BACK TO THE FUTURE

THE MAV 52ND ANNUAL CONFERENCE

3 & 4 December 2015, La Trobe University, Bundoora

www.mav.vic.edu.au

THE MATHEMATICAL ASSOCIATION OF VICTORIA

Editorial team:
Dr Dona Martin, Ms Tina Fitzpatrick, Dr Simon Smith, Dr Christopher Lenard, Dr Sharyn Livy, Emeritus Professor Terence Mills, Dr Jonathon MacLellan, Ms Diane Itter, Dr Katherine Seaton, Ms Kym Barbary, Ms Cindy Tassone

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Published by:
THE MATHEMATICAL ASSOCIATION OF VICTORIA
FOR THE FIFTY SECOND ANNUAL CONFERENCE
3–4 December 2015

Published December 2015 by
The Mathematical Association of Victoria
"Cliveden"
61 Blyth Street
Brunswick VIC 3056
Designed by Idaho Design & Communication

National Library of Australia Cataloguing-in-Publication Data:

ISBN: 978-1-876949-59-4

Should readers wish to contact any of the authors, full addresses are available from the Professional Officer, the Mathematical Association of Victoria.

FOREWORD

The 52nd Annual Conference Proceedings of the Mathematical Association of Victoria contains a wonderful collection of articles on a wide selection of mathematical topics. The works, by teachers, teacher educators and researchers, offer a clear indicator that continued innovation is taking place in mathematics classes in our schools and universities. Back to the Future offers readers many thought provoking ideas through a breadth of engaging articles.

The editorial team is representative of the wide variety of MAV members. We enjoyed working together as much as we did with the authors, whom we thank individually for their dedicated commitment to progressing mathematics education.

We thank the MAV conference organisation team for their professionalism and encourage each of you to continue in your support of their most important work. The professional development focus and opportunities offered by MAV remain as valuable as ever in progressing mathematics education.

The Review Process for the Mathematical Association of Victoria 52nd Annual Conference Proceedings

Papers were submitted for double-blind review, peer review or as summaries. The Editors received 11 full papers for the double blind review process, for which the identities of author and reviewer were concealed from each other. Details in the papers that identified the authors were removed to protect the review process from any potential bias, and the reviewers' reports were anonymous. Two reviewers reviewed each of the 11 blind review papers and if they had a differing outcome a third reviewer was required. Ten of the 11 papers were accepted for publication. In addition, we received 14 full papers for the peer review process, where the names of the authors were identified to reviewers; 11 were accepted for publication as peer-reviewed papers and two were accepted as summary papers. Eight papers submitted as summary papers were reviewed by a combination of external reviewers and the editorial team. Seven of these were accepted for publication (one as a peer reviewed paper).

In the Conference Proceedings, double-blind and peer reviewed papers are grouped together and arranged in alphabetical order of author names. Double-blind reviewed papers and peer reviewed papers are indicated by ** and * respectively following the paper title. Summary papers follow the double blind and peer reviewed papers.

Of the total of 34 papers received, 33 papers are published: ten double-blind reviewed papers, seventeen peer reviewed papers and six summary papers. Altogether, 21 reviewers assisted in the review process, all of whom provided thoughtful feedback and were outstanding in responding quickly to our invitations.

CONTENTS

***** Double blind reviewed paper**
**** Peer reviewed paper**
*** Summary paper**

USING THE MODEL METHOD TO ASSIST PRIMARY STUDENTS TO SOLVE WORD PROBLEMS

Lei Bao

Leopold Primary School

The aim of this study is to investigate whether the model method is effective to assist primary students to solve word problems. Three Grade 4 classes were asked to complete a pre-test before a two-week trial on model method. It contains 5 questions involving part-whole relationship, comparison and multiplicative structure. The terminologies such as 'more than", 'fewer than' and 'as many as' are used in the questions. Students then were asked to use the 'bar model' to solve five parallel questions after the trial. The rate of success increased in each question in the post test. The bar model method not only provides students with a visual aid but also helps students to present the problem situations and apply relevant concepts by drawing the rectangular bars. The findings suggest that drawing the bar model is an effective method to assist primary students to solve word problems.

Introduction

Word problems have been a major part of primary school mathematics. However many primary students throughout schooling have difficulties with word problems, particularly multi-step word problems. Teachers have been seeking effective teaching methods to support student learning in problem solving. Researchers (Fong & Lee, 2005; Ng & Lee 2009; Englard, 2010 Chan & Foong, 2013) believe that the model method can address this need. Since the model method was introduced in Singapore in the early 1980s, it has made a great contribution to student learning. In this approach, students draw diagrams in which they represent the problem situations and relevant concepts using rectangular bars. Rectangular bars are used because they are easy to draw, divide, represent numbers and display proportional relationship (Ng & Lee 2009).

Though the model method has been practised in Singapore and other countries for more than a decade, the potential of this method seem to have escaped the notice of primary school teachers in Australia. This study attempts to examine the effectiveness of using the model method with Grade 4 students. It will shed a light on following questions:

1. Is there a significant difference between the problem solving performance before and after the trial?
2. Is the model method effective to assist students to solve multi-step word problems?

Literature Review

Several studies (Cheong, 2002; Ng & Lee, 2005; Ng & Lee, 2009; Englard, 2010) used word problems involving the relationship of part-whole, comparison and multiplicative structures to trial the model method. Different lengths of rectangular bars are used to represent different numbers: a relatively longer rectangular bar for a bigger number, a short one for a smaller number, or a bar of arbitrary length for an unknown number. The comparison model shows the relationship between two or more quantities when they are compared. The terminologies such as 'more than", 'less than' and 'as many as' are used in these studies for the construction of comparison models. These models not only serve to explain and reinforce the concepts such as addition and subtraction but assist students to deal with multiplicative structure problems. Englard (2010) highlights that with the visualised bar model, it is easier for students to identify the equivalent units and partition the quantities into smaller and equivalent units, thus to overcome the barrier of the concept of multiplicative structure.

Several publications (Cheong, 2002; Ng & Lee, 2005; Ng & Lee, 2009; Englard, 2010; Chan and Foong, 2013) report that drawing the model not only permits students to visualise the abstract information given in the problem but also helps students to use rectangular bars to represent relationship between the known and unknown numerical quantities and to solve problems related to these quantities. Chan and Foong (2013) explain that the model method shifts the focus from the result to the working process and from calculations to relationships among the known and unknown quantities and the comparison units. Moreover, the model method helps students identify the correct operations and steps that are needed to solve a problem (Englard, 2010). She explains that the nature of the bar model provides students with the visual aid which makes it easy to see when to add or subtract and to keep track of what the outcome of each operation. It is believed that if students can draw the bars realistically and represent all the relevant quantities, thus to identify the relationships among the known and unknown quantities and the comparison units, they have already understood the structure of the problem (Cheong, 2002; Ng & Lee, 2009).

Method

Participants

Three grade 4 classes from a public school in Victoria participated in a two-week trial on the model method. Students had not been taught the bar model method before.

Instrument

A pre-test with five questions involving part-whole relationship, comparison and multiplicative structure was administrated before the trial. After the two-week trial of the model method, a post-test (Table 1) with five parallel questions was administrated.

Table 1 – *The Five Word Problems of Post Test*

1	Pizza Hut and Domino's sold 150 pizzas on Sunday night. Domino's sold 70 pizzas on that night. How many pizzas did Pizza Hut sell on Sunday night?
2	Jordan has 37 footy cards. Oscar has 23 more cards than Jordan. How many cards does Oscar have?
3	130 students took part in an art competition. There were 40 fewer boys than girls took part. How many boys took part in the competition?
4	Katelyn and Paige have collected 60 coins. Katelyn has three times as many coins as Paige. How many coins has Katelyn collected?
5	A zoo keeper weighed some of the animals at Melbourne Zoo. He found that the lion weighs 90kg more than the leopard and the tiger weighs 50kg less than the lion. Altogether the three animals weigh 310kg. How much does the lion weigh?

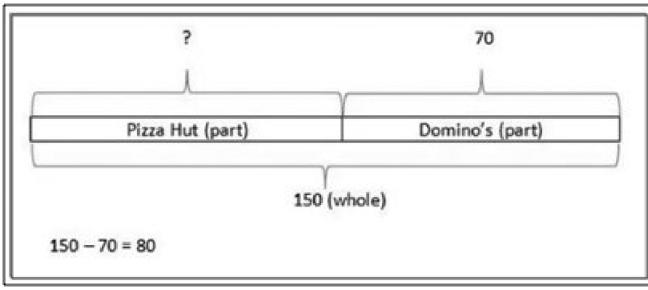
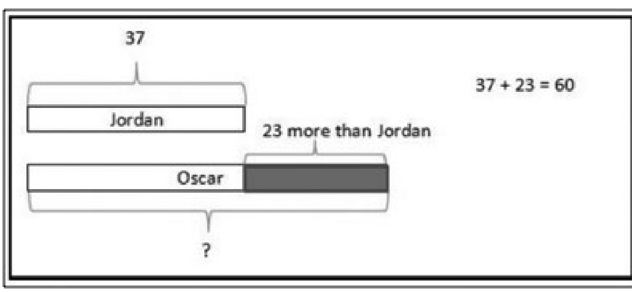


Figure 1.



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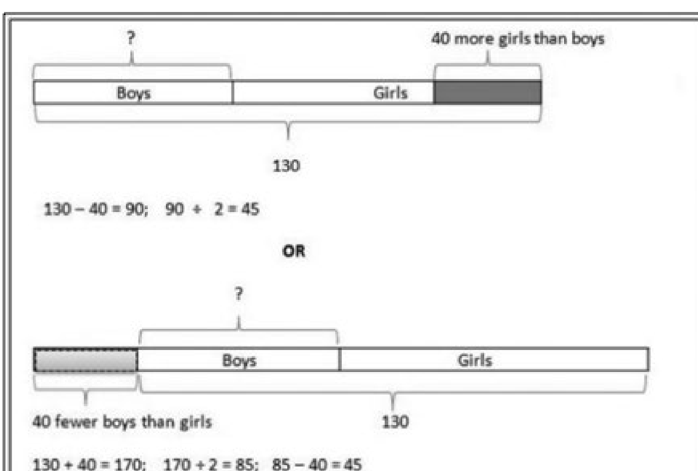
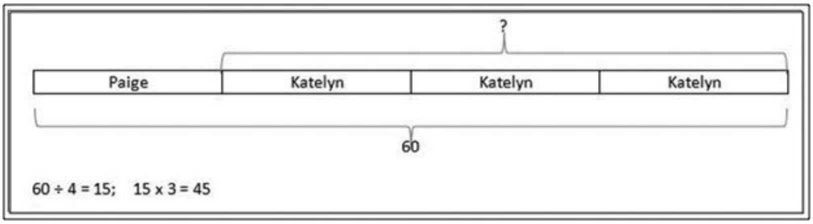


Figure 3.

The part-whole model (Figure 1) is used in question 1. The comparison model (Figure 2, Figure 3, Figure 4 and Figure 5) is used in question 2, 3, 4 and 5. It shows the relationship between two or more quantities when they are compared. The varying lengths of the rectangles show that one quantity is more than another and the difference between the quantities is indicated by the difference in lengths of rectangles. The multiplicative structure can be found in questions 3, 4 and 5. Question 3 involve the concept of halving after taking away 40 from the girl's or adding 40 onto the boy's. The terminology such as 'as many as' in question 4 indicates that the concept of equivalent bars should be applied in the model (Figure 4). Although question 5 does not show the equivalent bars initially, such structure can be revealed after adding the fictitious quantity (Figure 5).



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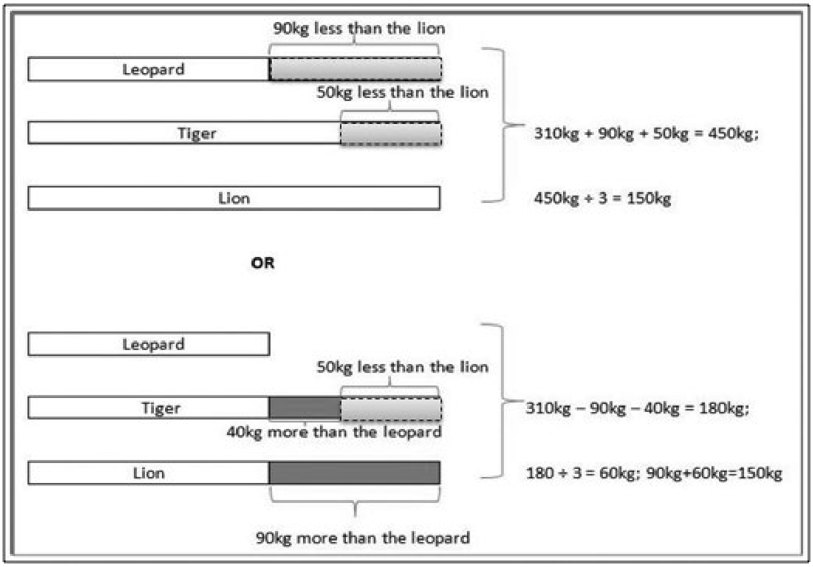

Figure 5.

Results

Figure 6 provides students' overall success rate in solving the problems before the trial. Questions 1 and 2 are one-step question. Students were more successful in solving these questions than questions 3, 4 and 5 as they are multi-step questions involving the comparison model and multiplicative structure.

Questions	Q1		Q2		Q3		Q4		Q5	
	Before	After	Before	After	Before	After	Before	After	Before	After
Class A (N=24)	14	23	20	23	0	0	0	12	0	1
Class B (N=24)	17	22	15	21	3	3	2	10	1	4
Class C (N=25)	16	22	18	23	0	2	2	13	0	1
subtotal	47	67	53	67	3	5	4	35	1	6
Percentage	68%	97%	76%	97%	4%	7%	6%	51%	1%	9%

Figure 6. Students' success ate in solving word problems before and after the trial

After the trial, students' overall success rate (Figure 6) is increased by 29% in question 1, 21% in question 2, 3% in question 3, 45% in question 4 and 8% in question 5. Both test results show students performed better in question 2 than question 1. Question 1 is a subtraction problem and question 2 is an addition problem. Students made little progress in both questions 3 and 5 but had an extraordinary improvement in question 4. Many students were able to draw two separate bars in question 3 and show the different length of bars between the boys and girls in the post-test. Some of them used dotted line and extended the bar on the boys' to indicate that boys are 40 fewer than the girls (Figure A). Some of them also identified the equivalent quantity based on the diagram after adding 40 to the boy's bar. But overall the success rate in solving question 3 is the lowest among all of problems. In question 4, most students drew 3 equivalent bars for Katelyn and another one for Paige, showing the concept of "3 times as many as" (Figure 8). Some students used 'guess and check' strategy to work out the correct answer; others recognised 4 equivalent quantities so they halved 60 to get 30 and halved 30 to get 15 (Figure B). A lot of students in the post test correctly constructed the bar models for question 5 (Figure C) and used the fictitious weights for the tiger and leopard; some of them identified the equivalent quantities and solve the problem correctly (Figure D); however, most of them failed to progress further to solve the division equation due to relatively large numbers involved in the problem.

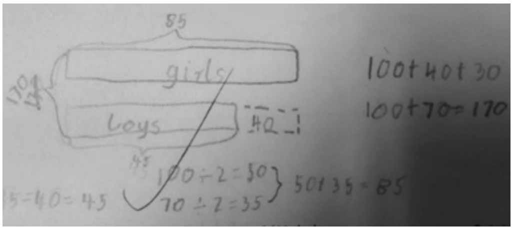

Figure A.

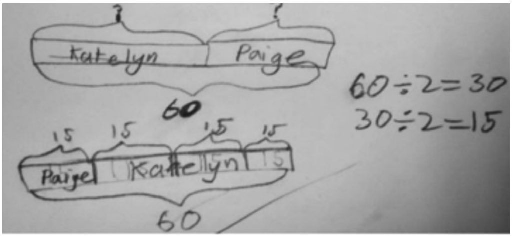

Figure B.

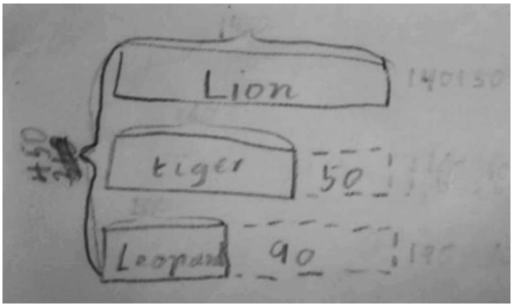

Figure C.

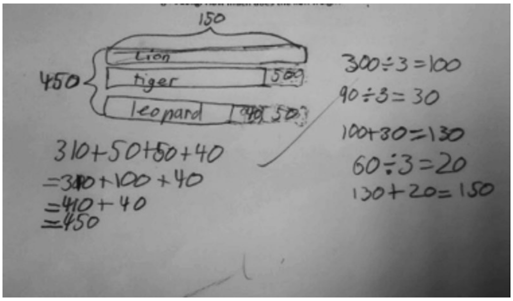

Figure D.

Findings

After the trial, students used the bar model and demonstrated their understanding of the part whole relationship in question 1. Students can clearly illustrate their sound understanding of the concepts involving ‘more than’, ‘less than’, ‘few than’ and ‘as many as’ in the post test. This study confirms that multi-step word problems involving additive and multiplicative structure create more barriers for the students to succeed.

The post-test results show that students were more likely to recognise the equivalent quantities in a multiplicative comparison model if the bars were drawn separately in a vertical way (Figure 7 and 8) rather than being drawn in a horizontal way (Figure 3 and 4). It also seemed easier for students to see the comparison units added onto or subtracted from the total quantity when the bars were vertically aligned. Students made the best progress in question 4 since the elements of a problem are presented including the concept of equivalent quantities. On the other hand, questions 3 and 5 are hard to deduce as students need to use fictitious quantities in order to create equivalent quantities. It is beneficial to use dotted lines for the comparison unit involving the terminology of “less than” or “fewer than” to connect the related unit; other coloured lines and braces could also be used to show the relationship among the known and unknown quantities and their comparison units, which will help students to understand the structure of the problem such as equivalent quantities.

After comparing the model drawing of correct solutions with those that resulted in wrong answers, it is evident that most students who accurately drew the bars and represented the problem situations on the bars successfully solved the problems. On the other hand, students who failed to draw the bars accurately could not utilise them to solve the problems.

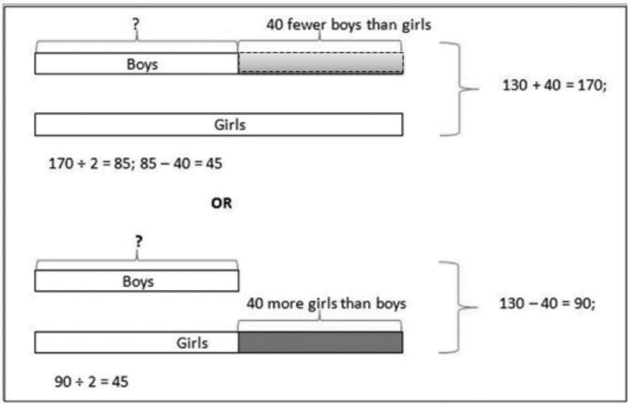

Figure 7.

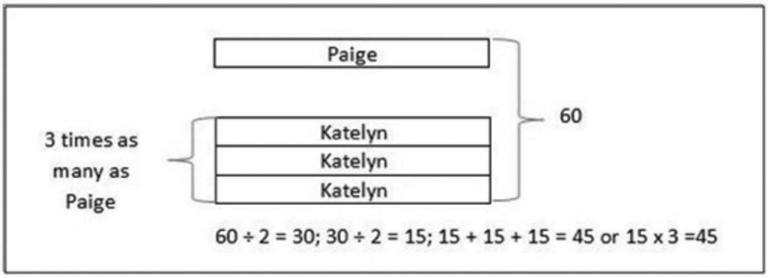

Figure 8.

Conclusion

The bar model method not only provides students with a visual aid but also helps students to present the problem situations and apply relevant concepts by drawing the rectangular bars. The purpose of drawing the models is not to teach students to follow specific rules but to provide students with a tool to understand the concepts involved in the problem, identify the operations they need and work out a strategy for finding the answer.

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IDENTIFYING A HIERARCHY OF REFLECTIVE PRACTICES

Yvonne Reilly and Jodie Parsons

Sunshine College

It is often the results of standardised test data which provide the evidence that a change in programs or practice may be required. Schools and systems analyse the available quantitative data to determine where changes need to occur and tracking of these data also allows schools to evaluate how successful the change has been. At Sunshine College this has certainly been the case, and whilst the quantitative data has been essential in initiating the change process and measuring success, it has been the collection, analysis and interpretation of qualitative data or as the authors describe it; Behavioural Indicator Data, which has driven the change across the school in both our mathematics program and our teacher practice. In our experience the reliability of the qualitative data is determined by the reflective capacity of the teacher and in this paper we discuss reflective practices and how we use team teaching to maximise their efficacy.

Introduction

The analysis of standardised test data such as NAPLAN (National Assessment Program - Literacy And Numeracy), PISA (Programme for International Student Assessment) and TIMSS (Trends in International Maths and Science Study) are often the incentive for education systems and individual schools to initiate changes which it is hoped will improve student outcomes. This data along with other standardised measures

(On-Demand, Victorian Curriculum and Assessment Authority) are then engaged to track the effectiveness of the change.

This was the case at Sunshine College; a multi-campus secondary school situated in the western suburbs of Melbourne. The college is situated within a community which suffers a high degree of disadvantage (SFO of 0.8 and an ICSEA OF 932). In 2007, a dedicated whole-school literacy program was introduced (SunLit) followed in 2009 by the introduction of a junior years maths program. The introduction of both programs heralded the beginning of a significant improvement in student learning outcomes as determined by NAPLAN, On-Demand and TORC (Test of Reading Comprehension) data.

