions*

Criterion	Class	Class	Class
	1	2	3
Identifies 400 as 10 times bigger than 40	43	18	25
Identifies 4000 as 10 times bigger than 400	33	21	25
Provided additive responses (shown in Set 3)	13	14	50

Two things stand out from Table 5. Firstly, even though nearly half of the students in Class 3 could explain conceptually what happened when a number is multiplied by ten, only a quarter of the class could articulate the ‘times bigger’ relationship in these two questions. Further interrogation of the data indicates that the students who did so were not the same students (with two exceptions) who provided a strong explanation of multiplication by ten. Secondly, a relatively high percentage of students in Class 3 provided an additive response to the ‘times bigger’ questions. On further interrogation of the data, three of those students did provide a conceptual explanation of multiplication by ten. As previously noted these data suggest that there may have been some explicit teaching around the concept of moving digits to places of higher value, and possibly of the ‘times bigger’ notion, but explicit connections between the two ideas have not been drawn.

### Theme 3 – Properties of 2x1 Digit Multiplication

Students were given the example  $6 \times 17$  and were asked to calculate it mentally and explain how they did it, and then to show a written method for working it out. We wanted to see if they used a standard place value partition either mentally or in their written method, that is, the answer was calculated by using  $(6 \times 10) + (6 \times 7)$ . The results are shown in Table 6.

Table 6. Summary of responses given to the  $6 \times 17$  questions

Criterion	Class 1	Class 2	Class 3
Correctly calculated answer to $6 \times 17$ using mental computation	60	46	29
Explained mental computation for $6 \times 17$ based on place value partition	57	25	11
Correctly multiplies $6 \times 17$ using a written method	0	3	39
Correctly uses standard algorithm and shows place value partition	0	3	0

Once again, there are clear differences between responses of students in different classes. It would seem that students in Classes 1 and 2 were familiar and comfortable with mental computation of such examples with many of them in Class 1 able to explain their thinking in terms of the standard place value partition. Less students in Class 2 and even less in Class 3 were able to do so. However, further interrogation of the responses from students in Class 3 indicates that 50% of them used the 'lattice method' for their written calculation but only half of them did so correctly. A further four students in Class 3 used an alternative partition or non-standard method of calculation. It is also worthy of note that no student in Class 3 who used the 'lattice method' was able to calculate  $6 \times 17$  mentally and provide an explanation in terms of the standard partition. The use of the lattice method was confined to Class 3 so it is likely that it has been explicitly taught to that class. However, the presented evidence suggests a lack of underpinning conceptual understanding to support its use.

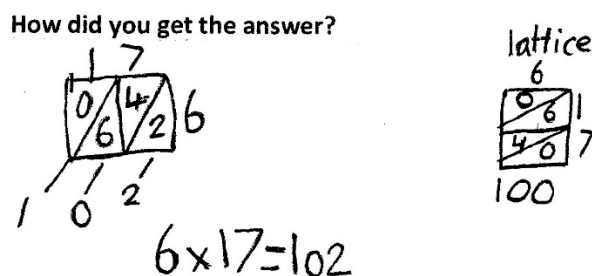


Figure 2. Example of correct and incorrect use of lattice method

### Conclusions

When considering the data presented here, it seems reasonable to come to some tentative conclusions about some of the pedagogies used in the three classes. With regard to the first theme, it is likely that there has been some explicit teaching around the use of arrays to depict the multiplicative situation but at the same time, the connection to the idea of a number of equal groups has not been made clearly in each of the three classes. Data from the second theme suggests that students in all three classes had been taught a procedure based on 'adding a zero' when multiplying by ten/s although the very strong conceptual explanation provided by some students in Class 3 suggests that they had been explicitly taught that digits move to a place of higher value when multiplied by ten. Responses to the 'times bigger than' questions suggest that many students have difficulty understanding the difference between that concept and the idea of 'how many more than', as indicated by the high number of additive responses from one class at least. Also, many students interpret 'times bigger than' questions in terms of the size of the bigger number rather than as a comparison. Finally, data from the third theme suggests that explicit teaching of mental computation strategies has occurred in some classes while students in Class 3 have likely been explicitly taught the 'lattice method' for multiplication as a procedure without the

associated conceptual understanding of partitioning. This paper reports on one aspect of our current research into children's multiplicative thinking. As part of the study it has been arranged to revisit the school and interview the three Year 4 teachers to provide a firmer base for drawing conclusions.

Overall, the presented data suggests that none of the students in the three classes has a broad understanding of multiplicative thinking but that they all have a partial knowledge. While this is reasonably expected at Year 4 level, explicit teaching based on connecting various aspects of multiplicative thinking as outlined here would be of benefit. Such teaching needs to be based on development of conceptual understanding as opposed to the use of procedures. The impact of 'Captain Zero' provides a good example of how the development of conceptual understanding can be hindered by the use of procedures which, in the long run will be found wanting.

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# CONNECTING YOUNG CHILDREN'S MATHEMATICS LEARNING: EARLY YEARS (P-3)

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*This presentation promotes an integrated approach to teaching and learning mathematics. Common underlying features that link mathematical concepts are based on an Awareness of Mathematical Pattern and Structure (AMPS). Spatial reasoning plays an integral role in developing mathematical concepts and processes, and the development of AMPS. Examples drawn from a series of research projects, the Pattern and Structure Mathematics Awareness Program, focused on 4 to 8 year olds, will demonstrate how this integrated approach aims to move mathematics teaching and learning beyond basic numeracy to discovering connected mathematical patterns and relationships. The presentation illustrates how the project assesses children's mathematical development through a Pattern and Structure Assessment (PASA) interview, and exemplifies an innovative pedagogy, the Pattern and Structure Mathematics Awareness Program (PASMMap), that focuses on connected mathematical ideas.*

## **Background**

The Australian Pattern and Structure Mathematical Awareness Project, now spanning over a decade, shows how young children develop key underlying mathematical ideas based on an awareness of mathematical patterns and relationships and organisational 'structures' such as arrays and grids (Mulligan, English, Mitchelmore, & Crevensten, 2013). The project is based on a growing body of new research that supports the idea that young children have the ability to reason and generalise if given appropriate challenges and scaffolds. Through the development of an interview-based assessment, the Pattern and Structure Assessment (PASA) and an innovative pedagogical program, the Pattern and Structure Mathematics Awareness Program (PASMMap), young children have shown that they can develop rich mathematical ideas much earlier than previously expected, and they are able to connect mathematical concepts in a variety of ways. Adopting an inclusive approach, the program assists those children who are at risk of not progressing mathematically so they can be more effectively assessed and supported.

## **In Search of an Awareness of Mathematical Pattern and Structure**

Over many years of research investigating young children's mathematical development, mathematics educators and psychologists have been focused on identifying common underlying mathematical features that children acquire to varying degrees. Our aim was to identify, describe and measure this underlying feature; we wanted to 'put our finger on' what it was that some children just 'got' easily while others were not focused on 'it' at all. After many studies delving into the depths of children's intuitive mathematical thinking we found evidence of a feature that we called 'Awareness of Mathematical Pattern and Structure' (AMPS). Children with this AMPS ability could readily see patterns and represent them logically and in well-organized ways. Students who didn't appear to have AMPS were disorganised in their representations - they lacked basic structural and conceptual aspects such as seeing the unit of repeat in an ABB pattern. At a practical level, we aimed to explain and describe the wide variation in early mathematical competence in order to develop more effective

pedagogical approaches. We raised the fundamental question of whether AMPS is innate or whether it could be developed through pedagogical intervention.

Figures 1 and 2 are examples of representations made by children with high and low AMPS levels, respectively. These examples are drawn from student profiles from the PASMMap studies. In Figure 1, a child with high level AMPS represents how number facts to 10 are interrelated. She uses spatial features of symmetry and clearly indicates her understanding of pattern and commutativity. Figure 2 provides an example of a child with low AMPS attempting to represent various quantities using ten frames. Here we see the child's inability to recognise the 2 by 5 quinary structure of the ten frame although there is an understanding that frames partitioned into equal parts can represent quantities. There is some developing awareness of equal parts and alignment structures, but co-linearity is not consistently developed.

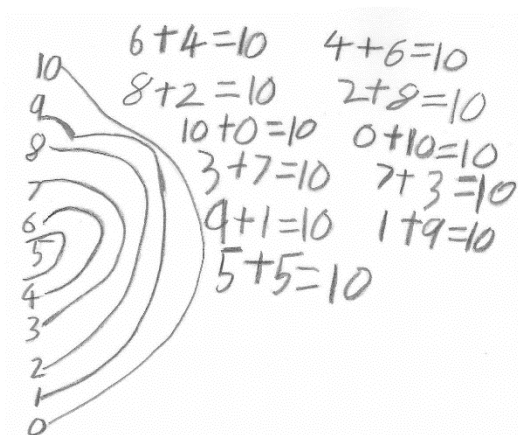


Figure 1. Child (aged 5 years) records her image of number facts represented as a rainbow of connected addition facts

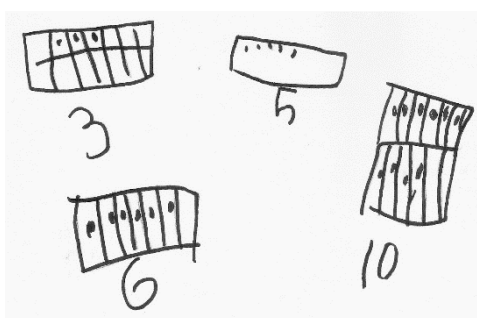


Figure 2. Child (5 years) draws inaccurate ten frames to represent quantities

## Assessing Awareness of Mathematical Pattern and Structure

The PASA assessment interview is designed in three levels for Foundation to Grade 2 or 3 students (Mulligan, Mitchelmore, & Stephanou, 2015). Each PASA level focuses on similar core concepts that underlie mathematical development ranging from 14 to 16 items from Foundation to Year (Grade) 2. There are a series of tasks to which the student provides a drawn response in the student booklet. There are five stages of structural development, the teacher's task is to ascertain which level of structural development they think the student's response represents. Descriptors are provided with examples of typical responses.

### Stages of Structural Development

- **Prestructural:** Representations lack evidence of relevant numerical or spatial structure
- **Emergent:** Representations show some relevant elements, but their numerical or spatial structure is not represented

- **Partial structural:** Representations show most relevant aspects but representation is incomplete
- **Structural:** Representations correctly integrate numerical and spatial structural features
- **Advanced:** Children show they recognise the generality of the underlying structure

This is where the insight of the teacher is more carefully developed; they can also question the student's understanding and their representations. A score is obtained and this equates to an AMPS score. Thus the teacher becomes aware of the level of AMPS and the student's stage of structural development. Changes can be tracked over time if the assessment is utilised again. The AMPS score is also equated to another measure of mathematical achievement, the PATMaths (Stephanou & Lindsey, 2013). By assessing children's AMPS structural scores using the PASA, teachers can select the most effective PASMMap learning pathways.

### What are the concepts and processes assessed in PASA?

The types of tasks included in the PASA include:

1. Partitioning Length into Thirds
2. Border Pattern
3. Triangular Array
4. Partitioning Money
5. Ten Frames
6. Counting by Threes: number track
7. Spatial Pattern Continuation
8. Square Array
9. Structuring/using Hundred Chart
10. The Analogue Clock
11. Grid Completion
12. Comparing Triangles
13. Growing Pattern Continuation
14. Making a Ruler
15. Constructing/Interpreting Bar Chart
16. Comparing Capacities

### Pattern and Structure (PASA) Items as Structural Groupings

AMPS levels can be described overall, or according to each of the following five individual structural groupings that represent the concepts embedded across the range of items.

1. **Sequences:** Repeating and border patterns; spatial pattern continuation, visual memory triangular array; growing patterns
2. **Structured counting:** Visual memory rectangular array; multiple count; grouping; ten frames; hundred chart
3. **Shape and Alignment:** Grid completion (other items on transformations, congruence, 2D-3D)
4. **Equal Spacing:** Distance/number line; structure of ruler, clockface; bar charts
5. **Partitioning:** Length (thirds); comparing triangles (embedding); money (base ten); capacity

These groupings are linked to the PASMMap Pathways of Learning. Essentially they show five core 'big ideas' that are aligned with most of the curriculum requirements.

### Pattern and Structure Mathematical Awareness Program (PASMMap)

PASMMap was developed and evaluated longitudinally (Mulligan et al., 2013). Kindergarten students engaged in PASMMap for the entire first year of schooling showed significantly higher levels of AMPS than for students in a regular program and these levels were retained one year later although the program was not continued during this year. PASMMap was trialed with young students aged 4 to 8 years, of wide-ranging abilities including those with mathematics learning difficulties (MLD) and those assessed as gifted in mathematics. Students engaged in the PASMMap also showed that they had developed spatial

reasoning, and relationships such as equivalence, as well as the ability to form emergent generalisations. Some PASMMap learning experiences focus on spatial aspects such foreground/background, alignment (collinear or axis), unitising the number line, space and shape, symmetry and transformations, and graphical representation of data. Spatial aspects are highlighted so as to develop spatial structuring. An example of this would be drawing a grid freehand with equal sized units.

PASMMap covers key concepts and processes (Proficiencies) that are central to the Australian Curriculum- Mathematics for the first years of formal schooling and to some extent beyond these. They are integrated into two books of Learning Pathways. The idea is that teachers can use the program in a flexible way and it can be used as a complete program or it can be integrated with existing programs and syllabi.

PASMMap's implementation is designed to give the teacher many options as they can target specific mathematical concepts with which the individual has most difficulty, or they can extend the learning for very able children. The program doesn't follow a lock-step compartmentalised approach; it is intended that the Pathways are interconnected and used flexibly depending on the ability of the students and the teacher's program.

Table 1. *PASMMap Pathways for Learning*

<b>PASMMap Book 1 Grades F-1</b>	<b>PASMMap Book 2 Grades 1-3</b>
Pathway 1: Repeating Patterns	Pathway 1: Multiplication Patterns
Pathway 2: Structured Counting	Pathway 2: Fitting Shapes Together
Pathway 3: Grid Structure	Pathway 3: Partitioning and Fractions
Pathway 4: Structuring Shapes	Pathway 4: Place Value
Pathway 5: Partitioning and Sharing	Pathway 5: Metric Measurement
Pathway 6: Base Ten Structure	Pathway 6: Patterns in Data
Pathway 7: Growing Patterns	Pathway 7: Angles and Direction
Pathway 8: Structuring Measurement	
Pathway 9: Structuring Data	
Pathway 10: Symmetry and Transformation	

Pathways give descriptions of learning experiences linked to central concepts as shown in Table 1. These have been outlined with explicit and comprehensive teaching scaffolds with examples of children's responses that teachers can apply to problem-based contexts.

The PASMMap program provides a highly integrated approach to developing five core structural features described earlier for the PASA. There is particular attention to developing mathematical relationships such as multiplicative structures, equivalence and commutativity, the relationship between metric units, transformations and pattern, and structuring data. The first book focuses initially on repetitions and growing patterns, the grid structure, two-dimensional and three-dimensional relationships; structuring base ten, partitioning and sharing, equal grouping, unitising in measurement, and symmetry and transformations. These are followed by more challenging tasks that link with the previous pathways and extend to multiplicative patterns, metric measurement, patterns in data, and angles, direction and perspective taking.

Next I describe the pedagogy of the Pattern and Structure Mathematics Awareness Program approach that builds on sound teaching strategies, but there are several key processes that extend the learning to visualising, justifying and generalising.

Essentially the ‘mantra’ of the program is to highlight and model opportunities for developing pattern and structure, i.e., What’s the same? What’s different? Can you show a different pattern with the same structure? Figure 3 gives an example of such processes.

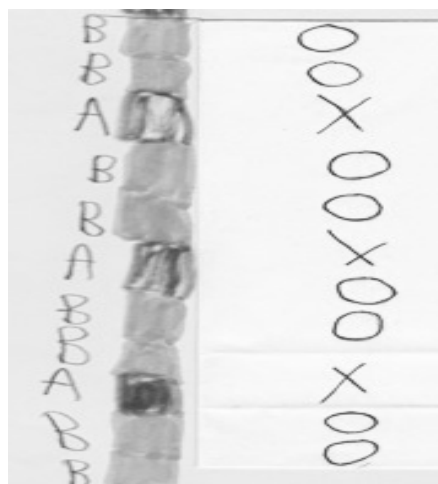


Figure 3. Symbolisation of AAB pattern structure

In this example the 5 year old child makes a tower of blocks in a green-green-brown pattern and then symbolises it using the letters BBA. When asked to find another way of showing the same pattern, using different symbols again, she produces the identical pattern structure as an OOX pattern, perhaps reflecting the game of noughts and crosses.

The tasks often have an explicit *focus on one aspect of structure* at a time, such as the unit of repeat in an ABB pattern. Further there is a direct attempt to help children make *connections* between components of pattern and structure. After children have used models and concrete materials the child is encouraged to explain and justify thinking, and then translate and generalise the pattern and structure. Tasks gradually become more complex and link more extensively to other concepts. The pedagogical model is summarised as follows in Table 2.

Table 2. *The PASMMap Pedagogical Model*

<b>Modelling</b>	Children copy, model and describe a pattern through completing a specified mathematical task, usually under teacher direction.
<b>Representing</b>	Children draw the pattern or a model of the pattern while it is still visible. Children isolate the essential features of the pattern or structure
<b>Visualising</b>	Children draw or symbolise the pattern or structure without being able to see it. Comparing their productions with the original pattern or structure highlights its essential features; process repeated until pattern or structure internalised.
<b>Generalising</b>	The teacher helps children (either individually or as a group) to make the pattern or structure explicit or to find similar examples in other contexts. Expressing and justifying thinking leads to generalising. What’s the same? What’s different? (make same pattern structure with different media)
<b>Sustaining</b>	Some suggestions are given for additional learning experiences that reinforce or extend the pattern and structure to other contexts. Relevant enquiry-based investigations are encouraged.

Some key questions are posed for the teacher in promoting children’s mathematics using PASA and PASMMap. With consistent implementation of the PASMMap approach answers to these complex questions will become easier to answer and the implications of these will become clearer.

- What are the structural features of the mathematical concept e.g. equal grouping?
- What are the structural features of the mathematical representations e.g. the number line?
- What are the spatial structural features e.g. square or cube; array or grid?

- Are there any patterns e.g. repetition, growing pattern, functional relationship; numerical or spatial?
- Is there a relationship between the structural features and the patterns?
- What evidence is there that the child has integrated or connected pattern and structure?

## Where to From Here?

In further studies there will be a review of how effective the PASA is in providing deep assessment of core mathematical concepts and how this informs practice. The PASMMap program is being adopted enthusiastically in many Australian classrooms and at an international level. But there needs to be more follow up of the impact of PASMMap on mathematical achievement longitudinally, with diverse samples. How would PASMMap influence other measures of mathematical achievement? What are the consequences for teachers who realise that their students are far more capable than what the curriculum or their school program mandates? Is there room to extend students using PASMMap while achieving required outcomes in an effective manner? A key implication is that once teachers see for themselves the impact of such a challenging assessment and program they may question their prior beliefs about what constitutes critical components of early mathematical development. The outcomes may require educators and the community to better understand, and consequently promote, deep mathematical structural development from an early age.

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# A TEACHER'S JOURNEY THROUGH MULTIPLICATIVE THINKING WITH EARLY SECONDARY STUDENTS

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*As part of the Reframing Mathematical Futures national project four WA secondary schools embarked on a journey to help students to become more able multiplicative thinkers, so that many areas of secondary mathematics would become more accessible to them. This involved targeted teaching within the regular mathematics classes. This paper is about the journey of one of these teachers.*

## **Reframing Mathematical Futures**

The Australian Department of Education and Training under the Mathematics and Science Partnership Program (AMSP) awarded two national grants, a one-year (2013) Priority Project, *Reframing Mathematical Futures* (RMF), and a three-year (2014-2017) Competitive Grant Project, *Reframing Mathematical Futures II* (RMFII), to Prof. Dianne Siemon from RMIT University who worked with a team of researchers across Australia. The RMF Project had the aim “to improve student outcomes in relation to multiplicative thinking and proportional reasoning in Years 7 to 10” (Siemon, 2014, personal communication). Multiplicative thinking was selected for this Priority Project as it had previously been identified as the area most responsible for the eight-year range in mathematical understanding from Years 5 to 9 (Siemon, Virgona, & Corneille, 2001). *The Scaffolding Numeracy in the Middle Years Linkage Project* produced a research-based Learning and Assessment Framework for Multiplicative Thinking, two formative test options and teaching resources (Siemon, Breed, Dole, Izard, & Virgona, 2006) and it was decided that these resources would be used in the RMF Project. It was seen as important for schools who joined the project in the RMFII Project stage to complete the RMF component before moving onto the RMFII stage, as the second stage of the project built upon the multiplicative thinking component.

## **Background Information**

Kalamunda Senior High School is a suburban school with 1100 students, in an area of Perth with a number of independent schools surrounding it. The school has a focus on Outdoor Education and the Arts, with a gifted and talented program in each of these areas. A nearby school has a gifted and talented program in Mathematics and Science. Kalamunda SHS has an Index of Community Socio-educational Advantage (ICSEA) value of 1005. Students seem content to achieve a pass mark. NAPLAN results were steady, but had not achieved the school's Business Plan target of exceeding the Australian mean in the past three years. It has long been an issue at the school that Year 10 grades in mathematics appear to fall without a well understood reason and there is a poor uptake of certified mathematics courses in Years 11 and 12.

Teachers at the school have identified that many students experience problems with mathematics and have had to cater for a large range of abilities for a number of years. The strategic approach to this was to stream students, typically starting in Year 9, into three ability groups following different courses of work. There is much research (e.g. Brophy, 1983; Hattie, 2012; Zevenbergen, 2003) to suggest that this is an unproductive strategy. Classroom management in the lower groups was generally an issue, with lower groups seldom provided with anything other than simplified worksheets, repetitive and memory based to try to improve the students' understanding. Teachers often felt powerless to know what to do but to go back to easier material. Learning Support staff were always willing to assist, but often endeavor to do so without adequate materials or development time.

## The Journey

Part of the attraction of involvement in the RMF Project was access and training in using diagnostic assessments and teaching materials in a targeted approach. As the nominated teacher in charge, I was trained over three days in the issues and difficulties faced by students with conceptual misunderstandings in numeracy and how to move students from being additive to multiplicative thinkers. I had not understood that many students would never achieve multiplicative thinking without specific intervention (Breed, 2011). When I returned to school I updated the team on the information and options available and we agreed to use the RMF materials on half the cohort in Year 7 and in Year 8. Together we predicted that between 20 to 25% of students would have difficulty with multiplicative thinking and we expected a five-year range in our Year 8 groups.

## Diagnostic Results

After the initial diagnostic assessment, we found that our school was normal! Our students had a range from zone 1 to zone 8 in Year 8 and ranged from zone 1 to zone 7 in Year 7 (see Figures 1 and 2). Around 44 % of our students were below the multiplicative thinking threshold (lower than zone 4), approximately 32% of the students were at the threshold (zones 4 and 5) and 23% were multiplicative thinkers (above zone 5).

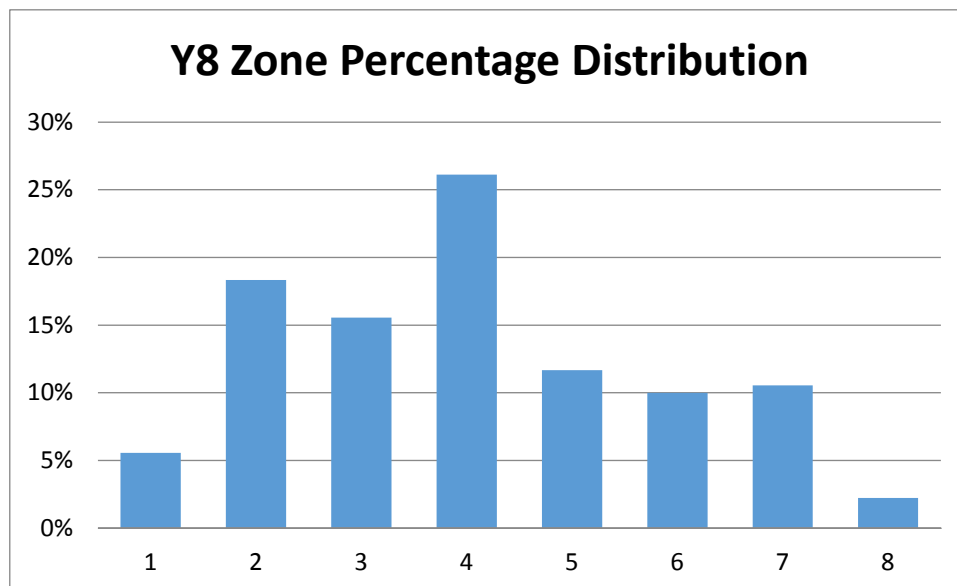


Figure 1. LAF zone distribution for Year 8 students.



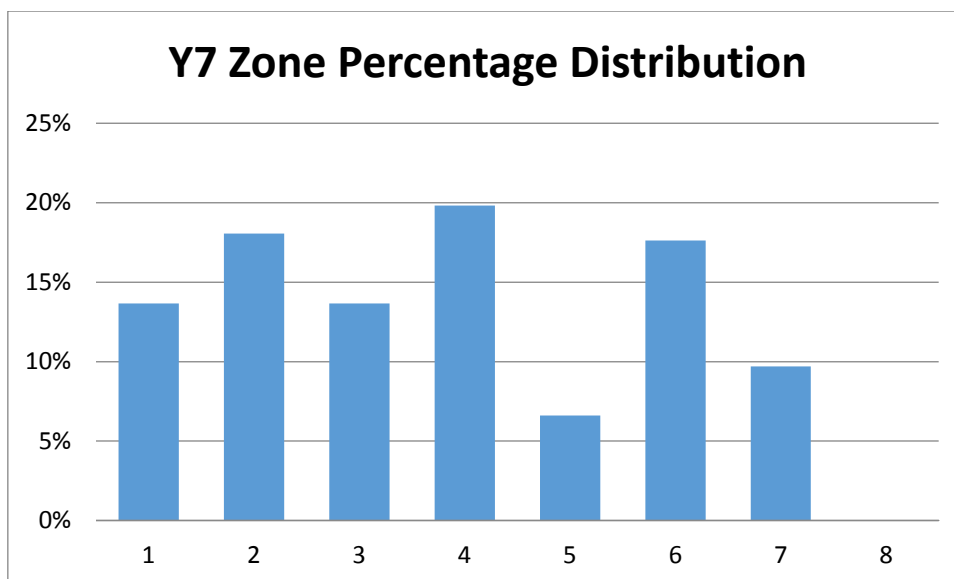


Figure 2. LAF zone distribution for Year 7 students.

These results were totally unexpected. The teachers had not predicted the depth of the lack of understanding of the students. We had to accept that the problem was bigger than we had expected or wanted to believe. The data did, however, offer an insight into the potential issues faced in our school. The challenge would be to take the information and use it to improve teaching and learning.

### The Challenge and Intervention

Now we had a better understanding of the depth of the issues faced by our students, the next phase was to design an intervention to assist in specific ways for individual students. A big challenge! The development of multiplicative thinking was seen as a benchmark in predicting future success in Years 9 and 10. Kilpatrick, Swafford and Findell (2001) considered proportional reasoning as being a gateway to higher levels of mathematical success, and Dole, Clarke, Wright and Hilton (2012) saw proportional reasoning as fundamental to further success in mathematics and science. Knowing what the range of difficulties students faced offered a chance to start a targeted intervention program to assist all students to be able to access mathematics in future years.

As a team, we spent time discussing the strategies and materials we would need to approach RMF in a way that would suit our students. We had very few manipulatives and tasks to exploit the use of concrete materials. Over the years our school has spent thousands of dollars buying newer and better differentiated text materials and yet our results had never reflected that investment. It appeared that our problems were deeper than any standard text book could accommodate and we had to design learning experiences to assist in the conceptual development of our students. RMF provided a series of structured tasks for each zone level which we decided to put into use. We appreciated that teacher involvement would be huge and perhaps chaotic, especially when running several tasks in a room. We understood that the entire process would need to be supported and managed well. The decision was made to have two four-week blocks dedicated to RMF.

We began by splitting students into zone groups. Teachers made themselves aware of the purpose of the progression for each set of activities and put them into operation. All went well for the first week! It soon became obvious that we could never maintain this level of operation in our classrooms. We used funding from the RMF Project to have two teachers in each room, but although this was a better solution it was still difficult to control and support and it would be impossible to sustain this approach once the Project funding was no longer available. Running multiple groups was too demanding for the teachers so we needed to adapt. After discussing what it was we were aiming to achieve, we decided to use a targeted approach to fundamental concepts suitable for early, middle and late zones, as we called them. Teaching specific strategies to a group put the teachers back into an area in which they felt comfortable.

Additive thinkers were encouraged to group and partition, two essential skills that were missing for many students. Those acquiring multiplicative skills were encouraged to develop multiplication strategies to better understand the process of multiplication. At no stage were any algorithms taught or used, rather students were encouraged to use arrays and partial product strategies with partitions determined by the students. Those who were multiplicative thinkers were stretched into using and describing ratio and proportion problems involving rates or movements.

### **Professional Development and Learning Teams**

One of the difficulties for secondary teachers of starting where the students are developmentally, is that they are not trained in primary mathematics. Many secondary teachers do not have the understanding of how early mathematics is conceptually developed and many do not have the language understanding to follow a primary developmental progression and understand it. Sessions were spent with the mathematics staff demonstrating the conceptual development necessary for numeracy. We drew on the teaching advice in the Learning Assessment Framework and other material available through the RMF Project and came to an agreement about how to approach specific topics, for example when teaching place value how decimal points remain static and it is the numbers that move when multiplying or dividing by ten. We have agreed on a set of multiplication strategies which has helped to reinforce strategies rather than memory skills as important in mathematics. There is still an emphasis on fluency, but with numbers other than one to ten. An important question we adopted was “Can you draw this for me to help me understand your thinking?”

As well as sharing strategies we agreed on a common approach in the classroom. Each lesson begins with five to ten questions using a particular RMF strategy, for example using a partial product when multiplying. The questions are left on the board so students may return to them as necessary. Teachers then take a particular group and make the targeted teaching point to be achieved in that session. This is repeated around other groups until every student is working on a specific task or series of tasks. Teachers circulate around the room checking on progress and questioning students to determine understanding.

Some of the teachers felt more confident than others when it came to the organisational aspects of teaching students in groups. This was a pedagogical leap for some of the teachers who had been used to teaching the entire class as a group. We have a few teachers with primary teaching experience and their experience has been beneficial in assisting to organise and structure the activity based learning in the classes. We determined that some lessons would be task based and others would be based on text book work. The use of task based mathematics offers teachers a way to engage students in reasoning activities and problem solving. Better tasks will have a low entry point allowing most students to begin the problem and have multiple exit points (Boaler, 2016; Day, 2012; Sullivan 2011) allowing teachers and students to access the useful mathematics embedded in the task. One of the problems some of the teachers faced, at least initially, was the transition from the task to the specific mathematical knowledge to be gleaned from the task. Making the mathematical point requires skill from the teacher in connecting the mathematical ideas and making the teaching point explicit. It was important for us to spend time on seeing what mathematics could be ‘mined’ from each task to ensure that there were opportunities for deeper learning (Boaler, 2016).

### **Results after Intervention**

After the two four-week blocks of intervention the question was whether our efforts had been worthwhile. Hattie (2012) suggested an effect size of 0.1 equated to a natural improvement of approximately one term of learning. Our figures indicated an average effect size of 0.55 after our two four-week blocks (see Figures 3 and 4). This was encouraging, and in our view worth pursuing further.

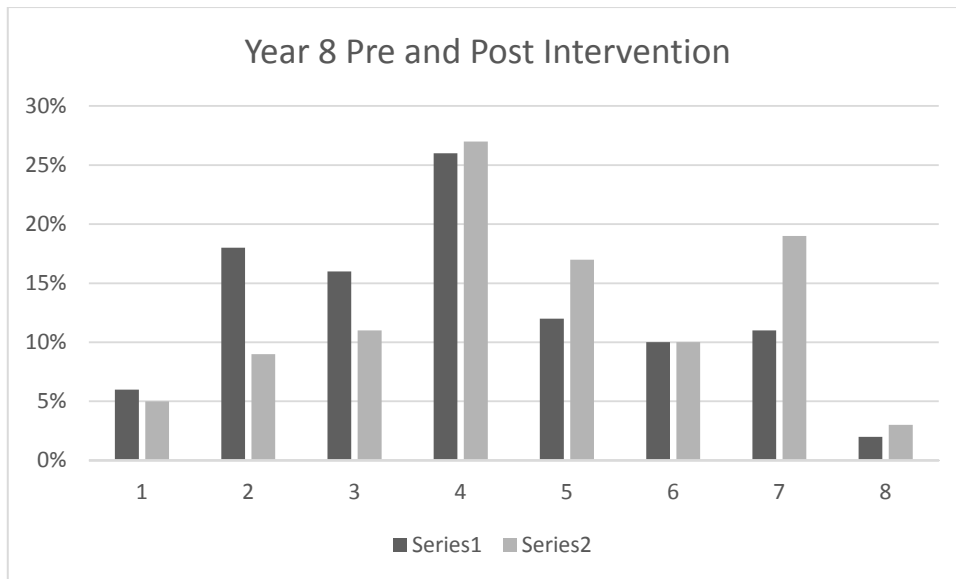


Figure 3. LAF zone distribution before and after intervention for Year 8 students.

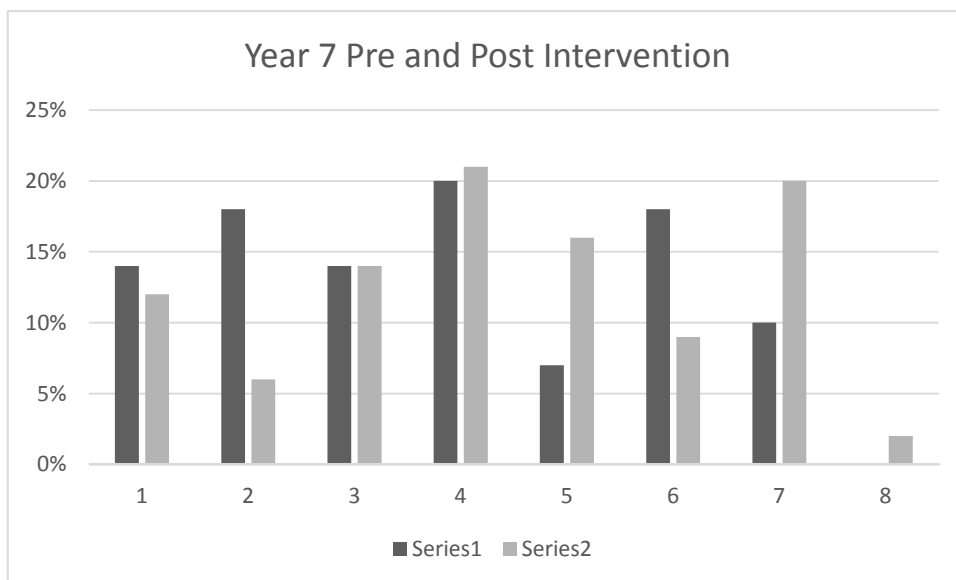


Figure 3. LAF zone distribution before and after intervention for Year 7 students

The graphs indicate that there has been a general shift to the right which is the desired outcome. Comparative percentages in the three bands of zones are provided in Table 1. Interestingly our data reinforce the national data that shows that zone 4 is the most difficult zone for students to move out of. Recent NAPLAN results for Year 9 were also encouraging, with us reaching our target of surpassing the national average for the first time in four years. The target has now been revised to surpass the state average, which is higher, in 2017.

Table 1.  
*Comparative Percentages Pre and Post Intervention*

	Pre Intervention	Post Intervention
Zones 1 - 3	44%	28%
Zones 4 & 5	32%	40%
Zones 6 - 8	23%	31%

### **Student Feedback**

Student surveys indicate that the students enjoy the practical, hands-on aspects of using tasks. They seem to enjoy the collaboration between groups and the challenge to solve particular problems. The students do not always appreciate why it is important to proceed past the original problem to generalise a solution, but that will hopefully come in time. Since adopting this methodology in the school we have seen better engagement with all levels of students. Students who are viewed as more able have often been surprised by the practical insights offered by those considered as less able, which has contributed to a better equality in the learning process. The students enjoy the freedom that tasks bring to their mathematics and they do not always want to practise the new skills they have learned by returning to text based exercises. From our perspective improving the students' understanding of the whole learning and doing process still requires work. Although most students prefer the task based approach, some still prefer a text based approach to learning mathematics through being taught specific algorithmic approaches. Interestingly, these students tend to sit in the lower to middle multiplicative thinking zones.

### **Sustainability**

Over the last two years, we have scheduled our task activities into two four-week blocks per year. Both teachers and students have suggested that this should be extended into three or four blocks next year, so we are currently planning for four blocks in 2017. Although it is tempting to change to more and more task based learning, it is important that we balance this temptation with patience. We have adopted the notion that it is better to do a small amount well than take on too much. The increased engagement with students needs to be balanced against the stress on the mathematics staff, who have found it tough going but rewarding.