During this same time period many other schools in the then Western Metropolitan Region (WMR) had the same priority and were exposed to a similar level of regional support, funding and professional development. Interestingly, however, very few schools were able to realize the same magnitude of improvement that Sunshine College achieved, being named as one of only five schools in Australia to "turn around" the effectiveness of the school (Grattan Institute Report 2014). The authors wanted to understand possible reasons for this disparity.

For the classroom teacher who is part of a school team striving to change student outcomes, the data most focussed on will be common assessment tasks, pre- and post- topic assessments, On Demand testing and NAPLAN data. Analysis of these data are essential components to driving change and justifying resourcing whilst providing a degree of accountability.

It is our contention, however, that it is not the application of the analysis of standardised testing data which contributes most to improved student outcomes, although this is undoubtedly an essential part of the process of change, but rather the analysis of, and response to, Behavioural Indicator Data (BID) which contributes most profoundly to improve student learning.

Behavioural Indicator Data

Behavioural indicators have been identified in many fields, such as learning difficulties, anxiety, stress, animal welfare, pain management, exploitation leadership, but very little, if any, work has been done which categorically identifies behaviours in learners which indicates learning is occurring or has occurred. This could be one of the reasons teachers rely so heavily on topic tests or other written assessments to determine whether a concept or process has been learned.

For the purposes of this paper we would like to clarify our working definition of BID. It encompasses all of the visible responses and non-responses that students display whilst in school. These include, but are not restricted to: punctuality, attitude to learning, levels of engagement, quality of completion of set tasks, conversations with peers or teachers, active listening, compliance with instructions and body language.

The very subjective, complex and subtle nature of BID makes it difficult to collect and even more difficult to categorise and agree on what it actually indicates. For example;

Behaviour	**Indicates**
Student swings on chair	Boredom
	or
	Inability to remain still
	or
	Defiance or non-compliance
	or
	Trying to reach something/someone

The reason for the behaviour could be any one of the above suggestions, but the response to it by the teacher will often reveal what the teacher believes about the cause of the behaviour. The teacher who "reads" the indicator as being of boredom is able to respond by providing something which alleviates the boredom. If, however the indicator is "read" as defiance or non-compliance, the teacher may respond by disciplining the student. An error in responding to this data can have a negative effect on the student-teacher relationship and potentially damage the student's belief in the teacher's efficacy.

Teacher beliefs, attitudes and values both influence and are influence by their knowledge of a subject and how it is taught and on the constraints and opportunities which arise whilst teaching (Sullivan *et al*, 2015). Ramsden (1992) described three main teacher types: the one who believes that teaching is a transfer of their own knowledge to students; the one who believes that teaching is about providing the student with an engaging activity and teacher who believes that teaching is about making it possible for students to learn. Askew (2002) finds similar teacher types in mathematics classes and names them Transmission, Discover and Connectionist respectively. He differentiates between these types depending upon the relationship displayed between teacher, student and mathematics knowledge.

The authors believe that a teacher's values and beliefs about teaching will determine their types of Behavioural Indicator Data which they are able to collect which, in itself is a facet of their reflection during and after each lesson. Our description of different reflector types is given in the next section

Reflective Practices

Reflection, as suggested by Moon in 2005, is applied to gain a better understanding of relatively complicated or unstructured ideas. It is such a widely held belief that reflective practice in the profession of teaching is of such singular importance that it not only constitutes part of every teacher education program but it is also a requirement of on-going professional practice.

Reflections should be

- Deliberate
- Purposeful
- Structured
- Able to link theory to practice
- About learning
- The catalyst to change and development (Scales, 2010)

The value and importance of teachers reflecting on what they are doing and how they are doing it has been expertly described by many (Dewey, 1933; Stenhouse, 1975; Korthagen, 1985; Gore, 1987; Schon, 1987;

Calderhead, 1989) but the purpose of this paper is to identify and organise common reflective practices in order to develop a hierarchy of their efficacy/importance as tools which elucidate behavioural indicator data.

In our experience working with teachers to read behavioural indicators we have noted three distinct types of teacher response to data that can be collected in the classroom; Compliance, Engagement and Learning.

The Compliance Reflector

This is where most teachers begin the development of their reflective practices, noticing only those behaviours which indicate that students are not complying with either teacher instructions or with school rules.

If a teacher requires all of their cognitive ability to ensure that the primary requirements of the students are being met in their classroom their remaining capacity to detect and decipher subtler levels of behavioural data that is available to be read is nil. This is often the case with pre-service teachers who literally cannot "see" an action which would be obvious to a more experience teacher. This type of practitioner appears to primarily value students getting the answer correct and following instructions.

However this type of reflector is not solely found among less experienced teachers with some experienced teachers believing that their professional obligation to teach has been fulfilled if, at the end of their lesson, no-one has been physically hurt. These teachers look to make changes in student behaviour and give feedback on flaws while praising compliant behaviours.

The Engagement Reflector

In our experience the vast majority of teachers are Engagement Reflectors. They are able to provide for the primary needs of their students with minimal effort and understand that students need to be engaged to learn. They are able to read compliance data, but also look for behaviours which indicate students are engaged in the task in hand. They look for activities to be completed and for their students to have fun. Engagement Reflectors look for behavioural indicators that students have enjoyed being in their class. This reflective practitioner will ask themselves why a student 'got it'.

Rollett (2001) describes the difference between an expert and a novice teacher as follows "Experts rely on a large repertoire of strategies and skills that they can call on automatically, leaving them free to deal with unique or unexpected events....The wealth of knowledge and routines that they employ, in fact, is so automatic that they often do not realize why they preferred a certain plan of action over another. However, when questioned, they are able to reconstruct the reasons for their decisions and behaviour." We would extend this to describe the difference between Compliance and Engagement Reflectors. These teachers also look to manage student behaviour, but in such a way that students enjoy being in their class.

The Learning Reflector

Learning Reflectors are teachers who not only read compliance and engagement data but are also actively seeking data on learning. These teachers not only question why this student 'got it' but also why another student did not. They do not look for tasks necessarily to be completed but instead value mistakes that are made and look for opportunities to pose questions which deepen student understanding.

Learning Reflectors use the four possible lenses, identified by Brookfield (1995) to evaluate their practices:

1. Autobiographically as learners and teachers
2. Their students' perspective
3. Their colleagues/peers perspective
4. Theoretical literature perspective

This contrasts with Compliance Reflectors who use only the first lens and the Engagement Reflector will use the first and the second lens on some students, but not all.

The main difference between the Engagement Reflector and the Learning Reflector is that the latter is actively looking for tasks which are not only engaging but directly address each learner's learning needs. This inevitably leads to the development of activities which are specifically tailored to an individual's needs which, in effect, means a differentiated classroom. These teachers look to make changes to self to improve what they bring to the class in order to improve the student outcomes.

The authors have previously described (Reilly and Parsons, 2011) how offering 'just right" tasks in a fully inclusive classroom is an effective way to differentiate lessons, but the ability to develop and refine resources such as "just right" tasks is dependent upon a teacher's ability to reflect.

Behavioural Indicator Data Collection and Analysis

The collection of BID and its subsequent analysis can be done by an individual teacher at any of the three levels but our observations of teacher reflections have shown that different partnerships often favour a particular type of reflective practice.

We observed that a non-expert or an inexperienced teacher who self-reflects or works with a mentor teacher is more likely to demonstrate Compliance Reflector attributes. More experienced teachers and those who work with a peer observer or a coach often demonstrate Engagement Reflector attributes, but it is not until you have either an outstanding practitioner or accomplished practitioners who are truly working in concert with each other, for example team teachers, that teachers really demonstrate Learning Reflector qualities.

We would emphasise that these levels do not necessarily follow the chronology of teaching years, but instead the development of a teaching belief system, where teachers who reflect on student learning as being a construct of student attitude are less likely to be looking for ways in which to change the activities or approaches (self) and more likely to believe any deficit is in the student, and look for ways to change the student such as; seating plans, behaviour cards, parent meetings, rewards, feel-good feedback.

Teachers who are trying to change self are more likely to offer individually tailored learning experiences, question students to promote understanding, give constructive feedback which leads to the next step i.e. a feedback loop, or subtly adjust approach to suit need.

A summary of our description of the ways teachers reflect can be seen in *Figure 1*.

Reflector	Data Collected	Partners	Response
Compliance	Answers are correct Class rules School rules	Self Mentor	Discipline Feedback on flaws
Engagement	Tasks are complete Students are engaged Students are having fun	Self Observer Coach	Cultivating relationships Feel-good feedback
Learning	Mistakes are valued Students are learning	Self Teacher Partner/ team	Differentiation Feedback Loop

Figure 1. A Hierarchy of Reflective Practices

Conclusion

The authors believe that teachers fall into one of three main categories of reflective practitioners; Compliance, Engagement or Learning, and that the key difference between these reflectors is their ability to collect, analyse and respond to behavioural indicator data.

Without ensuring that the dispositions of compliance and engagement are present with students, it is impossible for reflectors to read the much more subtle data that are available to Learning Reflectors. When reflecting on learning it is most important to consider why some students are learning whilst others are not and what the teacher can do differently. The Learning Reflector rarely concludes that a student is the reason that learning did not occur.

Teachers collecting, understanding and responding to the behavioural indicators of their own classrooms is the only way to determine what is really going on. They are best qualified to propose "...locally-based theories that recognise the idiosyncrasies of site-specific circumstances, and that acknowledge the integrity and worth of knowledge won by people at the workface" (described By Schon in Smyth 1993).

Becoming a Learning Reflector is a process of professional development which is ongoing and informed by educational research. The professional conversation between a teacher and an observer may get to the crux of the matter in a classroom, but the conversation between professionals who have a shared responsibility for their students' learning and have an equal stake in improving that learning for students is an additional impetus to uncover the real truth of the behaviour displayed and increase the chance of selecting the response which most benefits learning. In the authors' experience when more than one educator in the classroom has responsibility for the learning of a cohort of students, the identification of behavioural indicator data for learning is augmented.

At Sunshine College.is has been the development of teacher capacity to read and respond to student behavioural indicators that has led to the substantial improvements in students learning in mathematics. It has been achieved by teachers working in pairs and larger teaching teams to reflect on the wealth of data that is available in a classroom. Our collective responses to that data, and the relationship between teacher immediacy and effectiveness (Anderson, 1979 and Christophel 1990) has been increased. In short, a change in the quality of teacher reflective practices has allowed an effective program to be developed.

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PROBLEM BASED LEARNING AND INCORPORATING SUGATA MITRA'S RESEARCH IN MATHEMATICS TEACHING

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My name is Julie Andrews and I have been specialising in teaching primary school maths for over ten years. Now an Assistant Principal, I have continued my passion for teaching teachers mathematics content, and teaching teachers how to recognise developmental turning points in children's mathematical knowledge.

In my current position, I have continued to support the use of professional learning teams to promote collective responsibility for student learning outcomes, and the use of individual learning plans for students where the goal is succinct and achievable, based on important mathematical concepts, and known to the student and their parents. I use the mathematics online interview as my main tool in facilitating this process, based on the idea that achieving zone of proximal development is critical to improved student learning outcomes. Anyone who knows me knows my level of commitment to the principles outlined in this potted history. Teachers who work or have worked with me are committed to this approach because it makes sense and makes a difference. Up to this point in my career, however, there has been nothing new or surprising about my journey to explore best practice in teaching mathematics. Current mathematical research concurs with my methods. Recently, however, something unexpected happened which has changed my approach to teaching mathematics. To appreciate the significance of this, I need to tell you a bit about my journey.

One of the reasons for being a teacher is the joy of discovery as the never ending journey toward knowledge unfolds, in particular the "ah-ha" moments. My first "ah-ha" moment in teaching maths was as a classroom teacher when I “discovered” the mathematics online interview. I have since used this as my baseline strategy for teaching mathematics, both here and in NY. Up until the point where I began using the maths online interview, I had always taught a deficit model, that is, looked up the curriculum for the level I was teaching, test that content, and teach the elements that students did not get correct. The maths online interview changed my thinking into a developmental approach – identifying students’ zone of proximal development using developmental signposts and teaching each student at their point of need. Teaching this to others was my focus as an Early Years Mathematics Facilitator. My next area of interest was developing a bank of activities that matched these researched developmental mathematical turning points. Initially as a coach, I did a lot of modelling in classrooms. The activities I chose were intuitive rather than from a text and I received many requests to share my lesson plans with other teachers. This did not, however, provide a systemic method that all teachers could use to have consistent teaching of maths school-wide. I was fortunate enough to be trained as a Western Australia First Steps Mathematics Facilitator and trained staff at primary and secondary schools across the region. This provided me with a systemic way to link big mathematical ideas to a bank of high quality activities and to provide professional development to strengthen teachers’ content knowledge.

In 2006 I moved to New York City to assist others to apply these sound principles in a different setting. At this time, NYC had a very prescriptive mathematics program. The other consultants with whom I worked were very critical of this program, however, I identified many useful elements: there was a sound explicit mathematical focus for every lesson; the activities provided linked to the lesson focus and were generally of a high quality; and differentiation activities were provided for students above and below the expected level. One criticism was that it was probably pitched 12 months above the expectations for the year level, but this was consistent with setting high expectations for students. This program, however, was in general carried out very badly: materials provided for lessons were often left in their plastic, un-opened; teachers did not read the lessons before teaching them; and teachers often did not understand the mathematical focus of the lesson. This raised more questions for me as an educator. The NYC system contained many of the elements that I would consider important, clear mathematics, high quality activities and a systemic approach, and yet failed spectacularly. To try and assist teachers I designed assessments based on developmental turning points and aligned with the state standards and lessons in order to set individual learning goals for students. It was at this point that I shifted my focus to ensuring that students were aware of their own learning goals and worked toward them. I also looked at ways for teachers to work collectively to build ownership of their curriculum and student learning outcomes.

On my return to Australia I worked as a mathematics coach across many primary and secondary schools at a regional level. I was feeling fairly confident that I had locked in a successful approach: develop teacher content knowledge, ensure the students are being taught at point of need and ensure that students and their parents were aware of an achievable short term goal to work toward. By this stage, I was beginning to identify common barriers to students' learning of mathematics, and I agreed with many researchers that subitising and multiplicative thinking were key to ensuring that primary school students experienced success in mathematics. A focus on partitioning and known tens facts consistently showed significant improvement in student learning outcomes. I continued to focus on ensuring ongoing professional development was available to teachers in mathematics content and in fostering the collective ownership of student progress through Professional Learning Team structures. Using this approach, I was able to show significant improvements in student learning outcomes through data such as NAPLAN and presented my findings at conferences.

On taking up my second position as Assistant Principal at a very large primary school, my focus on mathematics lessened. I continued to support staff through the structures I set in place, however, tended to have a less hands on approach, rather, coaching my middle level leaders to coach others. I was given the opportunity, through the resourcefulness of committed young teachers on my staff, to take part in a trip to NYC as part of a mathematics study tour. I agreed as a mentor to my staff, hardly expecting that this would be a turning point in my thinking about best practice in teaching mathematics. I was introduced to a consultant, Chris Coombes, who was presenting his ideas based on a publication called "5 Practices for Orchestrating Productive Mathematics Discussions" *By Mary Kay Stein, Margaret Schwan Smith*. Initially, this did not strike me as a new way of thinking that would impact my practice significantly, however, I had the opportunity to see Chris present this material several times and in conjunction with the extended conversations I was having with the teachers on the New York trip about setting high expectations for students, I began to formulate a new hypothesis about best practice in teaching mathematics.

Chris proposed that central to the success of countries like Singapore in achieving success in the international tables, was the notion of having students struggle with mathematical problems, and hence students being active participants in the construction of knowledge. Essential to this end, therefore, was the teacher setting a problem that would actually challenge students. Philosophically, this was problematic for me as I was firmly entrenched in my belief that teachers should ensure that students were working in their zone of proximal development. During the professional development, Chris gave us difficult worded problems to complete. On one problem, which I had found particularly challenging, Chris asked me, "Would grade 5 students be able to solve this problem?" I replied that the problem was too difficult for grade 5 students. On subsequent days presenting, Chris asked me the same question and my reply was the same. On the third day I wondered, "Do I have low expectations of my students?"

As we further unpacked the mechanics of what a problem based lesson would look like in a classroom, I remembered a speaker I had seen at a Principal Conference some years earlier, Sugata Mitra. (His talk is available online as a Ted Talk for free at: http://www.ted.com/talks/sugata_mitra_the_child_driven_education?language=en) I decided that I would run an experiment of my own, combining the Sugata experiment and the "5 Practices for Orchestrating Productive Mathematics Discussions". To this end, I chose the problem Chris had given me and I had determined was too hard for grade 5 students: *Sarah had $179. She bought a necklace, a scarf and a note pad. The scarf cost three fifths the price of the necklace and the note pad cost one sixth the price of the scarf. After shopping she had $34.50 left. What price was each item?*

I asked the teacher I was with if I could come into her class to do this experiment. She was a little bit reluctant, telling me that the whole class had just failed their Fractions and Decimals Online Interview assessment, but agreed. I told quite a few people about the upcoming experiment, noting that it was doomed to failure and that I must be crazy. I had never been in this classroom before. I began the lesson by telling the Sugata story and explaining the hypothesis for my experiment: that collectively, children could solve hard problems that they would not otherwise be able to solve. I told them they could use the internet, calculators and move around as they wished. I said that if they chose to work in a group or work independently, that was their choice as either option would prove or disprove my hypothesis. I told them that the teachers had difficulty solving the problem and I did not expect them to be successful. I asked them to focus completely on the task at hand for the allotted 25 minutes and not give up. I gave them the problem written on a slip of paper with no clarification and told them to start work. I did not allow the teachers or integration staff present to read the question for those students unable to read the question independently. I had planned prompts, of "what are you doing well", based on the Sugata "grandmother" model; they were not needed. All students were fully engaged and on task, needing no prompting or redirection. The class contained a wide range of students all of whom were fully engaged. Several students googled "what is three fifths?" With 30 seconds left on the clock, one group solved the problem. It was a group of three girls, 2 who would usually be in the "medium" group, and one who would usually be in the "low" group. The children traditionally in the "high" group were not able to solve the problem.

Elated, but confused, I continued to explore these concepts with other classes and indeed the same class over the coming weeks. Using the combined "Sugata / 5 Practices" method, I had done virtually no teaching. I summised that minimal scaffolding to this process would improve student learning outcomes, however, in the following weeks this did not prove to be the case. Now I am left only with questions: I wonder about the importance the narrative played in inspiring these students? I wonder about the significance of the grouping of three students in solving the problem? I wonder if teachers have indeed created a culture of over-scaffolding and ignored the importance of contructivism? I do know that based on what I saw in that first lesson, I will never teacher mathematics the same way again.

When I have discussed my dilemma with others, they answer, "well surely you can do both?" i.e. have explicit teaching and problem based learning. "Why don't you do problem based learning once or twice a week and explicit teaching at students' point of need on the other days?" I am not sure, however, that I can justify wasting my students time in this manner. Clearly, there is the potential for students to make much greater gains than they are currently making and it would be negligent to not provide students with maximum opportunity to learn. The only way I could test this, however, would be to do problem based lessons every day for an extended period and then measure student learning outcomes. Would this result in an extended period where children fell behind in their learning for the sake of an experiment?

I think that there is a sensible third option. After the problem based lessons, students are very motivated to continue with the problem. They go home and tell their parents and their parents send me solutions. They seek me out when I am on yard duty to discuss the problem with me. It is in these discussions that students have "ah-ha" moments. In the 5 Practices model the teacher developmentally sequences the student work samples in the lesson conclusion and draws out the relevant mathematical concepts. It makes sense that once students are engaged with the content they are more likely to take on board explicit instruction. This is important, but not the only place teaching happens. During the problem based lessons I took as many photographs as possible at different stages of the process to inform my teaching. By analysing student work samples in detail later, I was able to gain a very good understanding of students' mathematical thinking, which gave me a much clearer idea of students' zone of proximal development than just their test results, even when the assessment tools previously used had been well research developmentally based assessment tools (e.g. mathematics online interview, the fractions online interview and the scaffolding numeracy in the middle years assessments). This meant I could set problems for the next lessons, which would allow students to focus on the particular areas they needed. Furthermore, it also allowed me to set individual goals for students so that I could praise them when I saw them exploring that area of mathematics in class.

The most tangible outcome thus far of the change in the approach to teaching mathematics has been the engagement and confidence of the students. The three children who solved the problem on the first day have now near celebrity status and enormously improved confidence. They definitely see themselves as mathematicians. Feedback from all the students has been extremely positive. Traditionally, mathematics teaching has eroded student confidence, resulting in a society where it is acceptable to say "I am not very good at maths". A change in this culture has got to be a change for the better.

I am writing this summary because I feel very strongly that it is important to record and share my experience. Watching the Sugata Ted Talk after my own experiences is very moving for me. Just as children learn better collectively and with discussion, so do we. It is my intention to contribute to the conversation. I believe that the greater the numbers of people that do so, the more likely that we can achieve success. In this new age of technology, given the number of teachers all focused on making a difference, we have a huge collective capacity to revolutionise education.

THE PRODUCT AND QUOTIENT OF TWO INDEPENDENT CAUCHY RANDOM VARIABLES

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The probability density function (pdf) of the product and quotient of two independent Cauchy random variables that each have a median equal to zero is calculated. The pdf of the product is calculated using both the method of distribution functions and the Change of Variable Theorem. The usual properties of a pdf are verified by direct calculation. Asymptotes and removable discontinuities ('holes') of the pdf are investigated. It is proved that the mean does not exist and that the cumulative distribution function is well-defined. The pdf of the quotient is calculated from the pdf of the product.

Introduction

A continuous random variable X has a Cauchy distribution if it has a probability density function (pdf) given by

$$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2 + (x - x_0)^2}, \qquad -\infty < x < +\infty \qquad \text{.... (1)}$$

where $x_0 \in R$ and $a > 0$ are constants (Evans, Hastings and Peacock 2000 p 48). The Cauchy distribution has a central peak and is symmetric. The location parameter x_0 specifies its centre and the shape factor a specifies its width. It can be shown that the median and the mode are both equal to x_0 and that the mean and variance do not exist (Kermond 2009 pp 112-113).

Let $C(x_0, a)$ denote a Cauchy distribution with location parameter x_0 and shape factor a; thus $X \sim C(x_0, a)$ denotes a random variable X with pdf given by equation (1). In this paper the pdfs of the product and quotient of two independent Cauchy random variables $X \sim C(0, a)$ and $Y \sim C(0, b)$, that is, two independent Cauchy random variables that each have a median equal to zero, are calculated. The pdf of the product is calculated using both the method of distribution functions and the Change of Variable Theorem and found to be

$$g(u) = \frac{2ab}{\pi^2 (u^2 - a^2 b^2)} \ln\left(\frac{|u|}{ab}\right), \quad -\infty < u < +\infty .$$

The properties $\int_{-\infty}^{+\infty} g(u)\, du = 1$ and $g(u) > 0$ of this pdf are verified by direct calculation. Asymptotes and removable discontinuities ('holes') of the pdf are also investigated. It is proved that the mean of U does not exist and that the cumulative distribution function (cdf) is well-defined.

The pdf of the quotient is calculated from the pdf of the product and found to be

$$h(u) = \frac{2ab}{\pi^2 (b^2 u^2 - a^2)} \ln\left(\frac{b|u|}{a}\right), \quad -\infty < u < +\infty .$$

Calculation of the PDF of the Product

Let $X \sim C(0, a)$ and $Y \sim C(0, b)$ be independent Cauchy random variables. Then the product $U = XY$ is a continuous random variable with support $-\infty < u < +\infty$. Since X and Y are independent, their joint pdf is given by

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y) = \frac{ab}{\pi^2} \frac{1}{(a^2 + x^2)(b^2 + y^2)} . \qquad \text{.... (2)}$$

The pdf of U can be calculated from equation (2) using either the method of distribution functions or the Change of Variable Theorem.

Method of Distribution Functions

The method of distribution functions involves finding the cumulative distribution function (cdf) of U and then differentiating it to get the pdf of U (Mendenhall, Scheaffer and Wackerley 1996 pp 231-240). To find the cdf of U it is helpful to consider the separate cases $u>0$ and $u<0$.

Case 1: $u > 0$.

The cdf of U is given by $G(u) = \Pr(U \le u) = \Pr(XY \le u)$.

Since $\Pr(X = 0) = \Pr(Y = 0) = 0$ it follows that if $XY \le u$ then $Y \le \dfrac{u}{X}$ if $X > 0$ and $Y \ge \dfrac{u}{X}$ if $X < 0$. Therefore

$$G(u) = \Pr(XY \le u) = \Pr\left(Y \le \frac{u}{X} \cap X > 0\right) + \Pr\left(Y \ge \frac{u}{X} \cap X < 0\right)$$

The two probabilities are calculated by integrating the joint pdf of X and Y over the regions of the xy-plane defined by the inequalities (i) $y \le \dfrac{u}{x}$ and $x > 0$ and (ii) $y \ge \dfrac{u}{x}$ and $x < 0$ respectively. These regions are shown in Figure 1(a) (note that $u > 0$). Therefore

$$G(u) = \int_{x=0}^{x=+\infty} \int_{y=0}^{y=u/x} f_{XY}(x,y)\, dy\, dx + \int_{x=-\infty}^{x=0} \int_{y=u/x}^{y=0} f_{XY}(x,y)\, dy\, dx. \qquad \text{.... (3)}$$

Substitute equation (2) into equation (3):

$$G(u) = \frac{ab}{\pi^2} \int_{x=0}^{x=+\infty} \int_{y=0}^{y=u/x} \frac{1}{(a^2 + x^2)} \frac{1}{(b^2 + y^2)}\, dy\, dx$$

$$+ \frac{ab}{\pi^2} \int_{x=-\infty}^{x=0} \int_{y=u/x}^{y=0} \frac{1}{(a^2 + x^2)} \frac{1}{(b^2 + y^2)}\, dy$$

$$= \frac{ab}{\pi^2} \int_{x=0}^{x=+\infty} \frac{1}{a^2 + x^2} \left(\int_{y=0}^{y=u/x} \frac{1}{b^2 + y^2}\, dy \right) dx + \frac{ab}{\pi^2} \int_{x=-\infty}^{x=0} \frac{1}{a^2 + x^2} \left(\int_{y=u/x}^{y=0} \frac{1}{b^2 + y^2}\, dy \right) dx$$

from a corollary to the strong form of Fubini's Theorem (Thomas and Finney 1996 p 1006). The pdf of U is given by:

$$g(u)=\frac{dG}{du}=\frac{ab}{\pi^2}\frac{d}{du}\int_{x=0}^{x=+\infty}\frac{1}{a^2+x^2}\left(\int_{y=0}^{y=u/x}\frac{1}{b^2+y^2}\,dy\right)dx$$
$$+\frac{ab}{\pi^2}\frac{d}{du}\int_{x=-\infty}^{x=0}\frac{1}{a^2+x^2}\left(\int_{y=u/x}^{y=0}\frac{1}{b^2+y^2}\,dy\right)dx$$

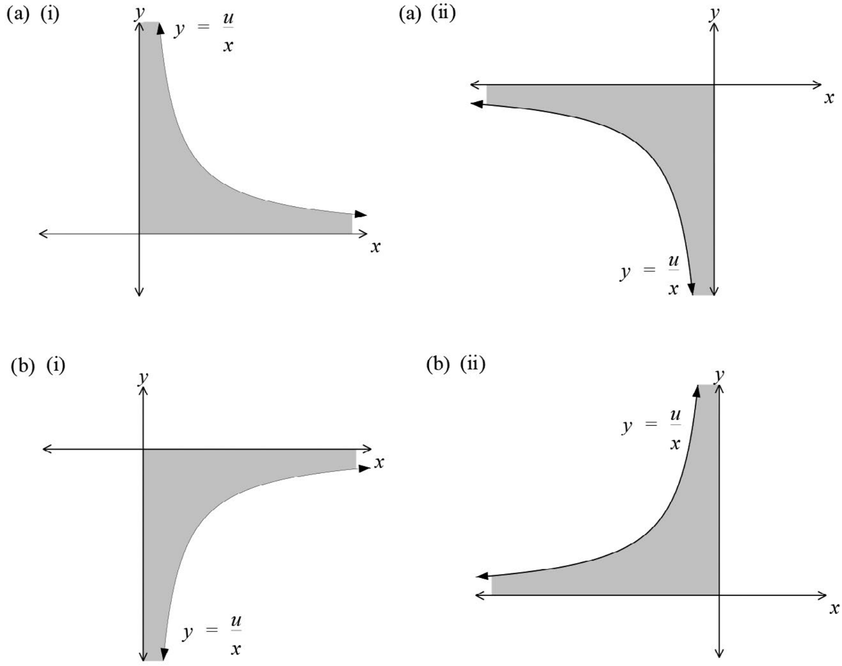

Figure 1. Regions (shaded) of the *xy*-plane defined by (a) (i) $y \le \frac{u}{x} \cap x > 0$ and (ii) $y \ge \frac{u}{x} \cap x < 0$ where $u > 0$, and (b) (i) $y \ge \frac{u}{x} \cap x > 0$ and (ii) $y \le \frac{u}{x} \cap x < 0$ where $u < 0$.

$$= \frac{ab}{\pi^2} \int_{x=0}^{x=+\infty} \frac{1}{a^2 + x^2} \frac{d}{du} \left(\int_{y=0}^{y=u/x} \frac{1}{b^2 + y^2} dy \right) dx$$

$$+ \frac{ab}{\pi^2} \int_{x=-\infty}^{x=0} \frac{1}{a^2 + x^2} \frac{d}{du} \left(\int_{y=u/x}^{y=0} \frac{1}{b^2 + y^2} dy \right) dx . \qquad \text{.... (4)}$$

The two derivatives in equation (4) can be calculated using the Fundamental Theorem of Calculus (Thomas and Finney 1996 p 333) and the chain rule:

- $$\frac{d}{du} \int_{y=0}^{y=u/x} \frac{1}{b^2 + y^2} dy = \frac{1}{b^2 + \left(\frac{u}{x}\right)^2} \cdot \frac{d}{du}\left(\frac{u}{x}\right)$$

$$= \frac{x^2}{(b^2 x^2 + u^2)} \cdot \frac{1}{x}$$

$$= \frac{x}{b^2 x^2 + u^2} . \qquad \text{.... (5)}$$

- $$\frac{d}{du} \int_{y=u/x}^{y=0} \frac{1}{b^2 + y^2} dy = -\frac{d}{du} \int_{y=0}^{y=u/x} \frac{1}{b^2 + y^2} dy$$

$$= \frac{-x}{b^2 x^2 + u^2} . \qquad \text{.... (6)}$$

Substitute equations (5) and (6) into equation (4):

$$g(u) = \frac{ab}{\pi^2} \int_{x=0}^{x=+\infty} \frac{x}{(a^2 + x^2)(b^2 x^2 + u^2)} dx + \frac{ab}{\pi^2} \int_{x=-\infty}^{x=0} \frac{-x}{(a^2 + x^2)(b^2 x^2 + u^2)} dx$$

$$= \frac{2ab}{\pi^2} \int_0^{+\infty} \frac{x}{(a^2 + x^2)(b^2 x^2 + u^2)} dx . \qquad \text{.... (7)}$$

Case 2: $u < 0$.

It will be convenient to express the cdf of U as $G(u) = 1 - \Pr(U > u)$. Then

$$G(u) = 1 - \Pr(XY > u)$$
$$= 1 - \Pr\left(Y > \frac{u}{X} \cap X > 0\right) - \Pr\left(Y < \frac{u}{X} \cap X < 0\right)$$
$$= 1 - \int_{x=0}^{x=+\infty} \int_{y=u/x}^{y=0} f_{XY}(x,y)\, dy\, dx - \int_{x=-\infty}^{x=0} \int_{y=0}^{y=u/x} f_{XY}(x,y)\, dy\, dx$$

where the integral terminals of the double integrals are suggested by the regions shown in Figure 1(b) (note that $u < 0$). Reversing the order of the integral terminals associated with the integration with respect to *y* gives

$$G(u) = 1 + \int_{x=0}^{x=+\infty} \int_{y=0}^{y=u/x} f_{XY}(x,y)\, dy\, dx + \int_{x=-\infty}^{x=0} \int_{y=u/x}^{y=0} f_{XY}(x,y)\, dy\, dx$$

and it obviously follows from equation (3) that equation (7) will be obtained.

Change of Variable Theorem

An integral that gives the pdf of U can also be found using the Change of Variable Theorem. The Change of Variable Theorem states (Mendenhall, Scheaffer and Wackerley 1996 p 185):

Suppose that X and Y are continuous random variables with joint pdf $f_{XY}(x, y)$. Let $U = U(X, Y)$ and $V = V(X, Y)$ define a one-to-one transformation between the points (x, y) and (u, v) so that the equations $u = U(x, y)$ and $v = V(x, y)$ may be uniquely solved for x and y in terms of u and v, say $x = x(u, v)$ and $y = y(u, v)$. Then the joint pdf of U and V is

$$g_{UV}(u,v) = f_{XY}[x(u,v),\ y(u,v)]\,|\,J(u,v)\,|, \quad J(u,\ v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix},$$

if the point (u, v) is in the range of the transformation and zero otherwise. $J(u, v)$ is the Jacobian of the inverse transformation.

Consider the transformation $\left.\begin{matrix} U = XY \\ V = Y \end{matrix}\right\} \Rightarrow \begin{matrix} X = \dfrac{U}{V} \\ Y = V \end{matrix}$ so that $x = \dfrac{u}{v}$ and $y = v$. Then the joint pdf of U and V is

$$g_{UV}(u,v) = f_{XY}[x(u,v), y(u,v)]\,|\,J\,| \qquad \text{.... (8)}$$

where $J = \begin{vmatrix} \frac{1}{v} & 0 \\ -\frac{u}{v^2} & 1 \end{vmatrix} = \frac{1}{v}$. .... (9)

Substitute equations (2) and (9) into equation (8):

$$g_{UV}(u,v) = \frac{ab}{\pi^2}\frac{1}{\left(a^2+\left(\frac{u}{v}\right)^2\right)(b^2+v^2)}\frac{1}{|v|}$$

$$= \frac{ab}{\pi^2}\frac{1}{|v|}\frac{v^2}{(a^2v^2+u^2)(b^2+v^2)}.$$

The pdf of U is given by the marginal density function of U:

$$g(u) = \int_{-\infty}^{+\infty} g_{UV}(u,v)\,dv$$

$$= \frac{ab}{\pi^2}\int_{-\infty}^{+\infty}\frac{1}{|v|}\frac{v^2}{(a^2v^2+u^2)(b^2+v^2)}\,dv. \qquad \text{.... (10)}$$

Substitute $\frac{v^2}{|v|} = \begin{cases} \frac{v^2}{v} = v & \text{if } v > 0 \\ \frac{v^2}{-v} = -v & \text{if } v < 0 \end{cases}$ into equation (10):

$$g(u) = \frac{ab}{\pi^2}\int_{0}^{+\infty}\frac{v}{(a^2v^2+u^2)(b^2+v^2)}\,dv + \frac{ab}{\pi^2}\int_{-\infty}^{0}\frac{-v}{(a^2v^2+u^2)(b^2+v^2)}\,dv$$

$$= \frac{2ab}{\pi^2}\int_{0}^{+\infty}\frac{v}{(a^2v^2+u^2)(b^2+v^2)}\,dv. \qquad \text{.... (11)}$$

The integrals in equations (7) and (11) are equal since by symmetry $U = XY$ is unchanged if a and b are interchanged. In fact, equation (7) is obtained from the Change of Variable Theorem if the transformation $\left.\begin{array}{l} U = XY \\ V = X \end{array}\right\} \Rightarrow \begin{array}{l} X = V \\ Y = \frac{U}{V} \end{array}$ is used.

The integral in equation (11) can be calculated using the technique of partial fraction decomposition (Stewart 2003 pp 496-503):

$$\frac{v}{(a^2v^2+u^2)(b^2+v^2)} = \frac{Av+B}{a^2v^2+u^2} + \frac{Cv+D}{b^2+v^2}$$

where $A = \frac{-a^2}{u^2-a^2b^2}$, $C = \frac{1}{u^2-a^2b^2}$ and $B = D = 0.$

Substitute this decomposition into equation (11):

$$g(u) = \frac{2ab}{\pi^2(u^2-a^2b^2)}\int_0^{+\infty} \frac{v}{b^2+v^2} - \frac{a^2v}{a^2v^2+u^2}\,dv.$$

This integral is improper (Stewart 2003 pp 530-537, Thomas and Finney 1996 pp 594-602) since the upper bound is infinite. Therefore:

$$g(u) = \frac{2ab}{\pi^2(u^2-a^2b^2)}\lim_{\beta\to+\infty}\int_0^{\beta} \frac{v}{b^2+v^2} - \frac{a^2v}{a^2v^2+u^2}\,dv$$

$$= \frac{ab}{\pi^2(u^2-a^2b^2)}\lim_{\beta\to+\infty}\left[\ln\left(\frac{b^2+v^2}{a^2v^2+u^2}\right)\right]_0^{\beta}$$

$$= \frac{ab}{\pi^2(u^2-a^2b^2)}\lim_{\beta\to+\infty}\left[\ln\left(\frac{b^2+\beta^2}{a^2\beta^2+u^2}\right) - \ln\left(\frac{b^2}{u^2}\right)\right]$$

$$= \frac{ab}{\pi^2(u^2-a^2b^2)}\lim_{\beta\to+\infty}\ln\left(\frac{u^2}{b^2}\frac{(b^2+\beta^2)}{(a^2\beta^2+u^2)}\right). \qquad \text{.... (12)}$$

The limit in equation (12) can be calculated as follows:

$$\lim_{\beta\to+\infty}\ln\left(\frac{u^2}{b^2}\frac{(b^2+\beta^2)}{(a^2\beta^2+u^2)}\right) = \ln\lim_{\beta\to+\infty}\left(\frac{u^2}{b^2}\frac{(b^2+\beta^2)}{(a^2\beta^2+u^2)}\right)$$

using a theorem on the limit of a composite function (Stewart 2003 p 129)

$$= \ln\left(\frac{u^2}{b^2} \lim_{\beta \to +\infty} \left(\frac{\frac{b^2}{\beta^2} + 1}{a^2 + \frac{u^2}{\beta^2}} \right) \right)$$

$$= \ln\left(\frac{u^2}{a^2 b^2} \right). \qquad \text{.... (13)}$$

Substitute equation (13) into equation (12):

$$g(u) = \frac{ab}{\pi^2 (u^2 - a^2 b^2)} \ln\left(\frac{u^2}{a^2 b^2} \right) = \frac{2ab}{\pi^2 (u^2 - a^2 b^2)} \ln\left(\frac{|u|}{ab} \right). \qquad \text{.... (14)}$$

Equation (14) is in agreement with the special cases $a = b = 1$ (Rider 1965 p 304) and $b = a$ (Springer 1979 p 158 Question 4.5).

The product has also been calculated by Kermond (Kermond 2010 pp 6-7) using the following theorem (Curtiss 1941 p 420, Epstein 1948 p 371, Rohatgi 1976 p 141):

Suppose that X and Y are two independent and continuous random variables with pdf's f(x) and g(y) respectively. Then the product $U = XY$ is a continuous random variable with pdf given by

$$h(u) = \int_{-\infty}^{+\infty} \frac{1}{|y|} f\left(\frac{u}{y} \right) \cdot g(y)\, dy = \int_{-\infty}^{+\infty} \frac{1}{|x|} f(x) \cdot g\left(\frac{u}{x} \right) dx.$$

An algorithm for implementing this result, along with an implementation of the algorithm in a CAS, has been given by Glen, Leemis and Drew (Glen, Leemis and Drew 2004).

Calculation of $\int_{-\infty}^{+\infty} g(u)\, du$

$$\int_{-\infty}^{+\infty} g(u)\, du = \frac{2ab}{\pi^2} \int_{-\infty}^{+\infty} \frac{1}{u^2 - a^2 b^2} \ln\left(\frac{|u|}{ab} \right) du$$

$$= \frac{4ab}{\pi^2} \int_{0}^{+\infty} \frac{1}{u^2 - a^2 b^2} \ln\left(\frac{u}{ab} \right) du$$

by symmetry. Substitute $u = abt$:

$$\int_{-\infty}^{+\infty} g(u)\, du = \frac{4}{\pi^2}\int_{0}^{+\infty} \frac{\ln(t)}{t^2 - 1}\, dt\,. \qquad \text{.... (15)}$$

The integral in equation (15) is improper (Stewart 2003 pp 530-537, Thomas and Finney 1996 pp 594-602) because the integrand is undefined at the lower bound $t = 0$, not continuous at the interior point $t = 1$ and the upper bound is infinite. Since $t = 1$ is an interior point of discontinuity the integral must be split up at $t = 1$:

$$\int_{-\infty}^{+\infty} g(u)\, du = \frac{4}{\pi^2}\int_{0}^{1} \frac{\ln(t)}{t^2 - 1}\, dt + \frac{4}{\pi^2}\int_{1}^{+\infty} \frac{\ln(t)}{t^2 - 1}\, dt\,. \qquad \text{.... (16)}$$

Consider $\displaystyle\int_{1}^{+\infty} \frac{\ln(t)}{t^2 - 1}\, dt$ and substitute $t = \dfrac{1}{w}$:

$$\int_{1}^{+\infty} \frac{\ln(t)}{t^2 - 1}\, dt = \int_{1}^{0} \frac{\ln\left(\frac{1}{w}\right)}{\left(\frac{1}{w}\right)^2 - 1}\left(-\frac{dw}{w^2}\right)$$

$$= \int_{0}^{1} \frac{\ln(w)}{w^2 - 1}\, dw$$

$$= \int_{0}^{1} \frac{\ln(t)}{t^2 - 1}\, dt \qquad \text{.... (17)}$$

changing the dummy variable of integration. Substitute equation (17) into equation (16):

$$\int_{-\infty}^{+\infty} g(u)\, du = \frac{8}{\pi^2}\int_{0}^{1} \frac{\ln(t)}{t^2 - 1}\, dt$$

$$= \frac{8}{\pi^2} I \qquad \text{.... (18)}$$

$$\text{where } I = \int_0^1 \frac{\ln(t)}{t^2 - 1}\, dt . \qquad \text{.... (19)}$$

The integral in equation (19) can be calculated by first noting that

$$\frac{1}{1-t^2} = 1 + t^2 + t^4 + \cdots + t^{2n} + \cdots = \sum_{n=0}^{+\infty} t^{2n}, \quad |\, t^2 \,| < 1 \Rightarrow -1 < t < 1 \qquad \text{.... (20)}$$

using the formula for the sum of an infinite geometric series. Substitute equation (20) into equation (19):

$$-I = \int_0^1 \sum_{n=0}^{+\infty} t^{2n} \ln(t)\, dt .$$

The series can be integrated term-by-term since it is uniformly convergent (Apostol 1981 p 226):

$$-I = \int_0^1 \ln(t)\, dt + \int_0^1 t^2 \ln(t)\, dt + \int_0^1 t^4 \ln(t)\, dt + \cdots + \int_0^1 t^{2n} \ln(t)\, dt + \cdots$$

$$= I_0 + I_2 + I_4 + \cdots + I_{2n} + \cdots , \quad I_m = \int_0^1 t^m \ln(t)\, dt . \qquad \text{.... (21a)}$$