Initially, it was a challenge to organise and create a suite of resources suitable for such a diverse range of abilities. Having access to the RMF materials was critical in this aspect. Not all tasks that we have tried have worked as well as we had hoped, some have failed to engage students and others have had an entry point that it too high for all students to access the task. Using this knowledge and the RMFII Project resources that move into algebraic, geometrical and statistical reasoning has led to an engaging collection of activities for students and for most staff members a renewal of spirit. Several staff have indicated that this is why they wanted to be teachers of mathematics in the first place.

To be most effective the teachers have recognised that often they need to use a concrete model to assist students to understand a concept and that these times are not limited to our task based learning blocks. It is become a priority to build a bank of resources in each mathematics classroom, which can become quite expensive. We have found that along with concrete experiences students need to develop a deep understanding of the mathematics and to this end, we have found that interrogating incorrect solutions to problems has been a useful tool to help to improve student understanding.

Teachers now having an awareness of the additive and multiplicative thinking required in the content strands of the Australian Curriculum: Mathematics [ACM] (ACARA, 2016), has benefited the teaching of mathematics in our school. Teachers are working better to differentiate lessons depending on where students are on the continuum of additive to multiplicative thinking. The real litmus test though has

been the level of engagement from the students. They have shown a willingness to solve practical problems, they are more confident in using numbers including fractions, and we are beginning to see these changes reflected in results. We are hopeful that our hard working staff see the benefits of their efforts over a long period of time.

## Conclusion

Our progress at Kalamunda Senior High School has been slow but marked. Our students tell us that they are better engaged and seeing mathematics as a problem solving and interesting subject. They have enjoyed the change of emphasis away from the memorization of facts into actually doing mathematics. Teacher skill in understanding the conceptual acquisition of number has improved our own professional development and often takes place over the staff room table. We had some teething problems initially, mostly in the setting up and organization of the program, but these have been overcome. The benefits in student learning, in doing some mathematics and enjoying it are much more important. Our progress is pleasing, but needs to be maintained and our results need to show a level of consistency. It is early days to predict long term success, but the signs are very encouraging.

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# LEARNING WITHOUT UNDERSTANDING

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*This paper describes how 100 trainee teachers in my institute who are fluent in performing addition, subtraction, multiplication and division of proper fractions performed in making multiple representations of these mathematical computations involving fractions. The representations were first the symbolic representation of the computation, then representing the computation using a diagram and finally creating a word problem based on the mathematical computation. The aim of this study was to determine if multiple representations of the mathematical computation and the ability to make connections between the representations is an indicator of conceptual understanding. The study also attempted through discussion to mediate thinking and conceptual understanding of multiplication and division of fractions.*

## **Introduction**

Competency in Mathematics is an invaluable skill throughout the globe. Every country engages in developing ways to improve the mathematical competency of their citizens so that they become proficient mathematicians. In schools, before the advent of computation tools, the ability to do algorithmic computation aptly was regarded as a display of mathematical competency and the speed to do a computation was the yardstick to determine proficiency in Mathematics. Later the difficulty to do word problems which was classified as problem solving emerged as a separate area that was considered to require emphasis. With this came the realisation that although an individual is capable of performing the context free computation (example  $3+5 = 8$ ) they were unable to relate the computation to a real word situation and hence the individual faced difficulty to solve problems in context (word problems) (Vula & Kurshumlia, 2015). Most children have difficulty writing a mathematical statement to represent the problem situation. Once the mathematical statement is written the computation was a mechanical process. The lack of ability to relate a word problem to a mathematical statement was regarded as inability to solve word problems (Reeds, 1999) and not as a lack of conceptual understanding of the meaningless algorithm learnt without understanding by the children.

In recent years, efforts have been made to focus on what is necessary for students to be proficient in mathematics and what it means for a student to be mathematically proficient. The National Research Council (2001) set forth in its document, *Adding It Up: Helping Children Learn Mathematics* a list of five strands to illustrate mathematical proficiency. One of the five strands is Conceptual understanding (comprehension of mathematical concepts, operations and relations). In the document, they claim that a significant indicator of conceptual understanding is being able to represent mathematical situations in different ways. It is stated that it is important to see how the various representations connect with each other, how they are similar, and how they are different. It is further stated that the degree of students' conceptual understanding is related to the richness and extent of the connections they have made. For example, suppose students are adding fractions, for example  $1/3 + 2/5$ , they might draw a picture or use concrete materials of various kinds to show the addition. They might also represent the number sentence as a story. The implication of these claims is that merely being able to do the computation correctly is not an indication of conceptual understanding but the ability to make multiple representations and being able to see the connection between these representations is the significant indicator of conceptual understanding.

## The Study

This study aims to determine if individuals who are fluent in basic mathematical computation are able to represent the computation using multiple representations (the indicator of conceptual understanding set by the National Research Council (2001)). The study hence employed individuals who can compute fluently and requested them to represent the mathematical sentence using multiple representations. An individual's ability to make multiple representations of the mathematical sentence was considered as an illustration of their ability to make connections between the representations and hence an indication of the extent of their conceptual understanding. If these experts face no difficulty to illustrate the mathematical sentences using multiple representations then the claims made by the National Research Council (2001) are questionable. The question that will arise is, "Is computational fluency an insufficient indicator of conceptual understanding?" However, if those who can compute fluently face difficulty to use multiple representations to illustrate the mathematical sentence, then it may be concluded that computational fluency alone cannot be used as an indicator of conceptual understanding. The study goes a step further to attempt through discussion to help the participants gain understanding of the mathematical sentences and then investigate the change in the participants' ability to use multiple representations for the given mathematical sentences. The operations selected for this study were addition, subtraction, multiplication and division of proper fractions.

## Participants

The participants of the study were 100 trainee teachers who are fluent in doing addition, subtraction, multiplication and division of proper fractions. These participants pre-degree mathematics courses reveal that the algorithm to perform the four basic operations on proper fractions was never a problem for the 100 selected participants of this study.

## Methodology

The study is a survey design (Schoenfeld, 2007) to determine the ability of individuals who are high achievers in addition, subtraction, multiplication and division of proper fractions to provide multiple representations of four simple fraction questions to illustrate their conceptual understanding of the given questions. They were requested to compute the answer of the context-free problems, then draw a diagram to represent the question and solution and to write a word problem question based on the mathematical sentence. No time limit was imposed on the participants to complete the task. Ninety individuals completed, what they claimed they were able to do, in less than 20 minutes while the other ten took about forty minutes. After the test, the responses were checked and ten participants were interviewed for their responses. The participants selected for interview consisted of three with all correct responses for both multiplication and division, two had correct responses for diagrammatic representation of multiplication and division of fractions, two who had correct responses for diagrammatic representation of multiplication and the remaining three had incorrect responses for diagrammatic representation and word problem representations of multiplication and division. During the interview sessions the participants were required to explain their responses. Discussion during the interview sessions were also aimed to aid the participants who were unable to provide multiple representations to gain an understanding of the mathematical sentences that posed difficulty for them. A group interview was then conducted after the individual interviews to aid everyone who was unable to provide multiple representations. The discussion to aid the participants to understand the mathematical sentences was also aimed to obtain data of the participants' ability to provide multiple representations of the mathematical sentences with the attainment of conceptual understanding.

The following were the four questions administered.

1.  $\frac{1}{3} + \frac{1}{6} =$       2.  $\frac{1}{3} - \frac{1}{6} =$       3.  $\frac{1}{2} \times \frac{1}{4} =$       4.  $\frac{1}{2} \div \frac{1}{4} =$

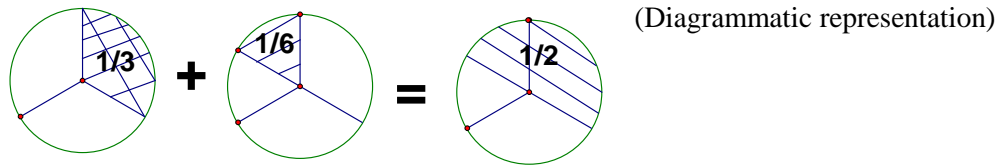
Expected responses

The following are the examples of expected responses for each of the questions.

Question 1.

$$\frac{1}{3} + \frac{1}{6} =$$

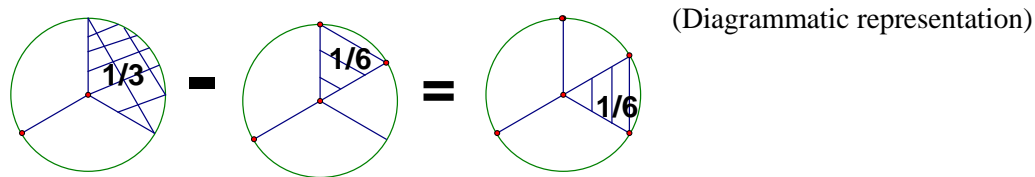
$$\frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2} \text{ (Symbolic representation)}$$



Ali ate  $\frac{1}{3}$  of a cake and John ate  $\frac{1}{6}$  of a cake. How much cake did they eat altogether? (Word problem representation).

Question 2.  $\frac{1}{3} - \frac{1}{6} =$

$$\frac{1}{3} - \frac{1}{6} = \frac{1}{6} \text{ (Symbolic representation)}$$



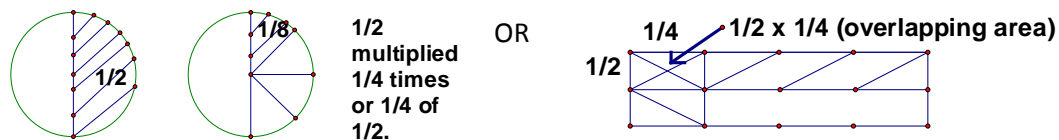
Ali had  $\frac{1}{3}$  of a cake. Ali gave John  $\frac{1}{6}$  of a cake. How much cake does Ali have left? (Word problem representation).

3. Question 3.  $\frac{1}{2} \times \frac{1}{4} =$

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8} \text{ (Symbolic representation)}$$

Interpretation 1: (Multiplicand x multiplier)

The following is the diagrammatic representation of the multiplication sentence.



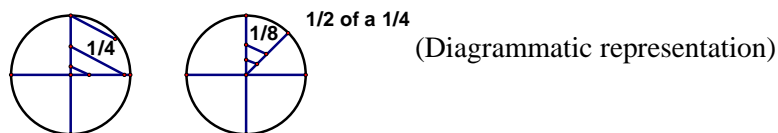
An example of a word problem is as follows.

Ali had  $\frac{1}{2}$  a cake. He ate  $\frac{1}{4}$  of the piece of cake he had. How much of the cake did he eat? (Word problem representation)



Interpretation 2:

The computation may be interpreted as  $\frac{1}{2}$  of  $\frac{1}{4}$ , since the multiplication operation is often read as ‘of’. In such a case of interpretation the multiplier is assumed to be  $\frac{1}{2}$  and the multiplicand is  $\frac{1}{4}$ . The following is the diagram for such a case of interpretation.



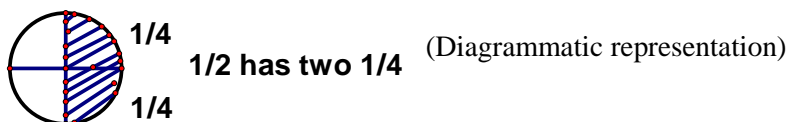
Ali had  $\frac{1}{4}$  of a cake. He ate  $\frac{1}{2}$  of the piece of cake he had. How much of a cake did he eat? (Word problem representation).

4. Question 4.  $\frac{1}{2} \div \frac{1}{4} =$

$\frac{1}{2} \div \frac{1}{4} = 2$  (Symbolic representation)

Division can be interpreted in two ways.

(i) For example when we write  $24 \div 8 = 3$ . This can represent a problem as follows: If you have 24 balloons and you give each child eight balloons, how many children will receive balloons? The dividend represents the quantity to be divided. The divisor represents the quantity in each group. The quotient represents the number of groups obtained (in this case number of children who receive eight balloons). Similarly the above division of fractions can be interpreted as follows:  $\frac{1}{2}$  represents the quantity to be divided. The divisor  $\frac{1}{4}$  represents the quantity in each group. The quotient represents the number of groups of  $\frac{1}{4}$ . Hence the diagram below shows the dividend  $\frac{1}{2}$  (the shaded area) which is divided into quarters and there are two quarters in the  $\frac{1}{2}$ .

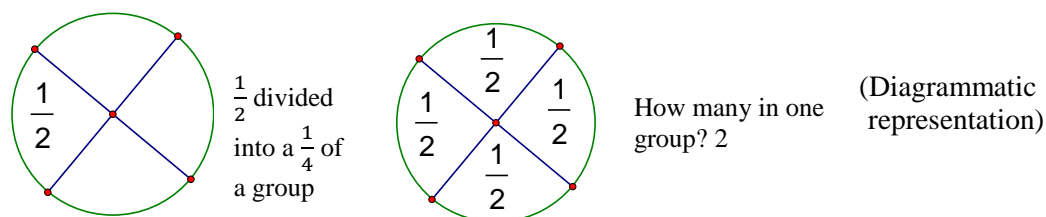


An example of a word problem is as follows.

Rani had  $\frac{1}{2}$  a cake. She made packs with a  $\frac{1}{4}$  cake in each. How many packs did she manage to make? (Word problem representation).

(ii) A second way to interpret  $24 \div 8 = 3$  is as follows: If you have 24 balloons and you share it among eight children equally, how many balloons will each child receive? The dividend represents the quantity to be divided. The divisor represents the number of groups. The quotient represents the quantity that will be received by one group (in this case number of balloons that would be received by one child). Similarly the above division of fractions can be interpreted as follows:  $\frac{1}{2}$  represents the quantity to be divided. The divisor  $\frac{1}{4}$  represents the number of groups (Note it is only a quarter of a group). The quotient represents the quantity one group will receive.

Hence, the diagram on the left below shows the dividend  $\frac{1}{2}$  in a quarter of the whole diagram (the whole square). The diagram on the right shows the quotient, which is the quantity one group will receive.



A  $\frac{1}{4}$  acre of land produces  $\frac{1}{2}$  a ton of a particular grain. How much grain will one acre of land produce? (Word problem representation).

## Data Analysis and Discussion

The responses from the 100 participants were analysed for the number of correct responses for each category: computation, diagrammatic representation and word problem. Table 1 shows the number of correct responses for the 100 participants of this study. The incorrect responses were also analysed to provide insight into the line of thinking of the participants and assistance was provided to help participants understand the mathematical sentences.

Table 1: Number of correct and incorrect responses for the four questions (✓ = correct response; X = Incorrect or no response)

Question \ Representation	1		2		3		4	
	$\frac{1}{3} + \frac{1}{6} =$		$\frac{1}{3} - \frac{1}{6} =$		$\frac{1}{2} \times \frac{1}{4} =$		$\frac{1}{2} \div \frac{1}{4} =$	
	✓	X	✓	X	✓	X	✓	X
Symbolic	100	0	100	0	100	0	100	0
Diagrammatic	100	0	100	0	16	84	5	95
Word Problem	100	0	100	0	3	97	3	97

The data shows that all 100 participants were able to provide the symbolic representation for all four fraction questions without any difficulty. All 100 participants were able to do the algorithm without any problem because it requires following fixed steps without demanding understanding of the algorithm. It also shows that all participants in this study did not face any difficulty to provide the symbolic representations of addition, subtraction, multiplication and division involving proper fractions.

Data from Table 1 also shows that illustrating the fraction question in diagrammatic form for the addition and subtraction of fractions' questions did not pose difficulties for the 100 participants. Figure 1 shows an example of the correct response for the addition question, question 1 (Figure 1: i) and two responses for the subtraction question, question 2 (Figure 1: ii and iii) of participants.

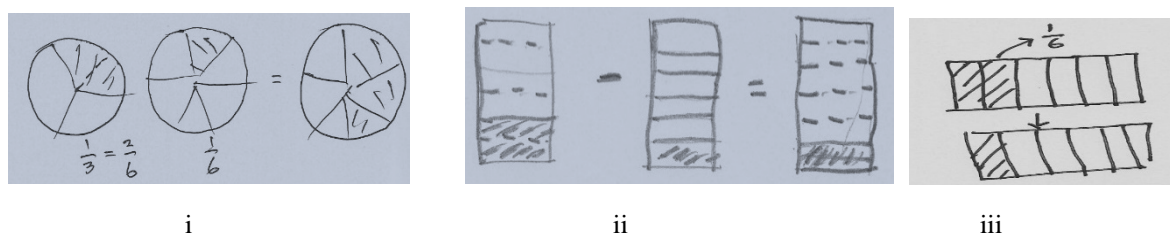


Figure 1. Example of correct responses for questions 1 (i) and 2 (ii and iii) using diagrammatic representations

All participants were able to provide correct word problem representations for both the addition and subtraction of fractions' sentences, that is, questions 1 and 2 respectively. During the interview the participants revealed that addition and subtraction of fractions were clearly understood by them. The data revealed that since the participants understood addition and subtraction of fractions clearly they were able to use multiple representations to illustrate the mathematical sentences in questions 1 and 2.

Data in Table 1 shows that only 16 participants were able to represent the multiplication problem using a diagram. Figure 2 shows two examples of the correct response for the multiplication question

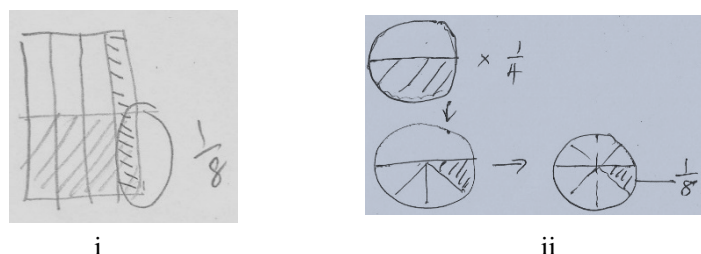


Figure 2. Correct responses for question 3 using diagrammatic representations

During the interview, the participant for the response on the left, Figure 2 (i), claimed that he first shaded  $\frac{1}{4}$  and then shaded  $\frac{1}{2}$ . He stated, " $\frac{1}{2}$  is the multiplicand and  $\frac{1}{4}$  is the multiplier and so taking  $\frac{1}{4}$  of  $\frac{1}{2}$  is  $\frac{1}{8}$ ". When asked if it can be multiplier first and then multiplicand the participant agreed that it can be because multiplication is commutative but they rather follow the rule. As for the response on the right (ii) in Figure 2, the participant explained, " $\frac{1}{2}$  so I drew  $\frac{1}{2}$  and then times  $\frac{1}{4}$ , so take  $\frac{1}{4}$  of  $\frac{1}{2}$  is  $\frac{1}{8}$ ". When asked if it can be multiplier first and then multiplicand the participant stated that he did not think along that line. He said, "I read the question from left to right and drew the diagram and did not pay any attention to the multiplicand  $\times$  multiplier rule. I do not think it is very important in the case of fractions". I then asked the second participant if he had  $\frac{1}{2}$  a cake and ate  $\frac{1}{4}$  of the cake he had, how much would he have left and he said, " $\frac{3}{8}$ ". Then I asked him if he had a  $\frac{1}{4}$  cake and he ate  $\frac{1}{2}$  of the cake he had, how much would he have left and he said, " $\frac{1}{8}$ ... Oh!!... It is not the same actually. The multiplicand represents what we have and the multiplier what we are taking from it for fractions". The discussion sessions with the interviewees helped them gain better insight and the participant at the end of the session stated, "I enjoyed this interview. It's an eye opener". Both participants' word problems were correct.

Data in Table 1 shows that 84 participants were unable to represent the multiplication sentence using a diagram. Figure 3 shows the incorrect diagrammatic representations from two participants who were able to provide the correct symbolic representation with the correct answer  $\frac{1}{8}$ .



Figure 3. Incorrect responses for question 3 using diagrammatic representations

Figure 3 (i) shows 4 times  $\frac{1}{2}$  and not  $\frac{1}{4}$  of  $\frac{1}{2}$  which is  $\frac{1}{2}$  times  $\frac{1}{4}$ . Figure 3 (ii) shows a bar shaded to represent  $\frac{1}{2}$  and another bar shaded to represent  $\frac{1}{4}$  with a multiplication operation in between the two fractions. The participant was unable to show the product in the diagram. Initially, during the interview, they claimed that they did not know how to represent the computation diagrammatically because they did not know what the mathematical sentence represented. Then during the interview they were asked to read the question and they read it as follows:  $\frac{1}{2}$  times  $\frac{1}{4}$ . They were then told to read it as ' $\frac{1}{2}$  of  $\frac{1}{4}$ '. Both participants were then able to draw a diagram to represent the mathematical sentence. They drew the multiplier  $\frac{1}{4}$  and divided it into two parts to obtain  $\frac{1}{2}$  of a quarter (multiplier  $\times$  multiplicand). They were then able to provide correct word problems for  $\frac{1}{2} \times \frac{1}{4}$ . Both participants then, during the interview

stated that they had a clearer picture of the question. One of them stated, “*I understand it better now, after all these years of not giving thought to it since I have never had problems when multiplying fractions but actually did not think about what it represented.*” It has been revealed that they had learnt multiplication of fractions without thought of what it represented, that is meaninglessly. The interview revealed that participants who had a clear conceptual understanding of the mathematical sentence were able to provide multiple representations of the multiplication of fractions and are able to explain the connection between the representations. On the other hand, those with a vague or no conceptual understanding, struggled or were unable to provide multiple representations of the multiplication of fraction sentence.

The 13 who were able to draw a correct representation but unable to provide a word problem claimed that they had guessed the diagram. All 13 had used the area model to represent the multiplication but were unable to explain the multiplication of two fractions in a meaningful manner. They stated that the length of the sides were  $\frac{1}{2}$  by a  $\frac{1}{4}$  metre long, and hence the area was  $\frac{1}{8}$  square meter. However, through discussion, understanding was achieved and these intelligent participants had no problem depicting the question with multiple representations and could then provide many examples of word problems.

Data in Table 1 shows that for the division question only five participants were able to represent it using a diagram. Figure 4 below shows 3 examples of correct responses. The other two responses were similar to Figure 4 (i).

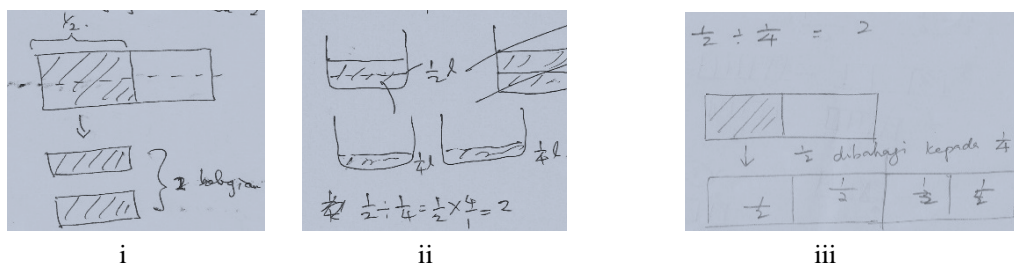


Figure 4. Correct responses for questions 4 using diagrammatic representations

During the interview it was revealed that participants of responses Figure 4 (i) and (ii) were clear and actually understood what was meant by the question. They also were able to provide correct word problems. The participant who gave the response for Figure 4 (iii) explained her diagram as follows:

“*Since  $\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{1}{2} \times 4$ , so the  $\frac{1}{2}$  is repeated 4 times*” She was unable to explain it as has been explained in the second interpretation of division, that is, if a  $\frac{1}{4}$  of a group has  $\frac{1}{2}$  therefore a whole group would have 2. She also was unable to provide a word problem for the question. I then asked her, “*If I gave you  $\frac{1}{2}$  a kg of flour and you are to pack it into  $\frac{1}{4}$  kg packets of flour, can you write a mathematical statement for it?*” She wrote  $\frac{1}{2} \div \frac{1}{4} = 2$  and stated that she can make 2 packets.

Then I drew a rectangle divided into quarters as shown in Figure 5.



Figure 5. A rectangle divided into 4 equal parts

I asked her, “*If you divide  $\frac{1}{2}$  kg of flour into  $\frac{1}{4}$  of this container, where would you put the  $\frac{1}{2}$  kg of flour given to you?*” She replied that she could not understand the question. I then asked, “*If 6 apples were to be divided into 2 containers, how many in each container?*” She replied, “3”. I then explained, you shared it because you have 2 containers and more than 2 apples. Then I asked her, “*Now let us say you have  $\frac{1}{2}$  an apple and you are to share it in a container having 3 equal compartments, but you only have to share it between  $\frac{2}{3}$  of the container. How would you do it?*” (I sketched a rectangle with 3 equal parts as in Figure 6).



Figure 6. A rectangle divided into 3 equal parts

She replied, “Well I guess cut it into 2 and put a  $\frac{1}{4}$  of the apple in two compartments of the container”. I then asked, “What have you done with  $\frac{1}{2}$  an apple?” and she replied, “Divided it into two parts”. I then told her, “Let us stop at  $\frac{1}{2}$  was divided”. Then I asked her, “What part of the container was filled with your pieces of apple?” and she replied, “2”. I then asked her, “2 out of how many parts in the container?” and she replied “3”. Then I asked her, “What fraction of the container did you fill with the pieces of apple?” and she replied, “ $\frac{2}{3}$ ”. Then I asked her, “Can you make a mathematical sentence with what you have done”. After some thought, she smiled and said, “Is it  $\frac{1}{2}$  divided by  $\frac{2}{3}$ ?” Then I asked her, “If you put another equal size piece of apple in the third empty compartment what is the total amount of apple in the container?” She replied, “ $\frac{3}{4}$ ”. I then told her to work out  $\frac{1}{2} \div \frac{2}{3}$ . She obtained a quotient of  $\frac{3}{4}$ . She then said, “O.K. I now understand what you were asking me. If  $\frac{1}{2} \div \frac{1}{4}$ , then each  $\frac{1}{4}$  of the container has  $\frac{1}{2}$  kg of flour and to fill the whole container I will require 2 kg of flour. Wow! I see it now”. She however stated, “I feel the division of fractions are used more for how many groups can you make if you divided for example  $\frac{1}{2}$  kg of rice into  $\frac{1}{4}$  kg packets and we get 2 rather than how many in one group”. (She meant  $\frac{1}{4}$  kg packets from  $\frac{1}{2}$  kg of flour).

Data in Table 1 shows that 95 participants were unable to provide correct diagrammatic representation of the division sentence. Figure 7 below shows examples of incorrect responses from two participants.

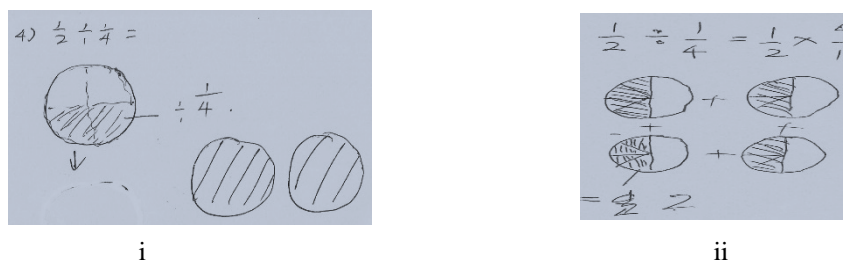


Figure 7. Incorrect responses for question 4 using a diagram

The participant whose response is shown in Figure 7 (i), when requested to explain his diagram during the interview stated, “I drew and shaded  $\frac{1}{2}$  and then divided the whole diagram into  $\frac{1}{4}$  because need to divide by  $\frac{1}{4}$ ”. I asked him why he had drawn two circles and shaded it. He replied, “Well the answer is 2”. I then asked, “So if I give you a cake and you divided it into 2 parts, each part is  $\frac{1}{2}$  and then you divide it into  $\frac{1}{4}$ , you can get 2 whole cakes?” He replied, “No I cannot but I don’t know why we get 2 as the answer”. Then I asked him, “If you had  $\frac{1}{2}$  a litre of water in a container, how many  $\frac{1}{4}$  litre bottles can you fill?” He replied, “2, of course, oh.... each  $\frac{1}{2}$  has two  $\frac{1}{4}$ , so when  $\frac{1}{2}$  is divided by  $\frac{1}{4}$  we have 2 pieces of  $\frac{1}{4}$  cake. I understand now how the two comes about. Thank you”.

The participant whose response is shown in Figure 7 (ii), when requested to explain his diagram during the interview explained exactly like the participant who gave the response for Figure 4 (iii). He explained, “Since  $\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{1}{2} \times 4$ , so the  $\frac{1}{2}$  is repeated 4 times” I then asked, “If you are given  $\frac{1}{2}$  a cake how does it become two cakes?” He replied, “The question wants us to get four half cakes and so it will be 2 cakes”. I then asked, “Why write it as  $\frac{1}{2} \div \frac{1}{4}$ ? It is clearer to write  $\frac{1}{2} \times 4$ , so what is the difference between the two operations?” He replied, “Well for normal multiplication that is whole number we repeat the multiplicand the number of times as stated by the multiplier. Division we take say 10 apples and divide into 2 if it is  $10 \div 2$ . For fraction and whole number multiplication is the same as normal multiplication. But division of fraction by fraction becomes fraction multiplying by whole number if the numerator is 1 for the divisor” It is obvious that here meaning was formulated using the step in the algorithm which made sense to him. The question itself was meaningless, hence the interpretation of the quotient was to match the numerical answer. I then asked him, “If you had  $\frac{1}{2}$  a litre of water in a container, how many  $\frac{1}{4}$  litre bottles can you fill?” He replied, “Twooo...” I then said, “Well what operation to describe the process of taking  $\frac{1}{2}$  a litre and making it into  $\frac{1}{4}$  litres?” He replied, “O.K, I see, it is  $\frac{1}{2}$  divided by  $\frac{1}{4}$  and it is 2. Yes it makes a lot of sense, I understand” He smiled

and stated, “I was wrong isn’t it?” I replied, “Not exactly” I then explained the second interpretation and he said, “O.K, so I was not totally wrong. The repetition of  $\frac{1}{2}$  four times is the total if we had a whole group. Not that  $\frac{1}{2}$  is repeated four times, that is  $\frac{1}{2} \times 4$ . I now see the difference”.

During the group interview all the participants could understand the first interpretation for division of fractions but had difficulty understanding the second interpretation. Hence they were all able to make word problems based on the first interpretation for the division of fractions, that is, how many groups, rather than how many in each group.

The interview revealed that participants who have a clear conceptual understanding of the mathematical sentence were able to provide multiple representations of the division of fractions and are able to explain the connection between the representations. On the other hand, those with a vague or no conceptual understanding, struggled or were unable to provide multiple representations of the division of fraction sentence.

Every participant who was interviewed and was guided to understand the mathematical sentences for multiplication of a fraction by a fraction and division of a fraction by a fraction expressed their joy in learning with understanding. Each and every one of them were then able to represent the mathematical sentence using a diagram and also were able to make a word problem without any difficulty.

Throughout the study participants who were able to provide symbolic representations but did not have conceptual understanding of the mathematical sentences were unable to provide multiple representations. All participants in the study had a clear conceptual understanding of the addition and subtraction of fractions sentences and hence were able to provide the multiple representations. In contrast, those who initially did not have a clear or no conceptual understanding of the multiplication and division of fraction sentences faced difficulty to provide multiple representations. However, through discussion the participants when they realised the meaning of the mathematical sentence, that is, attained conceptual understanding, were able to then provide multiple representations of the multiplication and division sentences.

## Conclusion

The study has revealed that the ability to make multiple representations of the mathematical sentences and the ability to see the connection between the representations is an indicator of conceptual understanding. Hence it is recommended that when students are learning computation using basic mathematics operations, it is vital to encourage them to provide multiple representations of the mathematical sentences and encourage discussion to aid the students to attain a better conceptual understanding of the mathematics they are learning.

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# INCLINING TO EXPLORE MATHEMATICALLY AND PEDAGOGICALLY: STUDENTS AND TEACHERS POSSESSING THE SAME CHARACTERISTICS

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*My research shows that students of mathematics who are inclined to undertake mathematical problem solving activity, and teachers who are inclined to experiment on-the-run share common personal characteristics. These characteristics are associated with what they do when something does not work. I have shown that such characteristic can be built in students as they undertake mathematical problem solving through my Engaged to Learn Approach. The focus of this paper is on recognizing, and building the characteristics that incline students to explore mathematically, and teachers to explore 'pedagogically'. These are the characteristics of optimistic/resilient people as defined by Seligman (1995). If (as my research suggests) these characteristics are a necessary part of exploring new ideas then we need to pay as much attention to building them as we have to increasing the mathematical knowledge of teachers, and increasing their knowledge about teaching approaches that build deep understandings.*

## **Introduction**

Performances on 'mathematical literacy' for Australian students are generally not improving and there are some indications that the performances of girls are dropping. The ability to work with unfamiliar mathematical problems in ways that engage students in the learning of mathematics and build deep understandings is crucial to the STEM (Science Technology Engineering and Mathematics) education agenda intended to retain students in the STEM 'pipeline' (Tytler, Osborne, et al., 2008) through senior secondary and tertiary studies. Being able to use mathematics in unfamiliar ways and experience excitement and surprise in doing so is key to retaining student interest in mathematics.

Have you ever wondered though why some students are inclined to explore new mathematical ideas and others will not step into unknown territory to do so? We need to find ways to incline these resistors to step into that space of not knowing so they can experience that excitement and surprise. What about the teachers who implement problem solving activity? Have you ever compared the characteristics of teachers who resist group work and problem solving to the characteristics of students who are not willing to explore new ideas? Have you thought about the types of comments each make and whether there are similarities between them? I have had the privilege of interacting with many students and teachers who are inclined to explore new ideas as a classroom teacher, head of mathematics, provider of professional learning, and a researcher. Through this time, there have also been those who resist engaging in problem solving approaches. The purpose of this paper is to examine differences in characteristics different teachers and students possess using 'resilience' in particular Martin Seligman's (1995) 'optimism' to examine them because I have found in my research that optimism is a characteristic of students and teachers who are willing to step into the unknown to explore new ideas and approaches.

## Defining Optimism

Optimism (Seligman, 1995) is one way resilience is described. It is an orientation to successes and failures whereby optimistic people have a greater likelihood of overcoming failures because of the way they respond to them. This is consistent with resilience defined as a “mechanisms and processes that lead some individuals to thrive despite adverse life circumstances.” (Galambos and Leadbeater 2000, p. 291). Seligman’s optimism differs to common usage of that term: “belief that good must ultimately prevail over evil” and “hopefulness and confidence about the future or the success of something” (The Oxford Dictionaries 2013, see < <https://en.oxforddictionaries.com/definition/optimism>>). Seligman’s definition includes personal contribution to bringing about that success. It involves the realistic assessing of situations to identify what can be changed to overcome problems rather than a general belief that all will be well. With regard to mathematical problem solving, optimistic activity can overcome adversities associated with encountering many failures on the way to achieving success. Seligman found that optimism could be built where people experienced success in flow situations (Csikszentmihalyi, 1992). Flow is a highly positive state that accompanies creative activity that occurs when a group of individual spontaneously decide to explore a challenge that is almost out of reach and develop new skills to do so.

## Students Inclined to Problem Solve Enact Optimism as They Do So?

My research has shown that students who are inclined to step into the unknown during problem solving situations are optimistic. This makes sense because mathematical problem solving can be considered a state of adversity where many failures may be encountered on the way to achieving successes. During this process, students spontaneously set up flow conditions when they want to find out more about a mathematical complexity that none of them are familiar with. They develop new skills and *concepts* during this process. In such situations ‘failure’ is defined as ‘not yet knowing’ and ‘success’ as ‘finding out’ (see Williams, 2014 for example). An *optimistic* mathematical problem solver sees *not knowing* as *temporary* and able to be overcome through the *personal effort* of looking into the specific situation at hand to work out what could be changed to increase the likelihood of *finding out* more. In doing so, they recognize what they cannot vary (what is *externally* controlled) and what they can vary. They see as permanent their knowledge of problem solving strategies and take this on as a characteristic of self “I can do this! I am good at finding out more’.

The intention of this section is to familiarize you with the dimensions of optimism so you are better able to identify optimistic actions of teachers. It includes the reflections and actions of various students that illustrate different indicators of optimism. Table 1 includes Seligman’s indicators of optimism as enacted by optimistic student problem solvers now described.

### Dean (Year 8)

Struggled in mathematics but knew that not knowing was a temporary state: “Well when I first get a sheet which I’ve never done before then I get a bit stressed [small laugh seems to be at self rather than anxiety] ‘cause the first time I do stuff I always don’t get it at *first*- it takes me like a little while- that’s why I go *over it and over it*” [see Table 1, Row 2, Column 2]. He developed his own correct way to find the sum of angles in polygons (using the table the teacher was generating from student responses). Dean looked in to the situation and focused on something different to what the teacher was drawing attention to and was able to find a way to proceed [Row 4, Column 2] that no one else in the class (including the teacher) mentioned. He could not work out how to find the sum of the interior angles of a polygon the way the teacher had directed the class to do so (by physically placing together the angles torn from each vertex of the polygon). Instead, he realized (whilst examining the table progressively developed on the board) that the number of triangles formed in the polygons by joining vertices and the number of 180 degrees in the sum were connected. He could just add 180 degrees for each triangle in the polygon to find the sum of the interior angles. His personal effort lead to him finding out more.

### Eden (Year 8)

Was perceived by his teacher to be average in mathematics. He showed in his interview that he was quietly and creatively developing new understandings for himself in class. The lesson focused on linear



equations and gradients using the Green Globes program. Green Globes is a game in which you try to hit as many Globes as you can with one equation. Globes are randomly positioned at integer coordinates on a Cartesian Plane and students were constrained to using linear equations. The general form of a linear equation had been given with no elaboration. Eden developed insight into a pattern common to each set of integer co-ordinates on the same line and was able to express this verbally and write the equation for each line as a result. He shifted easily between different representations (graphical, tabular, verbal, algebraic) when explaining to me what he had found. He developed these new ideas related to gradient during the Green Globes game, before the teacher introduced gradient in class. He helped his peers (who gained higher scores than Eden on class tests) with the unfamiliar questions in the exercise because Eden was more able to adapt what he had developed for himself to the unfamiliar questions. He described the importance of developing ideas for himself in his interview [Row 2, Column 3]. Although his teacher considered Eden had only average for mathematical ability was dropping him from the accelerated program the following year, Eden displayed as a creative mathematical thinker who developed deep mathematical understandings in his interview with me. It would thus appear that the class tests were not providing opportunity for Eden to show what he was capable of. Eden displayed optimistic indicators in the way he dealt with the teacher’s perception of his mathematical ability. He qualified his response to the question about whether he was really good, pretty good, about average, or what ... at mathematics with ‘in this class’ and accented that qualification. Eden was able to examine a situation in which his mathematical ability was in question from an external source and retain his mathematical problem solving identity: I like to work things out for myself [Row 3, Column 2].

Table 1. Actions of mathematical problem solvers fitting Seligman’s (1995) dimensions of optimism

	Failure: <b>Not Yet Knowing</b>	Success: <b>Finding Out</b>
<b>Dimension of Optimism</b>		
Temporary — Permanent	Not Knowing as Temporary Dean: “Well when I first get a sheet which I’ve never done before then I get a bit stressed [small laugh seems to be at self rather than anxiety] ‘cause the first time I do stuff I always don’t get it at <i>first</i> - it takes me like a little while- that’s why I go <i>over it and over it</i> .”	Being Able to Find Out More: Permanent Eden (Year 8) stated in his interview that “you try to work everything out for yourself because (pause) that way (pause) you know everything (pause) sort of (pause) you will be able to think clearer (pause) for <i>tests and whatever</i> .” Eden perceived that what he learnt through working things out for himself was permanent and he would draw on it as needed.
External — Personal	Not Within Your Control: External Influence Kerri: “[Last year I had more trouble understanding because] he would try and show us that while we were still learning [the first idea].”  Negative External Assessment: External Eden sees his ‘not being good at maths’ as limited to <i>the particular class</i> and the teacher’s assessment. He thought carefully about the wording he used as indicated by the pauses and the emphases: “Well I (pause) in the class I think I am (pause)	Finding Out: Personal Dean found a way to proceed (adding another 180 each time) through the <i>personal effort</i> of looking into the situation of not knowing to see what else he could attend to that might increase his chances of finding out. He linked the table on the board with the number of triangles in the polygon.

	pretty much average <i>in the class</i> (pause) because (pause) I've- always get pretty average scores on the tests so (pause) I'm basically just average I (pause) no way to explain it- I mean ... yeah."	
Specific — Pervasive	Finding Out: Personal, Specific Dean found a way to proceed (adding another 180 each time) through personal effort: <i>looking into the situation of not knowing to see what else he could use to increase his chances of finding out.</i> He linked the table on the board with the number of triangles in the polygon.	Finding Out As Pervasive Lenny: "Instead of just going (pause) 'I don't know', I (pause) [now] sit there and I <i>really really</i> (pause) think about it ..." When asked how that happened, Lenny said: "I don't really know ... it just made me think more (pause) It was probably actually <i>doing</i> the tasks ..."

### Kerri (Year 8)

Had been in a class for students identified as gifted since Grade 3. Her interview showed she was always thinking about new mathematical ideas, sometimes ahead of the teacher and sometimes peripheral to the mathematical focus of the lesson. She perceived herself as highly mathematically able and a good problem solver and demonstrated this by exploring new ideas that interested her in the interview. She explained that she had not done as well as usual in mathematics the previous year because the teacher did not allow enough time to think [Row 3, Column 2] showing she was aware of where external influences contributed to her not learning fast in the previous year.

### Lenny (Grade 5)

Was a student who performed at an average level on his class tests. He reported that at the start of the research period he used to give up when he could not see what to do, and that towards the end of the second year of undertaking unfamiliar problem solving tasks through the Engaged to Learn Approach, he persevered with tasks. He had developed an optimistic indicator in changing the way he approached situations when he did not know what to do. He attributed this change to just doing the tasks [Row 4, Column 3]. By experiencing some success with the tasks (with groups or thinking individually), Lenny became willing to persevere: he employed personal effort as he looked into the task to see what he could change to increase his likelihood of finding a way to proceed (see Williams, 2014). Lenny saw being able to find out more as permanent, and that by personal effort and thinking about what to change he could overcome his temporary state of not knowing and find out more [Column 3, Row 4].

## Optimistic Indicators Displayed by Teachers

I have interacted with many teachers both in Australia and internationally during my teaching and research careers. These include teachers I have taught with in schools, teachers at schools where I was the coordinator of mathematics, teachers at conferences with whom I have shared what I have learnt about deep understandings students can develop through problem solving activity, teachers who return to discuss ideas they have tried at subsequent conferences, teachers in clusters and schools where I have provided professional learning and 'whole schools of primary teachers' I have interacted with over several years as they have developed and refined the ways they implement problem solving activity in their classes. Through these multiple sources, I have developed understandings about the characteristics of teachers who are inclined to explore new ways to proceed, and have seen teachers develop and strengthen their personal characteristics as they work with school teams and myself. A culture of evidence-informed experimentation, and reflection has strengthened the capacity for innovation.

Teacher problem solving activity is now examined by drawing on interactions with teachers over several years at MAV conferences. In doing so, I elaborate further on what optimistic pedagogical problem solving activity could look like. The 'quotes' I use in this section contain the substance of what many teachers say rather than their actual words.

There are teachers who leave various MAV conferences ready to try tasks and ideas we have explored and find out how their students might respond and what they might learn. At the conference the following year, and perhaps for several years in a row, a teacher may stop to talk to me and state ‘I tried that task you showed us last year and it went really well *until* I got to ‘such and such’ part and then it all fell apart, I was wondering if you could give me some ideas of what I might try at that point?’ We would then discuss, and I would suggest, a few possibilities about why the process did not continue to work, and the teacher would go away ready to try different strategies in the light of our discussion. During that discussion there was sometimes mention of a lack of support for problem solving at their school and a willingness to persist in spite of this. The next year, they might approach me again, ‘Now I have the task working really well at that part where it was falling apart last year, but when we get further into the task ‘such and such’ starts to happen. Have you got any ideas about what might be happening and what strategies I could try?’ With some of these teachers, there comes a time when with excitement or intensity they share their progress: ‘I have become much better at working out what to do now’, ‘When I try a new task now’, ‘I can generally think on my feet about what to do’, ‘I have become much better at problem solving- at knowing how to proceed’. These teachers are displaying indicators of optimism: they see not knowing how to proceed as temporary and able to be overcome by looking into the situation to see what they could change to increase their likelihood of success (finding a way to proceed). They do not want to be told what to do, but rather be given some ideas of evidence-based ways to proceed so they can try them and reflect on what happens (to help them make decisions about what to explore next). In this way, they are seeing ‘not knowing how to proceed’ as specific to the situation at hand and able to be overcome through the personal effort of trying ideas that could increase their likelihood of finding a way to increase learning opportunities for their students. With the successes they achieve over time, like Lenny, they consider they will be able to find out what to do in other situations because they have developed problem solving capacity. Being able to find out has become permanent because they have taken on their successes as characteristics of self.

Table 2. Seligman’s indicators of optimism as enacted by MAV conference participants

<b>Dimension of Optimism</b>	<b>Failure: Not Yet Knowing How to Proceed</b>	<b>Success: Finding Out More</b>
Temporary — Permanent	<b>Not Knowing As Temporary</b> ‘I was wondering if you could give me some ideas of what I might try <i>at that point?</i> ’	<b>Being Able to Find Out More is Permanent</b> ‘When I try a new task now, I can generally think on my feet about what to do’.
External — Personal	<b>Recognise External Impeding Factors</b> Recognition of a lack of support for problem solving at their school and a willingness to persist in spite of this.	<b>Finding Out Requires Personal Effort</b> I was wondering if you could give me some ideas of what <i>I</i> might try at that point?
Specific — Pervasive	<b>Not Knowing Requires Specific Varying</b> I was wondering if you could give me some ideas of what <i>I might try</i> at that point?	<b>Taking On Successes As Part of Self: Pervasive</b> ‘I have become much better at problem solving how to proceed’.

The single statement ‘I was wondering if you could give me some ideas of what I might try at that point?’ includes enactment of overcoming not yet knowing because it is temporary, and can be overcome through the personal effort of looking in to the situation to make decisions about what could be changed to increase the likelihood of finding out what could work. Not all teachers have these optimistic characteristics though.

## Teachers Not Inclined to Explore

If a person possesses any of the following non-optimistic characteristics, my research indicates they will not be inclined to explore new ideas. I have also shown that these non-optimistic characteristics can be changed to optimistic characteristics in students. For example, consider Lenny in Table 1 who became a better problem solver and developed optimistic indicators as a result of his problem solving

successes. My current research is examining what teacher enactment of optimism might look like where they are trying to implement problem solving in their mathematics lessons. Firstly though, let's examine what the actions of a non-optimistic teacher might look like.

If a person possesses all non-optimistic characteristics they are pessimistic. Not all non-optimistic people are pessimistic but if a person possesses some indicators of lack of optimism, they are unlikely to engage in the process of developing new ideas for themselves.

Someone who is pessimistic sees *not knowing* as *permanent* and that the only way they can *find out* how to proceed is to consult an *external* source. They do not have *figuring things out* as part of their vocabulary. Instead they talk about learning through such *external* sources as the teacher, textbooks, and the Internet, without working with the ideas presented in ways that enable them to develop their own understandings. When they do *find out* something more, they see being able to achieve this as *temporary* and something they may not necessarily be able to do again. They take on their failures (not knowing) as part of themselves: perceiving himself or herself as dumb, or an idiot for example rather than seeing the failure as specific to the situation at hand. Table 3 uses statements I have heard teachers make over my teaching career and in professional learning sessions. I have also included statements made by students.

Of interest is those non-optimistic high performing students who resist entering an unknown space and instead take up all the talk time telling others something they already know. These students have disabling confidence. They see finding out as permanent but what is permanent is being able to do procedures they have learnt from external sources and getting high marks on tests as a result. I wonder if there are similar parallels in teachers? This is something I still need to explore.

Table 3. Indicators of pessimism displayed by teachers and students

Dimension of Optimism	Failure: <b>Not Yet Knowing How to Proceed</b>	Success: <b>Finding Out More</b>
Permanent — Temporary	<p><b>Not knowing as Permanent</b> Teacher: "these students can't do this and they are never going to be able to do it." "I tried group work once and it did not work." Student: "I can't do this and I am never going to be able to do this."</p>	<p><b>Finding Out as Temporary</b> Teacher: "The group work went well today but this does not mean it will necessarily work next time." Student: "I can do that maths today but probably won't always be able to do it."</p>
Personal — External	<p><b>Not Knowing as Personal</b> Teacher: "I always mess it up" Student: "It is just me, I am not able to work it out"</p>	<p><b>Finding Out as External</b> Teacher: "I need someone to help me to run group work. I cannot do it alone." Student: "I learn from the teacher, books, and the internet."</p>
Pervasive — Specific	<p><b>Not Knowing as Pervasive</b> Teacher: "My students will never be able to do this because they are dumb." Student: "I can't do this, I am just dumb."</p>	<p><b>Finding Out As Specific</b> Teacher: "The group work went well today they must have used all their energy up at lunch time." Student: "I passed that test, I must have been just lucky."</p>

Students and teachers who see the successes they achieve as due to something external (e.g., luck, something else that happened) rather than due to personal effort do not take their successes on as characteristics of self because they do not see themselves as personally involved in achieving these successes.

Study of non-optimistic teachers is not the focus of my study. I am more interested in those characteristic of optimistic teachers and how being optimistic is associated with 'on-the-run' activity in class.

## **The Engaged to Learn Approach**

I gradually developed the Engaged to Learn Pedagogical Approach as a teacher (see Barnes 2000; Williams 2002), then as a researcher I theorised extended and empirically supported it. It provides opportunities for students to creatively develop mathematical insights during flow conditions. Students undertake 3-4 problem-solving cycles during a task that extends generally over 1-3 sessions of sixty to one hundred minutes. This approach has been employed from Foundation to Year 12 and in university pre-service teaching in mathematics education. The time taken depends upon the task, and the progress the class are making. Work on the task stops when the teacher decides the balance between learning and time taken is such that it is no longer productive to continue with the task. At that stage, the teacher draws ideas together, and draws on them where appropriate several times through the year where learning from the task fits with aspects of the curriculum. Each task generally includes four cycles. Each cycle includes:

- Group work (3-4 group members) (10-15 minutes);
- Teacher suggestions on what reports might focus on
- Group priming of their reporter
- Reports from a member of each group (2-3 minutes a report)
- Comments / questions without contradicting or asking beyond what was presented
- Brief opportunity to draw attention to anything in the reports

Adaptations to the approach include some early years classes working in pairs, and occasions where student work is placed on the board and students consult it and think further about it long after the problem solving group work has finished. When further insights have developed, the teacher then undertakes the summary lesson.

Consistent with creative activity, *teachers do not* provide mathematical input during whole class discussion, answer mathematical questions with a definitive answer, or give hints, agree or disagree with the mathematical ideas the group is generating, *during* creative intervals. Finding out what teachers *can do* to promote and sustain creative mathematical activity is a focus of this paper. Herein, I focus only on the 'on-the-run' activity of the teacher in class. Other optimistic activity does occur though (e.g., selecting and modifying the task, considering mathematics that might emerge, and planning questions that might be useful). Undertaking 'on-the-run' activity shows the teacher sees not knowing how to proceed as temporary, and able to be overcome by the personal effort of looking into the situation to see what could be changed to increase the likelihood of finding an appropriate way to proceed.

There are many part of the Engaged to Learn Approach that require the teacher to be able to think on their feet, and flexibly change direction when what they try does not work.

## **Engaged to Learn Approach: Optimistic Teacher Pedagogical Problem Solving Displayed**

Students and teachers who are willing to step into the unknown to engage in such processes display optimistic indicators. They step into the unknown during every problem solving lesson they implement *if* they respond to the directions students take rather than control the direction of the problem solving activity. In this paper, I limit the identifying of teacher enacting of optimism to those instances that happen 'on-the-run' in class. It also occurs during the planning of the task and reflections about it but there is not scope to cover such activity here. I have selected one teacher's actions and reflections to use to illustrate optimistic enactment of teacher problem solving as he implements the Engaged to Learn Approach. Although often not so explicitly stated, in my research there are similar illustrations across Foundation to Year 12 Specialist Mathematics. I have selected Earl as the teacher because he has recently developed his own practice associated with the Engaged to Learn Approach and instances of such deep focus on the mathematics and meanings of it in the Early Years shows what can be possible.

Earl was a staff member in a school that was committed to engaging their children in the learning of mathematics and in doing so improving their mathematical understandings and subsequently their mathematical performances (where performance includes the ability to problem solve). Earl implemented problem solving activity in his classes at least once a week and implemented more complex problem solving tasks like the one discussed here several times through the year. In this school, there were two Foundation / Year 1 composite classes but for problem solving the two teachers decided it was better to divide the class into a Year 1 class (taken by Earl) and a Foundation class for mathematical problem solving activity. They made this decision because they found it was more difficult to implement complex problem solving activity in a class where there was a greater diversity in children’s mathematical development.

The video of Earl’s class demonstrates the high quality of student thinking he has been able to elicit. He has developed this practice over three years of professional learning about the Engaged to Learn Approach for his school. His Grade 1 students were able to sit and listen to the reports from other groups for more than fifteen minutes and ask and respond to questions about the meanings of the mathematics they presented. When I discussed with Earl how he considered he had been able to develop such practice, he responded:

*“It’s something that I’ve put a lot of thought into- It’s not come over night- I’ve had to chip away at it and keep coming back to that same question in my head ‘Did I lead that or did they lead that?’ And the more chances I give them to explain their thinking- the more scenarios we’ve put in front of them to work together- the easier it is to step back and the easier it is for me to- you know- take the time and craft my questions.”*

Earl’s practice of examining his questions and deciding whether the children were developing the ideas or he was leading them was a crucial part of Earl’s practice:

*“[I am always] just reflecting upon whether I’ve had the input that’s taken us to the end result or whether the kids have. And just very much being mindful of that ‘So did I teach that or did the kids learn that from themselves?’ Sort of that question back on myself.”*

Table 4. Earl’s Enactment of Optimism ‘On-The-Run’ in class

<b>Dimension of Optimism</b>	<b>Failure: Not Yet Knowing How to Proceed</b>	<b>Success: Finding Out More</b>
Temporary — Permanent	<b>Not Knowing How to Add Twist: Temporary</b> “having that open twist is something that the problems I’ve been coming up with lack at the moment.”	<b>Asking Questions ‘On-The-Fly: Permanent</b> “So, in terms of questions on the fly, just being really conscious of not driving it.”
External — Personal	<b>External: Diversity in Development of Children</b> Recognising the developmental levels in the class are beyond the teacher’s control. Early Years teachers find a lateral solution to this. This is not always possible where there is an external factor. Instead it is accepting and working around it.	<b>Finding Out How to Implement Engaged to Learn Approach: Personal</b> “It’s something that I’ve put a lot of thought into- it’s not come over night.” “I’ve had to chip away at it [the questioning] and keep coming back to that same question in my head “Did I lead that or did they lead that?”
Specific — Pervasive	<b>Examining Specific Situation to Find What Needs to Change</b> “just reflecting upon ... whether I’ve had the input that’s taken us to the end result or whether the kids have ... sort of that question	<b>Pervasive Characteristic of Self: I Can Problem Solving ‘On-The-Run’</b> “the more chances I give them [the class] to explain their thinking- the more scenarios we’ve put in front of them to work together- the easier it is to step back and the easier it is for me to-

	<p>back on myself ... ‘yep, you’ve just blown it there’ or ... ‘I shouldn’t have said that’ or ‘could have said another way’- quite often it’s a similar question that I need to ask or want to ask, but it’s just peel back a few layers so that- I’m not explicitly dropping the clues in.”</p> <p>“I’ll go and talk about...I’ll go and think about the next bit of the script and really dissect that.”</p>	<p>you know- take the time and craft my questions.”</p> <p>“And coming up with the questions on the fly, not to give it away, is something that I’ve improved in. And I suppose that’s giving me the chance to really think about my questioning.”</p> <p><b>Pervasive Characteristic of Class: Together We Can Questions ‘On-The-Run’ to Find Out</b></p> <p>“The questions from the floor often do that for me. The questions that the kids ask of each other- “Can you say it again?” or “What do you mean by?” So that does a lot of the questioning for me.”</p>
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Earl was continually reflecting on and developing his questioning technique to further achieve this:

*“So, in terms of questions on the fly- just being really conscious of not driving it”.*

*“[I sometimes say to myself] ‘Yep, you’ve just blown it there’ or you know ‘I shouldn’t have said that’ or ‘could have said another way.’ Quite often it’s a similar question that I need to ask or want to ask, but it’s just peel back a few layers so that, I’m not explicitly dropping the clues in”.*

He has taken his successes with questioning on as a characteristic of self. He knows he can problem solve on-the-run to respond to children’s ideas and draw out more of them through the questions he asks:

*“And coming up with the questions on the fly- not to give it away- is something that I’ve improved in- and I suppose that’s giving me the chance to really think about my questioning.”*

He also sees the class as having contributed to this process through their questioning capacity:

*“The questions from the floor often do that for me- the questions that the kids ask of each other- they won’t say ‘Can you explain it in a different way because I didn’t understand it?’ but a lot of the times they’re putting their hand up to draw out exactly ‘Can you say it again?’ or ‘What do you mean by?’ So that does a lot of the questioning for me.”*

He makes use of the supportive environment within which his teaching occurs:

*“I’ll go and talk about...I’ll go and think about the next bit of the script and really dissect that.”*

He is aware of aspects of the Engaged to Learn Approach that he is still working on:

*“having that open twist is something that the problems I’ve been coming up with lack at the moment” and his inclusion of ‘at the moment’ shows he considers not knowing this as temporary.*

There is a lot to learn from Earl’s articulation of what he does when he is problem solving, and the thinking he continues to do ‘on-the-run’ (and before hand and afterwards) as he implements it. Table 4 includes some of the quotes from Earl under various indicators but you will see that some of these quotes also address other indicators too.

## In Conclusion

Pedagogical problem solving involves complex actions and thus it is not surprising that it takes teachers time to develop their capability to implement such activity in ways that enhance student opportunity to develop new mathematical understandings. This paper provides insight into how the characteristics that teachers possess are crucial to whether or not they will be able to implement such problem solving *at this stage*. What is important though is that these characteristics can be developed in students and teachers. What is clear to me is that supportive teams who explore such ideas together can contribute to developing such characteristics. There are teachers like Earl who already possess optimistic characteristics so just need to apply them to developing their expertise through the Engaged to Learn

Approach or another evidence-based approach to problem solving or an approach they develop through experimentation and reflection.

Based on past experience where I have introduced the indicators of optimism in presentations at conferences and professional learning in schools, there are teachers who immediately begin to analyse their own ways of responding when something does not work, and go away ready to build some characteristics. It is hoped that this paper leads to many discussions about the teacher characteristics identified here in, and results in teacher reflections about what characteristics they presently possess, and what characteristics they may be able to build through small successes resulting from experimentation. Where such experimentation is encouraged and supported in schools through a whole school approach as described in Clarke, Duncan, and Williams (2013) and implicit in Williams, Harrington, and Goldfinch (2012) there is increased opportunity for such changes to occur. That said, such changes can be developed by teachers thinking alone as illustrated in Williams (2002) who was teaching in a country school at the time she first started to develop the Engaged to Learn Approach. Working together we should be able to strengthen and expand the optimistic characteristics we possess, and thus increase our potential to develop an approach to problem solving that engages young people in the learning of mathematics and deepens their understandings of mathematical meanings. In doing so, we all contribute to the STEM agenda.

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**Maths e<sup>x</sup>plosion**

**Peer Review**

# THE LOST LOGIC OF ELEMENTARY MATHEMATICS AND THE HABERDASHER WHO KIDNAPPED KAIZEN

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*Euclid's multiplication definition from Elements, (c. 300 BCE), continues to shape mathematics education today. Yet, upon translation into English in 1570 a 'bug' was created that slowly evolved into a 'virus'. Input two numbers into Euclid's step-by-step definition and it outputs an error. Our multiplication definition, thought to be Euclid's, is in fact that of London haberdasher, Henry Billingsley who in effect kidnapped kaizen, the process of continuous improvement. With our centuries-old multiplication definition revealed to be false, further curricular and pedagogical research will be required. In accordance with the Scientific Method, the Elements of western mathematics education must now be rebuilt upon firmer foundations.*

## **Multiplication Defined**

Euclid's definition of multiplication first appeared in 1570 as: *A number is said to multiply a number, when the number multiplied, is so oftentimes added to itself, as there are in the number multiplying units and another number is produced,* (Billingsley, 1570). Today, the *Collins Dictionary of Mathematics* states: *to multiply a by integral b is to add a to itself b times,* (Borowski & Borwein, 2012).

We read, (Harel & Confrey, 1994):

*In book VII, Euclid defines multiplication as 'when that which is multiplied is added to itself as many times as there are units in the other' ... and, Mathematically, this [i.e. multiplication] can be represented as repeated addition (a definition found in Euclid, for example), and, ...multiplication, as defined by Euclid, is repeated addition.*

Similar 'number added to itself multiplier times' multiplication definitions are attributed to the 11th century mathematician, Abraham bar Hiyya, and the 13th century mathematician, Jordanus de Nemore. Repeated often enough we read, (Hoffmann, Lenhard & Seeger, 2005):

*Multiplication seems to be a simple and conceptually and epistemologically unproblematic, innocent notion. This widespread assumption is reinforced when one consults the chapter on multiplication in the famous Tropicke for algebra,... The main content of the short conceptual paragraph [on multiplication] is given by a reference to Euclid's definition of multiplication as repeated addition, in Book VII, definition 15; A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced (Heath 1956 II, 278; Tropicke 1980, 207-208). Multiplication thus seems to present the case of a stable notion with a meaning remaining identical over millennia.*

If he were alive today, Euclid would be *astounded*. How could people believe the above definition was his? If Euclid could speak to us he would want everyone to know the above definition of multiplication is not his idea. Instead, it is the illogical invention of Henry Billingsley, a London haberdasher.

*'T is strange, - but true; for truth is always strange;  
Stranger than fiction: if it could be told.  
(George Gordon Byron, 1844)*

## Paradoxical Products

Consider the question, ‘How, for example, does one add  $5/8$  to itself  $3/4$  times,  $d$  to itself  $\pi$  times, or  $-2$  to itself  $-3$  times?’ (Davis, 2012). The answer is, you don’t. In  $ab$ , if  $b = 1$ , then  $a \times 1$  or  $a$ , cannot be  $a$  added to itself one time, because  $(a + a)$ , equals  $2a$ , not  $a$ . In  $ab$ , if  $b = 0$  then  $a \times 0$  or  $0$ , is not  $a$  added to itself zero times, because  $a$  added to itself zero times is  $a$ , not  $0$ . Yet, we read, (Strogatz, 2012):

*These humbling sessions have prompted me to revisit multiplication from scratch. And it’s actually quite subtle, once you start to think about it. Take the terminology. Does “seven times three” mean “seven added to itself three times”? Or “three added to itself seven times”?*

We also read, ‘The product  $3 \times 5$  could be defined equally well as  $3 + 3 + 3 + 3 + 3$ , i.e., 3 added to itself five times, but we have chosen to use the other convention instead: 5 added to itself three times’, (Wu, 2011). So, paraphrasing Dr. Strogatz, does ‘two times one’ mean ‘2 added to itself 1 time’ or ‘1 added to itself 2 times’? We know two times one ( $2 \times 1$ ) equals two, yet *see* two added to itself one time equals four ( $2 + 2$ ) and *see* one added to itself two times equals three ( $1 + 1 + 1$ ). Similarly, three added to itself five times equals eighteen and five added to itself three times equals twenty. Commendably, Dr. Wu has since corrected his explanations of multiplication, (HREF1). Algorithms are a ‘precisely-defined sequence of rules telling how to produce specified output information from given input information in a finite number of steps’, (National Research Council, 2005). Astonishingly, Billingsley’s algorithmic definition of multiplication cited since 1570 is false, because it neither commutes nor computes.

## Subtracting ‘Added to Itself’ from Multiplication

‘Added to itself’ did not appear in Euclid’s multiplication definition. Therefore add, added, addition and the like do not appear in any English translation of Euclid’s Book VII proposition reliant upon a definition of multiplication, (HREF2). Euclid’s multiplication definition had been translated into Italian, German and French by experts without a problem, yet Billingsley chose to translate the Greek  $\sigma\nu\nu\tau\epsilon\theta\eta$  not as ‘put/placed together’, but ‘added to it self’, (HREF3). Notably, the Greek philosopher Sextus Empiricus, (c. 200 CE), had written, *To what, I ask, is at added? It cannot be added to itself, since what is added is different from that to which it is added and nothing is different from itself*, (Bury, 1933).

In 1809 William Saint suggested a correction to Euclid’s definition of multiplication and in 1990 John Searle explained how to multiply correctly, yet both men were mocked, (HREF4). Instead of translating truthfully, for centuries people have been robotically inserting Billingsley’s definition into mathematics texts. Just as Euclid never wrote ‘added to itself’ in his multiplication definition, bar Hiyya, Jordanus and Tropicke never wrote ‘added to itself’. Similarly, writings of Isaac Barrow and Christian Wolff were correct yet ‘scholars’ inserted Billingsley’s ‘added to itself’ phrase into their translations, (HREF5).

## From MIRA to IMRA

In 2007 Keith Devlin, (HREF6), began a MIRA (Multiplication Is Repeated Addition) debate. Dr. Devlin wanted *Teachers* to stop saying ‘MIRA’. Yet *Teachers* said multiplication and addition connect via the distributive law. The fact *Mathematicians* point to (with parents, employers and politicians) is mathematics rankings don’t lie. English speaking countries are falling behind Asian countries, (HREF7). After the debate subsided I asked *Is Multiplication Repeated Addition?* (Table 1).

Table 1: *Is Multiplication Repeated Addition? (IMRA)*

Yes	16 votes	76.2%
No	5 votes	23.8%

2013 *Math, Math Education, Math Culture*, LinkedIn Group poll.

Several MIRA meme multiplication algorithms were then analysed. (Table 2.)

Table 2: *Multiplication Involving Repeated Addition: How Do You Calculate the Product  $ab$ ?*

$ab = a$ added to itself $b$ times	52 votes	52.0%
$ab = a$ added to zero $b$ times	26 votes	26.0%
$ab = a$ added to itself $b - 1$ times	22 votes	22.0%

2013 *Math, Math Education, Math Culture, The Math Connection and Common Core State Standards – Mathematics*, LinkedIn Group polls

Despite being historical and mathematical nonsense, Billingsley’s algorithm dominates. The choice  $ab = a$  added to zero  $b$  times should have emerged by the 17th century, yet Euclid had no concept of the number *one* let alone *zero*. Infinity and zero (cypher) as numbers were problematic for European churches. Only God was infinite and the void was the realm of the devil. So to avoid cipher (zero) the ‘work-around’  $ab = a$  added to itself  $b - 1$  times, emerged.

## Chinese, French and American Corrections

Soon after the emergence of the ‘Billingsley Virus’, (BV1570), the 17th century Chinese *Suan Fa Yuan Ben* (Elements of Calculation) was published. Euclid began his Book VII definitions with, ‘An unit is that by virtue of which each of the things that exist is called one’ and ‘A number is a multitude composed of units’, (Heath, 1908). The Chinese version read, ‘One is the root of numbers’ and ‘A multiplicity of ones combined together is called a number’. Contrary to the writings of countless westerners, the Chinese text says three added to itself two times is nine and ‘...adding two to itself three times must be equal to eight’, (Jami, 2012).

Seductively simplistic, BV1570 spread as if a Trojan virus hidden under the cloak of Euclid. Thus the wrong idea of multiplication spread throughout the colonies. England, an early adopter of printing presses, built a thriving export industry in the form of (infected) mathematics textbooks. Meanwhile the French disinfected the English multiplication definition. On the example of  $16 \times 4$  we read 16 is to be repeated four times or added to itself three times, (Lacroix, 1804). After the American War of Independence the flood of infected textbooks from England slowed to a trickle. Arithmetical instantiations of multiplication increasingly took the form  $a \times b$  was  $a$  taken/repeated  $b$  times, or  $a$  added to itself  $b - 1$  times, (HREF8).

## The Multiplication of Lost Logic

Mathematics promotes logical argument and deductive reasoning like, *Socrates is a man, all men are mortal, therefore Socrates is mortal*. Euclid was the master of reasoning and proof. Unlike most fields of science, Euclid’s definitions and theorems are as true today as they were 2300 years ago.

Yet the *1<sup>st</sup> Law of Marketing* caused a problem and children are given worse arithmetical foundations than 1400 years ago in India, when Brahmagupta documented the rules of arithmetic that featured zero, negative numbers and ‘Laws of Sign’. While Europe struggled with negative numbers in the 18th century, China had been using negative numbers for 2000 years.

Our multiplication definition has three added to itself twice being equal to six. Yet adding a number to another the same as itself is doubling and three added to itself once equals six. **Three added to itself twice equals three added to itself once!** When fractional multipliers appear, children led to believe multiplication makes more, must ‘unbelieve’. How is  $2 \times \frac{1}{2}$  explained? When two is to be added to

itself ‘half a time’ we arrive at  $2 + 1$ , not  $\frac{1}{2}$ . Similarly, children are led to believe ‘division makes less’ which also needs undoing when fractional divisors less than one get introduced.

Just as ‘Multiplication Is Repeated Addition’ (MIRA), it should be said ‘Division Is Repeated Subtraction’ (DIRS). Multiplication distributes over repeated addition with  $2 \times +3$  equaling  $2 \times (0 + 1 + 1 + 1)$ . Yet multiplication also distributes over repeated subtraction because  $2 \times -3 = 2 \times (0 - 1 - 1 - 1)$ . So multiplication is also repeated subtraction. When  $a = b$  and  $b = c$ , then logically  $a = c$ , so we must either accept the statement ‘repeated addition is division’ is true, or accept Billingsley’s fabricated multiplication concept is false.

## How Laws of Marketing Defeated Laws of Logic

The first two laws of marketing, (Ries & Trout, 1993), are said to be:

1. It’s better to be first, than it is to be better, and
2. If you can’t be first in a category, set up a new category you can be first in.

Billingsley was first to translate Euclid’s multiplication definition into English. Yet he was fourth to translate Euclid’s multiplication definition into a ‘modern’ language after Italian, German and French. It is possible he converted Euclid’s geometrical and proportional multiplication definition into an arithmetical definition to differentiate his product. Billingsley’s definition wasn’t the first, yet it was the first printed and imprinted in English minds. In 1570 Roman Numerals dominated and Hindu Arabic mathematics had a steep learning curve. So rather than reveal multiplication to be Proportional Covariation, Billingsley rebranded Euclidean multiplication as repeated addition.

## Billingsley’s Binary Bug

When numbers were placed together as many times as a multiplier had units in the 16th century, they were repeated not across the page, but down the page. This aligned each digit for summation according to the new concept of base ten positional notation. While it is obvious now that there are two additions in  $4 + 4 + 4$ , such a summation would have been done as shown here. Without the sign  $+$ , we can understand how ‘added to itself’ slipped through. People couldn’t see the error! In four multiplied by three, written  $4 + 4 + 4$ , four is added to four (itself) two times, not three times as said for centuries. ‘Placing’ is unary involving one number a time. Yet ‘adding’ is binary involving two numbers a time. Billingsley’s ‘Binary Bug’ has the number of *times* the addition Operation is done being the same as the number of *Terms*, which is impossible. In binary arithmetical expressions  $n$  Operations require  $n + 1$  Terms.

So should we update Euclid’s fourth century BCE multiplication definition with India’s seventh century arithmetic with one, zero and negative integers all accepted as numbers? The Scientific Method demands a YES response. Therefore, the calculation of  $a \times +b$  equals  $a$  added to zero (not itself)  $b$  times in succession. For integral multiplication,  $a \times +b$ , we can, according to the sign of  $b$ , either add  $a$  to zero  $b$  times in succession or subtract  $a$  from zero  $b$  times in succession. Such lost logic is found in, for example, the *Encyclopædia Britannica*, (Bell & Macfarquhar, 1768-1771). Under the entry for Algebra, we read:

*Multiplication by a positive Number implies a repeated Addition: But Multiplication by a Negative implies a repeated Subtraction. And when  $+a$  is to be multiplied by  $-n$ , the Meaning is that  $+a$  is to be subtracted as often as there are Units in  $n$ : Therefore the Product is negative, being  $-na$ .*

## Euclid’s Multiplication Concept

So what did Euclid do? The answer is simple. Euclid preserved proportional relationships between four terms. As the *Unit* is to the *Multiplier*, the *Multiplicand* is to the *Product*. When *Multiplicand*  $a$  is to be multiplied by *Multiplier*  $b$ , how do we arrive at *Product*  $c$ ? From Euclid’s original multiplication definition,  $a$  is placed as many times as there are *Units* in  $b$  and the number  $c$  is *Produced*. Euclid’s Book VII propositions involving multiplication saw proofs emerge via the creation of proportional line segments. Billingsley’s definition of multiplication on the positive integers, corrected and clarified, is:

*A number [the Multiplier] is said to multiply a number [the Multiplicand] when that which is multiplied [the Multiplicand] is **placed together** as many times as there are units [placed] in the other, [the Multiplier] and thus some number is produced [the Product].*


Henry Billingsley was understandably a little sloppy in his translation and conversion of Euclid's multiplication definition from geometry to arithmetic. So we will look at how Euclid (implicitly) multiplied geometrically without symbolic numbers. From a line depicting the Real numbers, line segments are proportional in length to the Real Number they represent. Therefore, Euclid's definition of multiplication also applies (anachronistically) to symbolic numbers.

## A Geometric Foundation for $3 \times 4$



Prior to Billingsley the application of Euclid's proportional multiplication definition had been extended from positive integers to fractions. At the start of the 16th century we find multiplication defined as *the creation of a number [product], being in proportion to a multiplicand, as the multiplier is to the unity*, (Huswirt, 1501). Even as the distinct nature of the multiplicand and multiplier began to blur, we still find four terms in a multiplication definition: *Multiplication is performed by two Numbers [multiplicand and multiplier] of like Kind for the Production of a Third, [the product] which will have such Reason [ratio] to the one, as the other hath to the Unit*, (Cocker, 1677). Isaac Newton also revealed how the proportional essence of multiplication extended well beyond the Naturals, *Multiplication is also made use of in Fractions and Surds, [irrational roots] to find a new Quantity in the same Ratio (whatever it be) to the Multiplicand, as the Multiplier has to Unity*, (Newton, 1720).

We read 'Multiplication is often "defined" as repeated addition, but this, I hold, is a confusion between a definition and an application', (Steiner, 2005). More specifically, if an explanation of multiplication fails to mention the four terms, *Unit/One, Multiplier, Multiplicand* and *Product*, it is an *application* of multiplication and not a *definitive Euclidean explanation* of multiplication.