I_m is not valid for $t = 1$ due to the restriction in equation (20) and is improper because the integrand is undefined at the lower bound $t = 0$. Therefore:

$$I_m = \lim_{\beta \to 0^+} \lim_{\delta \to 1} \int_\beta^\delta t^m \ln(t)\, dt . \qquad \text{.... (21b)}$$

I_m can be calculated using integration by parts (Stewart 2003 pp 475-479):

$$\int t^m \ln(t)\, dt = \ln(t) \frac{t^{m+1}}{m+1} - \frac{1}{m+1} \int t^m \, dt$$

$$= \ln(t) \frac{t^{m+1}}{m+1} - \frac{1}{(m+1)^2} t^{m+1} + C .$$

Substitute this anti-derivative into equation (21b):

$$I_m = \lim_{\beta \to 0^+} \lim_{\delta \to 1} \left[\ln(t) \frac{t^{m+1}}{m+1} - \frac{1}{(m+1)^2} t^{m+1} \right]_\beta^\delta$$

$$= \lim_{\beta \to 0^+} \lim_{\delta \to 1} \left(\ln(\delta) \frac{\delta^{m+1}}{m+1} - \frac{1}{(m+1)^2} \delta^{m+1} - \ln(\beta) \frac{\beta^{m+1}}{m+1} + \frac{1}{(m+1)^2} \beta^{m+1} \right)$$

$$= \lim_{\beta \to 0^+} \left(\frac{-1}{(m+1)^2} - \ln(\beta) \frac{\beta^{m+1}}{m+1} + \frac{1}{(m+1)^2} \beta^{m+1} \right)$$

$$= \frac{-1}{(m+1)^2} - \frac{1}{m+1} \lim_{\beta \to 0^+} \left(\beta^{m+1} \ln(\beta) \right). \qquad .... (22)$$

The limit in equation (22) has the indeterminant form $0 \cdot \infty$ and can therefore be calculated using l'Hospitals Rule (Stewart 2003 p 308):

$$\lim_{\beta \to 0^+} \left(\beta^{m+1} \ln(\beta) \right) = \lim_{\beta \to 0^+} \left(\frac{\ln(\beta)}{\frac{1}{\beta^{m+1}}} \right) = -(m+1) \lim_{\beta \to 0^+} \left(\frac{\frac{1}{\beta}}{\frac{1}{\beta^{m+2}}} \right)$$

$$= -(m+1) \lim_{\beta \to 0^+} (\beta^{m+1}) = 0. \qquad .... (23)$$

Substitute equation (23) into equation (22):

$$I_m = \frac{-1}{(m+1)^2}. \qquad .... (24)$$

Substitute equation (24) into equation (21a):

$$I = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \mathrm{L} + \frac{1}{(2n+1)^2} + \mathrm{L} = \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2}. \qquad .... (25)$$

The sum in equation (25) can be calculated in a non-rigorous way using the famous result $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Finch 2003 p 40, Kalman 2002):

$$I = \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2}$$

$$= \sum_{n=1}^{+\infty} \frac{1}{n^2} - \sum_{n=1}^{+\infty} \frac{1}{(2n)^2}$$

$$= \frac{3}{4}\sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{\pi^2}{8}. \qquad \text{.... (26)}$$

Substitute equation (26) into equation (18): $\int_{-\infty}^{+\infty} g(u)\,du = 1.$

The sum in equation (25) can be calculated in a rigorous way using Fourier analysis. The Fourier series for the function $h(t) = |t|$, $-\pi < t < \pi$, is

$$h(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2}\cos\left([2n-1]t\right)$$

(Solymar 1988 pp 23-24) and the result given in equation (26) immediately follows from $h(0) = 0$.

The integral in equation (19) can also be calculated using the amazing relationship between the Gamma function $\Gamma(s) = \int_0^{+\infty} x^{s-1}e^{-x}dx$ (Havil 2003 pp 53-54) and the zeta function $\varsigma(s) = \sum_{n=1}^{+\infty}\frac{1}{n^s}$ (Havil 2003 p 37):

$$\int_0^{+\infty}\frac{x^{s-1}}{e^x - 1}\,dx = \Gamma(s)\varsigma(s), \quad s > 1 \qquad \text{.... (27)}$$

(Havil 2003 pp 53-54). Substitute $x = -\ln(t)$ into equation (19):

$$I = \int_0^{+\infty}\frac{xe^x}{e^{2x} - 1}\,dx. \qquad \text{.... (28)}$$

The expression $\frac{e^x}{e^{2x}-1}$ can be re-written as

$$\frac{e^x}{e^{2x}-1} = \frac{(e^x+1)-1}{(e^x-1)(e^x+1)} = \frac{(e^x+1)}{(e^x-1)(e^x+1)} - \frac{1}{(e^x-1)(e^x+1)}$$

$$= \frac{1}{e^x-1} - \frac{1}{e^{2x}-1}. \qquad \text{.... (29)}$$

Substitute equation (29) into equation (28): $I = \int_0^{+\infty}\frac{x}{e^x-1}\,dx - \int_0^{+\infty}\frac{x}{e^{2x}-1}\,dx.$.

Make the substitution $z = 2x$ in the second integral:

$$I = \int_0^{+\infty} \frac{x}{e^x - 1}\, dx - \frac{1}{4}\int_0^{+\infty} \frac{z}{e^z - 1}\, dz .$$

Change the dummy variable of integration in the second integral:

$$I = \int_0^{+\infty} \frac{x}{e^x - 1}\, dx - \frac{1}{4}\int_0^{+\infty} \frac{x}{e^x - 1}\, dx$$

$$= \frac{3}{4}\int_0^{+\infty} \frac{x}{e^x - 1}\, dx . \qquad \text{.... (30)}$$

Comparing equation (30) with equation (27), substituting $s = 2$ and using $\Gamma(s) = (s-1)!$ when s is a positive integer (Havil 2003 p 53) gives

$$I = \frac{3}{4}\Gamma(2)\varsigma(2) = \frac{3}{4}(1!)\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{8}.$$

Proof that $g(u) > 0$

To prove $g(u) > 0$ for $-\infty < u < +\infty$ it is sufficient to prove the inequality $\dfrac{\ln|t|}{t^2 - 1} > 0$ for $-\infty < t < +\infty$ (since $a > 0$ and $b > 0$). $\dfrac{\ln|t|}{t^2 - 1} > 0$ when either:

- $\ln|t| > 0$ <u>and</u> $t^2 - 1 > 0 \Rightarrow t > 1$ or $t < -1$, or
- $\ln|t| < 0$ <u>and</u> $t^2 - 1 < 0 \Rightarrow -1 < t < 0$ or $0 < t < 1$.

Furthermore, $\lim\limits_{t \to 0} \dfrac{\ln|t|}{t^2 - 1} = +\infty$ and it is proved later that $\lim\limits_{t \to \pm 1} \dfrac{\ln|t|}{t^2 - 1} = 1$. It follows that for $\dfrac{\ln|t|}{t^2 - 1} > 0$ for $-\infty < t < +\infty$.

Features of the Graph of the PDF

Asymptotes

Vertical Asymptotes

The line $u = 0$ is a vertical asymptote of the pdf since $g(u)$ is undefined at $u = 0$:

$$\lim_{u \to 0} g(u) = \frac{2ab}{\pi^2} \lim_{u \to 0}\left[\frac{1}{u^2 - a^2 b^2} \ln\left(\frac{|u|}{ab}\right)\right] = +\infty .$$

Horizontal Asymptotes

The u-axis is a horizontal asymptote of the pdf. Consider the limit

$$\lim_{u \to \pm\infty} g(u) = \frac{2ab}{\pi^2} \lim_{u \to \pm\infty}\left[\frac{1}{u^2 - a^2 b^2} \ln\left(\frac{|u|}{ab}\right)\right].$$

Substitute $u = abt$:

$$\lim_{u \to \pm\infty} g(u) = \frac{2}{ab\pi^2} \lim_{t \to \pm\infty}\left[\frac{\ln|t|}{t^2 - 1}\right].$$

This limit has the indeterminant form $\frac{\infty}{\infty}$ and can therefore be calculated using l'Hospital's Rule:

$$\lim_{u \to \pm\infty} g(u) = \frac{2}{ab\pi^2} \lim_{t \to \pm\infty} \frac{\frac{1}{t}}{2t} = \frac{1}{ab\pi^2} \lim_{t \to \pm\infty} \frac{1}{t^2} = 0 .$$

Removable Discontinuities ('Holes')

$g(u)$ has the indeterminant form $\frac{0}{0}$ when $u = \pm ab$, indicating the possibility that $g(u)$ has removable discontinuities ('holes') at $u = \pm ab$ (Stewart 2003 p 125, Thomas and Finney 1996 pp 87, 226). Consider the limit

$$\lim_{u \to \pm ab} g(u) = \frac{2ab}{\pi^2} \lim_{u \to \pm ab}\left[\frac{1}{u^2 - a^2 b^2} \ln\left(\frac{|u|}{ab}\right)\right]$$

$$= \frac{2}{ab\pi^2} \lim_{t \to \pm 1} \frac{\ln|t|}{t^2 - 1}.$$

This limit can be calculated using l'Hospital's Rule:

$$\lim_{u \to \pm ab} g(u) = \frac{1}{ab\pi^2} \lim_{t \to \pm 1} \frac{1}{t^2} = \frac{1}{ab\pi^2}.$$

Since this limit is finite, $g(u)$ has removable discontinuities ('holes') at $u = \pm ab$.

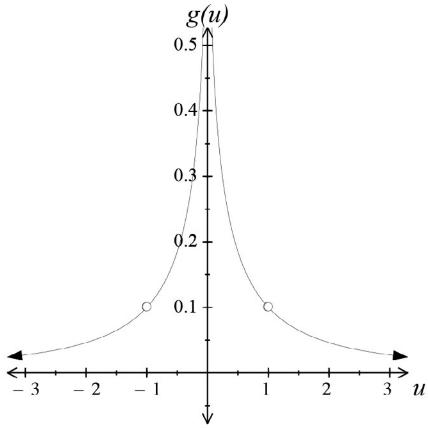

Figure 2. The graph of $g(u) = \dfrac{2}{\pi^2(u^2-1)} \ln|u|$.

A graph of the pdf for the case $a = b = 1$ is shown in Figure 2. It has removable discontinuities ('holes') at the points $\left(1, \dfrac{1}{\pi^2}\right)$ and $\left(-1, \dfrac{1}{\pi^2}\right)$.

Note that although $g(u)$ is undefined at $u = 0$ and indeterminant at $u = \pm ab$, there is no requirement that the pdf of a random variable be continuous over its support (Romano and Siegel 1986 p 29).

Mean

By definition:

$$E(U) = \int_{-\infty}^{+\infty} u \cdot g(u)\, du$$

$$= \frac{2ab}{\pi^2} \int_{-\infty}^{+\infty} \frac{u}{u^2 - a^2b^2} \ln\left(\frac{|u|}{ab}\right) du.$$

Substitute $u = abt$:

$$E(U) = \frac{2ab}{\pi^2}\int_{-\infty}^{+\infty}\frac{t\ln|t|}{t^2-1}\,dt. \qquad \text{.... (31)}$$

It is tempting to think that the integral in equation (31) is equal to zero (the Cauchy principle value) since the integrand is an odd function. However, for this integral to exist, both the improper integrals $\int_{-\infty}^{b}\frac{t\ln|t|}{t^2-1}\,dt$ and $\int_{b}^{+\infty}\frac{t\ln|t|}{t^2-1}\,dt$ must be finite for all $-\infty < b < +\infty$ (Apostol 1981 p 277) and this is not the case. For example, $\int_{1}^{+\infty}\frac{t\ln|t|}{t^2-1}\,dt$ is divergent by the comparison test (Stewart 2003 p 536):

- $\frac{t\ln|t|}{t^2-1} = \frac{t\ln(t)}{t^2-1} > \frac{t\ln(t)}{t^2} = \frac{\ln(t)}{t}$ for $t > 1$.
- $\int_{1}^{+\infty}\frac{\ln(t)}{t}\,dt = \int_{0}^{+\infty} w\,dw = +\infty$

where the substitution $w = 1\text{n}(t)$ has been made.

· Therefore $\int_{1}^{+\infty}\frac{t\ln|t|}{t^2-1}\,dt$ is divergent by the comparison test.

It follows that the integral in equation (31) is divergent and so the mean of U does not exist. It also follows that the variance of U does not exist (since variance is defined with respect to the mean).

Cumulative Distribution Function

The cdf of U is

$$G(u) = \int_{-\infty}^{u} g(w)\,dw$$

$$= \frac{2ab}{\pi^2} \int_{-\infty}^{u} \frac{1}{w^2 - a^2 b^2} \ln\left(\frac{|w|}{ab} \right) dw.$$

Substitute $w = abt$:

$$G(u) = \frac{2}{\pi^2} \int_{-\infty}^{u/(ab)} \frac{\ln|t|}{t^2 - 1} \, dt.$$

Although $g(u)$ is indeterminant for $u = \pm ab$ and undefined for $u = 0$, it follows from symmetry and equations (15), (19) and (26) that $G(0) = \frac{1}{2}$ (since the pdf is an even function), $G(ab) = \frac{3}{4}$ and $G(-ab) = \frac{1}{4}$. For example:

$$G(ab) = \frac{2}{\pi^2} \int_{-\infty}^{1} \frac{\ln|t|}{t^2 - 1} \, dt$$

$$= \frac{2}{\pi^2} \int_{-\infty}^{0} \frac{\ln|t|}{t^2 - 1} \, dt + \frac{2}{\pi^2} \int_{0}^{1} \frac{\ln(t)}{t^2 - 1} \, dt$$

$$= \frac{2}{\pi^2} \int_{0}^{+\infty} \frac{\ln(t)}{t^2 - 1} \, dt + \frac{2}{\pi^2} \int_{0}^{1} \frac{\ln(t)}{t^2 - 1} \, dt$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} g(t) \, dt + \frac{2}{\pi^2} I$$

$$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

It follows that the cdf is continuous over $-\infty < u < +\infty$ and therefore well-defined.

Calculation of the PDF of the Quotient

It is well known that if $X \sim C(0,a)$ then $\frac{1}{X} \sim C\left(0, \frac{1}{a}\right)$ (Evans, Hastings and Peacock 2000 p 50). This result can be readily proved using either the method of distribution functions or the Change of Variable Theorem. It has been proved by Kermond using the fact that the quotient of two independent normal random variables each with a mean equal to zero is a Cauchy distribution with a median equal to zero (Kermond 2010 pp 9-11).

Therefore the quotient $U = \frac{X}{Y}$ can be considered as a product of the independent random variables $X \sim C(0,a)$ and $\frac{1}{Y} \sim C\left(0,\frac{1}{b}\right)$. It follows from equation (14) that the pdf of the quotient is

$$h(u) = \frac{\frac{2a}{b}}{\pi^2\left(u^2 - \frac{a^2}{b^2}\right)} \ln\left(\frac{b\,|\,u\,|}{a}\right)$$

$$= \frac{2ab}{\pi^2(b^2u^2 - a^2)} \ln\left(\frac{b\,|\,u\,|}{a}\right) \qquad \text{.... (32)}$$

with support $-\infty < u < +\infty$. Equation (32) becomes $h(u) = \frac{2\ln|\,u\,|}{\pi^2(u^2-1)}$ when $a = b$.

The pdf of U can also be found using either the method of distribution functions or the change of variable theorem. Both methods give integrals that are equivalent to $h(u) = \frac{2ab}{\pi^2}\int_0^{+\infty} \frac{y}{(a^2+u^2y^2)(b^2+y^2)}\,dy$ from which equation (32) follows.

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BEYOND THE RATIONAL

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Rational and irrational numbers arise in Year 10A of the Australian Curriculum. What should students learn about irrational numbers? Why should students learn about irrational numbers? What interesting exercises about irrational numbers are suitable for Year 10A? How can we connect irrational numbers to other parts of mathematics, or other fields of study? We consider these questions in light of the aims of the Australian Curriculum. Also, we explain why irrational numbers are fascinating.

Introduction

According to the Australian Curriculum, students in Year 10A are expected to be able to "define rational and irrational numbers and perform operations with surds and fractional indices". (All quotes in this paper are from the Australian Curriculum v. 7.4 at http://www.australiancurriculum.edu.au/.)

Let us recall the definitions of rational and irrational numbers (Garner et al. (2012, p. 26).

A rational number is a number that can be expressed as m/n where m,n are integers and $n \neq 0$. An irrational number is a real number that cannot be expressed as the ratio of two integers.

In this paper we will address the following questions. What should students learn about irrational numbers? Why should students learn about irrational numbers? What interesting exercises about irrational numbers are suitable for Year 10A? How can we connect irrational numbers to other parts of mathematics, or other fields of study? We consider these questions in light of the aims of the Australian Curriculum. Finally, we will share our fascination with irrational numbers.

What Should Students Learn about Irrationals?

Definitions are important in life, and mathematics is one subject where good use is made of definitions. The above quote from the Australian Curriculum makes it clear that students ought to be able to define rational and irrational numbers. It is worthwhile to emphasise that irrational numbers are defined as numbers that do not have a particular property: they cannot be expressed as a ratio of two integers.

Do Irrational Numbers Exist?

Irrational numbers do exist. Since the proof that $\sqrt{2}$ is irrational is well known, we present a different example. Consider $\log_{10}(3)$ (the logarithm of 3 to the base 10). This number is positive. If it is rational, then there exist positive integers m,n such that $\log_{10}(3) = m/n$, and so $10^{(m/n)} = 3$, or $10^m = 3^n$. This is impossible because 10^m is an even number (when $m > 0$) and 3^n is an odd number. Therefore $\log_{10}(3)$ cannot be rational; it must be irrational. This cute argument is suitable for Year 10A students once they have encountered logarithms.

In general, it is not straightforward to show that a particular number is irrational. You cannot do it on a calculator. For example, we know that π and e are irrational – but proving these facts is beyond Year 10A. And, surprisingly, nobody knows whether $\pi + e$ is rational or irrational! Mathematicians have proved that e^{π} is irrational, but nobody knows whether π^e is rational or irrational.

Once irrational numbers have been defined, it is reasonable to expect that students know that irrational numbers exist. Someone should prove to them that irrational numbers exist. Perhaps they can be convinced by the argument above about $\log_{10}(3)$ or the classical arguments about $\sqrt{2}$. However, asking students whether a given number is rational or irrational is not suitable for Year 10A students. If the number is rational then the question is trivial; if the number is irrational then students cannot be expected to understand why the number is irrational except in a few special cases.

Surds

A "surd" is usually defined as an irrational number that is a root of a rational number; see for example Garner et al. (2012, p. 26). The ability to "perform operations with surds and fractional indices" is important, but it is part of arithmetic and algebra, and not connected with irrational numbers *per se.*

Why Should Students Learn about Irrationals?

Irrational numbers do not seem to be useful in any application that we might regard as practical. Even the irrationality of π is not relevant to its many applications. There is a challenge in demonstrating to students that irrational numbers are relevant to "their personal, work and civic life".

The Number Line

On the other hand, the concepts of rational and irrational numbers illustrate the beauty and complexity of the simple number line.

The number line, sometimes called the real line, contains all the real numbers. Every point on the line represents exactly one real number, and every real number is represented by exactly one point on the line. At first sight, the real line seems to be a rather boring object.

Between any two rational numbers on the real line, you will find another - just take the average of them. Indeed, any interval on the real line will contain infinitely many rational numbers. Rational numbers are almost everywhere on the real line.

Suppose that you remove the point representing the integer 7 from the number line. What do you see? Although there is a gap at 7, you would not notice it because it is infinitely small. Now remove another point, say the one corresponding to the integer 5. Still, you cannot notice the difference. In fact, if you remove all the points corresponding to integers you will not notice any difference in the real line.

Now here is an astonishing fact. Even if you remove all the points corresponding to rational numbers, you will not notice the difference!

We have an amazing state of affairs. On one hand, every interval on the real line, no matter how tiny, contains infinitely many rational numbers. On the other hand, if you remove all the rational numbers from the real line, you do not notice any gaps in the line. Almost every real number is irrational. If you remove all the points corresponding to irrational numbers, the line would disintegrate! Here we have been using ideas about measure, topology and infinity informally. Stillwell (2010, Chapter 1) provides a more formal description of the situation.

The concepts of rational and irrational numbers make the real line more complicated – and more interesting – than it first seems. They illustrate that "mathematics has its own value and beauty".

Even a brief reference to these ideas in the classroom contributes to realising one of the aims of the Australian Curriculum, namely "to ensure that students ... develop an increasingly sophisticated understanding of mathematical concepts".

Exercises on Irrationals

The study of irrational numbers cannot be conducted to any substantial depth in Year 10A: the subject gets too hard too quickly. Even the question "Is π rational or irrational?" is not suitable for Year 10A students because they cannot be expected to be able to explain or understand why this number is irrational. The challenge is to find exercises on irrational numbers that are mathematically sound, and suitable for Year 10A students.

It has been often said that proof is the glue that holds mathematics together. We can use irrational numbers to introduce students to this essential part of mathematics. Here we suggest some exercises about irrational numbers that may be suitable for the Year 10A classroom. Most of these problems involve very simple proofs. A teacher might show the students how to solve these problems and then let them reproduce it. The satisfaction comes from seeing, comprehending, sharing and reproducing a short proof in its entirety.