**To multiply a Multiplicand (3) by a Multiplier (4), we follow Euclid.**

*Multiplicands* and *Multipliers* are numbers composed from a given *Unit* of length.  Where the *Unit*  $u = 1$ , draw  $a \times b = c$  where the *Multiplicand*  $a = 3$ , the *Multiplier*  $b = 4$  and the unknown *Product*  $= c$ .

Units placed 3 times  compose the *Multiplicand* 3 

Units placed 4 times  compose the *Multiplier* 4 

**The *Multiplicand* is placed together as many times as there are *Units* in the *Multiplier* to create the *Product*.**

Whatever is done with a *Unit* (1) to make the *Multiplier* ( $b$ ), we do with a *Multiplicand* ( $a$ ) to make the *Product* ( $c$ ). Just as the *Unit* was placed together four times to make the *Multiplier*, the *Multiplicand* is placed together four times to make the *Product*. Via Proportional Covariation, (PCV), the three given inputs (Unit, Multiplier and Multiplicand) output the fourth term, the *Product*.



Euclid's concept is: **As the Unit is to the Multiplier, so is the Multiplicand to the Product**, written, Unit : Multiplier :: Multiplicand : Product. Algebraically, the four terms of the proportion are written

$1 : b :: a : c$  and read, “As 1 is to **b**, so is **a** to **c**”. Multiplication, (as Proportion Covariation), means whatever we do to vary the Unit to make the Multiplier, we do to the Multiplicand to make the Product. So, with lines proportional in length to numbers, from a geometric instantiation of three (**a**) multiplied by four (**b**) we also arithmetically solve for the missing fourth term of the proportion, (**c**).

Q. What did we do to the *Unit* 1 to make the *Multiplier b*?

A. We placed the *Unit* 1 together four times 1, 1, 1, 1.

Q. So what do we do to the *Multiplicand 3* to make the *Product c*?

A. We place the *Multiplicand 3* together four times 3, 3, 3, 3.

So as 1 is to 1, 1, 1, 1, so is 3 to 3, 3, 3, 3. Put simply, 1 is to 4 as 3 is to 12 and the proportion is written  $1 : 4 :: 3 : 12$ . Grade 2-3 children are not ready for the proportion theory of multiplication, yet they are ready to play a proportional game of multiplication. From abstract lines we can switch to physical manipulatives. Just as a *Unit block* placed together four times in a row makes a *Multiplier*, a *Multiplicand* three *Units* long, placed together four times makes a *Product*.

### Why $3 \times 4 = 4 \times 3$

So what happens if we reverse the number of *Units* in our *Multiplicand* and *Multiplier* and draw four multiplied by three instead of three multiplied by four? Will the answer be the same? With four (**a**) multiplied by three (**b**) we again solve for the missing fourth term of the proportion, (**c**).

Q. What did we do to the *Unit* 1 to make the *Multiplier b*?

A. We placed the *Unit* 1 together three times 1, 1, 1.

Q. So what do we do to the *Multiplicand 4* to make the *Product c*?

A. We place the *Multiplicand 4* together three times 4, 4, 4.


As 1 is to 1, 1, 1, so is 4 to 4, 4, 4. Put simply, 1 is to 3 as 4 is to 12 and the proportion is  $1 : 3 :: 4 : 12$ . Therefore, it does not matter if, with the same *Unit*, we reverse our two conceptually different factors and write  $3 \times 4$  or  $4 \times 3$ . That is why Euclid proved, as Proposition 13 in Book VII of Elements, *If four numbers are proportional, they will also be proportional alternately*, (Heath, 1908). Our proportion, (equality of ratios), that was  $1 : 4 :: 3 : 12$  became  $1 : 3 :: 4 : 12$ .

Mnemonics such as ‘Minus times minus results in a plus, the reason for this, we need not discuss’ are often contrived and bad pedagogy. Yet when  $a \times b = c$  is understood via the mnemonic, Multiplying can be fun you’ll **c**, Do to **a** as 1 made **b**, (Crabtree, 2015), the ‘multiplication makes more’ MIRAGE vanishes. With  $12 \times \frac{1}{2}$  we took half the Unit (1) to make the Multiplier  $\frac{1}{2}$  so we take half the Multiplicand 12 to make the Product 6. As 1 is to  $\frac{1}{2}$  so 12 is to 6 and the proportion is  $1 : \frac{1}{2} :: 12 : 6$ . With  $-8 \times -\frac{1}{2}$  we took half the Unit (1) which is  $\frac{1}{2}$  and changed its sign to make the Multiplier  $-\frac{1}{2}$ . So we take half the Multiplicand ( $-8$ ) which is  $-4$  and change its sign to arrive at the Product 4. Proportionally as 1 is to  $-\frac{1}{2}$  so  $-8$  is to 4, written  $1 : -\frac{1}{2} :: -8 : 4$ . The products of the outer terms equal the products of the inner terms.

### A Chinese Connection $-3 \times -4$ (200 BCE)

As mentioned, the Chinese were comfortable with negative numbers around 2000 years before western Europe. The pedagogy of their *rod numeral* arithmetic was simple. Red rods were positive and black rods were negative, which is the opposite of our accounting convention today. Just because it may not have been done, there is nothing to stop us blending Chinese arithmetical pedagogy with Euclidean proportion theory. All we do here, for simplicity, to reveal how negative three multiplied by negative four equals positive twelve, is depict black segments as positive and red segments as negative.

### How $-3 \times -4 = +12$ Can be Depicted Geometrically

Unit   
 composes   
 Multiplier   
 (*Units placed four times with a change of colour/sign.*)

*Multiplicand*

composes

*Product*

(*Multiplicands placed four times with a change of colour/sign.*)

With the *Unit* fixed as positive, as 1 is to  $-4$ , so  $-3$  is to  $+12$  and the proportion is  $1 : -4 :: -3 : +12$ . (Confirmation of multiplicative commutativity is again left as an exercise for the reader.)

## The Aftermath of BV1570

Billingsley, knighted for services as London's Lord Mayor, never wrote again on mathematics. Yet the *Billingsley brand* of 'multiplication is repeated addition' remains number 1 throughout the English speaking world. If it wasn't for Dr. Devlin, it is unlikely Billingsley's defective definition of multiplication would have been removed from the draft Common Core State Standards (CCSS) of the USA, (HREF9). Yet just as  $2 + 2 + 2$  is addition, the discrete area and equal group models are in the domain of addition. Only by going back to the future to 'debug' and update Euclid, will complexity be displaced with simplicity and concepts such as negative multipliers and divisors be clarified. In four-term proportions, which also alternate, the product of the two outer terms equals the product of the two inner terms. (Check with  $1 : 3 :: 4 : 12$  and  $1 : 4 :: 3 : 12$ .) Yet as strange as it may seem, because Euclid did not consider the *Unit* to be a number and because Billingsley changed (and broke) Euclid's multiplication definition, the foundational logic of the *Unit* and *proportion* was lost.

Of course, most of the English-speaking world copes with mathematics well enough, despite Billingsley. MIRA will likely remain along with 'equal groups' and 'array' pedagogies, yet preferably within TAOMIRA, (The *Application Of Multiplication Involves Repeated Addition*). Via symmetry and the mathematics of the East, the true TAO, or *way* of mathematics reveals many new games and fun lesson ideas that are as profound as they are simple. Some of these have been demonstrated. PCV is a 'missing link' from both the *Naturals* to the *Reals* and elementary mathematics to physics. Arguably, the only equal to *Elements* in scientific impact is *Principia*, (Newton, 1687), which presented laws of motion, universal gravitation and more. Just as addition evaporates as proportion is revealed in Euclidean multiplication, in *Principia*, Newton mentions addition 44 times and proportion 396 times, (HREF10).

## Heresy or Prophecy?

If this article appears heretical, historically, powerful people feared heretics, not because the heretics might be *wrong*, but because the heretics might be *right*. René Thom wrote: 'There is no case in the history of mathematics where the mistake of one man has thrown the entire field on the wrong track', (Thom. 1971). Yet René was wrong. He didn't know about Billingsley and neither have curriculum developers. Just as Roman Numeral arithmetic was replaced with India's, we must not rest. Generation Z will have new problems to solve so why not gift them new insights?

Because education budgets are a perennial problem, future politicians and principals may be forced to embrace the best code in class to Save Your Self/Schools/Staff/Students Time Energy and Money, (S.Y.S.T.E.M.). Just as future generations may condemn us for inaction on climate change, past generations of mathematics educators may stand condemned for their inaction on updating and fixing foundations of elementary mathematics. So, as current or future education leaders, it is our duty to *apply the Scientific Method* and pursue kaizen, or continuous improvement, to help move the human race forward, via the mathematics we teach, towards peace, prosperity and truth, for all. Thank you.

*All great truths begin as blasphemies.*  
(George Bernard Shaw, 1919)

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**Note**

For the conference presentation given that reveals the lost logic of: zero, negative numbers, negative multipliers, negative divisors and more, with child-friendly examples, feel welcome to email the author, via [research@jonathancrabtree.com](mailto:research@jonathancrabtree.com)

# GRADE 3/4 DEVELOPING BIG MATHEMATICAL IDEAS: SHARON AND HER LITTLE RED BOOK

## **Sharon Goldfinch**

*Brunswick South West Primary School*

## **Gaye Williams**

*Deakin University, The University of Melbourne*

*Sharon used to find many students were not interested in mathematics, did not engage, were bored, saw maths as irrelevant and / or found it difficult. She has found this is no longer so in her classes. Her students now look forward to problem solving lessons and relish the challenge. Over that time of change she progressively created her Little Red Book as she scribbled notes while working with Gaye, and referred to these notes in discussions with other teachers. Gaye saw Sharon scribbling away intently during classes they team-taught, referring often to this resource. Over time Gaye came to see that this resource was integral to the problem solving expertise Sharon had developed.*

## **Introduction**

Sharon describes what has happened over the past five years “during which time I have had the wonderful opportunity to work with Dr Gaye Williams with her ‘Engaged to Learn’ approach to problem solving”. She states that her Little Red Book “has pride of place in my teaching resources”. She reflects on the need for *all* elements of the ‘Engaged to Learn Approach’ (Williams, 2009) to problem solving be used for mathematical problem solving to be effective, and how exciting it has been for her to find that various elements of this approach are now embedded in her teaching in general (not only in her mathematics teaching). She draws attention to how her practice has shifted significantly and the benefits this has had for student learning. She discusses the importance of handing control to her students by developing their metacognitive understanding of the different elements of the problem solving process, and she points to ways she has made the approach her own. Gaye draws attention to ways Sharon has adapted the approach to make it her own, and points to refinements of Sharon’s she draws upon during professional learning sessions in other schools.

## **The Beauty of the ‘Same Pace of Thinking’ Approach**

### **Sharon’s Perspective**

The parts of Gaye’s approach that most appealed to me jump off the pages as I read through the many notes in my Little Red Book. What first excited me about participating in Gaye’s professional learning sessions was her idea of ‘maximising resilience (optimism) to support creative mathematical problem solving’ (Williams, 2010). The beauty of this approach is that it allows students to explore different mathematical concepts in a supported environment where collaboration is encouraged. Students work in small groups on mathematical tasks and concepts that are just beyond their current understanding. Academic performance takes a back seat when forming groups and an appreciation of different thinking styles is fostered. An emphasis on the thinking involved in solving each problem differs from the usual race to find a correct answer. Students are instead encouraged to take risks and uncover a multitude of ways to come to possible solutions. When participating in this style of problem solving, I found that

students begin to build confidence in themselves as learners and this results in a significant change in student's perceptions of themselves as mathematicians. I have also observed a change in mindset where students who complained of a dislike of maths begin to see it as something exciting and state that they cannot wait for the next task. A student recently made a comment that captured the appeal of the 'Engaged to Learn' approach. He stated that it is more challenging than the maths he did during his time overseas. He went on to explain that the tasks have made him think in different ways and find more than just one answer, which was very different to his usual math experiences.

Comments such as these are common when students are participating in the 'Engaged to Learn' approach to problem solving. The tasks are often referred to as a point of reference in later maths lessons. One such task that is still spoken of by the students with great fondness is the 'Cookie Task' (see Figure 1) from 2015.

This task was one of the most successful maths experiences I have had in my teaching career. The real life scenario coupled with the hands on aspects and steps leading to the cooking session, made the task purposeful and highly engaging to all students. Establishing the students as mathematicians set a tone where the students were determined to fulfil what was being asked of them and created a sense of excitement throughout the task. This task is described in the next section.

The benefit of implementing the 'Engaged to Learn' approach is that it creates a learning experience that differs markedly to the traditional maths lesson where the teacher is 'imparting knowledge and wisdom' (Williams, Harrington, & Goldfinch, 2012). I have found that the approach involves a range of strategies that have become for me effective teaching and learning practices, and that combining these strategies in their entirety maximises peer learning and teaching opportunities while allowing all students to achieve success within the task.

### **Gaye Reflects On Sharon's Similar Pace of Thinking Groups**

For several years, Sharon and Gaye have worked together during professional learning sessions. During that time we have each developed further understandings of useful grouping practices. This way of exploring to learn more about grouping practices is described in Williams (2013). Given the expertise Sharon has developed over time (supported by her interrogation of her Little Red Book), it is not surprising that the Grade 3/4 students became more resilient as the year progressed. Their resilience was displayed as students came to realise that trying new ideas when something was not working (rather than giving up) often led to further progress. Students explained that they were now more likely to keep trying to find a way to proceed because of the successes they had achieved as they explored and progressed their understandings in previous tasks. The resilience developed by this class was enabled by Sharon allowing groups the autonomy to explore in the ways they wanted to, and her expressiveness when they presented new ideas. For example, "Oh! (pause) Oh [softer like thinking about it and encouraging others to think]" with a look around the class. Listening to and responding to group ideas in a way that encourages others to engage with them whilst not judging these ideas is a skill that takes time and effort to develop. Sharon has achieved this through her own style of teaching.

### **The Brunswick Giant Choc Chip Cookie Factory Task**

As mathematicians, you have been consulted by the Brunswick Choc Chip Cookie Factory to advise them on what they need to do to keep their Giant Choc Chip Cookie Promise: "Every choc chip cookie contains at least 10 choc chips"

The Brunswick Choc Chip Cookie Factory cookie-making machine will hold enough cookie dough to make 6 Giant Cookies at a time. Your problem is:

What advice will you give the Brunswick Choc Chip Cookie Factory about how many chocolate chips to put in the dough to make sure that their Giant Choc Chip Cookie Promise is kept (that each cookie has at least 10 chocolate chips)?

*Figure 1.* The Brunswick Giant Chocolate Chip Cookie Factory Task

By the time students undertook the cookie task (see Figure 1) mid-year, they no longer perceived problem solving as a process requiring significant teacher input, but rather as opportunities to explore ideas together. They were expert at collaboratively developing ideas communicating what their group had found, and sharing their group's thoughts about what they had found.

Sharon's expertise in forming 'same pace of thinking' groups was evident in the way groups immediately engaged with the cookie task and continued to work together building new ideas over the days in which the task was explored. Sharon's Little Red Book captured her progressive development of ideas on how to achieve this. Sharon drew on this resource for the 2012 paper we wrote (Williams, Harrington, & Goldfinch, 2012).

### **Implementing The Brunswick Giant Choc Chip Cookie Factory Task**

1. Make predictions about the number of choc chips needed in each batch to keep the promise.
2. Group participate in a class simulation using rice as dough, and small coloured balls as choc chips.
3. 'Dough' was ladled 'equally' onto six plates (representing six cookies) under the watchful eyes of scrutineers.
4. Group scrutineers monitored the equality of cookie size (same amount of rice in each cookie).
5. Groups counted the number of 'choc chips' in their 'cookie' and wrote their result on the board.
6. Groups refined their predictions and tested these predictions with further simulations (using dice, counters, and six pictures of Choc Chip Cookies).
7. Once students decide how many choc chip cookies they considered are needed for the Brunswick Choc Chip Cookie Factory to keep their promise, they tested their finding with the assistance of much appreciated parents: The groups baked cookies.
8. Each group baked a batch of cookies using the number of choc chips they decided were needed.
9. Each group presented their findings.
10. Groups developed recommendations for the company.

*Figure 2.* Implementing The Brunswick Giant Chocolate Chip Cookie Factory Task

Through work with this task (see Figure 2 for how task was implemented), students developed and consolidated understandings about selecting and formulating statistical displays and interpreting their contents. They were surprised when they saw how different the result of the first simulation was from the initial predictions they made. They engaged more deeply as a result of this surprise, and developed

an understanding of randomness and variation that will stand them in good stead in the future (inside and outside school). Without Sharon's ability to group these students in 'same pace of thinking' groups, the opportunities she gave them for autonomous action, and her interest and expressiveness in responding to their ideas in ways that encouraged others to contribute ideas, this marked increase in inclination to problem solving would not have occurred.

Sharon and Gaye were delighted with the lateral thinking shown by several groups in their final report when they realised the enormity of the uncertainty about which cookie each choc chip might land in. Group comments like "Even if you had five hundred choc chips you could not be sure, it is possible that they could all land in the one biscuit", "We are going to tell the choc chip factory that we quit because it is impossible to be sure that there will be at least ten choc chips in each biscuit:" and "We agree with what the other groups have said about it not being possible to be sure there will be at least ten choc chips in each cookie so we decided to advise them that they need to change their promise." These Grade 3/4 children had developed big ideas about variability.

### **Vital Elements: When the Whole is Greater Than the Sum of its Parts**

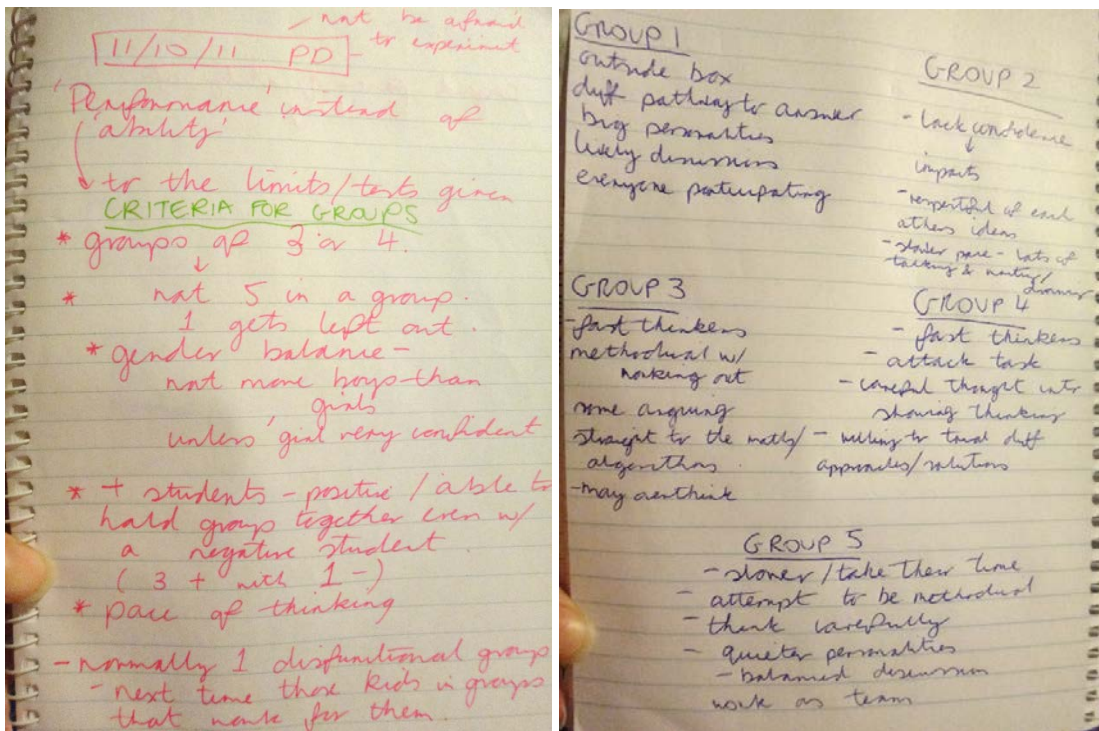
Sharon's detailed understanding of the interdependence of elements in the Engaged to Learn Approach (described by her below) arose from her conscientious and meticulous recording of what was happening in classes implemented by me, her continual returning to these ideas over time to reflect further, her experimenting with ideas, and her communicating of ideas from her Little Red Book to other teachers. She points particularly to the composition of the groups, student autonomy in organising their group and the thinking they undertake, opportunities for groups to question the ideas the reporters present within constraints that keep the reporter emotionally safe, emphasis on the process of developing ideas not answers, teacher selection of the reporting order, and students leading the development of language. These ideas are expressed in ways that make them very easy to understand. In the future, I may need to quote Sharon sometimes when describing elements of my approach.

### **Sharon Describes Key Elements**

As part of the Engaged to Learn Approach, my students work in groups of 3 or 4 on problem solving tasks that are carefully worded to ensure they are accessible to everyone. As the teacher I grouped students according to a set of criteria devised by Gaye (see Williams, Harrington, & Goldfinch, 2012);

- Similar pace of thinking or approach to problem solving tasks,
- Gender balance (or never more boys than girls. If there are more boys in the class have some 'all boy' groups),
- Balance of positive to negative personalities, and
- If possible separate friendship groups so the social aspect does not interfere with group problem solving dynamics.

Grouping students according to their 'pace of thinking' can seem daunting but I have found it becomes easier as your experience builds. I now smile as I look back at my notes in my Little Red Book starting from 2011 (see top left Figure 3) where I was so worried about 'getting it right' with the 'correct' number of students in each group and the appropriate balance of gender. My scribbles across the years show how it becomes almost second nature to isolate the different types of thinking within the class while balancing personalities and gender. I now look at my class list and instantly know what combinations will work within the pace of thinking approach. I have found that different dynamics are created by grouping student's with like personalities, similar perceptions of themselves as mathematicians, and similar mindsets when working in a small group (see top right Figure 3).



<p style="text-align: center;"><u>Criteria for Groups</u></p> <ul style="list-style-type: none"> <li>* groups up to 3 or 4</li> <li>* not 5 in a group 1 gets left out</li> <li>* gender balance – not more boys than girls unless girl very confident             <ul style="list-style-type: none"> <li>• + students – positive / able to hold group together even if a negative student ( 3 + with 1 - )</li> </ul> </li> <li>* pace of thinking             <ul style="list-style-type: none"> <li>- normally one dysfunctional group</li> <li>- next time those kids in groups that work for them</li> </ul> </li> </ul>	<table border="0" style="width: 100%;"> <tr> <td style="width: 50%; vertical-align: top;"> <p style="text-align: center;"><u>Group 1</u></p> <ul style="list-style-type: none"> <li>- outside box</li> <li>- different pathway to answer</li> <li>- big personalities</li> <li>- lively discussions</li> <li>- everyone participating</li> </ul> </td> <td style="width: 50%; vertical-align: top;"> <p style="text-align: center;"><u>Group 2</u></p> <ul style="list-style-type: none"> <li>- lack confidence capacity</li> <li>- respectful of each others' ideas</li> <li>- willing to try each other's ideas/ solutions</li> </ul> </td> </tr> <tr> <td style="vertical-align: top;"> <p style="text-align: center;"><u>Group 3</u></p> <ul style="list-style-type: none"> <li>- fast thinkers</li> <li>- methodical with working out</li> <li>- some arguing</li> <li>- straight to the maths / algorithms</li> <li>- may overthink</li> </ul> </td> <td style="vertical-align: top;"> <p style="text-align: center;"><u>Group 4</u></p> <ul style="list-style-type: none"> <li>- fast thinkers</li> <li>- attack task</li> <li>- careful thought into showing thinking</li> <li>- willing to trial different approaches / solutions</li> </ul> </td> </tr> <tr> <td colspan="2" style="vertical-align: top;"> <p style="text-align: center;"><u>Group 5</u></p> <ul style="list-style-type: none"> <li>- slower / take their time</li> <li>- attempt to be methodical</li> <li>- think carefully</li> <li>- quieter personalities</li> <li>- balanced discussion</li> <li>- work as team</li> </ul> </td> </tr> </table>	<p style="text-align: center;"><u>Group 1</u></p> <ul style="list-style-type: none"> <li>- outside box</li> <li>- different pathway to answer</li> <li>- big personalities</li> <li>- lively discussions</li> <li>- everyone participating</li> </ul>	<p style="text-align: center;"><u>Group 2</u></p> <ul style="list-style-type: none"> <li>- lack confidence capacity</li> <li>- respectful of each others' ideas</li> <li>- willing to try each other's ideas/ solutions</li> </ul>	<p style="text-align: center;"><u>Group 3</u></p> <ul style="list-style-type: none"> <li>- fast thinkers</li> <li>- methodical with working out</li> <li>- some arguing</li> <li>- straight to the maths / algorithms</li> <li>- may overthink</li> </ul>	<p style="text-align: center;"><u>Group 4</u></p> <ul style="list-style-type: none"> <li>- fast thinkers</li> <li>- attack task</li> <li>- careful thought into showing thinking</li> <li>- willing to trial different approaches / solutions</li> </ul>	<p style="text-align: center;"><u>Group 5</u></p> <ul style="list-style-type: none"> <li>- slower / take their time</li> <li>- attempt to be methodical</li> <li>- think carefully</li> <li>- quieter personalities</li> <li>- balanced discussion</li> <li>- work as team</li> </ul>	
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Figure 3. Sharon's Little Red Book pages: 2011 (left top), transcript of 2011 (left bottom), more recent (right top) and transcript of more recent (right bottom)

There are other features of the Engaged to Learn Approach that I have found contribute to the quality of group interactions. Students choose roles within the group such as reporter, recorder, encourager and timer. These roles are switched after each report. Groups are given limited materials such as one piece of poster paper and a marker, to promote discussion and collaboration. Concrete aids are available but appropriate resources often need to be identified by the group rather than the teacher telling the class what to use. See Williams, Harrington, and Goldfinch (2012) for ways in which Sharon and other teachers in her school orchestrated this. Each group reports at intervals throughout the task. The students select who reports within their small group each time. Groups are given two minutes to 'prime' their reporter and to ensure they represent the key ideas. The students decide how to present their information in the best possible way so that everyone understands their ideas. Emphasis is placed on showing the thinking not just an answer.

The students are able to ask questions at the end of each report but cannot pose a question beyond the reporter's current level of understanding. They are also not allowed to contradict or correct another group's report. Groups are also discouraged from adding further information to another group's report. They are reminded to keep the information for their group to think about during the next part of the task. Subsequent groups are able to state that they disagree with an idea presented earlier though, making the focus purely on the maths and not on the person presenting. This provides a safe reporting environment where all students are expected to have a turn reporting on behalf of their group.

The teacher carefully selects the reporting order by observing each stage of the task with the idea that each group builds on the previous report. Feedback is given to each group about something they did well or how they have contributed to the whole. The teacher may highlight an interesting strategy that was used or draw attention to mathematical language introduced in a report. Comments such as "I really like the way you used the ..." validate the students' thinking and let them know that they have made a positive impact on the task as a whole. Through listening to each report the groups are able to select what is important to them and use it during the next stage of the task. During the final stages of the 'Cookie Task' there were many 'light bulb' moments where choruses of exclamations from the class were heard during group reports. The idea of not being able to meet the promise was one such idea that prompted a great deal of debate and led to groups rethinking their predictions.

Looking back through my notes I can see how my view of the role of the teacher has changed dramatically. From the onset I could see that the role of the teacher is significantly different to the standard 'teacher at the front of the room' lesson. Over several sessions I realised that the role of the teacher is to simply pose questions throughout the process to prompt deeper thinking and discussion without leading to a direct answer. The idea of the teacher 'asking not telling' was something I have repeatedly scrolled in my notebook. Questions such as "How can you show what you are thinking?" and "What will you try next?" are so simple yet had such a profound impact on my students.

The students lead development of language with the teacher giving the mathematical terms only once other more common language means of communicating ideas are established. The role of the teacher may appear passive at first (see Williams, Harrington, & Goldfinch, 2012) but involves the complex task of listening and forming questions relating to each group's report. It also takes great restraint by the teacher, to not give any hints or clues as to whether the students are on the 'right track', through facial expressions. In revisiting my Little Red Book, I remember a student in the early stages of working with Gaye who would constantly seek reassurance for his ideas and strategies used. I recall him asking me if a strategy would work. Gaye's quick interjections of 'this is your task. If it doesn't work try something else' and 'I don't know will it?' put the emphasis back on him taking control of his thinking and building independence as a mathematician.

### **Gaye's Reflects on Sharon's Descriptions of the Approach**

Sharon has captured parts of the process that are normally not noticed by teachers as they begin developing an understanding of the Engaged to Learn Approach. Sharon identifies resilience / optimism building parts of the process in her attention to listening and working out how each report can be valued in terms of what it contributes to the class as a whole (see for example Williams, 2009). These small successes that are amplified through teacher valuing build student resilience over time. Both Sharon,



and Judy Harrington (see Williams, Harrington, & Goldfinch, 2012) have been at Brunswick South West Primary School within this professional learning program and have discussed together how important it is to include all parts of the process. These discussions have helped to crystallise Sharon's thinking as she developed her own version of the Engaged to Learn Approach.

### **Sharon: Making the Process My Own**

It was only after speaking to Gaye recently that I realised that I have made changes to the original format of the 'Engage to Learn' approach and have therefore started to make it my own. I had always given recognition to Gaye for certain aspects of how I implement her approach, not remembering that I had actually developed them myself to suit my class and my style of teaching.

One simple adaption was the seating of the same gender students in each group on a diagonal instead of side by side or opposite each other. This came about in an early task when I observed students of the same gender chatting and working in pairs instead of as a group when in close proximity to each other on opposite sides of a single desk. The diagonal seating encouraged cross gender discussions and made the group work more productively.

Another change of mine that Gaye alerted me to was the addition of a class reflection at the end of each lesson. Through the use of a strategy called 'Glow and Grow', we highlight something that the class did well and isolate an area of improvement for the next session. This acknowledges the significant gains of the students within the process each session while ensuring we continue to look for ways to improve the overall learning experience.

Two Class Team Teaching is another my adaptations, that my teaching team have found very successful. When working with my team teaching colleagues, we work through a problem-solving task with our double class of approximately 50 students in an L shaped three-classroom-space that has two sets of concertina doors. This allows us to give simultaneous instructions to both classes from the room in the middle of the L shape, both monitor group work across the combined space, yet run our reporting sessions separately. We both rove around the groups, listening to the discussions and jotting down notes to assist us to choose a reporting order, then meet mid-room and quickly compare our notes and decide on the order of reporting.

### **Sharon and Her Team: Metacognition**

At the beginning phase of introducing each new step, the team discusses its importance and I explain the reasoning behind it. I then revise this throughout the year. Gottfried (1990) defines academic motivation as 'enjoyment of school learning characterised by a mastery orientation; curiosity; persistence; task-endogeny; and the learning of challenging, difficult, and novel tasks'. Students are motivated when they are prompted to reach within themselves, not give up when the task seems difficult, and develop a deeper understanding of the task through experimentation with a range of possible solutions. Part of my success in implementing Gaye's 'Engaged to Learn' approach in my classroom has been ensuring the students understand each part of the process and how these parts combine to make such a valuable learning experience for them. They are determined to 'master' each part of the process and pride themselves on being able to explain the purpose and reasoning behind it. This has had impact on the degree of focus students display during the progressive reporting of ideas, they now know that each report may hold valuable information that will lead to success in future stages. Students are also aware of the significance of, and reasoning behind, not asking the reporter a question beyond their current level of understanding. They now use self-talk to decide whether or not a question is suitable to ask after each report and openly state why a question might go beyond a classmate's report, therefore alleviating the risk of putting their peers in a difficult situation.

### **Sharon Handing Over the Reigns**

"The art of teaching is handing over control of the lesson to the students; while teaching the class everything that they need to know" (Artzt, Armour-Thomas, Curcio and Gurl, 2016, p74). A key part of my implementation of the 'Engaged to Learn' approach to problem solving has been relinquishing more control to the students, moving away from the traditional model of the teacher being at the centre

of the learning. Each year I have been excited to see my class take a more active role in the execution of the different aspects of each session. Once the students develop a metacognitive understanding of the various features of the 'Engaged to Learn' approach, they feel a sense of ownership and begin to take control of the process as a whole. This has been demonstrated on a few occasions where I have failed to adhere to the usual format and have been inundated with cries of protests and timely reminders of what is to happen next. The students are now, amongst other things, capable and ready to assume the responsibility of writing the reporting order on the board, remind each other to pin their work up for everyone to see and have input the amount of time they think each step will take just to name a few.

### **Sharon: The Transferability**

I have found that the different elements of Gaye's 'Engaged to Learn' approach to problem solving are all essential to effective teaching practice, making them easily transferrable to other areas of the curriculum. These strategies observed while working with Gaye are now part of my everyday teaching across different learning domains. One feature that has had a significant impact on my teaching is the idea of reporting throughout a task instead of the standard reflection or sharing at the end of the lesson. The ongoing reporting requires students to verbalise their thinking throughout the task while being able to listen to others' ideas and use what they see fits. It also allows teachers to break up the learning into smaller and more manageable parts between each report.

There are a multitude of other strategies that can be transferred beyond the mathematics classroom. Strategies such as:

- The teacher not giving hints that lead students directly to an answer, instead asking questions that prompt deeper thinking.
- Valuing independent and creative thinking over finding an immediate solution to a problem.
- The use of 'think aloud' to prompt deeper thinking and to model effective questioning.
- Allowing opportunities to learn through discovery and experimentation rather than expecting learning to occur through teacher transmission of information.
- Using open-ended tasks that allow students to think outside the box and find a multitude of answers.

Not only have I changed my way of implementing problem solving in mathematics, I have developed a whole new orientation to teaching and learning in general and both the children and myself are benefitting from and enjoying the process. My Little Red Book has been crucial in this process.

### **Concluding Remarks**

Gaye: This paper draws attention to what Sharon has achieved through the process of constructing and reflecting on her Little Red Book. She has not only progressively changed her practice through her engagement with this self-developed resource but has also been able to articulate her process of change for others to consider. I have included Sharon's strategy of seating same gender students diagonally opposite each other (with acknowledgement to Sharon) in sessions I provide for teachers. Sharon has gained a great deal through opportunities to reflect on and discuss her professional learning with myself and others over time. Her passion for learning more about the Engaged to Learn Approach and her ability to share the benefits of different features of the approach with her students has contributed greatly to the successes Sharon and her students have achieved.

Sharon: Writing this paper with Gaye has encouraged me to revisit my Little Red Book and celebrate the changes I have made in my mathematics teaching over time. By selectively recording observations over several years and revisiting records to answer questions for myself, I have searched for ways to move forward, and share my thinking with others. Such a process could be useful to others wanting to change their teaching practice. Not only have I changed my practice, I can also describe why these changes are useful and have engaged my students in unpacking their process of learning through the Engaged to Learn Approach. These experiences have prompted me to look for further ways to improve mathematics-learning experiences for my students. The benefit of collaborating with an expert such as

Gaye has been a source of valuable professional learning for me. It has had a profound impact on my ability to design what I have found to be a highly effective mathematics teaching and learning program.

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# PLANNING TEACHER PROFESSIONAL LEARNING TO FOSTER INNOVATION IN MATHEMATICS TEACHING

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*This is the report of an aspect of a system-supported teacher professional learning initiative that focused on an innovative approach to the teaching of mathematics. The initiative included the study of a theoretical approach to the teaching of mathematics, supported by a set of resources, as well as school based collaborative implementation of the proposed approaches. Data suggest that the initiative had an impact on teachers' practices. Key aspects of the initiative seemed to be the focus on innovation, the provision of quality resources, and the mix of external presentation of theory and school based implementation.*

## **Introduction**

Ongoing learning is part of the expectations of all professionals, none more so than for teachers. Therefore university teacher educators and those with systemic responsibility for teacher professional learning (TPL) are interested in ways of making such learning effective. At least part of the challenge in designing TPL from a system perspective is balancing the cost and complexity against the potential impact on teaching practices and student learning. At least part of the challenge in researching the effectiveness of TPL initiatives focusing on mathematics is that the pathway from the TPL initiative to teacher planning to classroom implementation to student learning is complex on a scale that is likely to be convincing to those with an interest in the funding of such initiatives.

The following is a report of the initiative of a system, a Diocesan Education Office serving a metropolitan region of NSW, Australia, that sought to examine the impact of a specific TPL initiative on student achievement, although only results related to the first step of the pathway, changes in teachers' reported practice, are presented below.

Using the topic of fractions, teachers were encouraged to implement the pedagogical approach and lesson structure recommended by Sullivan, Askew, Cheeseman, Clarke, Mornane, Roche, and Walker (2014). This approach is based on the use of tasks that are appropriately challenging, posed with limited initial instruction, and are adaptations of the experience for students experiencing difficulty and for those who solve the tasks quickly. The teachers manage whole class discussions of the students' explorations of the tasks, a key aspect of which is the projection of student work samples. The next step is for further tasks, appropriately varied to consolidate learning, to be posed. As part of the TPL, participating teachers were given a resource of suggestions for this teaching. There were eight suggestions, each of which included a rationale for the particular focus and the ways it connected to the Australian Curriculum, task solutions, pedagogical considerations, a challenging task, a suggested introduction, enabling and extending prompts, and follow up tasks.

There were two professional learning days, at the start and end of the initiative, and the students completed an online assessment prior and subsequent to engaging with the lessons. At the first TPL

day, the teachers were given outlines of eight lessons on the topic of fractions from which they could choose, and the majority of the first day involved working through the lesson suggestions with the pedagogical approach being modelled. The participating teacher teams were encouraged to meet to review the pre-test results, again during the lesson implementation and then again after receiving the post-test results.