- Explain why the following numbers are rational numbers:
  $\frac{22}{7}, 4, -\frac{5}{10}, \frac{2\sqrt{3}}{7\sqrt{3}}, \frac{(\sqrt{3}+1)(\sqrt{3}-1)}{4}, 1.21, 1.\dot{2}, 1.\dot{2}\dot{1}.$
- Prove that the sum of two rational numbers is a rational number. [Hint: Show that $2/3 + 4/5$ is rational; then generalise to $m/n + p/q$.]
- Prove that it is possible for the sum of two irrational numbers to be a rational number. [Hint: You may use the fact that $\sqrt{2}$ is irrational.]
- In the previous two problems, replace "sum" by "product". [Hint: You may use the fact that $\sqrt{2}$ is irrational.]
- Prove that the average of two rational numbers is a rational number.
- You have just proved that the average of two rational numbers must be a rational number. Use this repeatedly to prove that, between any two given rational numbers, there are infinitely many rational numbers.
- Prove that it is possible for the average of two irrational numbers to be a rational number. [Hint: You may use the fact that $\sqrt{2}$ is irrational.]
- If we assume that π is an irrational number, prove that $\pi + 5$ is also an irrational number.
- The number $\varphi = (1+\sqrt{5})/2$ is a famous irrational number, sometimes called the golden ratio. Prove that $\varphi^2 = \varphi + 1$. Then use this to show that $\varphi^3 = 2\varphi + 1$ and $\varphi^4 = 3\varphi + 2$.
- Prove that $\varphi = 1 + \frac{1}{1+\frac{1}{\varphi}}$. This, and many other identities, can be found in Posamentier & Lehmann (2012).

- A calculator says that $\sqrt{2} = 1.4142136$. Prove that this is not correct by calculating the exact value of $(1.4142136)^2$.
- Prove that $\log_{10}(5)$ is irrational. Prove that $\log_{10}(2)$ is irrational. Then observe that $\log_{10}5 + \log_{10}2 = 1$ which is not rational.
- Write a short letter to a Year 9 student in which you explain the difference between rational and irrational numbers.

This is a short list – but irrational numbers form only a small part of Year 10A.

Collectively, these problems cover several general capabilities recommended for Year 10A in the Australian Curriculum such as "comprehend texts", "understand learning area vocabulary", "compose texts", "identifying, exploring and organising information and ideas" and "develop ICT capability". The problems also offer opportunities for students to "work collaboratively in teams" which is a personal and social capability listed for the curriculum. Giving students in Year 10A a little experience in reproducing "previously seen simple mathematical proofs" prepares them for more advanced work in Specialist Mathematics.

Irrationals and Other Topics

One of the aims of the Australian Curriculum is to ensure that students "recognise connections between the areas of mathematics and other disciplines". How can we connect the study of rational and irrational numbers to other topics?

We have shown how knowledge of rational and irrational numbers increases our understanding of the number line.

In the language of the Australian Curriculum, rational numbers have decimal expansions that are either "terminating" or "recurring and non-terminating". Irrational numbers have decimal expansions that are non-terminating and non-recurring. Consideration of irrational numbers in Year 10A deepens the understanding of decimal representations encountered in Year 8.

The exercises listed above demonstrate the power of algebra.

The example about $\log_{10}(3)$, and some of the exercises above, link irrational numbers to logarithms, and indices both of which are encountered in Year 10A.

Thinking about rational and irrational numbers leads to thinking about infinity. Although the Year 10A classroom is not the place to discuss infinity, a discussion of rational and irrational numbers is on the verge of touching on this subject that has intrigued human beings for thousands of years. Stillwell (2010), and, to a lesser extent, Rucker (1982) discuss the association between rational and irrational numbers, and infinitely small and infinitely large numbers.

In her biography of Pythagoras, Ferguson (2010), explains why the discovery of irrational numbers was so shocking; Fritz (1946) presents a mathematical, but readable, account of the discovery of irrationals; Havil (2012) describes the history of irrational numbers from ancient Greece to modern times. Studying irrational numbers affirms that mathematics "has its origin in many cultures".

As shown in one of the exercises above, the golden ratio $\varphi = (1 + \sqrt{5})/2$ is a solution a quadratic equation – another topic in Year 10A. It also has connections with many fields quite different from mathematics; see Rossi (2004) and Posamentier & Lehmann (2012).

Advances in knowledge often stem from making connections between ideas. The topic of irrational numbers allows students to make such connections. Connecting ideas brings strength and cohesion to our understanding of the world.

The Fascination of Irrationals

In this section, we share our fascination with irrational numbers and stray from our emphasis on the Year 10A classroom.

Irrational numbers play a deep role in other branches of mathematics. A number of results in mathematics are based on the irrationality of some certain numbers. Irrational numbers are a main part of the system of real numbers, which is the foundation of analysis. The proof of irrationality of particular numbers usually requires deep methods from analysis, and hence, provides an impetus for advances in mathematical analysis.

While irrational numbers have almost no application in practice, they provide examples of numbers that have no pattern in the digits in their decimal representation. This motivates computer scientists to attack the problem of computing their digits to a large number of decimal places. Such problems provide an impetus for advances in computer science. For other applications in information sciences see Schroeder (1997).

Conclusions

Rational and irrational numbers form a small topic in Year 10A of the Australian Curriculum in Mathematics. Discussion of these numbers can advance several aims and aspirations of the Australian Curriculum. By considering one topic, one gains a deeper appreciation of the Australian Curriculum.

We conclude with an exercise. Let $z = x^y$. There are four possibilities: x might be rational or irrational; y might be rational or irrational. Find examples that show that, in each of the four cases, z can be rational or irrational. [Hint: There are many solutions. We used only the facts that $\sqrt{2}$ is irrational, $\sqrt{3}$ is irrational, $\log_2 3$ is irrational, and the laws of indices and logarithms. See also Jones & Toporowski (1973).]

Acknowledgements

The second author thanks Patrick Mills (Year 10, Girton Grammar School Bendigo) for inspiring his interest in this topic. We thank Christopher Lenard for helpful discussions and suggestions. Bendigo Health is supported by the Victorian government.

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MAKING THE MOST OF THE MATHS IN THE OLYMPICS

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Teachers aim to embed school mathematics in relevant and engaging contexts. The Olympic Games provides an ideal opportunity for students to experience the use of mathematics content that they are learning to an event beyond the classroom in sporting and other contexts, as well as making connections across the curriculum. This article will outline how most aspects of mathematics and the proficiencies can be included as part of a program focusing on the Olympics.

Why Use the Olympic Games in Your Program?

Embedding school mathematics in relevant and engaging contexts for students is advocated in the Australian Curriculum - Mathematics (Australian Curriculum, Assessment and Reporting Authority, 2013). What could be more relevant and engaging for students in Australia, which is such a sporting nation, than an International sporting event of such diversity as the Olympic Games.

There will be approximately 205 nations participating in the Olympic Games at Rio de Janeiro. The countries range from the world's largest populations, such as China and India with populations of around 1.4 and 1.2 billion, to some of the world's smallest populations such as St Helena, a 308 square kilometre island in the Atlantic, with a population of 7000.

Sports at the Olympic Games

There will be 28 different sports and 306 different events, including Archery, Athletics, Badminton, Basketball, Beach Volleyball, Boxing, Canoeing, Cycling – BMX, Cycling – Mountain Bike, Cycling – Road, Cycling – Track, Diving, Fencing, Equestrian events, Equestrian – Dressage, Football, Golf, Gymnastics, Hockey, Judo, Netball, Pentathlon, Rowing, Rugby, Sailing, Shooting, Softball, Squash, Swimming, Synchronized swimming, Table Tennis, Taekwondo, Tennis, Volleyball, Water Polo, Weightlifting, Wrestling.

The sport and mathematics connection provides a wonderful opportunity for students to experience the use of the content that they are learning at school to the society beyond the classroom. This event is also a rich context for the application of mathematics topics not necessarily related to sport, for example the exploration of the countries competing in terms of their populations, area, team size in relation to population etc. Virtually every aspect of mathematics could be included as part of a unit focussing on the Olympic Games, as well as the potential to integrate this with all learning areas across the curriculum.

Mathematics and the Olympic Games

All mathematics domains and topics can be linked to the Olympic Games. Here are some suggestions.

Statistics and Probability

- Predict whether there are families from countries participating in the Olympic Games in the school community. Survey and graph nationalities of students in the school/year level/class.
- Students develop questions to collect data on favourite sports/games; sports played by students; favourite event in the Olympic Games. Explore ways of representing data.
- Students predict the number of medals certain countries will win (students choose countries) and justify their prediction. Keep a medal tally over the Games. Compare at end and discuss.
- Students collect personal data during sport (running, jumping, swimming, goals shot/shots on goal, ...) and represent graphically comparing these with elite athletes.

- Investigate the schedule of events (available on the Rio de Janeiro Olympic website).

Measurement and Geometry

- Estimating, then measuring lengths (100m, 200m, 400m, 1500m ..., long/high jump, shot put, javelin records) and time/measure themselves 'having a go';
- Younger students could start with 10/20/50m and conduct races, time student races and compare with Games' records;
- Students estimate the height/length of the Games record jumps;
- Make sets of weights (appropriate to age/size of students), conduct weight lifting events and compare with records – what is the difference?
- Dimensions and volume of the Games swimming pool;
- Distance travelled and time taken for athletes to travel from their country to Rio de Janeiro (include time zones);
- Shapes of different arenas and the reasons for these shapes;
- Shapes of track and field event areas;
- Design of equipment – shapes involved;
- Designing logos and flags;
- Location of Rio de Janeiro and each country on a world map (include longitude and latitude for older students).

Number and Algebra

- Ordinal number can be explored when young students are performing in a sports day, or having races with toys (there are some picture story books to use as stimulus for this);
- Tallying medals; combinations of gold, silver, bronze if there are 'x' medals;
- Order numbers of athletes in each country's team;
- Compare/order populations of the Olympic countries;
- Cost of tickets;
- Explore decimals by comparing results of competitors in an event (timing, height/length/mass); comparing winners with past events; comparing with students' results;
- Investigate scoring of different events;
- Numbers of volunteers/officials needed in the various sports;
- Explore catering quantities by using local data (canteen/catering outlet ...);
- Explore waste management by investigating this at school level first;
- Speed of athletes/swimmers/cyclists – how can this be determined?
- Investigate fitness measures (pulse rate, heart rate at rest and when exercising);

- Investigate gearing on bikes.

Links to the Australian Curriculum – Proficiency strands can also be made using this context. For example:

Understanding: Aim to build robust knowledge of adaptable and transferable mathematical concepts, connect related concepts and develop the confidence to use the familiar to develop new ideas, and the 'why' as well as the 'how' of mathematics.

Fluency: work to develop skills in choosing appropriate procedures, carrying out procedures flexibly, accurately, efficiently and appropriately, and recalling factual knowledge and concepts readily.

Problem solving: Aim to develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively.

Reasoning: Aim to develop increasingly sophisticated capacity for logical thought and action – analysing, evaluating, explaining, inferring, proving, justifying and generalising.

This list is designed to trigger other ideas. I'm sure by working together with your Professional Learning Team, the list will expand. There is also the possibility of developing integrated units, such as 'Plan a day at the Olympic Games' OR 'Plan a trip to Brazil to attend the Games', which could include many of the above, with students being personally involved in the decision making as an individual, or planning as a group.

Available Resources

There are many resources that could be used to support the development of mathematics and integrated units.

- *Prime Number*, Volume 15 No.2, July 2000, Mathematical Association of Victoria. This was a special Olympics edition of this journal with many ideas and resources that could be used or adapted.
- *Prime Number*, Volume 19 No.2, June 2004, Mathematical Association of Victoria. This journal included more ideas to be added to those in the earlier publication.
- *Prime Number*, Volume 29, No.3. 2014, Mathematical Association of Victoria. This journal included ideas to be used for the Commonwealth Games that are also suitable for the Olympics.
- *MCTP Professional Development Package. Activity Bank Volume 1*, Curriculum Corporation, 1992. This invaluable resource includes the use of Olympic events such as 100m and 200m running focusing on pace length and running pace, men's and women's marathon, long jump, high jump, swimming and more.
- *Maths300*, if your school subscribes to this web-based resource, includes activities that can be related to the Games: 'Potato Olympics'; 'Pulse Rates'; 'Country Maps' ...

- Children's literature, a great context for maths investigations, includes titles that can be linked, for example *Koala Lou*, Mem Fox; *The Possum Creek Olympics*, Dan Vallely ...
- Media – local and metropolitan newspapers, magazines, television and radio are reporting on the build-up to the Games. Collecting these from now on could provide useful material to be used both before and during the Games.
- Rio 2016 Olympics: http://www.olympic.org/rio-2016-summer-olympics
- International Olympic Committee: http://www.olympic.org/ioc
- Australian Olympic site: http://corporate.olympics.com.au/

With all the above ideas, added to by those of your own and other teachers at your school, you are encouraged to start planning early to make the most of this world event. Go for gold!

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PREPARING SECONDARY MATHEMATICS TEACHERS: A REVIEW OF RESEARCH

Gregory Hine

The international literature base concerning the preparation of pre-service secondary mathematics teachers has grown steadily over the past two decades, yet there appears to be no consensus on a best practice approach. A review of three research projects (2 American; 1 Australian) that focus on different aspects of secondary mathematics teacher preparation provides consideration for universities wishing to strengthen existing programs.

Introduction

The tertiary training of pre-service secondary mathematics teachers is pivotal in their professional preparation and formation as qualified educators. With large numbers of Australian secondary mathematics teachers set to retire in the next decade (Weldon, 2015), schools are already experiencing difficulties replacing those leaving with suitably qualified staff. In a recent review of research, Hine (2015) noted that over the past two decades there has been a growing body of literature concerning the prepara-tion of pre-service mathematics teachers (PSMTs). Specifically, research into teacher education programs have examined how pre-service teachers (PST) are prepared with respect to their pedagogical content knowledge (PCK) (Beswick & Goos, 2012) and mathematical content knowledge (MCK) (Meany & Lange, 2012). Other efforts have scrutinised how to best support pre-service primary and secondary teachers' PCK (Aguirre, del Rosario Zavala, & Katanyoutanant, 2012) and MCK (Stohlmann, Moore, & Cramer, 2013), the effects of mathematical content units on PST (Matthews, Rech, & Grandgenett, 2010), and the effects of mathematical pedagogy units on PST (Sowder, 2007). Moreover, various commentators herald the importance of reinforcing theory and practice within teacher-education programs (Emerick, Hirsch, & Berry, 2003; Miller & Davidson, 2006; TEMAG, 2014). For instance, the Teacher Education Ministerial

Advisory Group (TEMAG) (2014, p. x) asserted that pre-service teachers must develop a "solid understanding of teaching practices that are proven to make a difference to student learning". Despite the extensive literature no consensus exists on how to adequately train pre-service mathematics teachers (Chapman, 2005). This paper will present key findings of three research projects which each examine a different aspect of mathematics teacher preparation. First, the work of Cox et al. (2013) examines how capstone courses in the United States have been recommended as a way for pre-service secondary mathematics teachers to connect the mathematics they learn at universi-ty to the mathematics they will teach to high school students. Second, Cavanagh and Garvey's (2012) investigation of an Aus-tralian collaborative learning community that influenced the professional development experiences of PSMTs. Third, Bonner, Ruiz, and Travis (2013) compared the results of alternatively and traditionally trained PSMTs completing a mathematics con-tent licensure examination within a large American university. The purpose of this paper is to present key findings from each project for the consideration of mathematics teacher-educators. Such consideration may inform and strengthen overall current approaches used in the preparation of PSMTs for the profession.

Capstone Courses

Commencing with the recommendation from the Conference Board of the Mathematical Sciences (CBMS) for PSMTs to complete "a 6-hour capstone course connecting their college mathematics courses with high school mathematics" (2001, p. 8), Cox et al. (2013) sought to adjudge the current status of capstone courses in secondary mathematics teacher education pro-grams. Specifically, these researchers surveyed departments of mathematics and education within the United States to deter-mine (i) the extent to which capstone courses were offered to PSMTs, and (ii) how such courses were designed. Within the study, the term *capstone course* was used according to the definition provided by Loe and Rezak (2006). Such a course must have been studied at the conclusion of a program of study for PSMTs, fulfilling at least one the following four criteria: (1) bridges content between upper-level mathematics courses, (2) makes connections to high school mathematics, (3) gives addi-tional exposure to mathematics content in which students may be deficient, and/or (4) provides experiences communicating with and about mathematics (Loe & Rezak, 2006).

From the 200 departments of mathematics and education surveyed, 73 responded to the online questionnaire developed by the researchers. Of the 73 responding departments, 42 indicated that they offered a course that satisfied at least one of the criteria for capstone courses. Moreover, the capstone courses offered in 26 of these 42 departments were classified as CBMS courses, insomuch as they aligned with the CBMS purpose to explore connections between college mathematics and secondary school mathematics. According to Cox et al. (2013), the collected data provided insight into three major aspects of the capstone courses: (1) Course Goals, (2) Logistics, and (3) Participants. Although the researchers noted a distinction between the goals of CBMS aligned and non-CBMS aligned courses, the most commonly stated course goal overall was for students to develop a deeper understanding to mathematics. To illustrate, specific course goals included students clearly writing and communicating mathematics (CBMS), and investigating a substantial mathematical topic and learning advanced mathematics independently (non-CBMS). A logistical review of capstone courses uncovered information regarding the prerequisites, topics, and resources of such courses. To commence, it was found that calculus and linear algebra were the most commonly listed prerequisites. Again, distinctions between CMBS and non-CBMS capstone courses emerged; 65% of CBMS courses required two or fewer upper division prerequisites (e.g. probability, calculus-based statistics, non-Euclidian geometry, abstract algebra, and real anal-ysis) while only 46% of non-CBMS courses required two or fewer. Topically, the CBMS courses examined more secondary mathematics content and pedagogical concerns than non-CBMS courses; all of the latter's courses addressed advanced mathe-matical topics. A variety of electronic (e.g. Dynamic Geometric Software, Microsoft Excel) and print (texts, Education Organi-sation Standards) resources helped to develop and support both CBMS and non-CBMS courses. However, the development of CBMS courses far exceeded that of non-CBMS courses through the guidance of national organisations (e.g. National Council of Teacher of Mathematics) and recommendations (e.g. National Advisory Board Recommendations), as well as by education-al organisation standards (Common Core State Standards).

A review of the capstone course participants revealed that all students in non-CBMS aligned courses were mathematics majors. For CBMS aligned courses, six capstone courses were described as being required specifically for PSMTs. Within most de-partments (both CBMS and non-CBMS), students intending to be mathematics teachers did not populate exclusively the cap-stone courses. As Cox et al. (2013, p. 7) asserted, "only six capstone courses (all CBMS) reported that they were exclusively for students seeking teaching licensure". These researchers found a variety of capstone university lecturer backgrounds, cate-gorised as hailing from: mathematics, mathematics education, both mathematics and mathematics education, or neither. Lec-turers of CBMS courses were more likely to have a mathematics education or both mathematics education and mathematics background, while non-CBMS lecturers had backgrounds in mathematics. Overall, Cox et al. (2013)

concluded that ten years after the initial CBMS recommendation, course alignment with this recommendation appears not to be widespread. Conse-quently these researchers conceded that capstone courses for future teachers may be difficult to implement in institutions serv-ing a small number of PSMTs. Nonetheless, a defining feature of the current state of capstone courses is the variety of forms – not least with reference to the goals, logistics and participants.

A Professional Learning Community

Cavanagh and Garvey (2012) examined how a group of nine Australian pre-service teachers (PSTs) and their supervisors par-ticipated in a professional experience learning community for secondary mathematics. Commencing with the notion of a learn-ing community as a group of people "involved in some kind of activity that learn together and, more importantly, learn from each other" (Ponte et al., 2009, p. 197), the participating PSTs engaged in various activities to develop their professional prac-tice. Prior to the commencement of the study, a secondary Head Teacher and the PSTs methods lecturer agreed to establish a learning community partnership between the school and the university. The researchers chiefly wished to investigate the par-ticipants' views about the learning community and to identify if it helped the PSTs to develop their professional practice. As Cavanagh and Garvey (2012, p. 61) noted, the teacher and lecturer " envisioned the learning community as an extended series of school visits (incorporating lesson observations, co-teaching, and sustained opportunities for discussion and critical reflec-tion) combined with complementary activities during the university methods workshops". Over the course of one academic year the PSTs were all completing a Graduate Diploma of Secondary Education at an Australian university. The PSTs and methods lecturer visited the school in groups for Year 8 mathematics lessons. Following each observed or co-taught class the teacher, lecturer and PSTs discussed the lesson for approximately 15 minutes. The methods lecturer attended all visits, and undertook a role of observation and facilitation. Specifically, this lecturer observed all lessons, participated in the post-lesson discussions, and used field notes (from school visits) to frame subsequent workshop discussions at university. Within two days following each lesson, the PSTs also wrote and posted online a personal reflection for perusal and commentary by the other learning community members. Neither the teacher nor the lecturer contributed to the online forum.

Cavanagh and Garvey (2012) anticipated that the school visits would be opportunities for PSTs to learn from observing an experienced mathematics teacher present a series of problem-solving lessons. During the data collection stage of this research (i.e. through questionnaires, focus group interviews and reflections) PSTs reported that much fruitful learning took place when they co-taught, observed and discussed each other's lessons. The reflection-based activities also received significant mention from the PSTs. Specifically, Cavanagh and Garvey (2012, p. 69) highlighted how high quality reflection was achieved "through a combination of individual and group tasks, both oral and written, which encouraged pre-service teachers to think more deeply about their experiences". In addition, these authors asserted how the synchrony of reflective activities was achieved through interweaving school visits with follow-up activities in methods workshops; such synchrony helped to mirror the classroom experiences and reinforce the practice of reflection. Participants also commented on how their cohort developed a greater appreciation of the importance of problem solving as a practical way of teaching mathematics. Furthermore, partici-pant testimony revealed that they felt encouraged to develop further their professional knowledge of mathematics teaching. Overall, the learning community activities assisted PSTs in establishing strong links between theory and practice, and opportu-nities for co-teaching and peer observation allowed participants to collaborate and support each other's learning.