## **The Framework(S) That Guided the Research**

The initiative was informed by prior research on effective TPL, by approaches to classroom based TPL, and by perspectives on teacher knowledge.

## **The Elements of Effective Teacher Professional Learning**

The overall design and delivery of the TPL initiative was informed by the principles proposed by Clarke (1994, p. 38), presented along with ways that we enacted these principles in this initiative. Clarke recommended that TPL ...:

- “address issues of concern and interest to the teachers”: the focus issues addressed were not only improvement in the teaching of fractions but also of the benefits of students’ cognitive activation (OECD, 2014) along with encouraging students to persist when engaging with problems;
- “involve groups of teachers from a school including the school leadership”: all teachers from the years 5 and 6 teams and leadership of the participating schools were invited to the face to face days;
- “recognise impediments to teachers’ growth”: these were assumed to be lack of access to resources of challenging tasks, lack of familiarity with the process of posing tasks with limited instruction, and ways of conducting whole class reviews of student work, each of which represent the innovation of this initiative;
- “model desired classroom approaches during in-service sessions”: all of the suggested lessons were modelled, using the proposed pedagogies, on the first professional learning day;
- “enlist teachers’ commitment to participate”: even though the teams of teachers were nominated by their principals, the collaborative nature of the initiative, the resource of suggested lessons and provision of feedback on the pre and post assessments were intended to enlist their participation;
- “that changes are derived largely from classroom practice”: this connected to the expectation that teachers implement the recommended lessons in their classrooms;
- “teachers should be allowed time to plan and reflect”: even though no additional teaching release other than the professional learning days was made available, the groups of teachers were encouraged to meet to plan and review their experience;
- “engage teachers as partners”: this was through the combination of research and professional learning being explained to the participants; and
- “recognise that change is gradual”: although this implies that longer time frames are needed, there were around 10 weeks of teaching time elapsed between the professional learning days.

Key considerations in the structuring of the initiative were the provision of arguably high quality teaching resources, the explicit articulation and modelling of the rationale and pedagogies of the innovative approach to teaching, and the design and presentation to the schools of their students’ pre and post assessment data.

## **Rationale for the Classroom-based Components**

Within the schools, the teaching teams were encouraged to approach the implementation of the recommended lesson suggestions collaboratively. Such approaches are common internationally. *Learning Study*, for example, is a process that engages groups of teachers in thinking about student

learning through studying specific examples of their own teaching (Runnesson, 2008). Similarly *Japanese Lesson Study* is a well known approach to sustainable and collaborative teacher learning that involves teachers in thinking about developing shared teaching-learning plans, encountering tasks that are intended for the students, and finally observing lessons and jointly discussing and reflecting on them. This has also been successfully adapted for the Western contexts (see Groves & Doig, 2014). Teaching teams in the project were encouraged to plan and review suggestions collaboratively and the option of observing lessons was proposed. Note that even though teachers were offered suggestions for the components of lessons, these were not a script and there was still a need for substantial planning to convert the suggestions to lessons.

### **Knowledge for Teaching Mathematics**

Even though commitment to the pedagogical approach ultimately involves changes to teachers' attitudes and beliefs, in the first instance, as suggested by Guskey (2002), we assumed that changes in practice would precede changes in beliefs. As a result, our focus was on experiences that would enhance teachers' knowledge that was intended to influence the disposition of the teachers towards the pedagogical approach.

We use the categorisation of teacher knowledge proposed by Hill, Ball, and Schilling (2008) to describe the focus of the TPL. They proposed two categories of knowledge: subject matter knowledge; and pedagogical content knowledge. The topic of fractions was suggested by school advisers who had judged that participating teachers might benefit from consideration of different perspectives on fractions. The resources were designed to incorporate the elements of fractions learning recommended by Empson and Levi (2011) using tasks that had potential to expose partial or misconceptions.

The main focus, though, was on what Hill et al. described as pedagogical content knowledge. They suggested that this includes: knowledge of content and teaching; knowledge of content and students; and knowledge of curriculum. The first of these, content and teaching, was addressed through discussions associated with cognitive activation, modelling of approaches to differentiation of learning opportunities, and the processes of managing whole class reviews of student explorations. The second aspect refers to the presentation of information on approaches to student persistence and resilience which addresses students' mindsets (Dweck, 2000). The third refers to the sequencing of the suggestions, each of which was connected to one or more curriculum descriptors.

The research questions addressed by the initiative were:

To what extent does the provision of specific resources support the implementation of innovative pedagogies?

To what extent do teachers, as an outcome of the professional learning, report starting learning with an initial challenge, allowing a degree of confusion and struggle, and differentiating tasks for particular student needs?

### **Data Collection and Analysis**

The data reported below were sought from participating teachers of Year 5 and 6 classes in schools serving communities across a variety of socio economic backgrounds. The overall project adopted a design research approach which "attempts to support arguments constructed around the results of active innovation and intervention in classrooms" (Kelly, 2003, p. 3). The intervention was the suggestions of the resources and pedagogical approach and the innovation was the notion of embracing confusion as the way of activating cognition. The project was also iterative in that up to eight suggestions, each of which was reviewed by the teaching teams, were implemented sequentially. While it is common for design research to include classroom observations, it was decided that it was threatening enough for the teachers to be implementing pedagogies with which they were not experienced, possibly observed by their colleagues, without adding further to the teachers' stress.

The TPL initiative gathered data through online surveys using Qualtrics (2015) from teachers at the start of each of the first and second face to face days, and through pre and post assessments of student learning. Only results from the teacher survey completed at the start of the second day after the teaching of the lessons are reported below. There were both fixed format and free format items on the survey.

The responses from the fixed format items were used to describe aspects of the project implementation. Some of these items with descriptive summaries are presented below. There were also some open responses items, the results from one of which are reported below. For this, the responses were inspected, categorised, sorted, re-examined for consistency and then re-categorised. The categories of responses with interpretations are also presented below.

## Results

The responses from the teachers are presented in three sections: the suggestions that they found helpful; some general aspect of their pedagogies; and specific reported changes to their practice. Note that in all cases, we were aware of the possibility that teachers might respond with answers they thought we were hoping for, and so the prompts were designed to be different from the conventional Likert type scales. Even though there would have been advantages in matching pre and post surveys, the surveys were anonymous to further increase chances that responses were reflections of the teachers' opinions and practices.

### Aspects of the Suggestions That Teachers Reported Finding Helpful

On the survey completed at the start of the second day of the professional learning, one of the items was "In terms of the fractions lessons overall, please select and drag the five aspects of what we did that were most helpful for your teaching of those lessons". Table 1 presents the elements from the list of 16 possibilities that were most frequently chosen as "most helpful" from among the 82 responding teachers.

Table 1: *Aspects Most Frequently Chosen as Most Helpful (n = 82)*

Option	Frequency
The suggested enabling prompts	68
The sequence of lessons in the booklet from which you could choose	53
The suggested extending prompts	53

Given that well over half of the teachers have chosen these three, this is a very strong indication that the teachers appreciated the provision of the resources which they could use to plan their teaching, especially the suggestions of ways of supporting students experiencing difficulty. This is one of the key insights from this initiative.

Table 2 presents the aspects that were next most frequently chosen.

Table 2: *Frequency of Other Aspects Chosen as Most Helpful (n = 82)*

Option	Frequency
The initial PD day	40
The overall approach to structuring lessons	37
The idea of starting learning by using an initial challenge	36
The description of the intent of the individual lessons	31

The “PD day” allowed teachers to work through the lessons, at least some of which would have extended the experience with fractions of some teachers. The other items refer to information that would assist in their planning and delivery of the lessons. It is suspected that teachers will appreciate it if specific suggestions are included in TPL initiatives.

It was also of interest, the aspects were not chosen as helpful. There were no teachers who selected “The feedback on the pre test” or “The discussion on the pretest feedback in your school”. This was information collected from their students, and returned expeditiously, with student by student item analysis. It would be interesting to explore why none of the teachers consider such information among the most helpful aspects. It is possible that teachers knew their students’ capabilities well already, although it is also possible that this is just a lower priority. Even the items about collaboration – “meetings with the planning team” (chosen 23 times) and “advice from colleagues” (chosen 11 times) were given a lower priority than the above factors.

### Reporting on Their Specific Practices

There was also an item that sought teachers’ responses on various aspects of the pedagogies. Using pull down menus, the teachers could choose one of five options for each of the suggested aspects: I already do this regularly and will continue to do so; I have not done this in the past and not do it in the future; I will do this more in the future; I will do it more if I can find the time; I need to know more about it.

These was a set of responses for which the majority of teachers claimed to be *already* enacting those pedagogies. These were:

- I start lessons without explaining to students how they should do the tasks or problems
- I prepare adaptations to the main task for students for whom that task is unreasonably challenging (enabling prompts)
- I prepare adaptations to the main task for students who complete that task quickly (extending prompts)
- I stop the class even though not all students have completed the main task and have a discussion using the strategies that some students have discovered
- I actively encourage students to persist and allow them time to persist
- I draw on the explanations of students before I show students what to do
- I make sure the students have time to experience the task and the challenge before I explain to them how to do it

One interpretation is that these teachers were already enacting many of the suggested approaches. Another interpretation is that, given that at least some do not seem commonly used pedagogies, it is possible that teachers were overstating their practices. Nevertheless, there were also other aspects proposed for which the majority of teachers chose “I will do this MORE in the future than I did in the past” as a result of the TRP. These were:

- I pose problems or exercises that most of the students are not able to do initially
- I allow students to struggle for quite some time before I intervene
- When students are stuck, I consciously hold back from telling them what to do, and instead ask another question
- I project student work when they are explaining their thinking
- I pose a similar (consolidating) task to the first one with the intention that the students apply what they have learned from listening to each other



I feel comfortable when students are struggling

I make sure the students have time to experience the task and the challenge before I explain to them how to do it

There were indeed key aspects of the proposed pedagogies, suggesting that the selections presented above were realistic. There were very small numbers who selected any of the other three options, with the one exception that 12 (out of the 82) teachers reported that they would not “allow students to struggle before I intervene”.

In seeking a different perspective on their practice, teachers were asked to “indicate the frequency of each of the following statements as they applied to your teaching of the fractions lessons”. The following were practices where the median, after giving responses a ranked score, indicated that they were used by the teachers overall between “2 or 3 times each week” and “every day”.

I talked to the students about the benefits of persisting in mathematics

I set students to work on a task without showing them how to do it

I projected students’ work so that the class could see it

I asked some students to demonstrate and justify their way of solving a problem on the board to the rest of the class

I read students’ work to make judgments on my subsequent planning

These teacher actions do represent pedagogical innovations. An interesting aside is that the teachers identified more frequently than once each week:

I noticed one or more students who usually find maths difficult who did very well

I noticed one or more students who usually find maths easy who found the tasks very difficult

This suggests that the innovation has the potential to activate a different sort of learning and learner from conventional teaching

### **Reporting on Specific Changed Approaches**

A further item seeking insights on specific changes to practices was “Describe one way that your teaching has changed as a result of the fractions project”. The advantage of this item is that it captures the teachers’ own words which gives some sense of the changes the teachers did experience. There were 4 major categories of responses, representing 75 out of the 82 teachers.

There were 37 responses whose main focus related to allowing the students to be “confused”. Some representative responses were:

Allow children to attempt the task independently and silently before working with peers and allowing them to be confused.

Trying to be more resilient in not supplying a solution to children who are struggling. The idea that confusion is good.

I loved giving students time to be confused and naming this as a good thing. I didn't actually call it zone of confusion - we made a rhyme that said 'get in the zone and think on your own!' The students also said they liked that time to think on their own before others gave them the answer, their way of thinking, or their explanation.

There were 14 comments that focused on differentiation:

I have thought more about the enabling and extending tasks when assisting my lower and top students

Allowing the students to experience the enabling prompt helped all students to engage in their own level of mathematical thinking ...some were able to continue to more difficult levels as the lesson continued.

There were a further 7 who combined comments about allowing students to be confused with addressing differentiation:

I now embrace the beginning of the lesson and the initial task outline. I allow the students to be confused, even those learning support students who I work with. It was helpful to have the enabling and extending prompts listed for teachers who find differentiating tasks on the spot a challenge. They enabled all students to access tasks at their level.

Allowing students more time to persist with the task. I wouldn't let the zone of confusion go for too long as I felt it rocked their confidence too much and a lot of the students just sat there puzzled not giving it a go. I would give them the enabling task and move on from there.

There were 6 who described having students explaining their thinking as the biggest impact:

I allow the students to question their understanding and come up with their own answers even though they say they don't get it. I encourage students to display their working out in front of the class or to a small number of students and compare their findings, then explain their similarities and differences in solving a task.

I am using more examples of the children's work and there is certainly more discussion about the various strategies used by various children. I like the layout of the task, i.e. the learning task, an enabling task and a consolidating task. I see myself more of a facilitator, however, it is a struggle for me not to model a strategy for them.

There were 12 comments that referred to allowing students time to engage with the tasks, such as:

I make a conscious effort to allow students more time on their own before I offer assistance. Reflection time is largely focused on the different strategies that students have used and different methods of getting to the same result.

Give students ample thinking time and stopping to reflect during the lesson when students are stuck or require a teaching point

It seems that the pedagogical aspects that were the focus of the TPL are represented in these comments, and the comments can be taken as an indication of the impact of the initiative on teaching practices.

As an aside, of the 12 teachers who earlier expressed reluctance to allow students to be confused, seven made comments about giving students more time, three focused on differentiation, and two on the student led reviews. This provides a check on the authenticity of the teachers' responses on the elements of the survey.

## **Conclusion and Implications**

This was a report of aspects of a TPL initiative that focused on an approach to teaching, termed cognitive activation, that involves posing challenging mathematics tasks to students, and structuring lessons with minimal introduction, allowing time for students to engage with the task, differentiation for students who need it, whole class discussions led by students and further tasks to consolidate the learning. The teacher professional learning consisted of two full days out of class for external input and school based meetings. The overall initiative included a mix of theory and practice and their connections. Additional resources were provided to the teachers. To evaluate the effectiveness of the TPL, data were collected from the teachers, some aspects of which are presented above.

In terms of research question one, a key finding from the data overall was that the provision of specific suggestions and supporting documentation contributed to the success of the initiative. It is stressed that the documentation did not act as a script, merely as suggestions of the elements that the teachers would subsequently craft into a lesson. One advantage of the provision of these suggestions was that it saved the teachers time in planning so they had more time to gather other materials and to plan the specific lessons. Another advantage was that it provided an independent focus of the discussions, so that it did not matter if teachers were critical of the resources or the approach within the team reflections. Indeed, it seems that any TPL that intends to inform or influence teaching practice will benefit from the

provision of high quality teaching suggestions, either sourced from the many resources of such suggestions or those purpose written.

In terms of question two, and recognising the limits of self-report data, it seems that teachers did augment their current practices and were comfortable to give students time to work on fewer tasks, to allow them to be confused, to differentiate the experiences, and to use student reports of explorations to facilitate the learning of other students.

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# MATHEMATICA – AN INTRODUCTION TO THE BASICS

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*This paper provides an introduction to some basic Mathematica functionality for numerical, graphical and symbolic computation for those with little or no previous familiarity with the software. A collection of examples and activities with some brief discussion and commentary related to the secondary mathematics curriculum are included.*

## **Introduction**

*Mathematica* is a general purpose computer algebra system (CAS) software that can be used for teaching, learning and working mathematically in the secondary school curriculum, and has been used in Victorian secondary schools from the early 1990's. It is also used in business, industry, research and academia. *Mathematica* has an extensive range of computation functionality: numerical, graphical, geometric, statistical and symbolic, covering many areas of mathematics and its applications to STEM, economics and other fields.

A *Mathematica* file is called a notebook (.nb) and can include text, graphical and mathematical content which is structured as a dynamic document. The fundamental construct of a notebook is a *cell* which can contain various types of objects that can be manipulated according to their properties. It is also linked to the web based knowledge and search engine *Wolfram | Alpha* and enabled for cloud platform [HREF1].

As part of the Victorian Government's initiatives for the *STEM in the Education State (VicSTEM)* plan launched in September 2016 [HREF2], the Department of Education and Training (DET) has made *Mathematica*, *Wolfram Alpha Pro* and *SystemModeler* available to all Victorian secondary schools over the next three years, with supporting resources and professional learning.

## **Natural Language Input**

There are two options in *Mathematica* for input where one doesn't know specific commands/options, but does know basically what one wants to do. The first of these is to use free-form input, for example to 'draw  $\sin x$ ' as shown in Figure 1. Using 'graph  $\sin x$ ' or 'plot  $\sin x$ ' would also work:

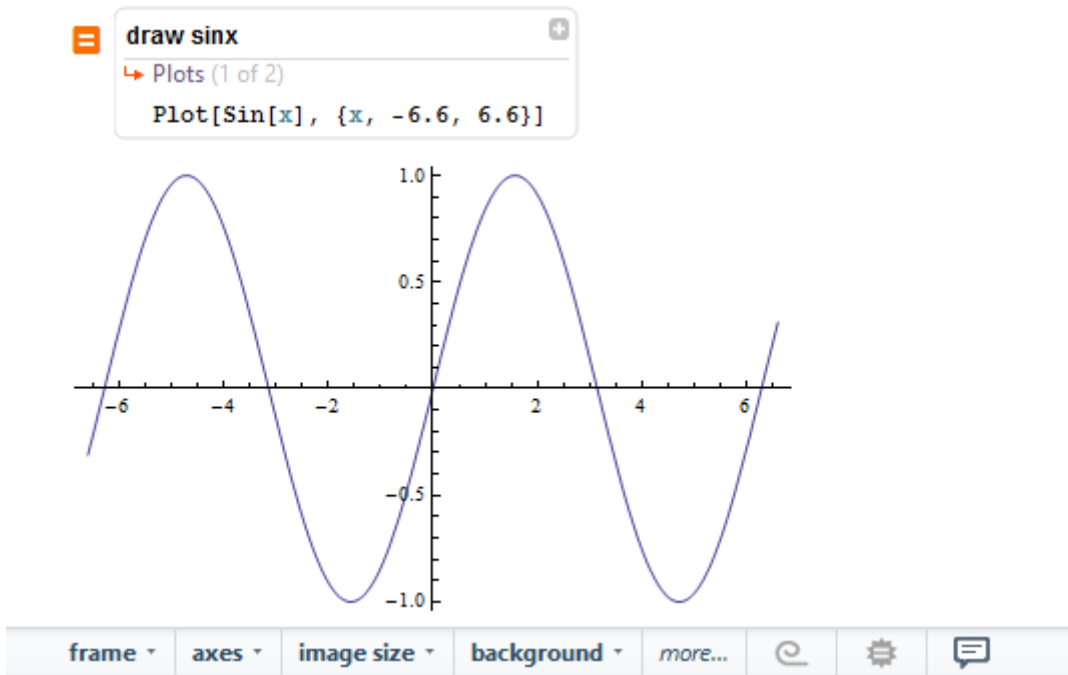


Figure 1. Free form input for the graph of  $\sin(x)$

The other approach is to use *Wolfram Alpha* from within *Mathematica*, for example to find factors of 209304, as shown in Figure 2, with cell brackets on the right hand side. Cell brackets and groups of cell brackets are inserted automatically, and can be hidden, visible, or visible on mouse-over. This would have worked equally well if the input had been ‘factorise 209304’ or simply ‘factor 209304’:

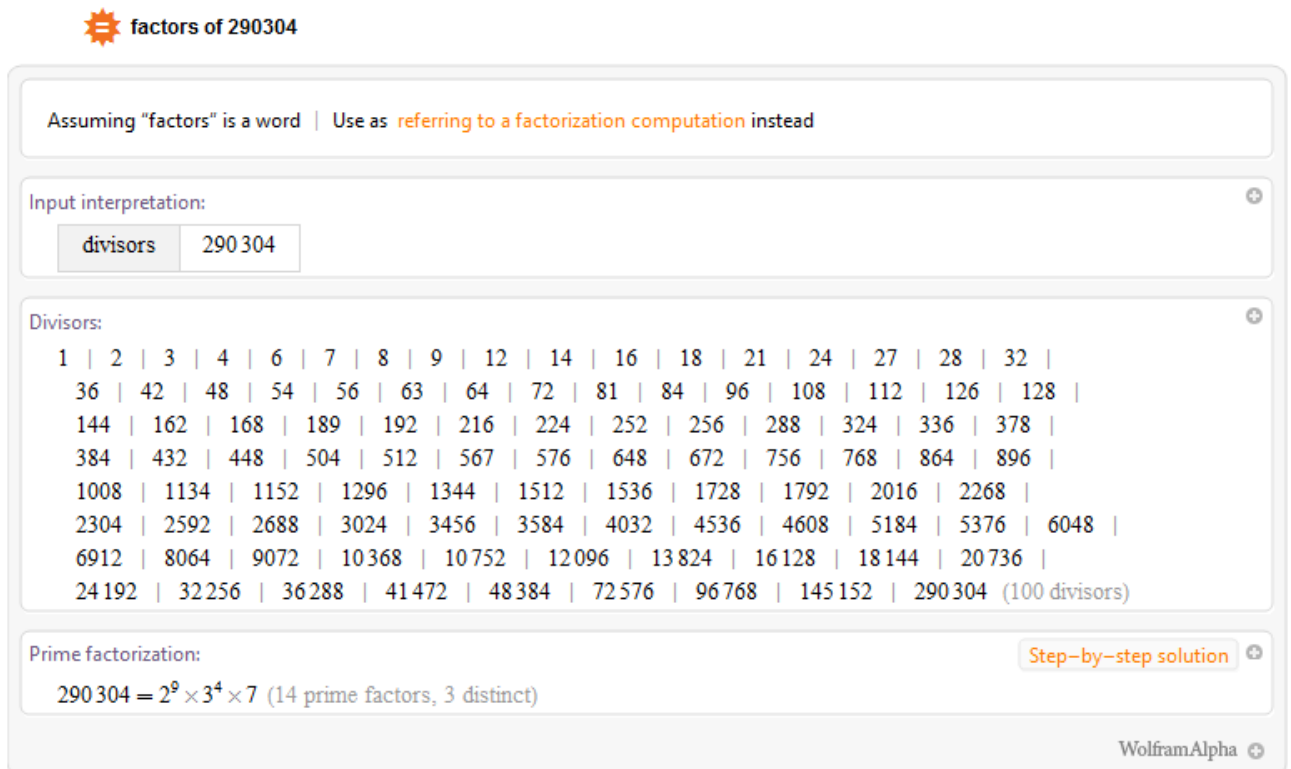


Figure 2. Using *Wolfram Alpha* to find factors of 290304

Just entering the number 209304, results in output of various information about the number, including its prime factorisation. Some sample activities related to prime numbers can be found in *Prime Explorations with Mathematica* (Leigh-Lancaster, 2013).

### Direct Operation Input

Input can also be done directly by using specific commands defined in the syntax of Wolfram language, using a combination of keyboard (there is an auto-complete process which list available functions indicated by initial letters of a command) and/or palettes. For example, typing in 'Com' causes a list of possible commands starting with these letters underlined to appear (Compile, CompleteGraph, Complex, Complement ...) and one selects the desired command, which is automatically inserted into a computation cell in the notebook. Each computation is accompanied by a set of predictive options for other possible related computations. For example, to produce the complete graph on 8 vertices as shown in Figure 3:

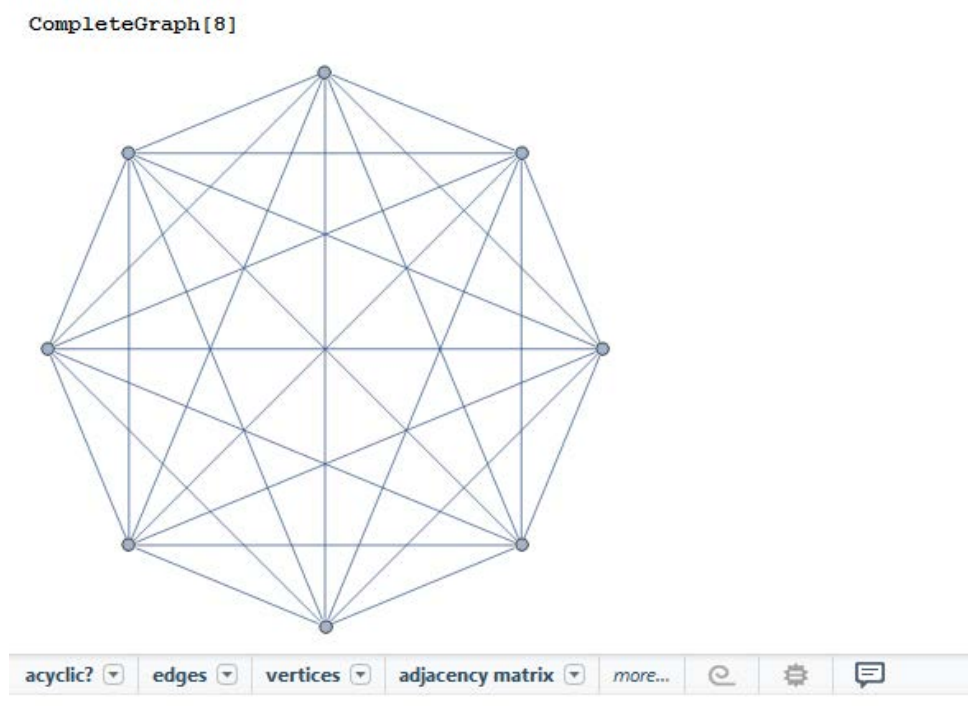


Figure 3. Mathematica command for the complete graph on 8 vertices

Alternatively, a palette, such as the **Basic Math Assistant**, part of which is shown for a Windows computer in Figure 4, could be used to automatically set up a template for a particular type of computation:

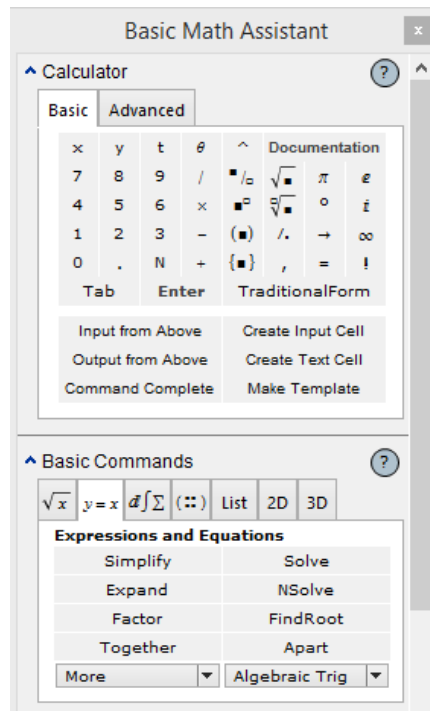


Figure 4. Part of the **Basic Math Assistant** palette

For example, Figure 5 shows how this can be used to solve a simple equation, just click on **Solve** in the palette and the following template appears and can be filled in accordingly:

$$\text{Solve} [ \text{lhs} = \text{rhs} , \text{var} ]$$

$$\text{Solve} [ 2x^2 - 3x = 7 , x ]$$

$$\left\{ \left\{ x \rightarrow \frac{1}{4} ( 3 - \sqrt{65} ) \right\} , \left\{ x \rightarrow \frac{1}{4} ( 3 + \sqrt{65} ) \right\} \right\}$$

Figure 5. Using the **Solve** template from the **Basic Math Assistant**

Palettes can be used both for mathematical expression and text typesetting, that is as an 'equation editor' or for computational input, depending on the nature of the cells (text or input) they are used within. Expressions can be copied and pasted between cell types, and edited/formatted as applicable.

## Some Examples Related to Simple Quadratic Functions, Their Graphs and Equations

### Example 1

*Mathematica* can be used to support inquiry based approaches in the classroom, for example, students could use *Mathematica* functionality such as **Plot** to establish connections between quadratic functions and their graphs.

Figure 6 shows how **Plot** can be used to draw the graphs of three quadratic functions together. Students can explore what happens to the graph of the functions as the value of the parameter inside the bracket is changed. Once students are familiar with this, the value of the parameter, and the effect of changes in its values can be illustrated dynamically using **Manipulate** functionality.

```
Plot[{x^2, (x + 2)^2, (x - 2)^2}, {x, -5, 5}, AxesLabel -> {x, y}, PlotLegends -> "Expressions", PlotRange -> {0, 10}]
```

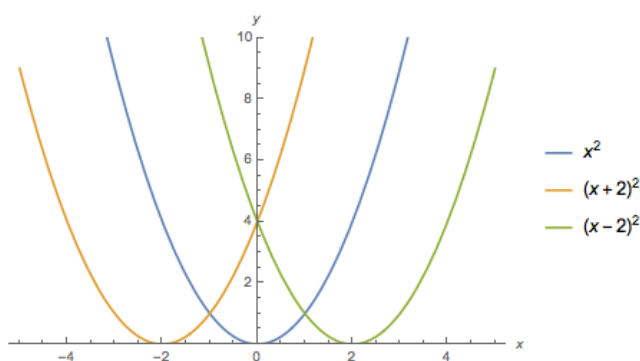


Figure 6. Using the **Plot** command to explore the connection between a quadratic equation and its graph

The **Plot** template along with other graphing options can also be found in the **Basic Commands** palette, part of which is shown for a Macintosh computer, in Figure 7:

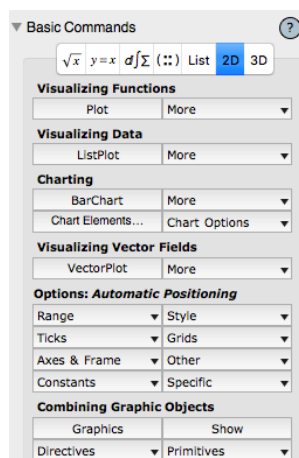


Figure 7. Finding the **Plot** template and other graphing features from the **Basic Math Assistant**

## Example 2

**Discriminant** along with **Plot** and **Solve** commands can be used to investigate how the value of the discriminant relates to the number of solutions/roots/ $x$  – axis intercepts. A variety of equations that have no solution, one solution or two solutions should be used when students are investigating this concept. A sample set of related computations is shown in Figure 8:



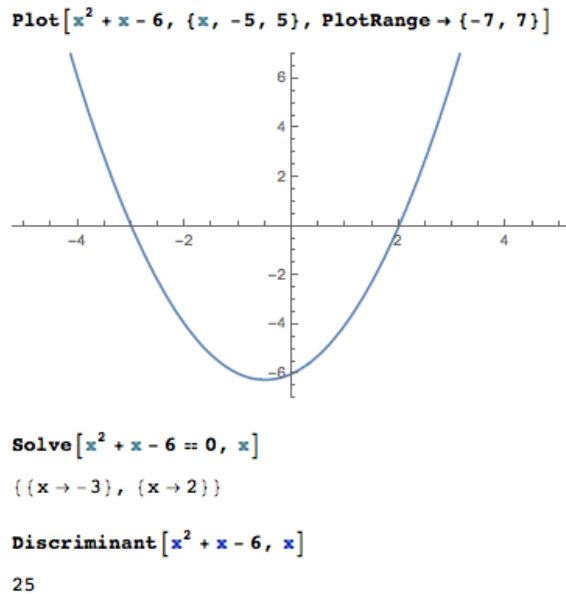


Figure 8. Using the **Plot**, **Solve** and **Discriminant** commands to investigate the number of solutions to a quadratic equation.

### Example 3

Using *Mathematica* at the beginning of a new topic allows the teacher to initiate discussion and introduce ideas about an application context even if students have not yet acquired fluency with the corresponding by hand skills.

For example, when introducing quadratic relationships, one could show a video of Robbie Knieval jumping the Grand Canyon. Make up a character such as dare devil Dave and give students the rule of a quadratic function representing the path of his jump, such as  $h = -\frac{11}{800}x^2 + \frac{11}{10}x + 8$ , where  $h$  metres represents the height of the motorcycle above ground level at a horizontal distance  $x$  metres from the launching point. A diagram such as the one shown in Figure 9 is also useful to give the students.

Students can **Plot** the graph of this relationship using *Mathematica* and discuss different features of the graph. How high was dare devil Dave when he first jumped? What was the maximum height he reached? How long was he in the air for? Did he make it safely to the other side of the canyon? What parts of the graph are not realistic?

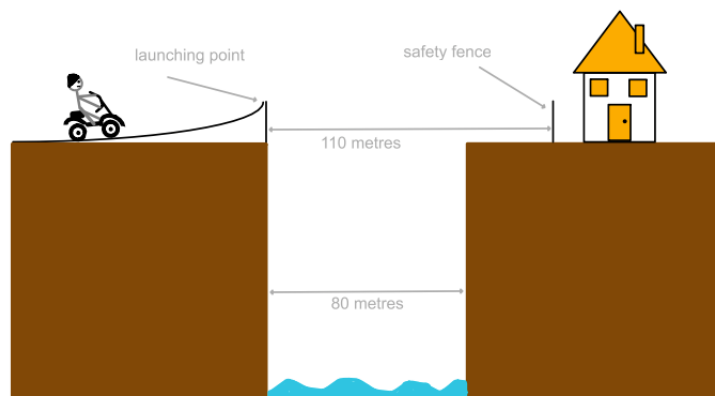


Figure 9. Diagram of dare devil Dave jumping a canyon

Figure 10 also shows other functionality that can be used to help answer some of these questions.

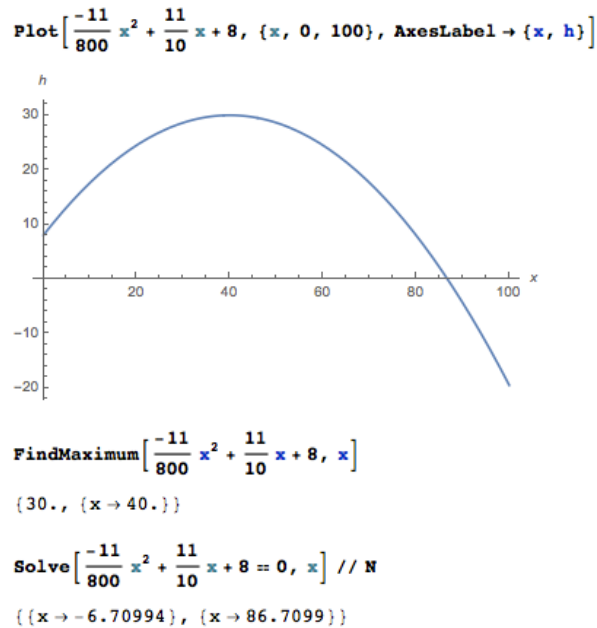


Figure 10. *Mathematica* commands to determine some key features of the graph of a quadratic modeling function

#### Example 4

*Mathematica* can be used to create a table of values using a given relation.

The **Table** command creates a list of ordered pairs as shown in Figure 11 below. In this example the values  $-3 \leq x \leq 3$  are substituted into the equation  $y = x^2$ .

```

Table[{x, x^2}, {x, -3, 3}]

```

```

{{-3, 9}, {-2, 4}, {-1, 1}, {0, 0}, {1, 1}, {2, 4}, {3, 9}}

```

Figure 11. Using *Mathematica* to create a list of ordered pairs

The list of ordered pairs can also be displayed in a table format by using the **TableForm** command as shown in Figure 12.

```

TableForm[{{-3, 9}, {-2, 4}, {-1, 1}, {0, 0}, {1, 1}, {2, 4}, {3, 9}}, TableHeadings -> {None, {"x", "y"}}]

```

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

Figure 12. Using *Mathematica* to create a table of values

If the table was to be produced for input values at half unit intervals then the iterator expression  $\{x, -3, 3, 0.5\}$  for decimal values or  $\{x, -3, 3, \frac{1}{2}\}$  for fraction form would be used. **Table**, **Plot**, **Solve** and **Discriminant** commands can all be employed together to help students develop a deep understanding of the connections between numerical, graphical and symbolic representations of quadratic and other mathematical relationships.

## General Comments and Remarks

*Mathematica* is a useful tool which assists with inquiry based learning, allowing students to explore and investigate beyond their immediate skill level as applicable. It introduces students to programming and provides an effective visual platform for students to model real life applications. These are important processes for STEM education.

*Mathematica* can also be used to create interactive worksheets and assessment tasks. A benefit of this is that the teacher can see the student calculations which normally are not evident on a hand written technology active test or exam.

The Victorian Curriculum now incorporates coding and algorithms in Mathematics [HREF3]. Teachers wishing to use the Wolfram language for this purpose may find further information from the resource, “An Elementary Introduction to the Wolfram Language”, written by Stephen Wolfram. [HREF4].

*Mathematica* also includes **Wolfram Documentation** under the **Help** tab in the tool bar at the top of every notebook, for example entering the query ‘complete graph’ returns the documentation shown in Figure 12.

### CompleteGraph

`CompleteGraph[n]`  
gives the complete graph with  $n$  vertices  $K_n$ .

`CompleteGraph[{n1, n2, ..., nk}]`  
gives the complete  $k$ -partite graph with  $n_1 + n_2 + \dots + n_k$  vertices  $K_{n_1, n_2, \dots, n_k}$ .

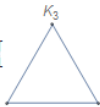
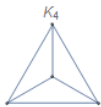
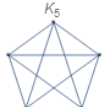

Details and Options

Examples (104)

Basic Examples (4)

The first few complete graphs  $K_i$ :

```
In[1]:= Table[CompleteGraph[i, PlotLabel -> Ki], {i, 3, 6}]
```

Out[1]= {  ,  ,  ,  }

This documentation is itself a notebook, so content such as the input command can be copied from this directly into the notebook one is working on when making a query, then edited and re-evaluated as applicable.