Traditional Versus Alternatively Prepared Content Knowledge

Bonner et al. (2013) investigated the extent to which Alternative Mathematics Teacher Education Programs (ATEPs) prepared teachers with the content knowledge needed to teach secondary mathematics. At one large, public university in the southern United States, the researchers reported on a quantitative analysis comparing scores between traditionally and alternatively pre-pared teachers on a secondary mathematics state licensure test. For this project, the ATEPs were defined as "teacher education programs that enrol non-certified individuals with at least a bachelor's degree offering shortcuts, special assistance, or unique curricula leading to eligibility for a standard teaching credential" (Adelman, 1986, p. 2). According to Bonner et al. (2013), ATEPs currently exist in all 50 states within the United States and have been a source of heated debate for some time. On one hand, supporters of ATEPs view such programs as viable means for universities and school districts alike to address teacher shortages and recruitment issues rapidly. Furthermore, Bonner et al. (2013, p. 1) contend that ATEPs may have the ability to recruit "highly skilled individuals (such as second career teachers or mathematics and science majors) who might not other-wise find their way into the field of education and can provide an expedited path to an advanced degree for in-service teach-ers". Conversely, opponents to these programs assert that by expediting the teacher preparation process, ATEPs provide a less rigorous teaching qualification which may undermine traditional programs and weaken overall the quality of the teaching pro-fession. Moreover, some teachers who have completed ATEPs report feeling underprepared for the classroom and perceive a lack of support from programs (Foote et al., 2011).

At the researchers' site, aspirant secondary mathematics teachers can follow one of two options for certification. The tradition-al certification plan is where undergraduate students major in mathematics while taking courses in the College of Education (COE) during their undergraduate degree. Following this traditional plan, students complete at least 45 hours of mathematics content courses, including "Modern or Abstract Algebra, Real Analysis, and a Capstone Course that focuses on connections between college and high school level mathematics" (Bonner et al., 2013, p. 4). These students are also required to fulfil various teaching certification requirements and complete practicum teaching experiences. Non-traditional students can seek certifi-cation through a post-graduate program or via ATEP, operated from within the COE. All students who elect for the non-traditional plan are college graduates with STEM or human-service related degrees and second career professionals. Upon ad-mission, students undertake 24-27 hours of teacher certification coursework, core graduate-level coursework in the COE, and 30 hours of field observations.

In summary, the researchers wished to discern any differences between the cohorts of traditionally and non-traditionally pre-pared mathematics teachers by comparing the performances of these individuals on the secondary mathematics certification examination. This examination is comprised of six mathematical domains: *Number Concepts, Patterns and Algebra, Geometry and Measurement, Probability and Statistics, Mathematical Processes and Perspectives,* and *Mathematical Learning, Instruc-tion, and Assessment.* The data were gathered from the examination results of 89 pre-service teachers (20 ATEP, 69 traditional). While the researchers noted that neither cohort had a passing mean for the examination overall, they found significant dif-ferences in scores in the following domains: *Total Score, Patterns and Algebra, Probability and Statistics,* and *Mathematical Processes and Perspectives.* As such, it was concluded that students who participated in a traditional program of study are more prepared for the secondary mathematics licensure examination. The researchers acknowledged that this conclusion was expected, as traditionally prepared secondary mathematics teachers generally have majored in mathematics and therefore have received more mathematical content knowledge. Nevertheless, while neither group had a mean passing score for the examina-tion, this cohorts' results were 9 points higher that the state average. The researchers felt that these results indicated three key findings. First, the teacher education programs currently offered are not preparing pre-service teachers adequately for the ex-amination. Second, the content represented on the licensure examination is not aligned well with content taught in teacher edu-cation program. Third, the licensure examination is nor a valid nor reliable measure of achievement.

Conclusion

The review of three research projects indicates that there are various means by which universities prepare pre-service second-ary mathematics teachers. Some factors that universities may consider in preparing the next generation of secondary mathe-matics teacher have been presented in the executive summaries of the three research projects presented. First, the introduction of a capstone course – as recommended by a national educational authority – could serve to standardise the preparation of mathematics teachers. Doing so might also assist in bridging the content of university mathematics courses closely with that content taught at secondary level, and in providing a final experience for student teachers to communicate with and about lin-gering mathematical deficiencies. Second, the implementation of a professional learning community within a teacher education program might provide mathematics teachers with an effective model of professional practice. Specifically, having students observe structured lessons taught by a master teacher and then engage meaningfully in a variety of collaborative learning activ-ities promotes skills and practices essential to quality mathematics teaching. Notably these activities include co-teaching op-portunities coupled with follow-up reflection and discussion, which have been found to drive a sense of professional develop-ment. Third, and while Australia continues to prepare secondary mathematics teachers through 4-year (BEd), 2-year (MTEach) and 1-year (Grad Dip Ed) university programs, there is consideration for a competency-based exit examination focussed on mastery of content knowledge. While literature suggests that higher scores on licensure tests lead to increased student achievement (Sawchuk, 2011) – and with all Australian graduate teachers set to undertake a mandatory competency test from 2016 (Literacy and Numeracy Test for Initial Teacher Education Students) – an additional examination might become a future certification requirement for secondary mathematics teachers. Following the findings of Bonner et al. (2013), for such an ex-amination to be a useful measure of mathematics teacher competency the content would need to be aligned closely with math-ematics taught both at university and secondary school level.

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INTENTIONALLY ENGAGING

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We all want to engage students and have them working on tasks enthusiastically with their consent. But are all engaging tasks good tasks and a productive use of class time? What is a good task? How does it draw out mathematical concepts related to the curriculum?

What Makes a Task Intentionally Engaging?

Woolfolk and Margetts (2007) indicate that students' interest in, enjoyment and excitement about what they are learning is one of the most important factors in education. They also indicate that when students' motivation levels are increased, they are more likely to find academic tasks meaningful. One of a teachers pressing questions is "How will I keep my students engaged and foster interest in learning Mathematics?" Choosing activities that are engaging, directly related to Australian Curriculum in a time proportional to the importance of the learning objective is the teacher's challenge. Some tasks are great fun but the level of mathematical challenge or link to Australian Curriculum may be very poor. Others just take more time than is available. Knowing how to incorporate engaging tasks that are meaningful, relevant and timely is our challenge.

This poses the question: what is an engaging activity?

Engaging activities are quite often in the form of a problem solving task or a game. The point we are making is that an engaging task must also have a learning objective with a direct link to the Australian Curriculum. To assess if a task meets this criteria you can use the "LIVENS" model. An engaging activity LIVENS your Mathematics class.

Each task should aim to have:

Learning Objective – Links to the curriculum

Inclusive – allows for the gender, academic and interests within a class

Variety of solution methods

Extends mathematical thinking and concepts

Need for teacher assessment

Social interaction

We will assess a selection of common classroom activities against the LIVENS model.

Celebrity Heads

Task

A student sits on a chair against the whiteboard and a mathematical word is written behind them. The student asks yes/no questions to the class in order to discover their word.

LIVENS

Celebrity Heads LIVENS the class and with a deliberate choice of words can introduce/revise a topic and address the importance of vocabulary.

For example, Year 7 teachers could choose vocabulary from the Measurement and Geometry strand to address the standard *Classify triangles according to their side and angle properties and describe quadrilaterals* or the Reasoning Proficiency… *applying known geometric facts to draw conclusions about shapes.* (The Australian Curriculum, Assessment and Reporting Authority)

Another engaging activity that fits the same vocabulary criteria but for the younger students is Guess my Number. The same approach is used but instead of a word you write a number on the board (or you could do it as a partner activity). By using charting on a hundreds chart to eliminate numbers this can help students to formulate their next question. This extends their thinking and encourages them to see rules and patterns developing.

Travel Project

Task

This task is commonly aimed at Years five to eight. Students are given a budget, a timeline and some estimated costs of basic expenses. They plan a journey that will be within their budget.

LIVENS

Students enjoy this activity but the rigor of the task and the content descriptors being satisfied are not obvious. It is essential to pose questions around the content descriptors shown in Table 1 to ensure the students do not get carried away with what their accommodation offers! Even with good questions and keeping the task constrained often a week of class time is set aside for this project. Given the limited number of content descriptors touched upon this task engages but in our opinion is not a good task. Table 1 below shows the content descriptors from Levels five to eight that could be addressed in such a project. Projects of this nature tend to have a limited scope for challenging and extending thinking of the more able students.

Table 1 *Australian Curriculum Mathematics (ACARA 2013)*

Level	Strand	Sub Strand	Content descriptors
5	Number and Algebra	Money and Financial Mathematics	Create simple financial plans
	Measurement & Geometry	Using Units of Measurement	Compare 12- and 24-hour time systems and convert between them
		Location & Transformation	Use a grid reference system to describe locations. Describe routes using landmarks and directional language
6	Measurement & Geometry	Using units of Measurement	Interpret and use timetables
7	Number and Algebra	Money and Financial Mathematics	Investigate and calculate 'best buys', with and without digital technologies
8	Number and Algebra	Money and Financial Mathematics	Solve problems involving profit and loss, with and without digital technologies
	Measurement and Geometry	Using units of measurement	Solve problems involving duration, including using 12- and 24-hour time within a single time zone

Greedy Pig

Task

Students all stand and a die is rolled. Rolling a 1, 2, 3, 4 or 5 becomes the score for that round.

Rolling a 6 means the total score becomes 0 for all the students standing. After each roll, students continue to accumulate their score. They then choose to either accept their current total by sitting down, or remain standing and add to their total the next number rolled (of course risking a 6 being rolled). The challenge is to maximise your score but sit down before a 6 is rolled. The winner is the student with the highest score.

This task is a simple game requiring only one die. The game is easy to learn and play, it is suitable for a range of levels as you could play with the younger years focusing upon simple addition mental computation or the strategies and learning objective can be more difficult and rely on probability. Students are continually making comparisons. The table below outlines some content descriptors that Greedy Pig addresses.

LIVENS

By extending the activity using the questioning below this is a great activity that really livens your class room.

When should you sit down? What is the average total before a 6 is rolled? What would happen if we changed the rules so you sat down when a 1 was rolled? What if we rolled 2 dice and you sat down when a double was rolled?

Extensions include assigning different point values to the numbers on the die (i.e. 1, 2, 3 are worth 2 points and 4 or 5 are worth 3 points), or playing with multiple die and using the sum of the die as the point value. These adjustments will change the probabilities and the expected value thus creating a new discussion.

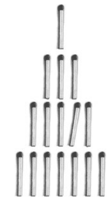
Table 2 *Australian Curriculum Mathematics (ACARA 2013)*

Level	Strand	Sub Strand	Content descriptors
3	Number & Algebra	Number & Place Value	Recall addition facts for single-digit numbers and related subtraction facts to develop increasingly efficient mental strategies for computation
5	Number & Algebra	Number & Place Value	Use efficient mental and written strategies and apply appropriate digital technologies to solve problems
	Statistics & Probability	Chance	List outcomes of chance experiments involving equally likely outcomes and represent probabilities of those outcomes using fractions
6	Statistics & Probability	Chance	Compare observed frequencies across experiments with expected frequencies
7	Statistics & Probability	Chance	Assign probabilities to the outcomes of events and determine probabilities for events

NIM

Task

NIM has been around for hundreds of years. It is said to have originated in China where it was called Jian – shizi, which means picking up stones. It was also found in Africa where it was called tiouk tiouk. The word NIM comes from the German verb nimm which means take. The rules of this game for two players are simple. Each player takes it in turns to take away either 1, 2 or 3 matches from a row. The aim is to leave your opponent with the last match. A simpler version we can all play involves starting with the number 21.



www.archimedes-lab.org/game_nim/matcho3.gif

Each player takes it in turns subtracting either a 1, 2 or 3 from the total.

The winner is the person who leaves their opponent with the number 1.

LIVENS

IVES - Inclusive, Variety of methods, Extension and Social interaction are well covered but maintaining the links to the Australian Curriculum and the Need for assessment require deliberate work on behalf of the teacher. Without the intervention of the teacher this task could be a time filler rather than a good task.

This task is engaging but can clearly meet learning objectives for problem solving with the appropriate line of questioning. After a few games stop the class and see if they have any strategies that are working. You could ask: How can we test that? Does it seem like it would always work? You would then ask are there other numbers that are important. What is the difference in the important numbers we need control over? How is this difference related to the original rules? Would it matter if we started at 25? Does it matter if we change the rules so we can subtract 1, 2, 3, or 4? What are the important numbers to control now? What about if we can subtract 2, 3 or 4? You could also build up to having more than one column of numbers and the number can be subtracted from either column. The most complicated NIM game with the 16 matches has a strategy that involves binary numbers which could really be a challenge for some.

It is very evident from these lines of questioning that this task satisfies the criteria for what makes a task engaging. It is very inclusive as it can simply involve simple subtraction for the less able students (aids such icy pole sticks, counters or blocks could assist if required) whilst the task can be extended to further challenge others by looking at strategy, patterns and algebraic expressions.

Solution

It quickly becomes evident to students that controlling the number 5 means they cannot lose.

For the one column game the difference between the numbers that need to be controlled is the one more than the largest number that can be subtracted.

Subtracting 1, 2, 3 game controlling numbers are 5, 9, 13, 17

Subtracting 1, 2, 3, 4 game controlling numbers are 5, 10, 15 and 20

Page Patterns

In order for a task to be engaging it does not have to be in the form of a game. The next task is an example of an individual activity which we have found to engage students whilst covering algebraic concepts.

Task

When books are made, large sheets are printed and then folded in half to make the pages of a book. In this task you will be investigating the sum of all page numbers in a book. Depending on the year level you may wish to simply give them the task with very little direction and see if they can devise an algebraic rule or you could scaffold the activity for the students following the procedure below:-

Step 1

Fold one sheet of paper in half to make a four page book

Step 2

Number the pages one to four. Open up the book. The page numbers on one side will be two and three and one and four are on the reverse side. Both have a total of five.

Step 3

Take your second sheet of paper, place it inside the first and fold a book, renumber the pages one to eight

Step 4

Open up the book and separate the pages. Add each pair of numbers appearing on the same side of each sheet. Record in the table. Use this information to find the sum of all the page numbers.

Step 5

Repeat step 4 with 3 & 4 sheets of paper (12 and 16 pages). Record your results in the table

Step 6

Look for a patterns in and across each column of the table

Step7

Using your patterns try to predict the sum of all pages for a book with 20 pages. Test this out, if it is correct try to write a rule

Step 8

Calculate the sum of the page numbers in a book with 200 pages

Table 3 *Solutions for Page Patterns*

Number of pages in book, n	Sum of numbers on same side of the sheet	Sum of all page numbers
4	5	$^4/_2 \times 5 = 10$
8	9	$^8/_2 \times 9 = 36$
12	13	$^{12}/_2 \times 13 = 78$
16	17	$^{16}/_2 \times 17 = 136$

Table 4 *Algebraic reasoning solutions for Page Patterns*

Number of pages in book, n	Sum of numbers on same side of the sheet	Sum of all page numbers
4	$4 + 1 = 5$	$^4/_2 \times 5 = 10$
8	$8 + 1 = 9$	$^8/_2 \times 9 = 36$
n	$n + 1$	$^n/_2 \times (n+1)$
200	$200 + 1 = 201$	$100 \times 201 = 20\,100$

Some students may also explain

(Last page number + 1) × half the number of pages. This is shown in the 200 pages above.

This relates directly to Gauss and adding up all the numbers from 1 – 100.

LIVENS

In our opinion this task meets all the criteria.

Locker Problem

This problem is a widely used task in schools. Most students begin by writing the numbers from 1 to 100 and then go through writing down each change as they go. This is tedious and generally they will make an error at some point which impacts on everything after that. This task is more engaging when students are encouraged to break it down to smaller parts and see if a pattern emerges.

Task

In a school of 100 students every student has a locker. Imagine that one student opens all the doors of all 100 lockers. A second student starts at the second locker and closes every second door. Then a third student starts at the third locker and changes the state of every third door (closes it if it was open or opens it if it was closed.)

And so on . . .

After 100 students have followed the same pattern, which doors will be open?

Table 5 *Pattern developed for 5 lockers*

	L1	L2	L3	L4	L5
Person 1	o	o	o	o	o
Person 2		c		c	
Person 3			c		
Person 4				o	
Person 5					c
Door open 1, 4					

At this stage ask students to predict.

If a student comes up with the square number answer ask if they have enough information to assume this yet. Suggest they need to test their theory further.

Table 6 *Pattern developed for 10 lockers*

	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10
Person 1	o	o	o	o	o	o	o	o	o	o
Person 2		c		c		c		c		c
Person 3			c			o			c	
Person 4				o				o		
Person 5					c					o
Person 6						c				
Person 7							c			
Person 8								c		
Person 9									o	
Person 10										c
Doors open 1, 4 and 9										

The answer is that all the perfect squares will be open.

Perfect squares have an odd number of factors

e.g. 20 – 20, 1, 10, 2, 4, 5 (6 factors)

36 – 36, 1, 18, 2, 12, 3, 9, 4, 6 (9 factors)

The only numbers with an odd number of factors are the perfect squares and this is because two of the factors are the same and count only once.

Doors open to 100 – 1, 4, 9, 16, 25, 36, 49, 64, 81, 100

LIVENS

This is an excellent task that meets all the criteria including links to outcome related to patterns, factors and perfect squares.

Sudoku

Sudoku is a puzzle that has become very popular in recent times. Some believe as it uses numbers it is therefore a good Mathematics task. To solve a Sudoku puzzle one needs to use logic, trial and error and resilience which are all integral to problem solving. One needs to question if it rigorous enough or it LIVENS up the class. There is no doubt it can be used if students complete their work early but it wouldn't be used as a targeted class with specific learning objectives.

If you want to use a Japanese logic puzzle then try "Area mazes" which has excellent links to the curriculum without losing the engagement.

Conclusion

The teacher is responsible for selecting the tasks to engage students with the content of the Australian Curriculum. Engagement of itself is not the aim and many engaging tasks are really just time fillers. The tasks we choose must stimulate thinking about mathematical ideas and develop skills. A good task must be engaging but must also have within it skill development that is directly related to the curriculum.

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DOING MATHS IS AWESOME – IPADS, ANIMATION AND ASSESSMENT

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Pyalong Primary School

Details the experience of using animation as an assessment tool in a Primary Mathematics classroom. iPads make creating animation a relatively easy task, achievable by even the youngest students. Asking students to prepare an animation that displays their understanding of Mathematics' concepts provides a visualisation of their thinking. This can be highly revealing of a students' thought processes and depth of understanding, enabling assessment and identification of misconceptions, especially when students work independently.

Introduction

Firstly, some background. I am a first year graduate teacher working in a small rural school with a total of 39 students. We operate 3 classes – Foundation and Year 1, Year 2 & 3, and Year 4, 5 & 6. Instead of traditional primary school classes with one teacher taking all lessons, this year we worked much like a high school, with teachers working in subjects rather than classes and I teach Mathematics to the whole school. Each class has 4 Mathematics sessions per week and each session is 90 minutes so we are providing an additional hour each week compared to a daily 1 hour Numeracy block.

Having an ICT background, and mindful of Standards of Professional Practice (AITSL, 2014), I was keen to incorporate ICT into my teaching program and we have a class set of iPads so that every student has access to an iPad. There are many apps available for Mathematics and many of these are designed for skills practice, which has its place, meaning the students are consumers of work created by others. I was more interested in seeing students create content, applying their knowledge and demonstrating their understanding.

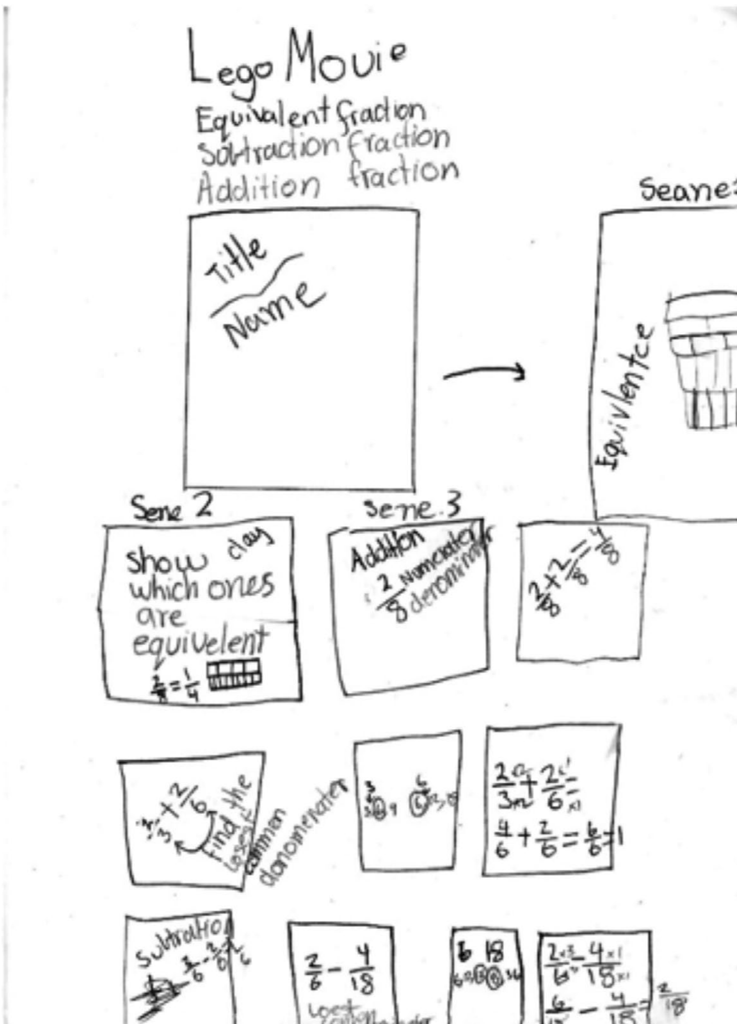
I was introduced to animation as a form of assessment during my training at university, where it was used for teaching and assessing Science concepts. Seeing the value of it firsthand, I resolved to trial it in the classroom to assess students' understanding of recently taught Mathematics concepts. Creating an animation provides a visualisation or representation of learning and provides assessment as, of and for learning and meets the principles for assessment encouraged by DET (Dept. of Education and Training, 2013). Additionally, it engages and excites students who are keen to create their own story, regardless of their previous interest (or lack thereof) in the topic being animated.

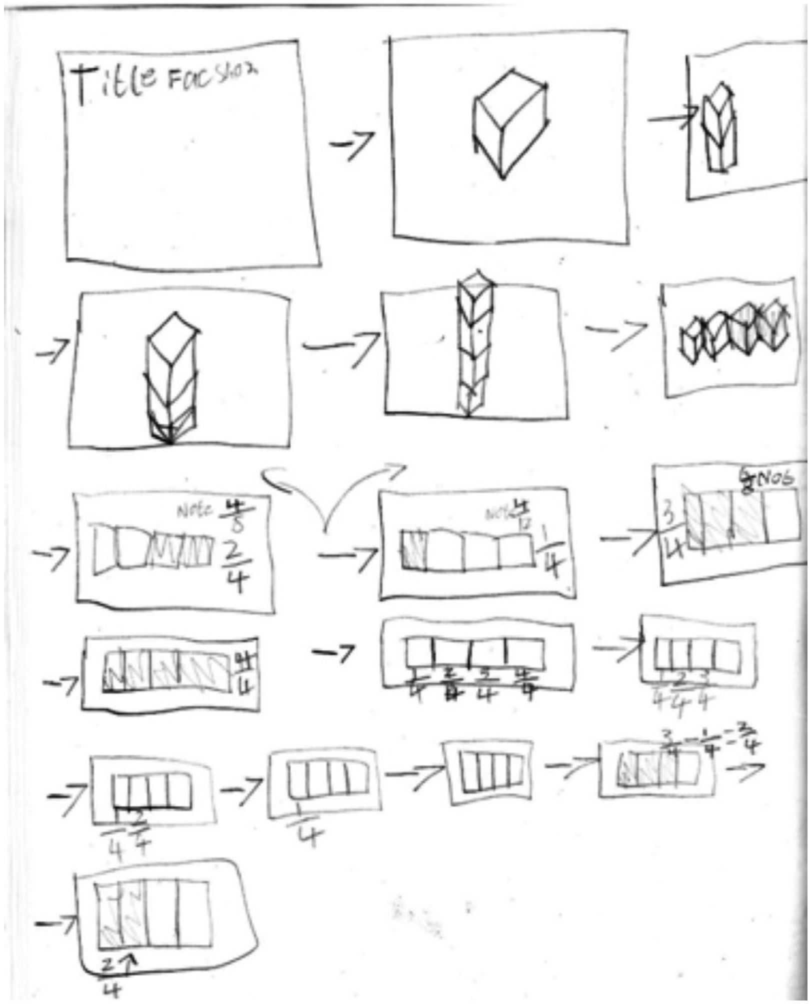
Of the large variety of animation apps available, Lego Movie Maker stands out as an ideal choice for a Primary classroom. It is a totally free app without ads or in-app purchases. It has many features allowing you to create and edit a complete movie within the one app, including: title screens; sound track using provided music or your own selection; the ability to add sound effects or a voice-over/narration in addition to the music track; onion skinning – where the last photo taken is 'ghosted' onto the screen to help alignment of the next shot; the ability to change the overall speed or the duration of individual frames; and a number of export options. It is very easy to learn how to use with students of all ages typically able to start an animation with less than 5 minutes instruction.

The first time I implemented this, I initially showed students a short animation I had prepared that demonstrated addition and subtraction using Unifix blocks but I was mindful that this may influence their resulting animations. In the F/1 class, this level of scaffolding was fine and their task was to create a similar movie using a combination of numbers of their choice. The 2/3 class had a similar task, although they were also required to demonstrate the connection between addition and subtraction. The 4/5/6 class had to demonstrate addition and subtraction of fractions and use of equivalent fractions.

To encourage creativity and distance from my example, the upper class were given a short time (approximately 30 minutes) to experiment with the app and produce a short animation of anything they desired. We then jointly developed an assessment rubric with the view that we would watch and assess the completed animations as a group. The rubric had categories for *maths content, presentation* and *other* so students had a good understanding of what was required in their animation. Differentiation was catered for by varying the task description, enabling assessment using the same rubric categories. For the

4/5/6 class the final task before commencing their animation was to prepare a storyboard, outlining their ideas and the maths content. Two examples of storyboards are shown in the accompanying diagrams.





The finished results were entertaining and enlightening. It was clear to see who had a good understanding of the relevant concepts and who needed extra work to develop their knowledge. The students were keen to watch their classmates' movies as well as their own and we shared them together in each class, projecting them on a whiteboard using Apple Air Play to wirelessly connect each iPad. This was done using a program called AirServer installed on a standard (non-mac) laptop and it is a simple matter for each student to display their work, mirroring their iPad screen on the laptop and the students are in control of their display. In the 2/3 and 4/5/6 classes we jointly assessed each movie using the previously developed rubric. This helped to reinforce the learning of the concepts being taught, with much discussion among the students to rate the content of each movie, and interpreting the different ways that the concepts were illustrated.

Using animation provided an effective method of assessment of learning as I was clearly able to judge each students' knowledge and understanding of the assigned topics. Their depth of understanding was revealed by the detail included in their animation. For example, some Year 2/3 students distinctly displayed their understanding that addition and subtraction were reciprocal activities whilst others demonstrated that they could add and subtract but did not show a link between the two. Furthermore, it provided an effective method of assessment as learning, when, during preparation, students discussed their animations with myself and their peers, and made changes to incorporate new understanding.

Feedback from the students was very positive and they are keen to employ this method of assessment again. They certainly enjoyed preparing the animations and in turn this brought enjoyment to the learning of Mathematics, developing interested and engaged students. They also enjoyed the popcorn we shared as we conducted the assessments and all agreed that 'doing Maths is awesome'.

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DEVELOPING PROPORTIONAL REASONING

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Proportional reasoning is one of the big mathematical ideas students will encounter. It applies to a wide range of contexts across all of the content strands and is considered a critical concept for success in secondary mathematics. It requires an ability to think multiplicatively and relationally, and is often problematic for students.

Introduction

Proportional reasoning is difficult (Weinberg, 2002). The ability to reason using proportional relationships is a complex form of reasoning that depends on many interconnected ideas and strategies (see *Figure 1.*) developed over an extended period of time. It takes many varied physical experiences to develop an understanding of what a proportional relationship is and then even more time to be able to deal with it abstractly (Cordel & Mason, 2000; Siemon, 2015; Van de Walle, Karp, & Bay-Williams, 2010).

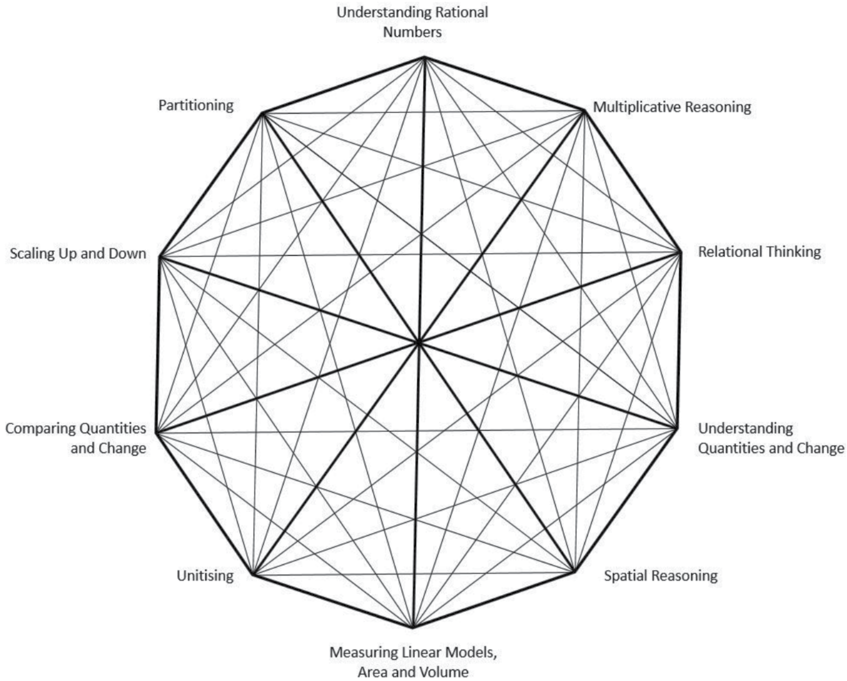

Figure 1. Interconnected concepts of proportional reasoning.
Adapted from Ontario Ministry of Education (2012).

To develop proportional reasoning is not the same as being able to complete an algorithm to solve a proportional problem. "It is important that students learn to solve proportional-reasoning problems using their own intuitive strategies before they are taught the cross-multiplication algorithm. In fact, even after students learn the algorithm, teachers should continue to talk about the students' informal reasoning strategies and how they result in the same answer as the algorithm." (Fazio & Siegler, 2011, p.19). Students use proportional reasoning as soon as they start to look at equivalent fractions and there is much literature which supports the position that students can often 'do' equivalent fractions without necessarily understanding what they are doing and why. In the parlance of the Australian Curriculum, we seem to be able to foster a degree of Fluency but may sacrifice Understanding, Problem Solving and Reasoning in its execution.

When working with proportional reasoning it is not unusual for teachers to teach 'cross multiplication' as a manner of finding a solution. What we would like to propose is that there are ways of developing and understanding concepts of proportionality, so that when cross multiplication is introduced, it is based on understanding.

What is Proportional Reasoning?

According to the work of researchers (Behr, Harel, Post, & Lesh, 1992; Lamon, 2006; Wright, 2005) proportional reasoning is the ability to understand (recognise, explain, make conjectures about, graph etc.) the multiplicative relationship inherent in situations of comparison. Dole (2010) explained that the research has shown that both students and teachers generally have a poor understanding of proportional reasoning.

The capacity to reason proportionally requires careful development (*Figure 2.*). Whilst we do not have the time and space to pursue the issue, we would like to suggest that without the careful development of mathematical thinking, it would be hard to imagine that a student would find the development of proportional reasoning to be possible.

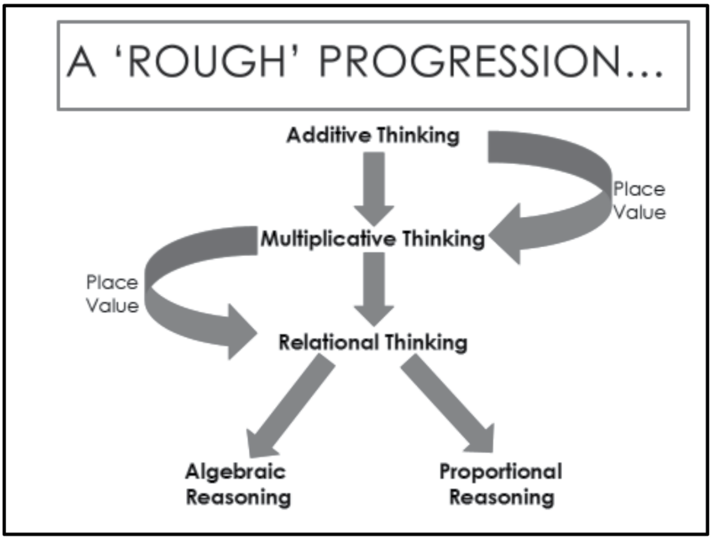

Figure 2. A 'rough' progression of mathematical thinking.

Is Proportional Reasoning Important?

According to Dole, Clarke, Wright and Hilton (2012), proportional reasoning is seen as being fundamental for success in the areas of mathematics and science. Kilpatrick, Swafford and Findell (2001) considered proportional reasoning as being a gateway to higher levels of mathematical success. Both Charles (2005) and Siemon, Bleckly & Neal (2012) included proportional reasoning amongst their 'big ideas'. Charles (2005, p. 10) defined a 'big idea' as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole".

From a perspective beyond school mathematics, proportional reasoning is important for such things as; calculating which of two items represents the best value; adjusting amounts in recipes; working with maps and diagrams; currency conversions and concentrations of mixtures, for example the oil to petrol ratio for two-stroke motors.

Developing Proportional Reasoning

While it is necessary to be cognisant of the fact that proportional reasoning is not an easy mathematical understanding to develop, teachers should celebrate that it is possible to do so. What is needed is a considered way of exposing students to proportional reasoning. In order to develop proportional reasoning we need to:

- Provide students with proportional situations that span a wide range of contexts and relate to their world.
- Offer problems that are both qualitative and quantitative in nature. Qualitative problems encourage students to engage in proportional reasoning without having to manipulate the numbers.
- Help students distinguish between proportional and non-proportional situations.
- Encourage discussion and experimentation in predicting and comparing ratios.
- Help students relate proportional reasoning to what they already know. E.g. connect how unit fractions and unit rates are very similar.
- Recognise that mechanical procedures for solving proportions do not develop proportional reasoning and that students need to be flexible in their thinking and acquire many strategies.

(Ontario Ministry of Education, 2012; Siemon, 2015; Van de Walle, Karp, & Bay-Williams, 2010)

Activities for Developing Proportional Reasoning

Baby in the Car

A very rich activity which can be used to develop proportional reasoning can be found in maths300 (http://maths300.com/). The lesson is called "Baby in the Car" (Lesson 111).

In this lesson the students are invited to consider a scenario which is a ‘real world’ context which has drastic ramifications, the context of babies being left unattended in poorly ventilated cars. It is a topic which unfortunately is raised too often in light of a baby being left in a car for a period of time and the consequences it can have on the health of that baby. The students are asked to consider why it is that if a baby and an adult are left in a poorly ventilated car, that the adult will perhaps suffer very minor effects whereas for the baby it may be catastrophic.

The students are then invited to work as mathematicians do and create some models for the situation. The implied message here (which should be made overt) is the purpose and power of mathematical modelling. The first model is quite simple and is used to make sure that all students understand the concept of surface area. The students are given one wooden cube and asked about the number of faces and therefore the surface area.

Once it is established that the volume (the number of cubes) is one and that the surface area (the number of exposed faces) is six, there is a discussion generated as to how this is representative of skin-area through which moisture can escape through evaporation, and that this single block represents a baby. The students are then invited to construct a cube which is twice as big in all directions to represent an adult. This then leads to recording the volume and the surface area of the baby and the adult and the surface area to volume ratio is introduced (Table 1).

Table 1 *Relationship between Volume and Surface Area*

	Volume	Surface Area	SA : Vol.
Baby	1	6	6 : 1
Adult	8	24	3 : 1

From this simple model and from this initial set of data, a more complex model is then constructed (model A required a deal of fraction work, which is mitigated in model B) and investigated for surface area (*Figure 3*). The students ‘build’ a baby and then an adult which is twice as big in every direction. Once again the volume and the surface area of both models is scrutinised.

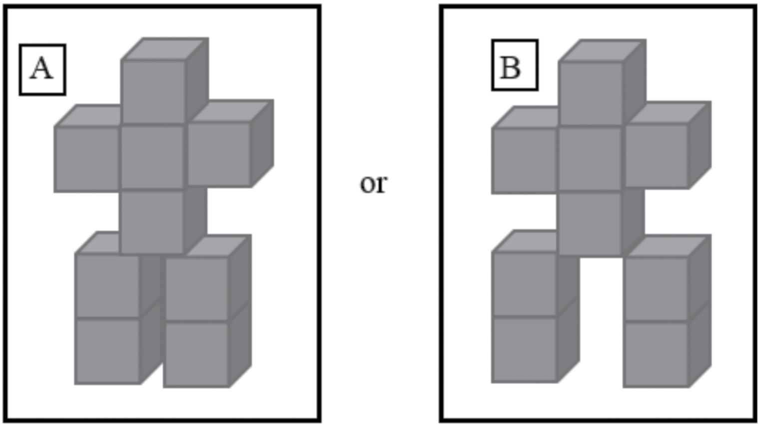

Figure 3. 'Complex' models of a baby constructed from blocks.

Eventually, and if required, an even more realistic model, using cylinders and spheres can be constructed and the data can be further refined.

This activity was employed in a professional learning session conducted for a group of twelve Secondary mathematics teachers, some of whom were teaching in-field (their area of specialisation) and some who were teaching out of field. What may have been of surprise was that on the first activity in building the simple model (one cube baby, eight cube adult) the notion of twice as big, caused more than one of the teachers to pause and consider what it meant. In fact, the first model constructed by some of the participants for the 'adult' was made from four blocks and was twice as big in only two of the three possible dimensions. This was soon amended by the other members of the group, and the need to provide correction and instruction came thick and fast, not unlike the occasions when this activity has been used with school aged students.

The rich discussion that accompanied the construction and calculation of the surface area of the 'complex' adult models highlighted the language of reasoning, problem solving and fluency as groups endeavoured to find the most efficient method in which to calculate the surface area. Several groups checked their calculations by trying a second method, just as a mathematician would.

Orange Soda

In this same professional learning session an activity was conducted that was adopted from the Mathematics Assessment Resource Centre (MARS) (available from http://mathshell.org/ba_mars.htm). This activity was based upon a scenario often used in the teaching of proportion but with the significant 'twist' of introducing manipulative materials in the form of unifx blocks, a consideration that improves accessibility to the task and to an understanding of proportions. The aim of this activity is to determine the relative strength of drinks. The participants were given a series of cards with different representations of proportions (*Figure 4*) and asked to arrange these cards in order from the 'least orangey' to the 'most orangey'.

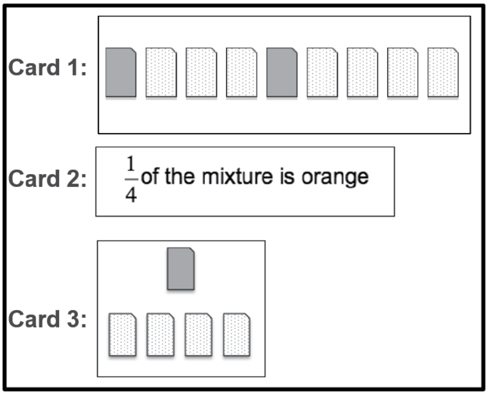

Figure 4. Cards with different representations for proportional reasoning.

In order to make comparisons as to the strength of the different drinks, it was necessary to interpret the various representations, meaning that the task was quickly elevated from a fluency task, to one much richer. Concrete materials, in the form of unifix blocks, helped participants to make sense of the problem which made it more accessible (*Figure 5*). The discussion that arose from the use and comparison of the different representations gave a meaningful and powerful insight to the thinking and reasoning of the participants.

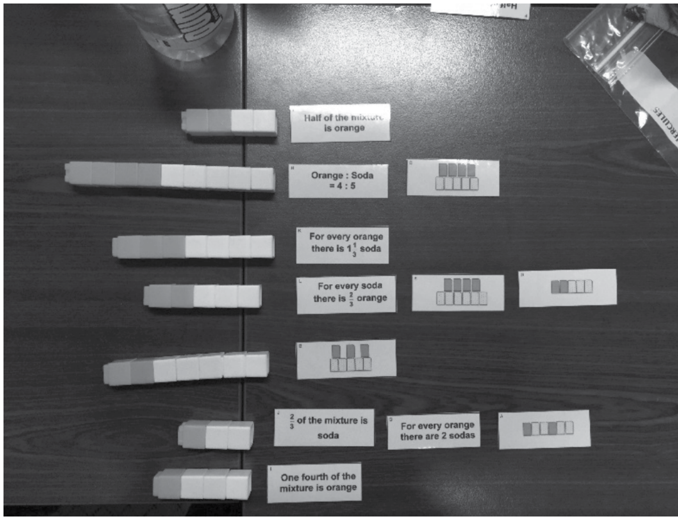

Figure 5. Using concrete materials to make sense of the problem.

This activity generated quite a deal of discussion with regard to the representation which best supported the understanding of proportional reasoning. Whilst some found that using the illustrations (Card A, *Figure 6*) on which to base all of their decisions, others preferred the cards with words and symbols (Card L, *Figure 6*), and others still, found the introduction of the unifix blocks as being the most useful representation. Although the direct question was not asked, no-one incidentally offered the ratio in symbolic form (Card H, *Figure 6*) as being the most 'helpful'.

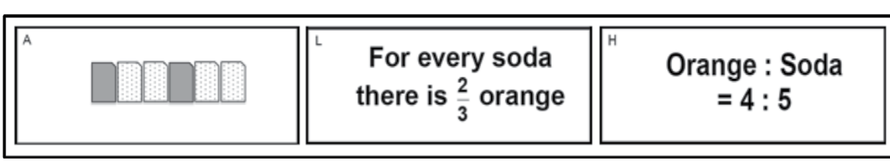

Figure 6. Multiple representations for proportional reasoning activity, Orange Soda.