## References

Leigh-Lancaster, D. (2013). Prime Explorations with Mathematica. *Australian Mathematics Teacher*, 69(3), 35-40.

### Websites

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HREF2: Department of Education and Training. (2016). *VicSTEM*. Retrieved September 6<sup>th</sup> 2017 from [http://www.education.vic.gov.au/Documents/about/programs/learningdev/vicstem/STEM\\_EducationState\\_Plan.pdf](http://www.education.vic.gov.au/Documents/about/programs/learningdev/vicstem/STEM_EducationState_Plan.pdf)

HREF3: <http://www.vcaa.vic.edu.au/Pages/foundation10/viccurriculum/maths/maths.aspx>

HREF4: <https://www.wolfram.com/language/elementary-introduction/preface.html>

# SCAFFOLDING STATISTICS UNDERSTANDING IN THE MIDDLE SCHOOL

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*The Reforming Mathematical Futures II Project (RMIT and AAMT) is building a learning and teaching resource to enhance mathematical reasoning in Years 7 to 10. One of the focus content areas is Statistics. Based on the results of assessment items completed by students in the project schools across Australia, tasks and scaffolding strategies are being suggested to improve student understanding across the years and to prepare for further study in Years 11 and 12. This paper includes a learning progression, suggested scaffolding questions for teachers, and examples of potential lessons.*

## **Introduction**

Assessments of various types help us as teachers make judgments about where students are “at” in their understanding of various topics and concepts in the mathematics curriculum. The tricky part is knowing a method of building students’ understanding to higher levels when necessary in ways that will be incorporated permanently and not just repeated or memorised temporarily. Although “telling” is often our last resort, surely our preference is to devise dialogue that highlights the current issue and provides scaffolding that will move the student forward. Unfortunately, there is no fool-proof method of doing this because each student is likely to be different, requiring a different strategy. We teachers hence need a rich treasury of pedagogical content knowledge that can be adapted to meet each situation.

One of the aims of the Reforming Mathematical Futures II project (RMFII, Siemon, 2016) is to extend work done previously on multiplicative thinking to algebra, geometry, and statistics. Based on student assessments and Rasch analysis, learning progressions are being proposed with descriptions of levels of understanding. These are then being used to assist teachers in interpreting outcomes from the initial assessments and planning learning sequences that will move students to higher levels. Recognising that a class is likely to include students at several different levels adds to the teacher’s load in helping the students currently performing at lower levels. One goal is for teachers to develop the ability not only to have dialogue with individual students, but also to devise class discussions that will involve other students in assisting with the scaffolding.

The first part of this paper provides a description of the characteristics identified by previous research associated with six levels of a Learning Progression for Statistics (e.g., Watson & Callingham, 2003). The second part makes some suggestions of possible teacher questioning that are hypothesised to help individual students move to higher levels. To employ these, teachers need to have the content knowledge of the various parts of the statistics curriculum as well as know their individual students and what pedagogical approaches may be successful. The third part models hypothetical classroom scenarios that exemplify what could happen in the classroom as part of discussion with a group of students.

## **Part 1: A Learning Progression for Statistics**

Table 1 outlines the levels of understanding typical of six levels or stages associated with statistics across the school years. Although the first two levels are predominantly found in primary classrooms, some students will arrive in high school with these beliefs or lack of procedural skill. These two levels are combined in the table.

Table 1

*Characteristics of Levels of a Learning Progression for Statistics (Watson, 2006, p. 253-4)*

---

**Levels 1 & 2: Idiosyncratic/Informal**  
*Context:* engagement personal, colloquial or informal  
*Sampling:* inappropriate beliefs or single elements employed  
*Representation:* basic graph and table reading; basic calculations from values observed  
*Average:* single or colloquial terms used  
*Chance:* inappropriate or colloquial interpretation, “anything can happen”  
*Inference:* imaginative explanations or single, non-central issues considered  
*Variation:* difference only for graphs, rigid predictions in chance settings  
*Math/Stat Skills:* one-to-one counting, select largest number, addition, subtraction

---

**Level 3: Inconsistent**  
*Context:* engagement selective or inconsistent  
*Sampling:* focus on inappropriate features  
*Representation:* interpretation of graphical details rather than context in graphs  
*Average:* colloquial interpretation on recognition of need for a formula  
*Chance:* limited interpretation of percent, conjunction, and conditional chance  
*Inference:* mainly non-central issues  
*Variation:* inappropriate attempts  
*Math/Stat Skills:* little change, qualitative chance statements

---

**Level 4: Consistent non-critical**  
*Context:* engagement often appropriate but non-critical  
*Sampling:* multiple elements but inconsistent  
*Representation:* partial recognition of context  
*Average:* straightforward application of mean and median  
*Chance:* mixed success depending on context  
*Inference:* inconsistent acknowledgement of central issues  
*Variation:* success in chance settings  
*Math/Stat Skills:* mean, simple probability, graph characteristics

---

**Level 5: Critical**  
*Context:* critical engagement  
*Sampling:* critical thinking in familiar contexts  
*Representation:* representation of bivariate association in context  
*Average:* consolidation of mean and median  
*Chance:* success on conditional tasks  
*Inference:* little change  
*Variation:* unsolicited acknowledgement in chance and graphs  
*Math/Stat Skills:* little change

---

**Level 6: Critical mathematical**  
*Context:* critical engagement including proportional reasoning  
*Sampling:* critical thinking in less familiar, subtle contexts  
*Representation:* summaries and rate calculations in context  
*Average:* recognition of biasing effect of outliers  
*Chance:* success when more sophisticated mathematics required  
*Inference:* appreciation of subtleties of uncertainty and cause-effect  
*Variation:* no change  
*Math/Stat Skills:* proportional reasoning, rates, multiplication principle for independent events

---

It is clear that skills and understanding from elsewhere in the mathematics curriculum are required to be successful with statistical investigations. These include arithmetic procedures, measurement skills and understanding, and proportional reasoning related to fractions, decimals, ratios, and percentages.

## Part 2: Scaffolding Improvement

A few illustrations are provided in this section related to the descriptions of levels in Table 1. These are imagined to be operating in a one-on-one situation with students rather than with an entire class, but it is likely they can be adapted for use with several students or even a class. In chance situations for example, it is often appropriate to ask students to test their suggestions in some manner. As the middle school curriculum moves into theoretical probability from intuitive contextual considerations of chance, using repeated trials to confirm theoretical claims is often a reinforcing initial step. The idea of “fair” is useful in talking about both chance and sampling. It leads to the belief that every person in the population or every outcome from a chance device has the same chance of being chosen or occurring. Later when events are introduced that are not equally likely to occur, care must be taken with the language use, and focussing on the context from which the probabilities arise can be helpful.

Often as understanding is developing partial responses are given to questions. Rather than just accepting these or moving on to the next student, it is possible to ask for more details, with “how”, “why”, and “what” questions. For data sets that the students collect themselves, we can have open discussion about which is the best way to describe the typical value in the data set: mean, median, or mode (reviewing their definitions). “What is the relationship among the three measures? Do they reinforce each other or not? How?” “Maybe a range of values describes the typical value better.” Such discussion is useful with students at several levels as they can be challenged to extend their current understanding. Table 2 gives some possible specific follow-up questions for students’ answers or comments at various levels.

Table 2  
*Examples of Teacher Scaffolding*

Level	Response from student	Possible questions or suggested actions
1,2	Answering questions on likelihood of chance events: “4 is my lucky number” or “the six dots on a die are heavier so it will be on the bottom”	My lucky number is 2, so how can we tell which one is really lucky? What is a way to test your theory about the dots on a die?
2	Answering questions on likelihood of chance events: “anything can happen”	Okay, so can you give me an example of one thing that might happen? How likely is it to happen? Compared to [an alternative outcome] do you think it has more or less chance?
3	Providing non-quantitative answers to chance questions that can be solved with frequencies: “It’s more likely to be blue” or “there are more boys than girls in the class”	What do you mean by ‘more’? How much/many more? How do you know? Can you be more specific and show me how you decided ‘more’?
1,2	Choosing a sample: “pick those who have finished their work” or “you have to pick everyone”	Would that be a fair sample? How do we get a sample that represents everyone? What can we do if the population is too large to pick everyone, like the whole school or the whole city or the whole country?
2,3	Graphing: reading individual values from a graph is correct but limited, and often labels are missing	So what are some of the nearby values? What do they tell you about the story the graph is telling? What do the labels on the graph tell you? Can you put a label on that axis so I’m sure about what the data are measuring?
3	Incomplete definitions for sampling: “a sample is a test” or “a sample is a little bit of something”	Why else is a sample going to help answer your question? What is the relationship between the data values in a random sample?”
3	Incomplete justifications about reasons for averaging: “my method is quick and easy to do”	But what does it tell you about the data and what is typical in the whole data set?
4	Believes all chance outcomes are equally likely: “when tossing two coins, HH, HT, and TT are equally likely” or “when	Can we draw a tree diagram to model the possibilities, starting with the first coin/die?

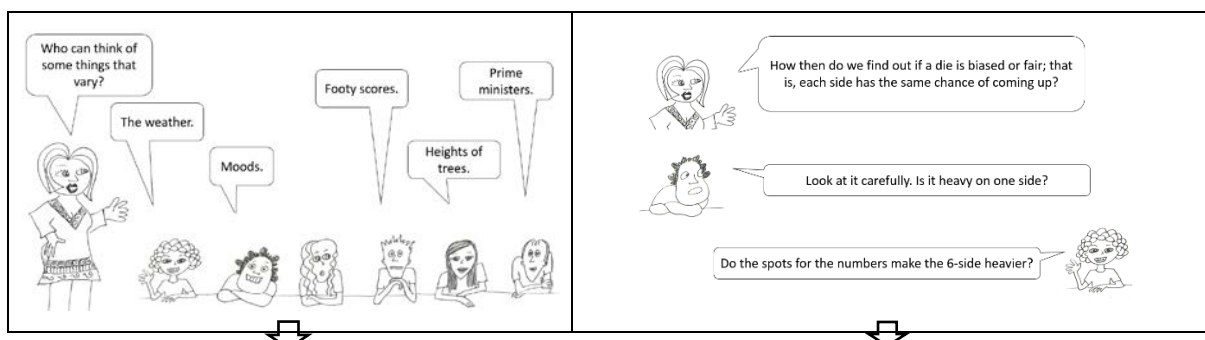
	<i>tossing and summing the outcomes of 2 dice, all numbers 2 to 12 are equally likely”</i>	Let’s do many trials in the classroom and see what happens.
5	Considering 2-way table problems: Decisions made without considering all information available	Do we need to consider all of the cells in the table? What does the information in the row and column totals give us?

By the time students reach Level 4, they possess most of the basic skills but not the critical thinking ability to rethink and question whether they have covered all of the implications of their decisions and whether other possibilities for analysis exist. Developing critical thinking is likely to be supported by critiquing the work of others, either their classmates or claims that are made in the media. Having led the class through questioning a claim in the media based on a phone-in survey or a misleading graph, these experiences can be used as reminders when students are asked to review their own work. In all situations it is a matter of constantly challenging students to think more deeply about their analyses. Statistical problem solving is likely not to have a definitive answer that students *know* is correct, like often occurs elsewhere in mathematics. Helping them to use the language of uncertainty rather than the language of proof is a goal in decision making.

### Part 3: Modelling a Classroom

In seeking to assist teachers to visualise what might take place in the classroom, the RMFII project is creating several scenarios of classroom lessons or segments discussing particular difficult problems. These scenarios are intended to illustrate the range of responses to be expected from students across the Learning Progression and how responses at higher levels are elicited by the teacher or volunteered by other students. Being produced in PowerPoint or video format the intention is for a teacher or a group of teachers to stop the presentation at various points and think about and/or discuss how they would expect students to respond next or what they would plan to say or ask to raise the level of response.

In the first scenario the purpose of the lesson is to expand students’ appreciation for two of the Big Ideas of Statistics: Variation and Expectation. These are integral to the AAMT’s *Top Drawer Teachers: Statistics* site <[topdrawer.aamt.edu.au/statistics](http://topdrawer.aamt.edu.au/statistics)> and in the lesson are used to help students appreciate the links between chance and data. The class begins with an introduction to the two ideas by exploring students’ understanding. Introducing a dice scenario, the teacher uses students’ initial responses to questions on expectations of outcomes to lead into predicting and then carrying out 30 trials of individual 6-sided dice. The experience of variation, often unexpected in the trials, leads to combining trials, and comparing the variation in trials of different numbers of tosses. This requires a recognition of the need for percentage rather than frequency to summarise results, and leads to some surprising numerical results. Two short sequences of three slides are shown in Figure 1, where student responses have been taken from actual student responses in research projects. A similar approach is described in Watson and English (2015).



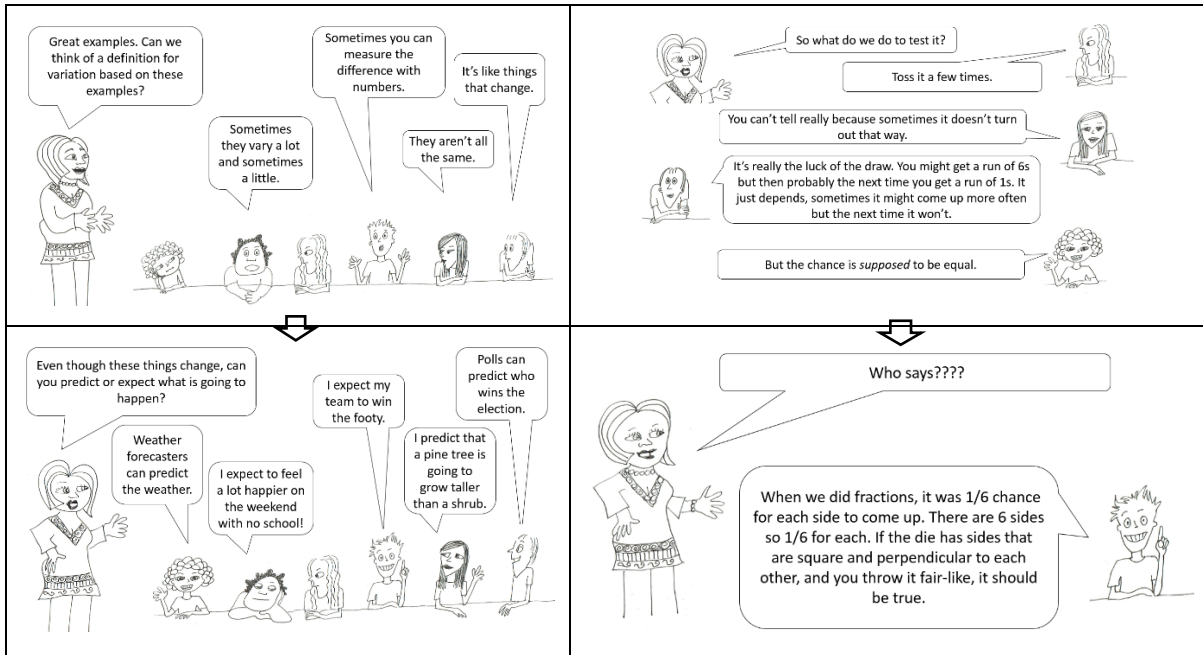


Figure 1. Six slides from a classroom scenario considering variation and expectation when tossing a die.

A different scenario, based on the weather, and again using student responses from research for the initial part of the “lesson”, is used to add Distribution to the other two Big Ideas of Variation and Expectation. In this case the teacher reviews the definitions and asks the students for examples of the Ideas. The context of the average daily maximum temperature in Hobart for a year is then introduced. Students are asked how the average value ( $17^{\circ}\text{C}$ ) might have been found and how they would describe Hobart’s weather to an exchange student. They are then asked to produce representations of the data set, which are shown and discussed. Moving on, the teacher introduces the possibility of comparing the average monthly maximum temperatures of Australia’s eight capital cities. This leads to considering distributions for Hobart and Canberra in the Summer and Winter, with characteristics of the distributions used to make comparisons. Figure 2 shows the description of Hobart’s weather and Figure 3 shows several student representations.

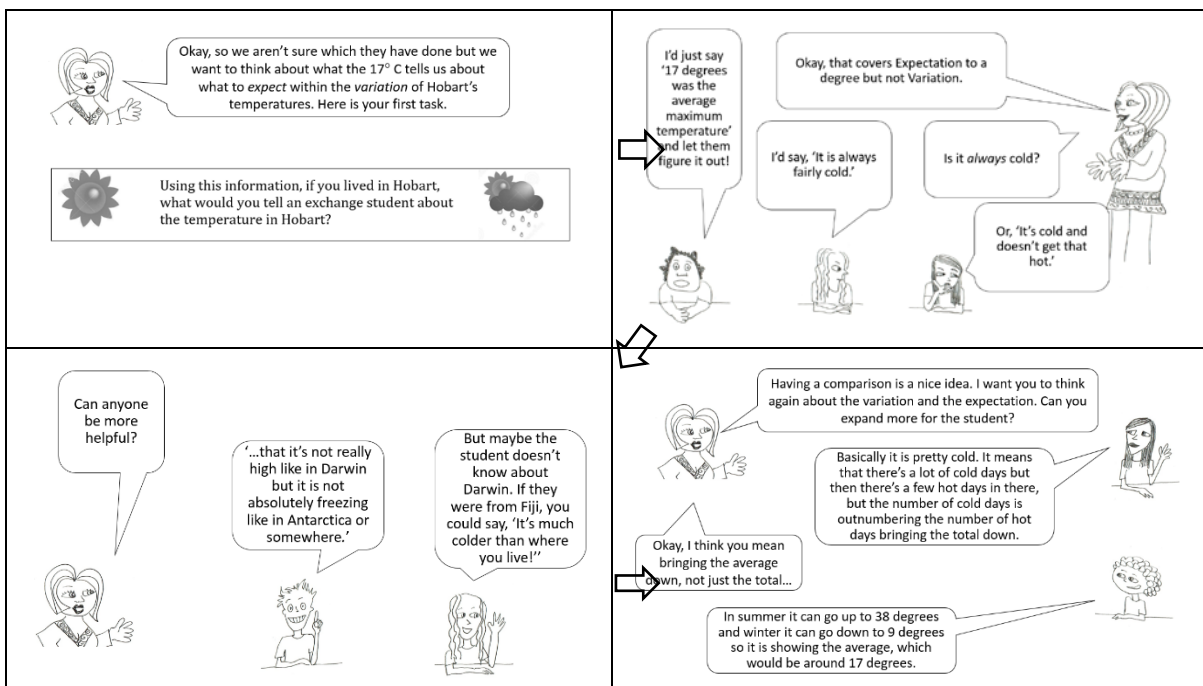




Figure 2. Describing Hobart's weather.

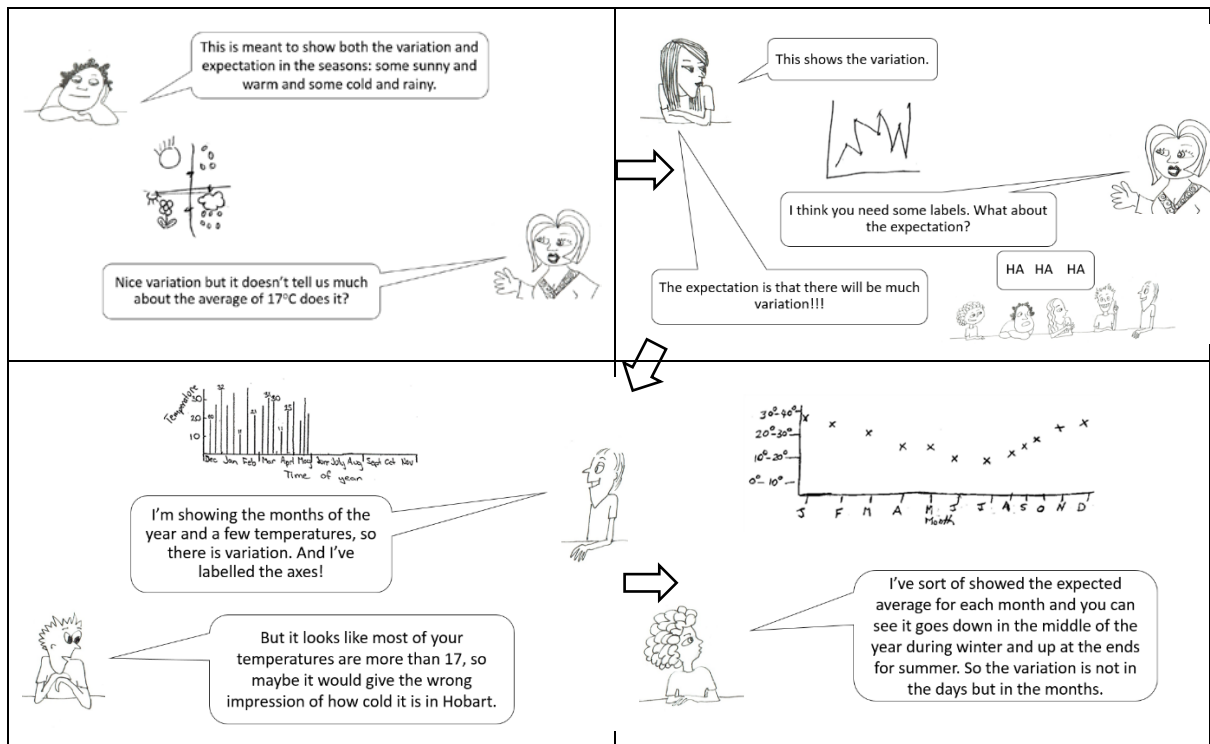


Figure 3. Student representation for a year's maximum temperatures in Hobart.

At the end of this lesson, rather complex comparisons of 54 years of data are used to discuss different levels of variation in data sets. This is an extension and the examples are created to show the potential for students to make comparative statements.

For the third example, a single problem requiring proportional reasoning is used to demonstrate the levels of response likely in a high school classroom. In the scenario different levels of response are suggested by different students and the dialogue has students included in the scaffolding as well as the teacher. If the teacher knows the students well this is a good strategy to use in avoiding being the only voice of knowledge in the classroom. All initial responses of the students were observed in research studies, illustrating the complexity of dealing with two-way tables and the proportional reasoning involved. Figure 4 introduces the problem and also shows some of the issues in linking context to the mathematics required to solve a problem. The second slide in the scenario, where the teacher explains the problem and asks the students to write a solution, is not included to allow for more student responses and interaction to be shown. She says, "As you can see data are given in the table for a hypothetical situation involving Smoking and Lung Disease. The data can hence be interpreted in this case as frequencies to represent the probability of contracting Lung Disease for those who smoke or do not. Looking at the frequencies we are asked to decide if lung disease depends on smoking or whether they are independent for this sample ... Be sure to explain your answers. When you hand me your answers I'll select some for us to discuss."

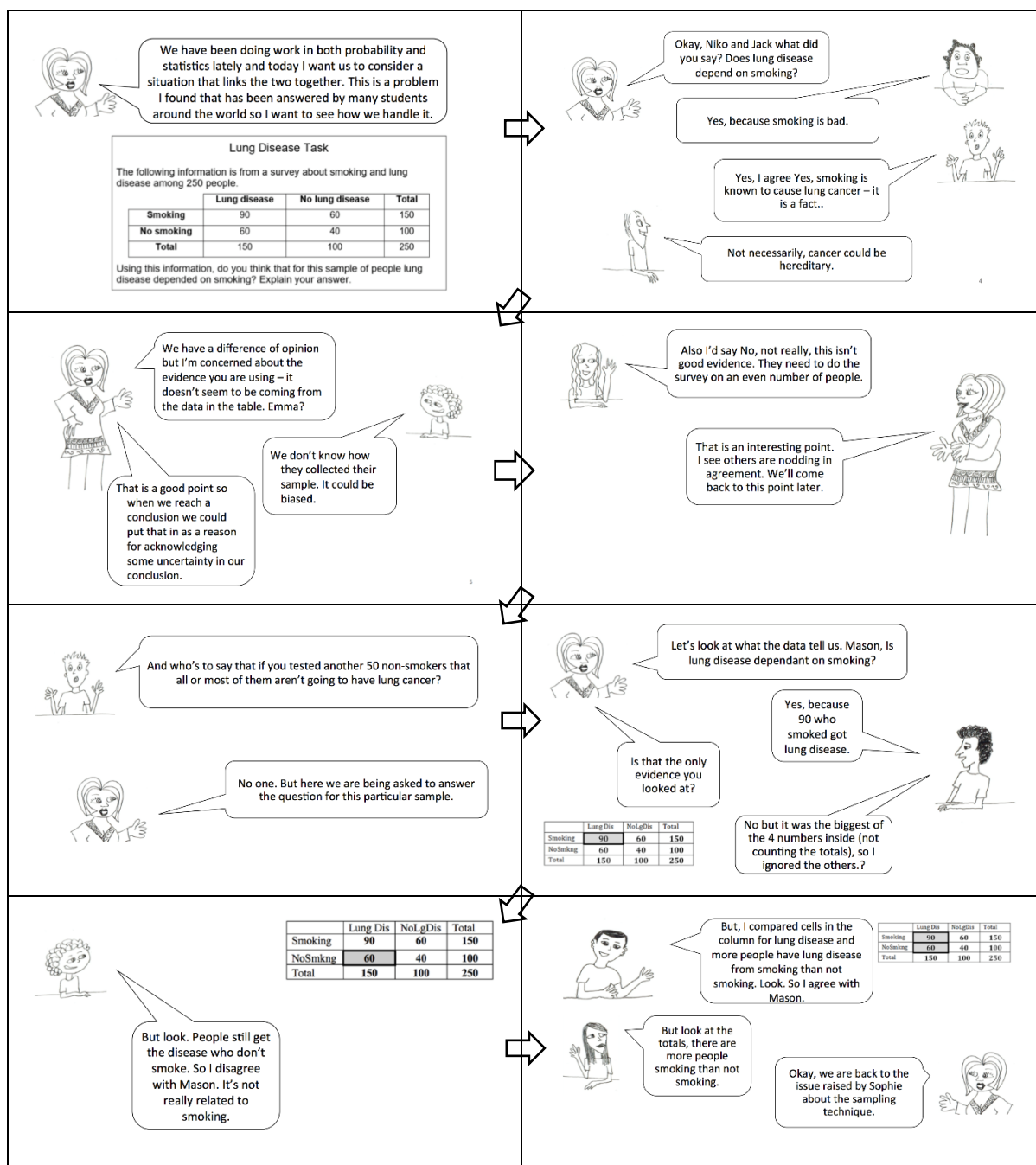


Figure 4. Beginning of a discussion on a two-way table problem.

The slides shown in Figure 4 begin to show the problem of not taking into account all of the cells in a two-way table for two categorical variables. A complete discussion of the issue for students and teachers is found in Watson and Callingham (2014).

## Conclusion

The idea of using classroom conversations to illustrate students' levels of response and how they can be used by teachers to scaffold development is further explored in Watson (in press) with scripts for scenarios (i) developing understanding of variability, (ii) describing distributions, (iii) using random sampling for decisions about a population, (iv) drawing comparative inferences about two populations, (v) investigating a probability model, and (vi) investigating association in bivariate data. Again these can be used as the basis for teacher collaborative discussion or individual study. Other background on middle school students' reasoning about probability is found in Watson (2005, 2016). Developing the

pedagogical content knowledge to scaffold a varied class of learners is a challenge that never finishes for all of us as teachers.

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**Maths e<sup>x</sup>plosion**

**Summary Papers**

# WHY WE SHOULD LOWER THE AVERAGE MARKS OF FEMALES IN STEM

**Felicity Furey**

*Machinam*

## **Introduction**

Only 11% of today's engineers are women. Imagine what our world might look like if that balance was fifty-fifty. How might our cities, school or hospitals be different?

90% of female engineering students have an ATAR 99, dramatically higher than their male counterparts. It starts with the perception that females need to be at the top of their class to select senior maths in high school.

We need to lower the average marks of females in STEM, not by lowering the top scores of our females, but by getting more females with lower scores to join their male counterparts.

At the risk of giving away the punchline too early, this seemingly controversial approach to increasing gender diversity in STEM subjects is not by lowering the top scores of our females, but by getting more females with lower scores to join their male counterparts.

## **Strategies for Increasing Gender Diversity**

Esentially, there are five key strategies for increasing gender diversity. These are: 1) Authentic learning, 2) Language and communication, 3) Participation and collaboration, 4) Role models, and 5) Relevance and meaning.

### **Authentic Learning**

One of the current buzz words in education is *authentic learning*. Authentic learning is an instructional approach that allows students to explore, discuss, and meaningfully construct concepts and relationships in contexts that involve real-world problems and projects that are relevant to the learner.

### **Language and Communication**

Language and communication refers to both: the way we talk to females about STEM, and the way we talk to females about themselves and their abilities.

- The way that we talk to females about STEM

Social science tells us that females are more likely to create and articulate their self-identity using adjectives, while boys are more likely to talk about themselves in terms of what they do, so using verbs.

If we think about how we usually hear STEM careers described, most of what we hear is centered around what scientists and engineers 'DO', so using verbs – like: build structures, write code, research genes, analyse data. And we rarely use adjectives to describe the attributes that are required of people in these jobs e.g. creative, organized, self-motivated, adaptable.

This means that females are less likely to recognise themselves in these professions.

- The way that we talk to females about themselves and their abilities

Student's mindsets - how they perceive their abilities - play a key role in their motivation and achievement. So students who believe their intelligence can be developed (a growth mindset) tend to outperform those who believe their intelligence is fixed (a fixed mindset).

There are a number of reasons that a fixed mindset about maths and science is more prevalent among females than males, one of which is called stereotype threat. Stereotype threat happens when members of a group perform according to stereotypes about them – for example, that females are just not as good at maths as boys.

The good news is that if we teach students about growth mindset, then they are less likely to be influenced by stereotype threat. Changing the way females see themselves and their performance in maths and science is one key to building diversity in STEM.

## **Participation and Collaboration**

In many cases, females are energised by the social part of STEM – working and learning together. We can encourage this by providing opportunities for small group work, and encouraging students to talk about their ideas and consider all possibilities before digging in and having a go.

## **Role Models**

Role models are an important aspect of demonstrating STEM for a variety of reasons, including to expand young people's view of the opportunities that are available out of different types of career pathways. But for many females, the most important thing about the role model element is about fitting in – seeing that these STEM pathways are open to people who are just like them.

Contrary to popular assumption, females don't necessarily need female role models to inspire them about STEM. Yet, this is often where the efforts of various programs lie. Evidence shows, that while female role models are critical in retaining women who are already pursuing STEM fields, when it comes to attracting them, male role models can be just as successful at influencing females to pursue STEM.

## **Relevance and Meaning**

### **Relevant to me, personally**

Research shows that females are more likely to consider studying a subject beyond age 16 if:

- They see that the subject keeps their options open
- They can see themselves working in that area
- They consider that they will 'fit in' and be working with people like them.

### **Relevant to the Real-world**

Providing real-world context in maths and science can be highly motivating and interesting. For primary school students, this could look like embedding maths into fantasy contexts, like saving the planet from an alien invasion, or searching for buried treasure. Whereas, for middle school students, linking what they are learning in maths and science class to a real-life context is more effective. Linking what students are learning in high school mathematics classes to the real world and future careers. It's answering that question that so many students have about maths 'Why do I need to learn this?'

### **Meaningful (Makes a Difference)**

Specific values like money, power, altruism, and family focus tend to be associated with specific occupations. Starting in the teenage years and moving into adulthood, social cues mean that a pattern of differences in values begins to emerge between genders. Where males, generally appear to value things like: achievement, challenge, and risk taking; females, generally, appear to emphasise altruism, interpersonal skills, family time, and knowledge development.

This is relevant because STEM fields are often perceived to encompass those values that are more broadly associated with masculinity: achievement, challenge, and independence for example. Females

and women are more likely to pursue careers in STEM when they can see that the work they do makes a positive social impact.

### **Increasing the Number, Lowering the Average**

So, how do you lower the average marks of females in STEM?

Not by lowering the top scores of our females, but by getting more females with lower scores to join their male counterparts.

# MATHS - NOT FOR PEOPLE LIKE ME

**Dr Jillian Kenny**

*Machinam*

## **Introduction**

Even though girls and boys sit side by side in classrooms all across the country, women are still much less likely to choose careers in STEM than men. The accepted response to these facts is that if only we can enthuse / inspire / encourage more girls to enter STEM fields, then the disparity will disappear.

This interactive workshop explores another avenue - the language we use to talk about STEM - and the impact it has on how people relate to the field. Attendees will work through two teaching resources that can be run with students to help engage and inspire girls about careers in STEM.

## **How We Talk to Girls About STEM**

Social science tells us that girls are more likely to create and articulate their self-identity using adjectives, while boys are more likely to talk about themselves in terms of what they do, so using verbs.

If we think about how we usually hear STEM careers described, most of what we hear is centered around what scientists and engineers 'DO', so using verbs – like: build structures, write code, research genes, analyse data. And we rarely use adjectives to describe the attributes that are required of people in these jobs e.g. creative, organized, self-motivated, adaptable.

This means that girls are less likely to recognise themselves in these professions. So, there is a conflict between their self-identity, or who they consider themselves to be, and their perception of a STEM-identity, who people in STEM fields are. This leads them to conclude that STEM is '*not for people like me*'.

So, the language we use to talk about STEM has a large impact on how girls relate to it. We need to use adjectives to describe the sort of people – their aptitudes – who work in STEM, in addition to explaining what scientists and engineers 'do', using verbs.

The first half of this workshop will involve everyone taking part in an activity that demonstrates the types of adjectives that are appropriate when talking to girls about STEM. The activity can also be used in your own classrooms to prompt girls to discover these elements themselves.

## **How We Talk to Girls About Themselves and Their Abilities**

This second concept around language is called growth mindset. Student's mindsets – how they perceive their abilities – play a key role in their motivation and achievement. So, students who believe their intelligence can be developed (a growth mindset) tend to outperform those who believe their intelligence is fixed (a fixed mindset).

Essentially, people with a fixed mindset believe their basic qualities like intelligence and talent are fixed traits. They also believe that talent alone creates success - without effort.

Alternatively, people with growth mindsets believe their abilities can be developed - that brains and talent are just the starting point.

Many teachers will have heard one of their students say something like, '*I'm just not a maths person*'. A study by the UK National Science Foundation in 2003 showed that in grades 4, 8, and 12, females were less likely than their male counterparts to agree with the statements, "I am good at maths" and "I am good at science". Ultimately, these viewpoints matter because if girls don't believe they are capable, they are less likely to succeed. This is an example of a fixed mindset.



For a variety of reasons, a fixed mindset about maths and science is more prevalent among girls than boys. So, changing the way girls see themselves and their performance in maths and science is one key to building diversity in STEM.

The second half of this workshop involves an interactive activity that explores the strategies to instill a growth mindset foundation among all students – girls and boys – in the classroom. Once this foundation is set, everything comes back to the language we use to communicate feedback, and the role of effort in the learning process.

# THE SENIOR MATHS CLASS CAN BE FUN TOO

**Lorna McClory**

*Staughton College, Melton South, VIC*

*Often, teachers and students find it hard to have fun in the senior maths class. The restrictions, both in time and content of the VCE course can often lead to the feeling that content must be plowed through. However, the teaching strategies used to make maths engaging in the lower years can still be applied to the senior mathematics class.*

## **Fun in the VCE Mathematics Class**

In my time teaching VCE Mathematics, I have often felt under considerable pressure to 'just get through the course'. This can often lead to very dry and boring 'chalk and talk' style lessons, resulting in lower engagement of students and even more questions about the relevance of what they are learning to the real world. While the challenges of completing the curriculum remain, I have become more focused on using projects and 'fun' lessons to engage students while deepening their learning.

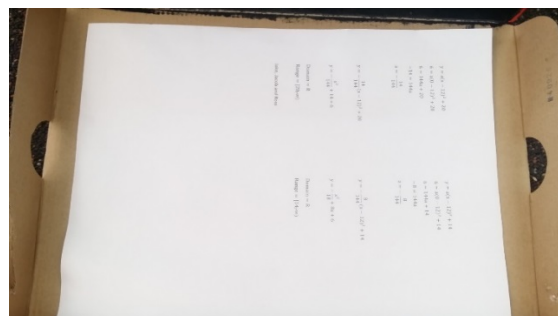
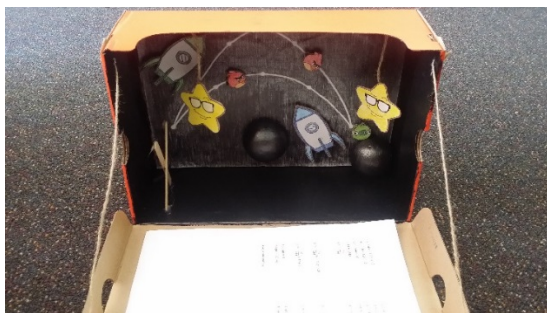
A challenge faced by many schools, including mine, is that students can arrive in your Mathematical Methods class missing many pieces of what should be prior knowledge. Our challenge is how to engage these students and fill that gaps as quickly as possible to then allow for teaching of the Methods course. Over the last few years I have incorporated a few alternatives to PowerPoints and teacher directed presentations.

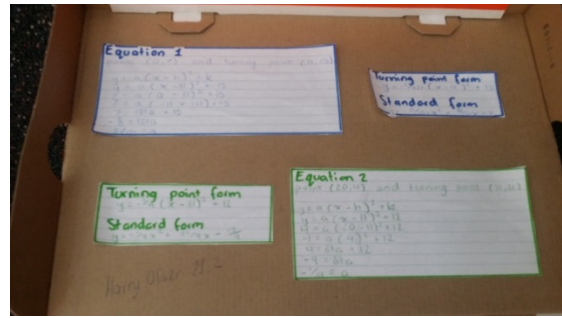
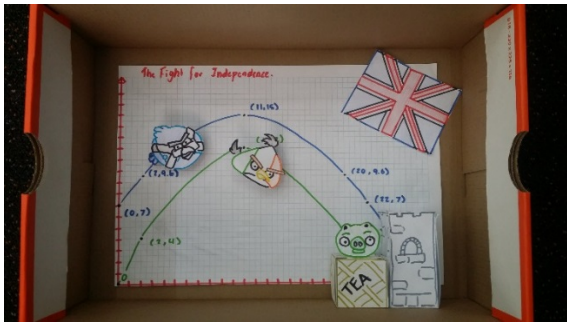
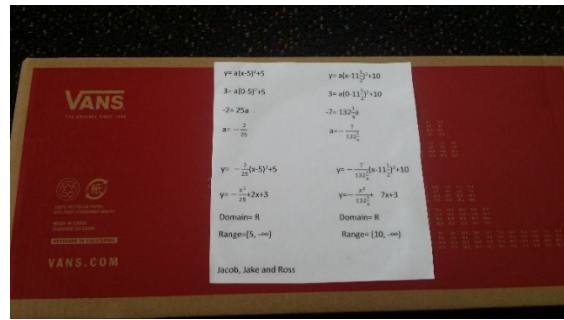
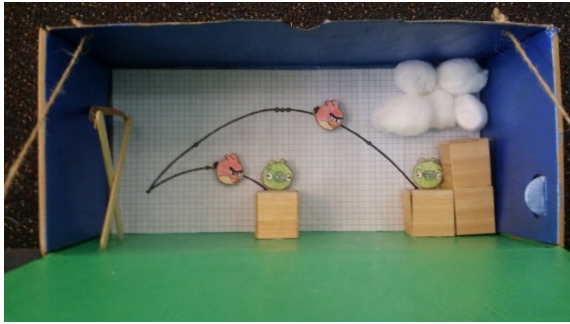
There are many great resources available online and a lot of my inspiration comes from teacher bloggers and Pinterest. Mathematics is a worldwide subject with few differences in the content taught. It is wise to use free resources from others rather than reinvent the wheel. As always, you can put your own spin on things to suit your own students.

Below are two examples of lessons I have delivered while teaching Mathematical Methods

### **Angry Bird Project - Quadratics**

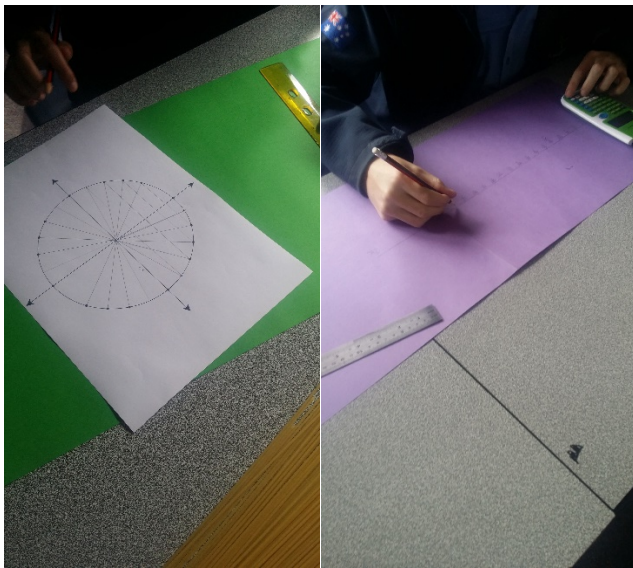
The Angry Bird Project involves the construction of an Angry Bird diorama with calculations around the flight path used by the birds. Students have the option of completing this individually or in a group of up to 3 people. While the focus is on the mathematics behind the flight paths, many students relished the opportunity to display their artistic skills.





### Spaghetti Graphs – Circular Functions

Using the unit circle and the ratio of *sine* and *cosine*, students are able to construct the graphs of each by measuring the length of the spaghetti for each ratio at 15° intervals and use this to sketch the basic shape of the curve for  $0 \leq x \leq 2\pi$ .



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Websites

<http://secondarymissrudolph.blogspot.com.au/2013/10/angry-birds-levels-shoobox-version.html>

# ANGRY BIRDS GRAPHING PROJECT

<http://secondarymissrudolph.blogspot.com.au/2013/10/angry-birds-levels-shoebox-version.html>

- You have been given the task of designing your own Angry Bird level(s).
- You have the option of working alone or in a group of two or three.
- The level should be illustrated/displayed colourfully and clearly to show the birds, pigs, blocks and obstacles.
- Flight paths will be drawn from the sling shot to the structure and then you will explain how the structure will fall.
- Each of the paths will model a QUADRATIC.
- The equation for each path will be written in both TURNING POINT and STANDARD form.
- A reasonable domain and range will be stated.

The project will be graded based upon the following criteria:

- **GROUP SIZE**
  - One Person: 2 flight paths
  - Two People: 3 flight paths
  - Three People: 2 levels (2 dioramas); 2 flight paths per level
- **TURNING IN**
  - Shoebox diorama of the level.
  - The “back” of the diorama box should feature the flight path (quadratic) of the bird along with three points clearly identified. (Graph paper can be used to assist with drawing the accurate path on the back)
  - Physical representations of objects and birds should be utilised
  - Colour and Uniqueness should be present!
  - Neatly written or typed work to show how you found the equations final equations (both TURNING POINT and STANDARD) and **domain** and **range** clearly identified.
  - Detailed explanation of how the structure will fall apart when the bird hits.



**PROJECT DUE** \_\_\_\_\_

# SPORTS GAMBLING

## Robert Money

*The Mathematical Association of Victoria*

Gambling is heavily marketed around sporting events and is readily available on-line. For many students it presents the first temptation to start gambling themselves – illegally if they are under 18. Research shows that many people who end up as problem gamblers started gambling in their teens.

This is one good reason why MAV has developed units of work on gambling for Years 9/10 (in 2015) and for VCAL (in 2016). The other reason is that gambling provides a real-world context through which a lot of mathematics and statistics can be learnt.

Here are a few excerpts from the sports gambling worksheets that are now available from MAV.

### Situation 1

Students simulate betting \$1 each on the toss of a coin and then decide that a fair payout for the winner should be \$2. After discussion they start the worksheet:-

1. A 'bookie' collects \$1 from each of two punters who are prepared to bet on the toss of a coin. If the bookie makes no profit, how much should be paid to the winner of the coin toss?
2. What are the fair payouts for \$1 bets on a win or a loss in a match between two equally good teams? Be prepared to explain your answer.
3. How much would the bookie expect to win if 1000 punters each bet \$1 on this game but the payouts were reduced to \$1.80? Be prepared to explain your answer.
4. I made some mistakes in copying these payouts from last week's newspaper.

Fremantle \$1.16	vs	Carlton \$5.25	Sydney \$1.22	vs	Essendon \$4.35
Adelaide \$1.19	vs	West Coast \$4.75	Richmond \$0.97	vs	GWS \$8.50
Gold Coast \$1.33	vs	St Kilda \$3.35	Nth Melb \$2.20	vs	Geelong \$1.68
Melbourne \$1.65	vs	Brisbane \$2.25	Hawthorn \$1.11	vs	Western Bulldogs \$6.75
Collingwood \$2.15	vs	Port Adelaide \$2.70			

If these figures were correct:-

- a) There is one game where you would lose money no matter whether the team you backed won or lost. Which team was that, and how much could you lose?
- b) There is one game where you could win money if you bet \$1 on each team. Which game was that, and how much could you win?

### Situation 2

Students simulate betting on win, lose or draw (soccer, hockey etc) in a situation, like the TAB, where the 'house' takes its profit before distributing payouts. Use this screenshot from the Sports Betting spreadsheet to answer the rest of the questions on this worksheet.

### Sports Betting Agency

Change the amounts bet by the punters on win, lose and draw.

Set the unfair payout percentages.

Recalculate to run the race again FN-F9

Place your bets	Win	Lose	Draw	Total	
for win,lose,draw	<b>\$50</b>	<b>\$100</b>	<b>\$50</b>	\$200	
Probabilities	0.250	0.500	0.250		
Fair payouts	\$4.00	\$2.00	\$4.00		
Unfair payouts at <b>75%</b>	\$3.00	\$1.50	\$3.00		Random number 128
Bet to payout ratio 1/payout	<b>0.33</b>	<b>0.67</b>	<b>0.33</b>	1.33	
Result		<b>loss</b>			For a win less than 50
Fair payouts	<b>\$0</b>	<b>\$200</b>	<b>\$0</b>		For a draw greater than 150
Unfair payouts	<b>\$0</b>	<b>\$150</b>	<b>\$0</b>		For in between, a loss
Bookies profit		<b>\$50</b>			

1. Based on the spreadsheet, is the Home Team more likely to win or to lose?
2. The bets are used to calculate the probabilities and the payouts. How did they get the 0.250 (1/4) for the probability of a Draw?
3. How is the fair payout of \$4.00 for a Draw calculated?
4. How is the Unfair Payout of \$3.20 for a Draw calculated?
5. Josie and Lin each started with \$50. Josie bet \$10 on a home team win and Lin bet \$10 on a home team loss. How much money would they each have left at the end of this Match?
6. In the next race the bets placed were \$200 for a win, \$500 for a loss and \$300 for a draw. Under the same arrangements, what will be the 'bookie's' profit?
7. The bookie must win and some of the punters must lose. Say why.

### Situation 3

From earlier work students have learnt that a betting situation is

- '100% fair if the probability of winning is equal to the bet to payout ratio
- Less than 100% fair, calculated by the formula probability x payout

Students then attempt some more challenging worksheet questions:-

- A bookie offers payouts of \$1.20 for a win and \$5 for a loss on Federer's next tennis match.

$$\frac{\textit{bet}}{\textit{payout}}$$

Calculate the sum of the bet to payout ratios for a win and for a loss and explain why these payouts are unfair.

- The betting agency offers Sharks \$2.50 vs Giants \$1.25. Based on this information what is the probability that the Giants will win the match?
- With Sharks \$2.50 vs Giants \$1.25, how much does the betting agency plan to gain from a total of \$10,000 in bets placed on the game?
- These are real figures (from last season): Melbourne \$1.65 vs Brisbane \$2.25.
- What percentage of total bets is the betting agency planning to pay back to the winners?
- Calculate the sum of the bet to payout ratios for the following two games and decide what percentage returns to the punters are planned for.

Match	Home team win	Home team loss	Draw
Sunderland v Manchester United	\$6.00	\$4.00	\$1.57
Manchester City v Newcastle	\$1.30	\$5.50	\$10.00

# WHAT HAPPENS WHEN CHALLENGING TASKS ARE USED IN MIXED ABILITY MIDDLE SCHOOL MATHEMATICS CLASSROOMS?

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*The topics of Decimals and Polygons were taught to two classes by using challenging tasks, rather than the more conventional textbook approach. Students were given a pre-test and a post-test. A comparison between the two classes on the pre- and post-test was made. Prior to teaching through challenging tasks, students were surveyed about their mindset in regards to mathematics and how they think they learn best. They were surveyed again at the completion of the project to see if there were any changes.*

## **Mixed Ability Classes**

As a practiced middle school mathematics teacher and a Mathematics Learning Area Leader, my experience supports the thought that mathematics is perhaps the most resistant discipline to change (Zevenbergen, Mousley & Sullivan, 2001). At the college I teach at, there is debate amongst mathematics teachers as to how we can best cater for individual differences in the learning of mathematics in the middle school (Years 7&8).

The document Numeracy in Practice: teaching, learning and using mathematics (DEECD, 2009) states that catering for a wide range of confidence and mathematical understanding is seen as one of the biggest challenges in teaching Mathematics. The report suggests that different strategies need to be employed so that students of all abilities are catered for. Clarke and Clarke (2008) report that ‘The research evidence is clear that generally any benefits which accrue from ability grouping are only to very high achievers, with a negative impact on average and low-attaining students’ (Clarke & Clarke, 2008 as cited in DEECD, 2009, p.39).

Mixed abilities can be catered for through presenting challenging tasks where students have a different entry and exit level depending on their ability or are able to select a their own task based on their interests. In this project I established a variety of assessments including challenging tasks, concrete materials and a classroom with respectful discussion.

## **The Project**

Two Year 7 Mathematics classes, 7B and 7C were observed in this project. There is a wide range of abilities in both classes. This study will observe the experiences and results of both classes as they do the topics of Polygons, Solids and Transformations and Decimals where both classes will be taught with incorporation of challenging tasks into their curriculum.

Prior to undertaking the challenging tasks approach the students responded to ‘If I could be granted one wish for my maths learning it would be...’

- To have more drive to succeed more
- Do harder Maths and get good at it

- To understand what the teacher says in class
- Sometimes I think one thing and write another
- Not doing as much board writing
- Doing it with a friend so I can study happily
- Doing Maths with a friend in a group
- For things to stick in my brain and not forget information later that has been learnt

This feedback sets the scene for implementation of challenging tasks. Students want to be able to collaborate with one another and they want to have the time to think about their work. Challenging tasks provide the opportunity to do harder mathematics and to pursue answers that require a high level of thinking

### **Polygons Challenging Problems**

In introducing the idea of challenging tasks, some informal challenges had been set as introductory tasks in class. For example, the following problem was written on the board and a discussion followed.

*The four angles of a quadrilateral are labeled A, B, C, D.*

*Angle A is one third of angle B.*

*Angle D is half of angle B.*

*What might be the size of angle C?*

*Draw what your quadrilateral might look like.*

Students readily tackled this task, shared their answers in small groups and with the entire class. They were able to draw their answers on the board and discuss their strategies. There were many correct responses.

The ‘Polygons Challenge’ task was done in a more formal manner, with a worksheet handed out to the class at the start of the lesson and collected at the completion of the exercise. Students were given a short survey after doing this activity.

When the task was initially conducted, the students incorrectly assumed that because there wasn’t a lot of writing or a lengthy list of questions, that they would do the challenge quickly and without much thought. Many of the sheets were returned within 10 minutes, but were not correct. After 20 minutes, there were only 4 fully correct responses from the entire class.

They were engaged during the task, having lots of conversations about what their triangles should look like. They were using words like ‘isosceles’ and discussions around what their angles should add up to. There was a working atmosphere and a general feel that this task was both enjoyable and challenging.

After they completed the activity they were asked ‘When I think about this task I prefer the questions we work on in class to be...’ and their responses are given in Table 1.

Table 1

*Response to ‘When I think about this task I prefer the questions we work on in class to be...’*

	Total
Much harder	4 (9%)
About the same	34 (77%)
Much easier	6 (14%)
Total responses	44

Less than 10% of students across both classes reported that they would like their class work to be ‘much harder’, which is little surprising since no students obtained the correct answer immediately. However, it is pleasing that 77% of students preferred questions to be at ‘about the same’ level of difficulty.



Students were also asked to complete the sentence ‘The best thing about this task was...’ to which they responded

- It was easy to understand
- You still had to consider what each angle could be and work out if all angles added to 180
- You had to use your imagination
- There was more than one way to find the answer
- Drawing and not writing
- It didn’t take long. It was fun
- The difficulty wasn’t too hard or too easy

Students were also asked to complete the sentence ‘The worst thing about this task was...’ to which they responded

- When you had trial and error
- The many failed answers
- I could not understand it at first
- The last one was hard
- Nothing. I thought it was fun

### Polygons Results

To observe how the teaching through challenging tasks affected the learning of students, I selected NAPLAN (National Assessment Program – Literacy and Numeracy) questions from the 2012 and 2013 papers that fitted in with the topic being studied. NAPLAN is an annual assessment for students in Years 3, 5, 7 and 9. The assessments are undertaken nationwide, every year, in the second full week in May. These questions were used because they assess at a national standard. National testing should not drive what we teach, but if we are teaching the same work that is being assessed in NAPLAN, using the questions that have been developed by experts seems to be logical.

The same questions were used on both the pre-test and post-test and the responses recorded. A discussion around the responses to one of these questions follows.

#### NAPLAN 2012 Question 19 Calculator active (ACARA, 2012)

19 Jason drew a shape that had six sides and exactly three lines of symmetry. Which of these could be Jason’s shape?

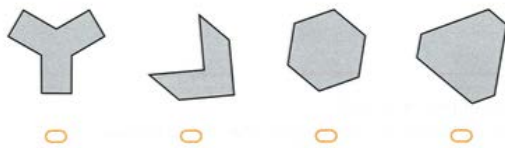


Table 2

*Correct Responses to NAPLAN 2012 Question 19 Calculator pre and post -test*

	Pre-test n=44	Post-test n=45
Total	16 (36%)	32 (71%)

There was improvement in this question from 36% to 71%. To support the learning of concepts connected with questions like this one, students were given a ‘Geogebra’ task where they had to draw lines with specified lines of symmetry. This was an open-ended approach. The most common incorrect

response was where the students chose the shape with 3 lines of symmetry but with 9 sides. Perhaps the hitch was with reading the question correctly rather than with the concept of lines of symmetry.

### Decimals Challenging Problems

The first exercise in textbooks for Year 7 decimals is usually along the lines of ‘What is the place value of the 6 in the number 4.65’. As an alternative to this, the students were posed the task of ‘Write a number that has a 6 in the tenths place’. There are a number of responses they can have to this. The first response was ‘5.6’, then students proceed to give answers that just changed the whole number part at the start, until someone realized they could also give an answer with two decimal places, then three, then students were giving responses with an excessive number of decimal places. By listening to each other students could develop their own correct response. The task developed further by asking ‘I am thinking of a number with a 7 in the hundredths place and a 6 in the units place. What might the number be?’ Again there are infinitely many answers, which was soon realized by the students.

To further consolidate place value, students were asked to write down 10 numbers between 3.01 and 3.1 and we shared this with the class. Communication verbally, rather than in writing was enjoyed by the students and was a response to the opinions given in an earlier survey where students indicated that that did not enjoy the amount of written work in mathematics.

When teaching rounding off of decimals, the usual process is to give the students a long list of numbers and get them to write them to specified number of decimal places. As an alternative to the textbook, I asked the class, ‘A number when rounded gives 5.8. What might the number be?’ Again, there is infinitely many correct answers and as the discussion progressed, a wide variety of suitable responses were given.

### Decimals Results

#### NAPLAN 2012 Question 32 Calculator (ACARA, 2012)

Barney has a bag of \$1 and \$2 coins.

The total mass of the coins is 71.4 grams.

Barney knows that:

- the mass of a \$1 coin is 9 grams and
- the mass of a \$2 coin is 6.6 grams.

What is the smallest mass of exactly \$3 worth of coins?

grams

What is the total value of the coins in the bag?

\$

Table 3

*Correct Responses to NAPLAN 2012 Question 32 Calculator pre and post -test  
First question*

	Pre-test n=41	Post-test n=35
Total	25 (61%)	24 (69%)

Table 4

*Correct Responses to NAPLAN 2012 Question 32 Calculator pre and post -test  
Second question*

	Pre-test n=41	Post-test n=35
Total	6 (15%)	15 (43%)

The first question showed only a small change from 61% responding correctly in the pre test and 69% in the post-test. However, the second question showed a greater improvement, from 15% in the pre-test to 43% in the post-test. The second question was more complex and required a higher level of thinking. More effort and persistence was needed and it may have been the challenging problems method of teaching that helped develop these skills, leading to improvement.

### Mathematics Self-efficacy

Whilst the responses to test questions show some promising results, investigation into the mindset around mathematics is also important. Dweck (2006) asked whether students viewed their intellectual ability as a gift or was it something that could be developed. She found that students who considered themselves to be 'smart' lost motivation when they experienced setbacks. Students who considered that their high results were due to hard work and effort were more able to deal with challenges. She claims that an evolving difference in mathematical achievement is the difference in coping with setbacks and perplexity. The students were asked pre and post-test to respond to the following statements.

- I can get smarter at maths by trying hard
- Learning more than one way to solve a maths problem helps me to understand better
- Tables 5 and 6 present the responses given by the students pre and post-test

Table 5

*Responses to 'I can get smarter at maths by trying hard'*

	Strongly disagree	Disagree	Undecided	Agree	Strongly agree
Pre-test (n=44)	1 (2%)	1 (2%)	4 (9%)	19 (43%)	19 (43%)
Post-test (n= 43)	0 (0%)	0 (0%)	4 (9%)	12 (28%)	27 (63%)

Even at the start of the program, most students believed that that they would experience more success by working hard. This is something that is a constant theme in our class. They are repeatedly reminded that the result of hard work is improved understanding. Initially, 43% strongly agreed that they could get smarter at maths by trying hard and 43% agreed. After the program, 28% agreed and 63% strongly agreed. This illustrates that the challenging tasks approach contributed to the students seeing that if they persist with tasks then they can get higher results.

Table 6

*Responses to 'Learning more than one way to solve a maths problem helps me to understand better'*

	Strongly disagree	Disagree	Undecided	Agree	Strongly agree
Pre-test (n=44)	0 (0%)	5 (11%)	12 (27%)	16 (36%)	11 (25%)

Post-test (n=43)	0 (0%)	1 (2%)	8 (19%)	19 (44%)	15 (35%)
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In completing the challenging tasks, students were able to try different approaches. Traditionally, the teacher would give instruction to the class as a whole, showing only one method. With the challenging task approach they were able to see more than one method or try more than one approach themselves. The data collected showed that prior to the approach 11% disagreed with statement 'Learning more than one way to solve a maths problem helps me to understand better', and afterwards only 2% disagreed with this statement. To further support the use of challenging tasks and trying different methods of solving them, 36% agreed and 25% strongly agreed with the statement prior to the unit and this increased to 44% agreeing and 35% strongly agreeing at the completion of the unit.

## Summary and Conclusion

Students were used to having a long list of repetitive problems with only one correct answer. The challenging tasks often had a number of correct answers. The general response to the tasks was positive with students believing the questions were at the right level of difficulty. They liked that there were some easy questions, they could draw and not write and that there was more than one way to find the answer. What they didn't like was that they got some 'failed' answers and that the 'last one was hard'. None of the students found the task boring or tedious.

To see how their understanding of mathematics changed the students did a number of NAPLAN questions prior to the topic and again after we had completed the unit. Student results improved for all questions, with the most improvement being in the questions that involved a deeper level of thinking and perhaps some trial and error.

Before and after the study, students were surveyed about whether they believed they had fixed mathematical ability or could they improve their understanding by working hard. There was a small shift towards students believing more so after the challenging tasks approach that they could change their mathematical intelligence. There was a slightly larger positive shift in believing that if they try hard they can get smarter at mathematics.

The aim of this study was to observe what happens when challenging tasks are incorporated into a middle school mixed ability mathematics classroom. It was observed that students experienced a more vibrant learning environment with mathematics that was both accessible and challenging. They had discussions, trialed lots of solutions and found multiple methods of solving the problems. They realised that sometimes there is more than one acceptable answer, they became more confident in sharing responses and were able to 'think outside the box' more readily. There was an improvement in the percentage of students that were able to respond correctly to NAPLAN test questions. Surveys indicated that they had more self-belief in their ability and that by working hard they could become 'smarter'.

One of the measures of success is how the study impacts future teaching. Rather than getting students to do 'drill and kill' activity sheets that provide reinforcement of processes rather than understanding, we should be giving them a variety of tasks that are open-ended, challenging and engaging. It is my hope that teachers will re-engage and that the students will become enchanted rather than disillusioned by mathematics.

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# THE BISECTION METHOD AND NEWTON'S METHOD

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In the current VCE Mathematics Study Design (2016 – 2018) for the subject Mathematical Methods, additional algebra material has been added as follows:

*Numerical approximation of roots of simple polynomial functions using the Bisection Method and Numerical approximation of roots of cubic polynomial functions using Newton's Method.*

The purpose of this article is to discuss these two methods and to illustrate their application with examples. In addition, exercises are also provided together with answers.

## The Bisection Method

The Bisection Method is a root-finding method that repeatedly bisects an interval and then selects a subinterval in which the root must lie. Consider the function  $y = f(x)$  which is continuous on the closed interval  $[a, b]$ . If  $f(a)f(b) < 0$ , the function changes sign on the interval  $[a, b]$  and, therefore, has a root in the interval. The bisection method uses this idea in the following way. If  $f(a)f(b) < 0$ , then we compute  $c = \frac{1}{2}(a + b)$  and test whether  $f(a)f(c) < 0$ . If this is so, then  $f(x)$  has a root in  $[a, c]$ . So  $c$  is now reassigned as  $b$  and we start again with the new interval  $[a, b]$  which is now half as large as the original interval. If, on the other hand,  $f(a)f(c) > 0$ , then  $f(c)f(b) < 0$  and  $c$  is now reassigned as  $a$ . In either case a new interval trapping the root has been found. The process can then be repeated until the required level of accuracy has been attained. Figures 1 and 2 illustrate the two cases discussed assuming  $f(a) > 0$  and  $f(b) < 0$ . The bisection method is sometimes referred to as the method of interval halving.

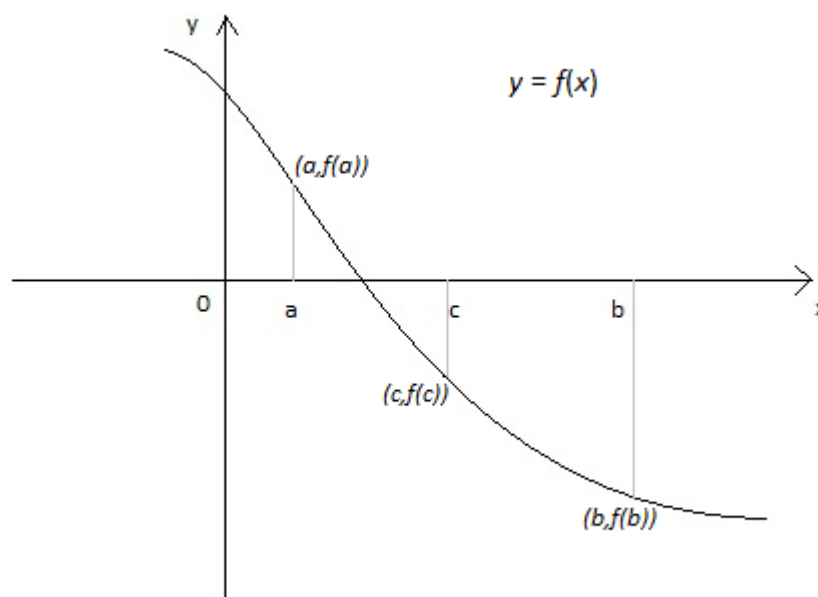


Figure 1: The bisection method selects the left subinterval. The root is to the left of  $x = c$  and  $f(a)f(c) < 0$ .

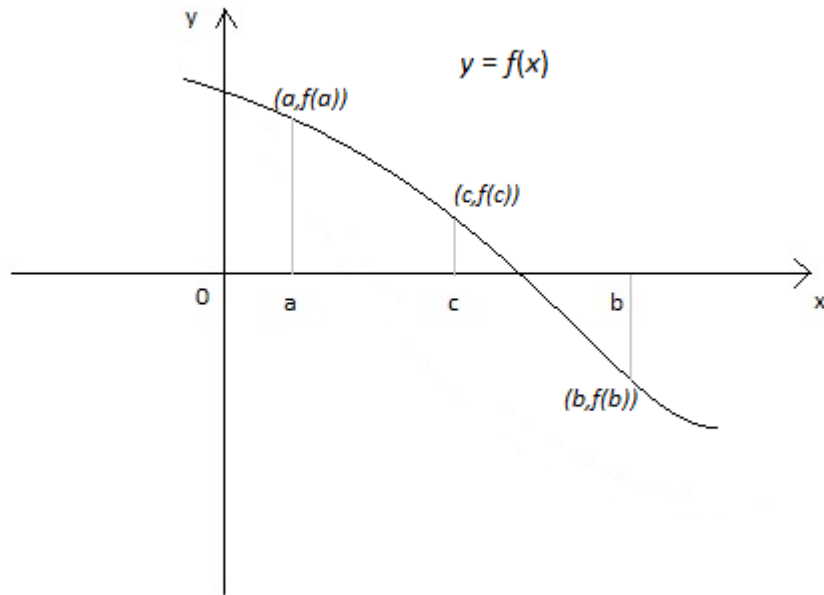


Figure 2: The bisection method selects the right subinterval. The root is to the right of  $x=c$  and  $f(c) > 0$ ,  $f(b) < 0$ .

**EXAMPLE 1.**

Consider the continuous function  $f(x) = x^3 + x - 1$ .

- (a) Evaluate  $f(0)$ .
- (b) Evaluate  $f(1)$ .
- (c) Determine the sign of  $f(0)f(1)$ .
- (d) What conclusion can you draw from (c)?
- (e) Use the bisection method to obtain the root of  $y = f(x)$  to four decimal places.

**SOLUTION TO EXAMPLE 1.**

- (a)  $f(0) = -1$
- (b)  $f(1) = 1$
- (c)  $f(0)f(1) = (-1) \times (1) = -1 < 0$ , i.e., the sign of  $f(0)f(1)$  is negative.
- (d)  $0 < x_{root} < 1$

(e)

$a$	$b$	$\frac{a+b}{2}$	$f(a)$	$f(b)$	$f(\frac{a+b}{2})$
0	1	0.5	-1	1	-0.375
0.5	1	0.75	-0.375	1	0.1719
0.5	0.75	0.625	-0.375	0.1719	-0.1309
0.625	0.75	0.6875	-0.1309	0.1719	0.0125
0.625	0.6875	0.6563	-0.1309	0.0125	-0.0610
0.6563	0.6875	0.6719	-0.0610	0.0125	-0.0248
0.6719	0.6875	0.6797	-0.0248	0.0125	-0.0063
0.6797	0.6875	0.6836	-0.0063	0.0125	0.0031
0.6797	0.6836	0.6817	-0.0063	0.0031	-0.0015

0.6817	0.6836	0.6827	-0.0015	0.0031	0.0009
0.6817	0.6827	0.6822	-0.0015	0.0009	-0.0003
0.6822	0.6827	0.6825	-0.0003	0.0009	0.0004
0.6822	0.6825	0.6824	-0.0003	0.0004	0.0002
0.6822	0.6824	0.6823	-0.0003	0.0002	-0.0001
0.6823	0.6824	0.68335	-0.0001	0.0002	0.0001
0.6823	0.68335	0.68233	-0.0001	0.0001	0.00001
0.6823	0.68233	0.68232	-0.0001	0.00001	-0.00002
0.68233	0.68232	0.68233	0.00001	-0.00002	0.00001
0.68232	0.68233	0.68233	-0.00002	0.00001	0.00001

TABLE 1. After 17 iterations, we can safely take  $x_{root}$  to be 0.6823 to four decimal places.

**EXAMPLE 2.**

Consider the function  $f(x) = x^3 + 10x^2 + 8x - 50$ .

- (a) Evaluate  $f(1)$ .
- (b) Evaluate  $f(2)$ .
- (c) Evaluate  $f(1)f(2)$ .
- (d) What is the sign of  $f(1)f(2)$ ?
- (e) What conclusion can you draw concerning your answer to part (d)?
- (f) Use the bisection method to obtain the positive root of  $y = f(x)$  to four decimal places.

**SOLUTION TO EXAMPLE 2.**

- (a)  $f(1) = -31$
- (b)  $f(2) = 14$
- (c)  $f(1)f(2) = -31 \times 14 = -434 < 0$
- (d) The sign of  $f(1)f(2)$  is negative.
- (e)  $1 < x_{root} < 2$
- (f)

$a$	$b$	$\frac{a+b}{2}$	$f(a)$	$f(b)$	$f(\frac{a+b}{2})$
1	2	1.5	-31	14	-12.125
1.5	2	1.75	-12.125	14	-0.0156
1.75	2	1.875	-0.0156	14	6.7480
1.75	1.875	1.8125	-0.0156	6.7480	3.3059
1.75	1.8125	1.7813	-0.0156	3.3059	1.6328
1.75	1.7813	1.7657	-0.0156	1.6328	0.8075
1.75	1.7657	1.7579	-0.0156	0.8074	0.3976
1.75	1.7579	1.7540	-0.0156	0.3976	0.1934
1.75	1.7540	1.7520	-0.0156	0.1934	0.0888
1.75	1.7520	1.7510	-0.0156	0.0888	0.0366
1.75	1.7510	1.7505	-0.0156	0.0366	0.0105
1.75	1.7505	1.75025	-0.0156	0.0105	-0.0026
1.75025	1.7505	1.75038	-0.0026	0.0105	0.0042
1.75025	1.75038	1.75032	-0.0026	0.0042	0.0012
1.75025	1.75032	1.75029	-0.0026	0.0011	-0.0005
1.75029	1.75032	1.75031	-0.0005	0.0011	0.0006
1.75029	1.75031	1.75030	-0.0005	0.0006	0.00003



1.75029	1.75030	1.750295	-0.0005	0.0003	-0.0002
1.750295	1.750295	1.750295	-0.0002	-0.0002	-0.0002

TABLE 2. After 17 iterations we can confidently deduce that  $x_{root}$  is 1.7503 to four decimal places.

## Exercises – The Bisection Method

### Question 1.

Use the bisection method to find the root of  $f(x) = 5x - 9$  to one decimal place. Take  $a = 1$  and  $b = 3$ .

[ANSWER:  $x_{root} = 1.8$ ]

### Question 2.

Using the bisection method, obtain the greater positive root of  $f(x) = x^2 - 6x + 7$  to four decimal places.

[ANSWER:  $x_{root} = 4.4142$ ]

### Question 3.

Obtain the greatest positive root to four decimal places of  $f(x) = -x^3 + 9x^2 - 20x + 6$  using the bisection method over the interval  $[5,6]$ .

[ANSWER:  $x_{root} = 5.6458$ ]

### Question 4.

Use the bisection method to obtain the co-ordinates of the point of intersection, to an accuracy of four decimal places, of  $y = -x^3 + 4x^2 - 3x + 2$  and  $y = 2x - 7$  over the interval  $[2,4]$ .

[ANSWER:  $(x,y) = (2.8637, 2.7274)$ ]

### Question 5.

Obtain, to an accuracy of four decimal places, the co-ordinates of the point of intersection of the cubic functions  $y = 2(x - 1)^3$  and  $y = -3(x - 2)^3$  using the bisection method over the interval  $[1,2]$ .

[ANSWER:  $(x,y) = (1.5337, 0.3041)$ ]

## Newton's Method

Newton's Method, also referred to as the Newton-Raphson Iteration Technique, involves less iterations than the Bisection Method since its convergence is quadratic rather than linear.

The basic idea is that if  $x_0$  is an approximation to the root,  $x_{root}$ , of the equation  $f(x) = 0$ , then a closer approximation will be given by  $x_1$  where the tangent to the graph at  $x = x_0$  cuts the  $x$ -axis at  $x = x_1$  as shown in Figure 3.

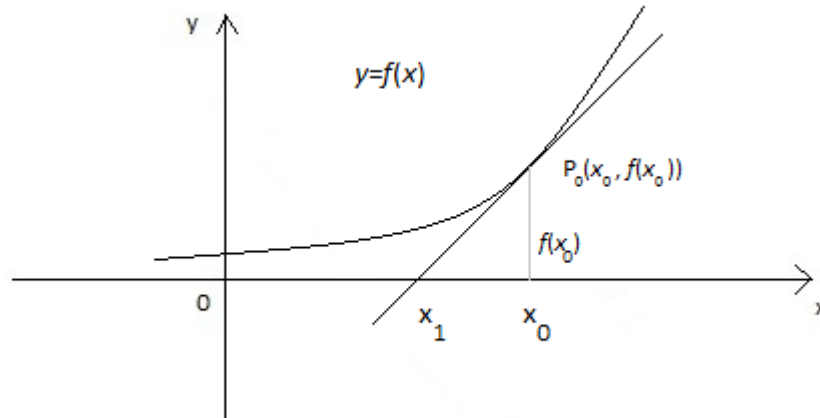


Figure 3: Newton's Method for finding roots.

Using the definition of derivative at  $x = x_0$

$$f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

$$\therefore (x_0 - x_1) f'(x_0) = f(x_0)$$

$$\therefore (x_0 - x_1) = \frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{- Equation}$$

(1)

More generally, we may write

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{- Equation}$$

(2)

where  $n = 0, 1, 2, 3, \dots$

This equation is known formally as the Newton-Raphson Iteration Procedure for obtaining an approximation to the root of  $f(x) = 0$ .

### EXAMPLE 1.

Use Newton's Method to find the positive root of  $f(x) = x^3 - 6x^2 - 9x - 1$  to an accuracy of four decimal places. Take the initial guess  $x_0$  to be 6.

### SOLUTION TO EXAMPLE 1.

$$f(x) = x^3 - 6x^2 - 9x - 1$$

$$f'(x) = 3x^2 - 12x - 9 = 3(x^2 - 4x - 3)$$

We shall take the initial guess to be  $x_0 = 6$ .

In accord with Equation (1), we define

$$k(x) = x - \frac{f(x)}{\frac{d}{dx}f(x)}$$

Initial guess is  $x_0 = 6$ .

We now carry out the following procedure using the Casio ClassPad II CAS calculator.

Define  $f(x) = x^3 - 6x^2 - 9x - 1$

Define  $k(x) = x - \frac{f(x)}{\frac{d}{dx}f(x)}$

6        EXE

$k(ans)$     EXE    8.0370

          EXE    7.3777

          EXE    7.2623

          EXE    7.2588

          EXE    7.2588

After four iterations we obtain the required root correct to four decimal places as follows:

$$x_{\text{root}} = 7.2588$$

Clearly, Newton's Method is more efficient and substantially faster than the Bisection Method.

### EXAMPLE 2.

Find the positive root of  $f(x) = 4x^3 + 12x^2 - 32x - 29$  to four decimal places using Newton's Method. Take the initial guess  $x_0$  to be 1.5.

### SOLUTION TO EXAMPLE 2.

$$f(x) = 4x^3 + 12x^2 - 32x - 29$$

$$f'(x) = 12x^2 + 24x - 32 = 4(3x^2 + 6x - 8)$$

$$\text{Initial Guess: } x_0 = 1.5$$

Again using the Casio ClassPad II CAS calculator, we carry out the following procedure:

Define  $f(x) = 4x^3 + 12x^2 - 32x - 29$

Define  $k(x) = x - \frac{f(x)}{\frac{d}{dx}f(x)}$

1.5        EXE

$k(ans)$     EXE    2.6774

          EXE    2.2706

          EXE    2.1872

          EXE    2.1837

          EXE    2.1837

After four iterations we obtain the positive root to this cubic function to four decimal places as follows:

$$x_{\text{root}} = 2.1837$$

## Exercises – Newton’s Method

### Question 1.

Use Newton’s Method to find the greatest root of  $f(x) = x^3 - 4x^2 - 2x + 4$  to four decimal places. Take the initial guess to be  $x_0 = 4.5$ .

[ANSWER:  $x_{\text{root}} = 4.2491$ ]

### Question 2.

Find the root of  $f(x) = 2x^3 - 4x^2 + 5x - 7$  to four decimal places using Newton’s Method. Take the initial guess to be  $x_0 = 1$ .

[ANSWER:  $x_{\text{root}} = 1.7263$ ]

### Question 3.

The function  $f(x) = x^3 - 7x + 7$  has two roots on the interval  $[1,2]$ . Find these two roots to four decimal places.

[ANSWER:  $x_{\text{root1}} = 1.3569$  and  $x_{\text{root2}} = 1.6920$ ]

### Question 4(a).

Find the square root of 17 to four decimal places using Newton’s Method.

Hint: Let  $f(x) = x^2 - 17$ . Then  $f'(x) = 2x$ . The iteration formula becomes

$$\begin{aligned}x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\&= x_0 - \frac{(x_0^2 - 17)}{2x_0} \\&= x_0 - \frac{x_0}{2} + \frac{17}{2x_0} \\&= \frac{x_0}{2} + \frac{17}{2x_0} \\&= \frac{1}{2} \left( x_0 + \frac{17}{x_0} \right)\end{aligned}$$

Take  $x_0$  to be 4 and start the iteration process.

[ANSWER:  $\sqrt{17} \approx 4.1231$  after only two iterations]

### Question 4(b).

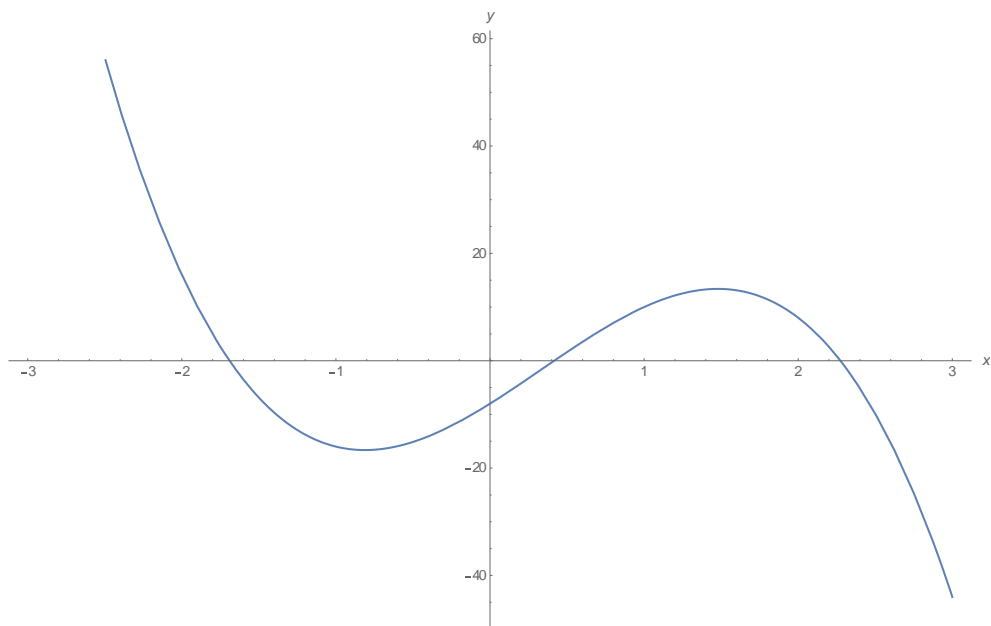
Find the cube root of 28 to four decimal places using Newton’s Method.

[ANSWER:  $\sqrt[3]{28} \approx 3.0366$ ]

**Question 5.**

Use Newton's Method to find all three roots to four decimal places of the function

$f(x) = -5x^3 + 5x^2 + 18x - 8$ . Take the initial guesses to be -2, 0.5 and 2. (See Figure 4)



*Figure 4*

$$f(x) = -5x^3 + 5x^2 + 18x - 8$$

[ANSWER:  $x_{\text{root}} = -1.6901, 0.4163$  and  $2.2738$ ]

# MATHS ACTIVITIES FOR STUDENTS WITH SPECIAL NEEDS

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*Teachers are required to build their capacity to teach learners with disability, as part of the Victorian Government Special Needs Plan. In this summary paper, activities that support the learning and participation of learners with disability are explored. A few activities that should meet some of the specific learning needs of students across a range of abilities follow. The activities and teaching strategies aim to cater for a wide range of abilities and recognise that all children can learn. Effective teaching of mixed ability groups is inclusive and applicable to most students with special needs.*

## Students with Special Needs

A case can be made that all students have ‘special needs’. Mathematics that each student is expected to learn needs to be both achievable and challenging. With the wide range of abilities within student groups, mathematics that is seemingly impossible to some students, is effortless to others. Both these student groups can become bored and disengaged. In the 1970’s a Year 8 group of students were being shown how to solve simple linear equations (Figure 1).

$$\begin{aligned}x + 2 &= 5 \\ \text{Take 2 from both sides} \\ x + 2 - 2 &= 5 - 2 \\ x + 0 &= 3 \\ x &= 3\end{aligned}$$

*Figure 1. Solving a simple linear equation.*

Understandably students asked, “*Why go through all the working when it is obvious that  $x$  is 3*”. The explanation, that showing this working allows more difficult equations to be solved, was met with scepticism. To prove this point, students were given  $\frac{4(5x+14)}{9} - \frac{3(x-12)}{7} = 17$  to solve. Two students

independently obtained the correct solution  $x = \frac{355}{113}$  (an accurate approximation to  $\pi$ ). These two clearly were not learning much, if anything, from by the earlier linear equation work. Their time was being wasted and they were able to pursue mathematics at a higher level with minimal teacher assistance. ‘Backtracking’ has been found to be more effective when introducing linear equations.

Bayetto (2006) writes “*It has also been suggested that doing more of the same low level tasks not only narrows the curriculum but that it does not enable a student to show what they truly know and can do. Instead of watering down, educators must program-up and have ambitious but achievable goals.*”

Trying to solve  $x = x + 1$  led to much discussion and interest with many students eventually realising there are no real values of  $x$  that satisfy  $x = x + 1$ . Further discussion ensued about whether  $x = \pm\infty$

were solutions. Solving  $2x+6=2(x+3)$  led to the conclusion that all values of  $x$  satisfy  $2x+6=2(x+3)$ . This unplanned diversion led to most students discarding the mistaken idea that there is only one right answer in mathematics.

## All Students Can Learn

Of course, students with disabilities have special needs. Common disabilities may include impairments in vision, hearing, movement or cognition. Teachers are likely to encounter students with allergies, anaphylaxis, asthma, attention deficit hyperactivity disorder, autism spectrum disorder, colour blindness, diabetes, dyslexia or epilepsy. Some of these may have little or no negative impact on learning mathematics. Those with disabilities also have abilities and it is important to recognise that all children can learn. It is important to concentrate on ability rather than disability. As part of the Victorian Government Special Needs Plan teachers are required to build their capacity to teach learners with disability (see Table 1).

Table 1.

*Relevant Standard Descriptors (Australian Institute for Teaching and School Leadership, 2016)*

### 1.5 Differentiate teaching to meet the specific learning needs of students across the full range of abilities

Graduate	Proficient	Highly Accomplished	Lead
Demonstrate knowledge and understanding of strategies for differentiating teaching to meet the specific learning needs of students across the full range of abilities.	Develop teaching activities that incorporate differentiated strategies to meet the specific learning needs of students across the full range of abilities.	Evaluate learning and teaching programs, using student assessment data that are differentiated for the specific learning needs of students across the full range of abilities.	Lead colleagues to evaluate the effectiveness of learning and teaching programs differentiated for the specific learning needs of students across the full range of abilities.

### 1.6 Strategies to support full participation of students with disability

Graduate	Proficient	Highly Accomplished	Lead
Demonstrate broad knowledge and understanding of legislative requirements and teaching strategies that support participation and learning of students with disability.	Design and implement teaching activities that support the participation and learning of students with disability and address relevant policy and legislative requirements	Work with colleagues to access specialist knowledge, and relevant policy and legislation, to develop teaching programs that support the participation and learning of students with disability	Initiate and lead the review of school policies to support the engagement and full participation of students with disability and ensure compliance with legislative and/or system policies

### 4.1 Support student participation

Graduate	Proficient	Highly Accomplished	Lead
Identify strategies to support inclusive student participation and engagement in classroom activities.	Establish and implement inclusive and positive interactions to engage and support all students in classroom activities.	Model effective practice and support colleagues to implement inclusive strategies that engage and support all students.	Demonstrate and lead by example the development of productive and inclusive learning environments across the school by reviewing inclusive strategies and exploring new approaches to engage and support all students.

## Mixed Ability Groups

Bayetto (2006) suggests it is worthwhile reflecting on the following collection of key teaching issues which may assist teachers to acknowledge the wide diversity in a group and decide what students can do and need to do next.

1. Use *brief, mini-lessons for specific skills.*
2. Provide opportunities to work *alone and together.*
3. Use *problem solving with divergent questions.*
4. Use *concrete materials.*
5. Confirm *student understanding of mathematical language.*
6. Have students keep *math journals.*
7. *Play games.*
8. *Use technology.*
9. Have a robust *pedagogical knowledge and positive attitude.*
10. *Choose published materials carefully.*
11. Assessment must be for learning, and of learning.

Clarke (2003) found the following two conclusions of interest in a meta-analysis using 165 studies, across a range of grade levels.

- Students working in small groups achieved significantly more than students not learning in small groups.
- The subject area made a difference: There were no significant differences between ability grouping and mixed ability grouping in mathematics, compared with significant differences in reading in favour of homogeneous groups.

He adds *“In the light of this research, I think that teachers need to think carefully about their reasons for choosing to place students into groups according to perceived ability. If the research is showing no significant differences, then we need to consider the potential impact upon students’ self-esteem, and also the potential for what Brophy (1963) calls ‘the self-fulfilling prophecy’, where students perform to the level expected of them by their teacher.”*

The activities that follow can be done in mixed ability groups or alone.

## Sheep and Emus

A farmer finds that emus have become mixed in with his sheep in a paddock. He counts 30 heads and 100 legs. How many sheep are there in the paddock? Hint: For every emu head there are 2 legs whilst for every sheep head there are 4 legs.

The following solutions are arranged roughly in order of difficulty.

### Picture Method



Draw extra pairs of legs changing emus to sheep until 100 legs

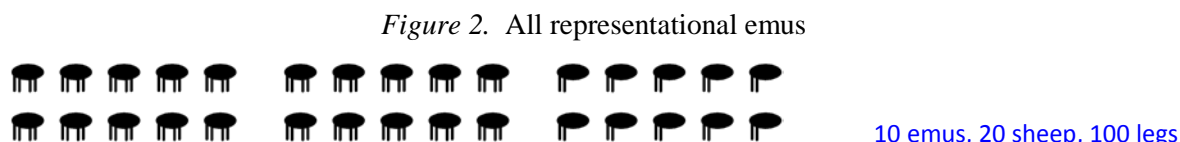
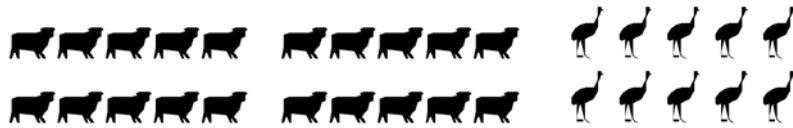


Figure 3. Representational sheep and emus





10 emus, 20 sheep, 100 legs

Figure 4. Realistic sheep and emus

### Primary Schoolgirl Method

If all the sheep stood up on their back legs there would be 60 legs on the ground (2 legs for each head). The extra 40 legs must be sheep front legs in the air, so there are 20 sheep (1 sheep for each pair of legs in the air).

### Equations

let  $e$  be the number of emus

let  $s$  be the number of sheep

$$e + s = 30 \quad \dots(1) \quad \text{head equation}$$

$$2e + 4s = 100 \quad \dots(2) \quad \text{leg equation}$$

Figure 5. Simultaneous equations

### Elimination Methods

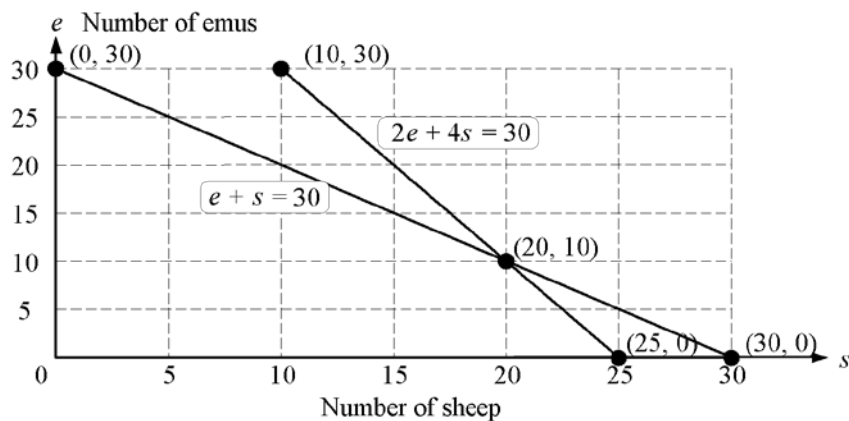
$2 \times (1) \quad 2e + 2s = 60 \quad \dots(1)'$ $2e + 4s = 100 \quad \dots(2)$ $(2) - (1)' \quad 2s = 40$ $s = 20$ <p>There are 20 sheep</p>	$4 \times (1) \quad 4e + 4s = 120 \quad \dots(1)'$ $2e + 4s = 100 \quad \dots(2)$ $(1)' - (2) \quad 2e = 20$ $e = 10$ <p>Sub. 10 for <math>e</math> in (1)</p> $10 + s = 30$ $s = 20$ <p>There are 20 sheep</p>	$(2) \div 2 \quad e + 2s = 50 \quad \dots(2)'$ $e + s = 30 \quad \dots(1)$ $(2)' - (1) \quad s = 20$ <p>There are 20 sheep</p>
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### Substitution Methods

$s = 30 - e \quad \dots(1)'$ <p>Sub. <math>30 - e</math> for <math>s</math> in <math>\dots(2)</math></p> $2e + 4(30 - e) = 100$ $2e + 120 - 4e = 100$ $-2e = -20$ $e = 10$ <p>Sub. 10 for <math>e</math> in (1)</p> $10 + s = 30$ $s = 20$ <p>There are 20 sheep</p>	$e = 30 - s \quad \dots(1)'$ <p>Sub. <math>30 - s</math> for <math>e</math> in <math>\dots(2)</math></p> $2(30 - s) + 4s = 100$ $60 - 2s + 4s = 100$ $2s = 40$ $s = 20$ <p>There are 20 sheep</p>
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Figure 6. Simultaneous equations can be solved in a variety of ways

## Graph Method



Graphs cross at (20, 10) i.e. 20 sheep, 10 emus

Figure 7. Graph solution (requires some point plotting)

## Matrix Method (detailed)

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} e \\ s \end{bmatrix} = \begin{bmatrix} 30 \\ 100 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} e \\ s \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 30 \\ 100 \end{bmatrix}$$

$$\frac{1}{2} \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} e \\ s \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 100 \end{bmatrix}$$

$$\frac{1}{2} \begin{bmatrix} 4 \times 1 + -1 \times 2 & 4 \times 1 + -1 \times 4 \\ -2 \times 1 + 1 \times 2 & -2 \times 1 + 1 \times 4 \end{bmatrix} \begin{bmatrix} e \\ s \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 \times 30 + -1 \times 100 \\ -2 \times 30 + 1 \times 100 \end{bmatrix}$$

$$\frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e \\ s \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 120 - 100 \\ -60 + 100 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e \\ s \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 20 \\ 40 \end{bmatrix}$$

$$\begin{bmatrix} 1 \times e + 0 \times s \\ 0 \times e + 1 \times s \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} e \\ s \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

Figure 8. A matrix solution (red working not necessary once method understood)

## TI-nspire CAS

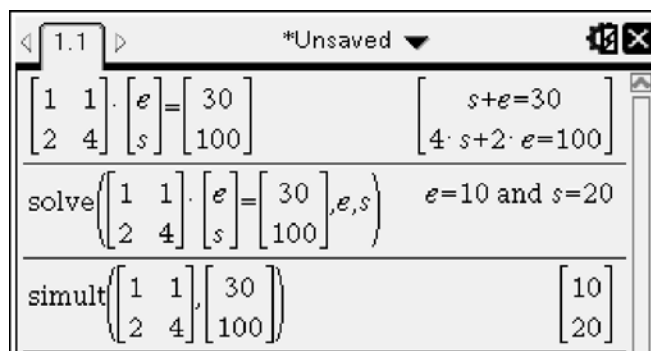


Figure 9. CAS calculator solutions

Whilst younger students might enjoy this problem and its ridiculousness, some older students may not appreciate the humour and see it as pointless and unrealistic. If the problem is changed to: ‘Tickets to a school function are \$2 for students and \$4 for adults. A total of 30 tickets are sold for \$100. How many adults have tickets to the function?’, then these students may see the problem as more meaningful. Creative students can be challenged to come up with different scenarios using the same and different numbers. Bicycles and tricycles is one such variation.

## Multiplication Race

Cooperation and competition motivate some students. A multiplication sheet similar to that shown in Figure 10 is handed to each student without answers. Students write their name on their sheet. They are told there is a pattern in the answers; they are allowed to use calculators and can work together but each student must complete their own sheet. The teacher will provide an answer if asked. Products of mixed numbers must be written as mixed numbers and products of decimals must be written as decimals. The top row continues across to  $90 \times 90$ , the first column continues down to  $1 \times 19$  and the ninth column tenth row has  $81 \times 99$ .

$10 \times 10 = 100$	$20 \times 20 = 400$	$30 \times 30 = 900$	$40 \times 40 = 1600$	$50 \times 50 = 2500$	...
$9 \times 11 = 99$	$19 \times 21 = 399$	$29 \times 31 = 899$	$39 \times 41 = 1599$	$49 \times 51 = 2499$	...
$8 \times 12 = 96$	$18 \times 22 = 396$	$28 \times 32 = 896$	$38 \times 42 = 1596$	$48 \times 52 = 2496$	...
$7 \times 13 = 91$	$17 \times 23 = 391$	$27 \times 33 = 891$	$37 \times 43 = 1591$	$47 \times 53 = 2491$	...
$6 \times 14 = 84$	$16 \times 24 = 384$	$26 \times 34 = 884$	$36 \times 44 = 1584$	$46 \times 54 = 2484$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure 10. Multiplication Race

The last three rows are similar to Figure 11.

$1\frac{1}{2} \times 1\frac{1}{2} = 2\frac{1}{4}$	$2\frac{1}{2} \times 2\frac{1}{2} = 6\frac{1}{4}$	$3\frac{1}{2} \times 3\frac{1}{2} = 12\frac{1}{4}$	$4\frac{1}{2} \times 4\frac{1}{2} = 16\frac{1}{4}$	$5\frac{1}{2} \times 5\frac{1}{2} = 30\frac{1}{4}$	...
$1.5 \times 1.5 = 2.25$	$2.5 \times 2.5 = 6.25$	$3.5 \times 3.5 = 12.25$	$4.5 \times 4.5 = 16.25$	$5.5 \times 5.5 = 30.25$	...
$15 \times 15 = 225$	$25 \times 25 = 625$	$35 \times 35 = 1225$	$45 \times 45 = 1625$	$55 \times 55 = 3025$	...

Figure 11. Last three rows of Multiplication Race

Experience has shown that virtually all students participate eagerly in this seemingly uninteresting task. Few, if any, discover or are aware of the numerical application of the difference of two squares. e.g.

$$\begin{aligned}
46 \times 54 &= (50 - 4)(50 + 4) & (5\frac{1}{2})^2 - (\frac{1}{2})^2 &= (5\frac{1}{2} - \frac{1}{2})(5\frac{1}{2} + \frac{1}{2}) \\
&= 50^2 - 4^2 & (5\frac{1}{2})^2 - \frac{1}{4} &= 5 \times 6 \\
&= 2500 - 16 & (5\frac{1}{2})^2 &= 30\frac{1}{4} \\
&= 2484
\end{aligned}$$

Figure 12. Arithmetic application of the difference of two squares

Most learn that  $\frac{1}{2} = 0.5$  and  $\frac{1}{4} = 0.25$  if they did not know this beforehand, and, can generalize from  $(5\frac{1}{2})^2 = 30\frac{1}{4}$  to  $5 \cdot 5^2 = 30 \cdot 25$  and  $55^2 = 3025$  as well as gaining some tables practice. This activity also provides an opportunity to explain the operation of the fraction key,  $a\frac{b}{c}$ , found on many school calculators.

## Point Plotting

### Robin

Learning to plot Cartesian coordinates is more rewarding if it results in a Robin as shown in Figure 13. This activity requires students to correctly plot and number points in pencil. They then connect groups of them in the order given. Some points involve decimals and a few involve fractions. The resulting drawing assists in correctly plotting these. A sheet with the points plotted is available for students whose disability makes them unable to plot points at this stage. Those who finish promptly usually enjoy colouring the Robin. Playing the game ‘battleships’ sometime beforehand is of assistance. By holding the students Robin plot up to a window (or light box) with the supplied Robin plot any incorrect plots can be quickly identified.

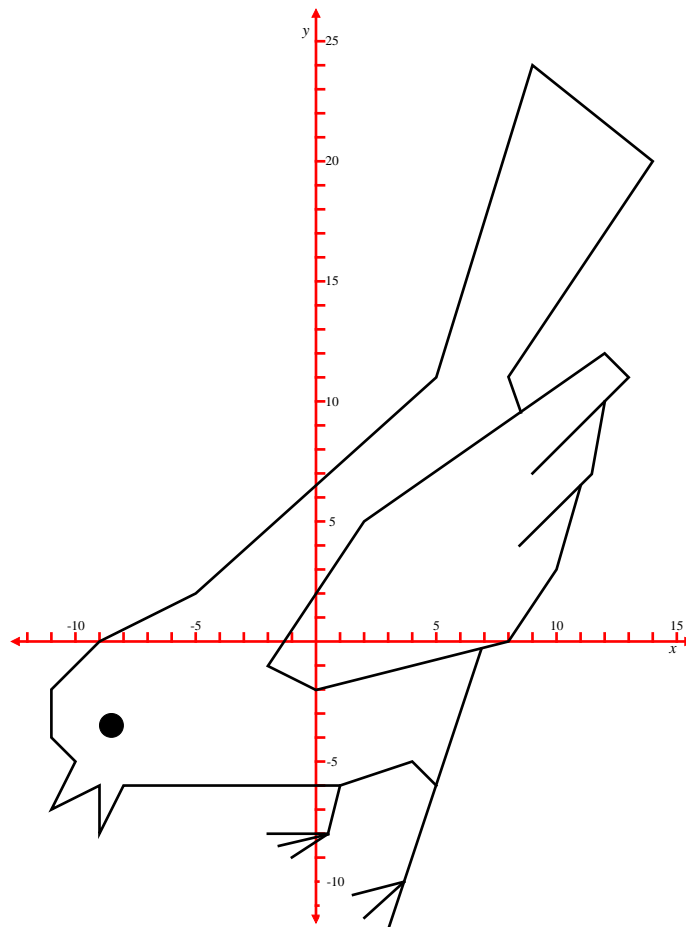


Figure 13. Connecting line segments between plotted points on Cartesian Axes.  
(Adapted from Boyle, P.J. (1971)<sup>4</sup>)

## Isometric Drawing

Many mathematically disengaged students spend much of their class time drawing. Getting them to participate in mathematics activities that involve drawing is considerably easier than getting them to engage in other mathematics activities. Isometric sketching on “dotty” paper is just one activity. This can easily lead to volume, surface area discussions as well as Euler’s Rule  $v + f = e + 2$ .

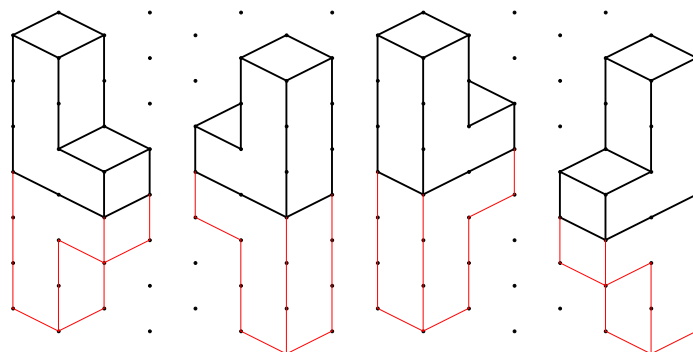


Figure 14. Isometric sketch and 3D reflection on ‘dotty’ paper

Students can trace the black “L” onto their dotty paper. Only a small number are able to copy these 3D shapes without tracing. Most will use a ruler. The red reflections are challenging to draw but as students have two sheets of “L” shapes the copied or traced sheet can be used first to acquire reflection drawing skills then done without area on the original printed sheet. Few students these days see the object on the right of Figure 15 as a Y inside a hexagon but see it as a cube.

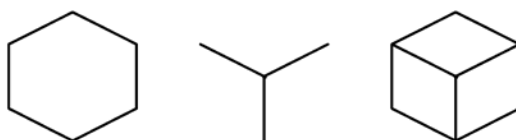


Figure 15. Hexagon, Y and cube.

Plastic Impuzzables (Figure 16) can be purchased or made by gluing wooden cubes together then painting. They are of varying difficulty and most students enjoy using them and improve their spatial thinking as a result. These are variations of the SOMA cube of widely varying difficulty. Whilst the aim is to put the pieces of each colour together to form a cube multiple other activities can be tried e.g. drawing pieces on isometric paper, reflecting, rotating and translating them; ; counting edges, faces and corners to see if Euler’s rule holds; measuring volume and/or surface area; learning that  $3^3 = 27$  and why raising to the power three is called cubing. This activity can raise some special needs students self-esteem and confidence. Concrete aids often prove invaluable in teaching mathematical concepts.

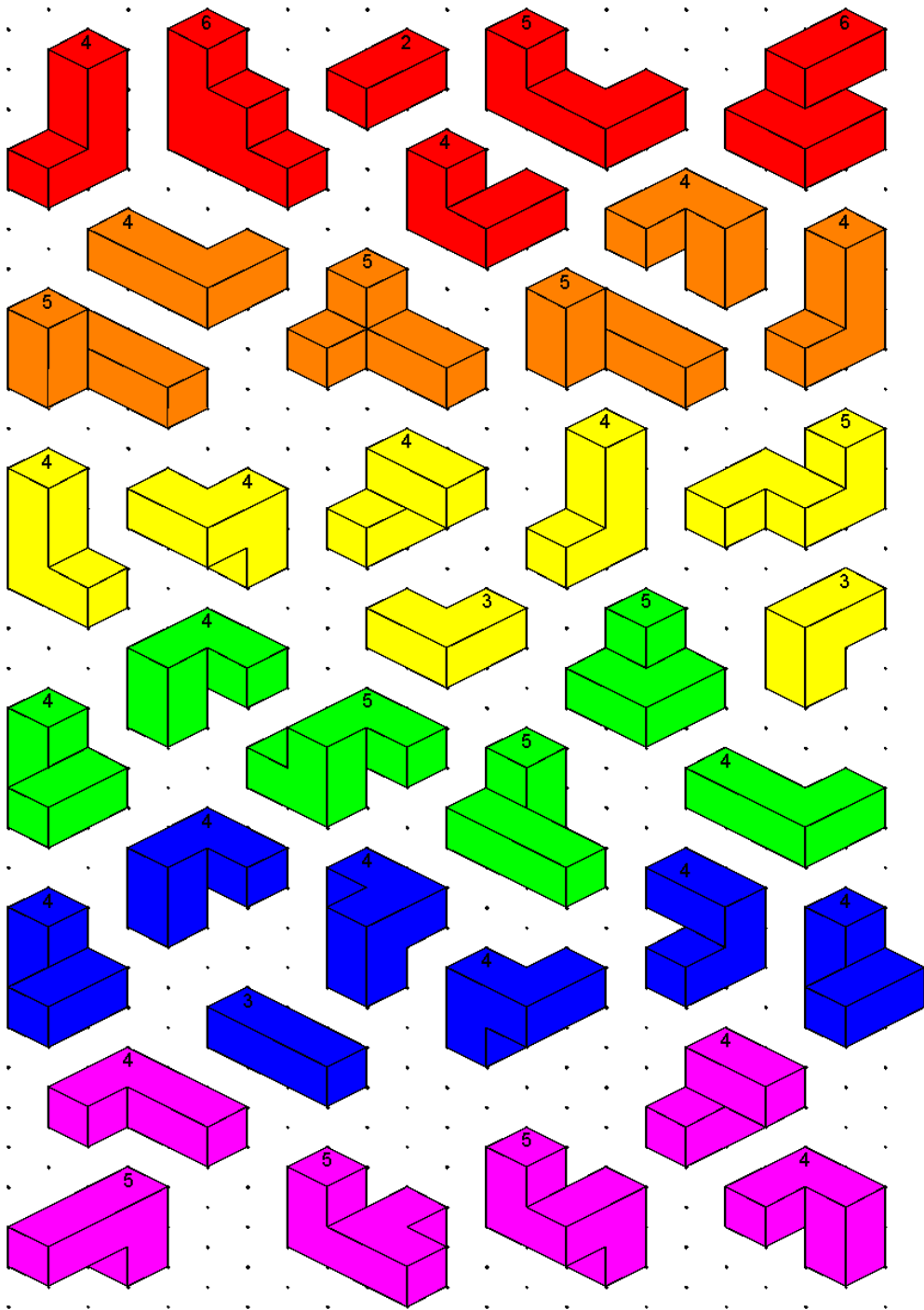


Figure 16. Impuzzables with the most difficult at bottom.

### Two Ladder Problem

Two ladders (2 metres and 3 metres) cross 1 metre above an alley as shown in the diagram above. How wide is the alley?

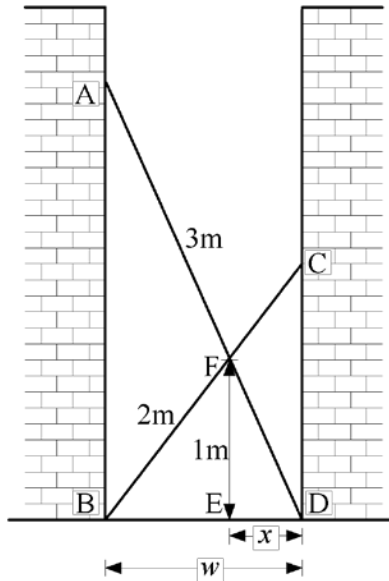


Figure 17. Two ladder problem.

This problem initially looks straight forward but is much more challenging to year 10 students and above.

$$\frac{x}{1} = \frac{w}{\sqrt{9-w^2}} \dots (1)$$

$$\frac{w-x}{1} = \frac{w}{\sqrt{4-w^2}} \dots (2)$$

substitute  $\frac{w}{\sqrt{9-w^2}}$  for  $x$  in (2)

$$w - \frac{w}{\sqrt{9-w^2}} = \frac{w}{\sqrt{4-w^2}}$$

$$1 - \frac{1}{\sqrt{9-w^2}} = \frac{1}{\sqrt{4-w^2}} \text{ since } w \neq 0$$

$$\frac{\sqrt{9-w^2} - 1}{\sqrt{9-w^2}} = \frac{1}{\sqrt{4-w^2}}$$

$$(\sqrt{9-w^2} - 1)\sqrt{4-w^2} = \sqrt{9-w^2}$$

Figure 18. Equation in  $w$ .

Current CAS calculators can numerically solve the equation in Figure 18 as shown in Figure 19.

$$\text{solve} \left\{ \left\{ \left( \sqrt{9-w^2} - 1 \right) \cdot \sqrt{4-w^2} = \sqrt{9-w^2}, \{w\} \right\} \right. \\ \left. \left. \begin{array}{l} w > 0 \end{array} \right\} \right.$$

$$w = 1.23119$$

Figure 19. TI-nspire CX CAS solution to equation in  $w$ .

$$w = \sqrt{\frac{1}{6} \left( 33 - \sqrt{3(37 + k_1 + k_2)} - \sqrt{222 - 3k_1 - 3k_2 + \frac{144}{\sqrt{\frac{1}{3}(37 + k_1 + k_2)}}}} \right)}$$

where  $k_1 = \sqrt[3]{25(1741 - 48\sqrt{39})}$  and  $k_2 = \sqrt[3]{25(1741 + 48\sqrt{39})}$

$w \approx 1.23119$  m

Figure 20. Exact solution to equation in  $w$  using Mathematica.

Students (and mathematicians) often proceed as in Figure 21, squaring to get rid of the square roots whilst realising that this may introduce extraneous solutions.

$$\begin{aligned} (9 - w^2 + 1 - 2\sqrt{9 - w^2})(4 - w^2) &= 9 - w^2 \\ 10 - w^2 - 2\sqrt{9 - w^2} &= \frac{9 - w^2}{4 - w^2} \\ 10 - w^2 - \frac{9 - w^2}{4 - w^2} &= 2\sqrt{9 - w^2} \\ \frac{40 - 10w^2 - 4w^2 + w^4 - 9 + w^2}{4 - w^2} &= 2\sqrt{9 - w^2} \\ \frac{31 - 13w^2 + w^4}{4 - w^2} &= 2\sqrt{9 - w^2} \\ \frac{961 - 806w^2 + 231w^4 - 26w^6 + w^8}{16 - 8w^2 + w^4} &= 4(9 - w^2) \\ 961 - 806w^2 + 231w^4 - 26w^6 + w^8 &= 576 - 352w^2 + 68w^4 - 4w^6 \\ w^8 - 22w^6 + 163w^4 - 454w^2 + 385 &= 0 \end{aligned}$$

Figure 21. Polynomial equation in  $w$ .

In the late 1980's students used a shareware numerical solver called Mercury (HREF4). This gave two positive feasible solutions to the equation in Figure 21,  $w = 1.23119$  and  $w = 1.87316$  as does the TI-nspire CX CAS calculator. Using Newton's calculus approximation i.e. for the solution to  $f(w) = 0$  where  $w = t$  is an approximate solution then  $w = t + h$  will be a closer solution where  $h \approx -\frac{f(t)}{f'(t)}$ .

Iteration gives  $w = 1.23119$  when  $t = 1$  is chosen as the first approximation and to  $w = 1.87316$  when  $t = 2$  is chosen as the first approximation. Although it is a straightforward exercise to substitute these values into the equation in Figure 18 most students arrived at the polynomial by squaring at a much earlier stage and were at a loss as to which solution was correct.

$$\begin{aligned} &\text{solve}\{w^8 - 22 \cdot w^6 + 163 \cdot w^4 - 454 \cdot w^2 + 385 = 0, w\} \\ &w = -1.87316 \text{ or } w = -1.23119 \text{ or } w = 1.23119 \text{ or } w = 1.87316 \end{aligned}$$

Figure 22. TI-nspire CX CAS solutions to polynomial equation in  $w$ .

A fellow student, earlier diagnosed with learning difficulties, solved this for them by constructing a scale model which clearly showed the width of the alley to be 1.2 m not 1.9 m.

## Conclusion



Students can be the neatest, fastest, most consistent, most creative, best explainers, most enthusiastic, most industrious, most determined, most cooperative, most considerate or most imaginative in their class. There are so many aspects to mathematics, that it is virtually impossible not to find at least one that a particular student could excel in. Students can be early or late developers, thorough or quick learners.

Whilst there are many resources to help students with a disabilities read, speak, listen or concentrate there are much fewer that directly deal with learning mathematics. However effective teaching of mixed ability groups is frequently applicable to students with disabilities.

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