Conclusion

What the two brief illustrations of activities in this article show is that, long before there is an expectation of 'calculating' proportions, there are opportunities to get students to engage with and develop proportional reasoning in contextually significant ways. The power of the physical act of manipulating the blocks or the unifix and cards and the ensuing conversations cannot and should not be underestimated. Activities such as these allow both the teacher and the students to employ informal and intuitive strategies which can be used to support understanding and reasoning for proportional reasoning which can be applied later when constructing algorithms.

If we relate the two highly accessible activities back to what is needed to be considered to develop proportional reasoning, four of the six features that the research (Ontario Ministry of Education, 2012; Siemon, 2015; Van de Walle, Karp, & Bay-Williams, 2010) indicates as important, are quite apparent. These four features are: providing students with proportional situations that span a wide range of contexts and that relate to the students' world; providing problems which are both qualitative and quantitative in nature; encouraging discussion and experimentation in predicting and comparing ratios; and helping students relate proportional reasoning to mathematics with which they are already familiar.

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PROFESSIONAL DEVELOPMENT & MATHS ENRICHMENT THROUGH MASSIVE OPEN ONLINE COURSES (MOOCS)

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The general opinion of academia is that Massive Open Online Courses (MOOCs) are merely a form of tertiary talking-heads inflated by the Internet. Nothing could be further from the truth. A hidden demographic are teachers, particularly in overseas countries, who have a degree and are not considered as part of tertiary education. They have very high pass rates because they are motivated to gain the skills that will improve their teaching and also demonstrate formal qualified professional development to their employer and professional institute.

MOOCs offer free professional development programs from the world's leading universities. Their syllabus are under your control, at home if you wish, with nobody looking over your shoulder.

The Web can give you a thousand answers to "What will I teach tomorrow that will make my students want to learn more?

And you can keep that student in the back row who knows more than you busy so he/she comes to you with, "Hey, look at what I did?" The amount of material available on the Web for maths teachers is staggering. Hyperlinks to two examples are given - the Khan Academy and Stephan Wolfram's Mathematica™.

All this is available for free to anyone anywhere there is an Internet connection. Discussion groups of you fellow students are available and responsive to any difficulties.
Unfortunately the economist Milton Friedman was right when he said, "There is no such thing as a free lunch." Most MOOCs require 5/6 hours a week for six/ten weeks and a certificate of completion is based on mastery learning of all of the material at the 70% level. A 'distinction' requires 85%. The most important factor in MOOC success on top of family and teaching is motivation.

This document makes extensive use of hyperlinks, the foundation of the Web. When downloaded from the Web, Ctrl+Click on any APA text citation retrieves the original source. Ctrl+W cancels the displayed Web page and returns to the document. Download instructions are given at the end of this proceeding.

Introduction

For those of you who are not acquainted with the MOOC world, the term MOOC was coined in 2008 by Dave Cormier, George Siemens and Stephen Downes.

Dave Cormier published three short YouTube presentations on his work which appear to be simple but contain the heart of why MOOCs are so successful.

- What is a MOOC? (4:26) (Cormier, 2010a, Dec 8)
- Success in a MOOC: (4.16) (Cormier, 2010b, Dec 1)
- Knowledge in a MOOC: (1:52) (Cormier, 2010c, Dec 1)

The sidebar to these videos contain about 50 additional YouTube and TED videos on this topic.

The first tertiary MOOC was triggered by Sebastian Thrun at Stanford University in late 2011 when he published that his AI course would be on the Internet for free in a local newsletter. University teaching has never recovered from the shock. See The Good MOOC Blog (Salmon, 2913, May 17) and (Salmon, 2012, Jan 23) for details.

Thrum's MOOC

- Had 160,000 people click on in the first few weeks.
- 23,000 earned a certificate showing 70% mastery of the entire material. 248 had a perfect score. None were in-house class.
- The in-house class started with 200 students. At the end all but 20 had shifted to the MOOC.

You would think that MIT's (OpenLearningCourseware, n.d.) and the response to Thrun's AI MOOC would ring alarm bells in every university. But it didn't. As an extreme example, Stanford University was home of the first MOOC and now ranks world number one with the 102 MOOC offered (Class Central, n.d.)

But the President of Stanford University, John L Hennessey, giving the 2015 Robert H. Atwell Lecture at the ACE's 97th Annual Meeting with the title Information Technology and the Future of Teaching and Learning spent 31 minutes (ACE, 2015, Mar15) telling 2000 university leaders that MOOC where educationally useless and their openness was letting the wrong sort of person access tertiary studies. His solution? Replace the 'massive' and 'open' with "LSOCs, or Large Selective Online Courses."

This reaction is natural when you consider MOOCs are the reverse of everything universities believe in. Students are in charge, the group-mind is a lot smarter than any academic and the 'teaching stars' are bringing in tens of thousands of students instead of research.

But the real danger is that most attend university to get a piece of paper that will get them job. MOOC are much better at this while being far cheaper than in-house classes. The Christensen Institute for Disruptive Innovation's *Hire Education: Mastery, Modularization, and the Workforce Revolution* details why. (Weise & Christensen, 2014)

Professional Development

Motivation

Who owns a MOOC? You do! And, as I heard Ron Barassi say at a MAV Annual Conference many years ago, "What you get out of education depends on what you put into it." A MOOC is a product looking for a customer who will pay in time which is far more valuable to a teacher than money.

Justin Reich, executive director of the MIT Teaching Systems Lab, has analised thousands of results from the Khan Academy, Udacity, Google Course Builder and HarvardX that show motivated students do much better than the others.

He proposes Reich's Law of Doing Stuff, "Students who do stuff in a MOOC or other online learning environment will, on average, do more stuff than those who don't do stuff, and students who do stuff will perform better on stuff than those who don't do stuff." (Reich, 2014, March 30)

A MOOC student must have a high motivation but not necessarily to pass. The President of Coursera, Daphne Koller, said there is a very large group who watch every video all the way through but do nothing else - teats, essays, discussion groups. They want the information but not any engagement. A very interesting letter on MOOC motivation, with figures based on half a million students, was sent to Science, AAAS, by J J Clark of the International Correspondence Schools in 1906. His opinions are confirmed by both Coursera and edX. (Clark, 1906, Sep 14)

The Web as a Peer-to-Peer Network

It is commonly assumed that in a class of thousands of students it would be difficult to be noticed. Not so. The Web is a peer-to-peer star network so every student can talk to any other. In effect the 'M' in MOOC can refer to the peer-review group of your work.

Question: In a class of 37,000 students with a lecturer on the other side of the world, what would be the chances of a face-to-face meeting in his office at the cost of buying him beer so I could pick his brains for two and one-half hours on why he was teaching a MOOC? Answer: Pretty good, and anyone on the Web can view a YouTube video proving it. I come in at 0:31. (Severance, 2013, Jun 25)

The Present and the Future

edX has released a 37-page PDF report, HarvardX and MITx: Two Years of Open Online Courses Fall 2012-Summer 2014. "In this "Year 2" report, we revisit these earlier findings with the benefit of an additional year of data, resulting in one of the largest surveys of massive open online courses (MOOCs) to date: 68 courses, 1.7 million participants, 10 trillion participant-hours, and 1.1 billion logged events." (Ho, et.al, 2015).

This report has some 40 references to 'teachers as learners' but they are just another countable category. However, Figure 6 on page 23 shows that HarvardX and MITx have become a meta-MOOC. It is obvious that edX, Coursera, (Future Learn, 2015, May 21) with its 380,000 students, and others will become a super meta-MOOC that will dominate the world's education.

One of the backrooms at MIT has given us a glimpse of the future in MOOC 4.0: The Next Revolution in Learning & Leadership. (Scharmer, 2015, April 5).

He says that, "Massive Open Online Courses (MOOCs) have evolved over the past three years. This is how we think about their evolution:

- MOOC 1.0 - *One-to-Many*: Professor lecturing to a global audience.
- MOOC 2.0 - *One-to-One*: Lecture plus individual or small-group exercises.
- MOOC 3.0 - *Many-to-Many*: Massive decentralized peer-to-peer teaching.
- MOOC 4.0 - *Many-to-One*: Deep listening among learners as a vehicle for sensing one's highest future possibility through the eyes of others."

In other words, a MOOC has a group-mind that will take the learning process far above a lecturer in a classroom.

Teachers as MOOC Students

Teachers have a particular motivation for gaining a verified certificate ($50-$100 for each subject - tax deductable) for each MOOC taken, namely proof of professional development for job security and promotion.

A group at MIT found that 'teachers as learners' in the Two Year Report was a poor description of the teaching profession when they noticed something odd about overseas teachers taking MOOCs. In their paper, Teacher Enrollment in MITx MOOCs: Are We Educating Educators?, they state:

"Surveys of 11 MITx courses on edX in the Spring of 2014 indicate 1 in 4 (28.0%) respondents identify as past or present teachers, while nearly one in ten (8.7%) identify as current teachers.

Despite representing only 4.5% of the nearly 250 thousand enrollees, survey responding teachers generated 22.4% of all discussion forum comments. More notably, 1 in 12 comments are from current teachers, and 1 in 16 comments are from teachers with experience teaching the subject." (Seaton, Coleman, Daries & Chuang, 2014, Oct 27)

The report goes on to indicate that teachers engaged in professional development may be one of the most important segments of MOOC acceptance.

A MOOC Example: Discovery Precalculus by the University of Texas

"This is an inquiry-based exploration of the main topics of Precalculus. The emphasis is on development of critical thinking skills." (Daniels, Lucas and Smid, 2015, Sept 1)

The MOOC takes 15 weeks. Usual MOOC is 5-6 weeks. Their estimated time is 10 hours a week but any MAV member should take less than half that time. To get an idea of the subject matter, click on the upper left YouTube video [Full screen]. 1:32 minutes.

The MOOC is free but teachers should pay a $50 fee, tax deductable, to receive a verified certificate and transcript endorsed by the University and EdX with hyperlinks to their database for professional development. (Ctrl+Click for sample certificate.)

Math Enrichment

Any teacher who has not completed a certificate should download this brochure with the inelegant title of The No Bullshit Guide to Succeeding at Online Courses: 9 Specific Steps to Getting the Most out of Online Courses and Dominating Online Education. (Skilledup, n.d.)

For a PowerPoint display of MOOC Suppliers and Course Indexes, 7 slides, Ctrl+Click on (Groves, 2015).

Mathematica™ was created by Stephen Wolfram in 1988 as an expensive computer algebra system for university graduate mathematics departments, corporate mainframes, financial 'rocket scientists' and industrial workstations, all with deep pockets. I remember a teacher at an AAMT Conference telling me that "I can't even cope with a programmable calculator and you expect me to use this in my teaching?"

Salman Khan (KhanAcademy, 2015) has a BS and two Masters from MIT and a MBA from Harvard so it was logical his day job was as a rocket scientist (Connective Capital Management). He was obviously qualified to tutor a cousin who needed help with math. Her friends and relatives join in so he decided to put his materials on YouTube. The rest is history.

The star map (KhanStarMap, 2015) has mathematics topics listed on the left hand side which locates it in the map. Clicking on a particular star gives the student a tutorial as part of a chain of topics listed on the left.

The (WolframEducationPortal, 2015) has a 2-minute video as an introduction to Mathematica™. Wolfram Research has a special arrangement with the Victorian Department of Education for access to Mathematica™ for teachers and students plus technology systems for (WolframHighSchools, 2015) and (WolframK-8Schools, 2015).

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Clicking on a citation brings up the original source on the screen to match the citation in the printed version.

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TEACHING RECURSION IN GENERAL AND FURTHER MATHEMATICS

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Recursion will be a compulsory topic in both General and Further Mathematics in the new VCE Mathematics Study Design. Approaches to presenting this topic at both year levels are presented, together with suggested teaching outlines.

Introduction

Recursion as a concept is currently encountered in two modules in Further Mathematics – difference equations in Number Patterns and Applications and transition matrices in Matrices. However, in developing this concept for the Recursion and Number Patterns topic in General Mathematics (HREF1, p. 20), and the Recursion and Financial Modelling module in Further Mathematics (HREF1, p. 56) for 2016 and beyond, a number of changes have been made in notation, style and presentation.

Recursion as a concept means that each new term in a sequence is generated from the previous term or terms by the application of a rule (e.g. each term of the well-known Fibonacci sequence is the sum of the two previous terms).

There are three different types of sequences that will be encountered.

The first is the linear growth/decay sequence in which each new term is generated by adding or subtracting a constant amount to the previous term.

The second is the geometric growth/decay sequence in which each new term is generated by multiplying the previous term by a constant amount.

The third is the combination sequence in which each new term is generated by multiplying the previous term by a constant amount and then having another constant amount added to or subtracted from this product.

General Mathematics

The Year 11 content, based on the first two types of sequences, could take some first steps towards the aim of modelling financial situations.

A recursive relationship contains a statement of a key term (usually the initial or starting term) and a rule for calculating the next term value given the current term value. This is similar to what is currently required for difference equations in General and Further Mathematics.

In setting up the recursive relationship, the definition of the general term is a key step.

I recommend the following definition where V_n represents the value of a term in a sequence:

"V_n represents the value of the nth term *after* n applications of the rule". The starting term would be allocated the label V_0, since the first term in the sequence V_1, would now only exist after one application of the rule.

Take, for example, the sequence 2, 5, 8, ...

Define term: V_n is the value of the nth term *after* n applications of the rule.

Starting value: $V_0 = 2$

Write the rule in words: The next value equals the current value plus three.

Write the rule in mathematical terms: $V_{n+1} = V_n + 3$

The recursive relationship is then written as: $V_0 = 2, \quad V_{n+1} = V_n + 3$

A third statement is implicit in this relationship, but is not required by the course design. This statement is the set of values that n could take in this situation ($n = 0, 1, 2, 3, \ldots$).

The use of a statement in words of the situation prior to assembling the recurrence rule is a process to assist students in correctly assembling the rule, as this is seen as the part that will cause the greatest difficulty for students.

Take, for another example, the sequence 4, 12, 36, ...

Define term: V_n is the value of the nth term *after* n applications of the rule.

Starting value: $V_0 = 4$

Write the rule in words: The next term equals the current term multiplied by three.

Write the rule in mathematical terms: $V_{n+1} = V_n + 3$

The recursive relationship is: $V_0 = 4, \quad V_{n+1} = V_n + 3$

Finding the rule for the nth term for either a linear growth or a geometric growth relationship can be easily achieved from first principles, by looking at the structure of successive terms.

This leads us to the rule $V_n = V_0 + nd$ for an arithmetic sequence, where V_0 is the starting value, n is the number of term and d represents the amount added or subtracted (the common difference).

This also leads to $V_n = R^n \times V_0$ for a geometric sequence, where R is the multiplying factor.

By starting with V_0 for the initial term we have developed more easily understandable rules for the *n*th term than those from more traditional approaches to sequences as has been taught in the past, and overcome difficulties with trying to identify what the correct term number is for a particular situation.

Modelling growth and decay in financial situations

Devise a recurrence relation for Shelley, who has invested $2500 in an account paying 4% p.a. interest compounded quarterly.

Define term: Let B_n be the balance in the account after n compounding periods.

Starting Value: $B_0 = \$2500$

Interest rate per compounding period:

$$\text{Rate} = \frac{\text{annual rate}}{\text{compoundings per year}} = \frac{4\%}{4} = 1\% = 0.01$$

In words: The next balance equals the current balance plus the interest earned

In mathematical terms: $B_{n+1} = B_n + B_n \times 0.01 = B_n(1 + 0.01) = 1.01 \times B_n$

The recursive relationship is then written as : $B_0 = \$2500, \quad B_{n+1} = 1.01 \times B_n$

The intention with these is not to set difficult modelling exercises, but to assist students to understand the process involved in the setting up and solution (hence the extra lines of working out shown in the example).

Fibonacci Sequence

The Fibonacci material comes from the current Further Maths module of Number Patterns and Applications. It focuses on setting up and finding values of terms in the sequence. The introduction of Lucas sequences is really about using a different pair of starting numbers for similar problems.

A suggested teaching outline for year 11 is outlined in Table One. The focus is on introducing key recursion techniques, processes and language, and the use of two lessons at each key concept is to give teachers time to teach the concepts clearly and to allow students to have working time with access to a teacher.

Table 1 *Suggested Teaching Outline for Year 11*

Lesson(s)	Topic(s)
1	Arithmetic sequences
2	Step-by-step, recursive relationships Rule for the nth term, graphing

3	Geometric sequences
4	Step-by-step, recursive relationships Rule for the *n*th term, graphing
5	Applications
6	Financial, Practical
7	Fibonacci Sequences
8	Step-by-step, recursive relationships Lucas sequences, graphing

Further Mathematics

It is important that there be a consistency of approach between years 11 and 12, particularly in the use of language and mathematical terms. However, the focus in Year 12 is the financial modelling, with less emphasis on the underlying basic theory (HREF1, p. 56).

Preparatory homework or assignments may be required, particularly where students have transferred from Mathematical Methods in Year 11 to Further Maths in Year 12 and will be lacking in the background for key areas. The curriculum allows for a review of the essential elements at the start of the topic.

The first applications are those of depreciation.

Prime cost or straight line depreciation is an example of linear decay with the common difference value representing the annual depreciation amount. For an asset costing $60 000 being depreciated annually at 12.5% prime cost, the annual depreciation amount would need to be calculated ($7500) and the book value relation (based on the *n*th term formulation above) becomes $V_n = \$60000 - \$7500n$

A reducing balance depreciation situation is an example of geometric decay. If our $60 000 asset was being depreciated at 20% p.a. reducing balance, then the asset's value each year would be (100% – 20% =) 80% of the value in the previous year. The book value relation (based on the *n*th term formulation for a sequence with geometric decay) becomes $V_n = 0.80^n \times \$60000$

Unit cost depreciation is a form of linear decay, where the key driver is not the number of years but the amount of use that the asset is put to. Examples of this include number of pages printed, number of kilometres travelled or number of bottle tops produced. Where examples provide details of average annual use, they become variations of the prime cost problems.

The new material for Year 12 is based around situations in which there is geometric growth followed by linear growth or decay. This generates a sequence in which each new term is generated by multiplying the previous term by a constant amount and then having another constant amount added to or subtracted from this product.

For example, Kate takes out a personal loan of \$2000 which charges interest at 12% p.a. compounding monthly. The loan is intended to be repaid with six equal monthly repayments of \$345.00.

(a) Write a recurrence relation to represent this situation.

Define term: Let B_n = the balance of the loan after n payments.

Starting value: $B_0 = \$2000$

Interest rate per month: Rate = $\frac{12\%}{12} = 1\ \% = 0.01$

In words: new balance = current balance + interest charged – payment

In mathematical terms: $B_{n+1} = B_n + 0.01 \times B_n - \$345 = 1.01 \times B_n - \$345$

The recursive relationship is written as: $B_0 = \$2000,\ B_{n+1} = 1.01 \times B_n - \$345$

(b) How much is still owing on the loan after four months ?

Using the calculator, \$680.37 see Figure 1.

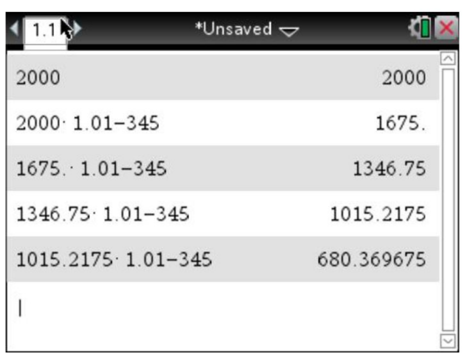

Figure 1. Calculator display of recurrence calculations

(NOTE– Graphic calculators work recursively!

In this case 2000 was typed in and the enter key pressed.

To perform the calculation, type in × 1.01 – 345 and then enter.

Press the enter key as many times as required to repeat the calculation)

(c) Is the loan paid out exactly after six months?

If not, how much will the last payment be to fully repay the loan?

The loan is not paid out after six months as there is still 60 cents owing. The usual bank practice is to add this small amount to the sixth payment which then becomes (345 + 0.60 =) $345.60

(This is a common feature to this type of loan, that the last repayment will be different to the previous ones because of the effects of rounding for the repayments)

(d) How much interest is paid in total?

Total amount of interest paid

= total repaid –the amount borrowed

= number of payments × payment amount – the amount borrowed

= 5 × 345 + 345.60 – 2000 = $70.60

(Repaying a loan quickly minimises the amount of interest charged)

There is a lot happening with this example.

The interest rate used in the recurrence relation has to be adjusted to allow for the frequency of compounding.

The recurrence relation is more complex than the previous examples in that there is both multiplication and subtraction occurring.

The term of this loan is such that students can track what is happening using the recurrence features of their calculators quite quickly.

The other new feature of this topic is the amortisation table. Table 2 contains an amortisation table for this $2000 loan.

Students do not need to construct an amortisation table for themselves, but have to interpret one that is provided for them. An amortisation table such as the one shown in Table Three will take a few minutes to construct using a spreadsheet.

The key features of a reducing balance loan stand out quite clearly in an example such as this one.

The interest charged decreases with each repayment, because it is charged on the outstanding balance. The principal reduction, representing the amount paid off the principal each time is increasing.

Table 2 *Amortisation Table for Kate's Loan*

Payment Number	Payment Amount	Interest charged	Principal reduction	Balance of loan
0	0.00	0.00	0.00	2000.00
1	345.00	20.00	325.00	1675.00
2	345.00	16.75	328.25	1346.75
3	345.00	13.47	331.53	1015.22

4	345.00	10.15	334.85	680.37
5	345.00	6.80	338.20	342.17
6	345.00	3.42	341.58	0.60
Total	2070.00	70.60	1999.40	

This table also shows that if only \$345.00 was paid for the sixth payment, then \$0.60 would remain.

The same principles apply when looking at larger and longer loans, such as housing mortgages.

For example, George borrows \$330 000 at 6% p.a. interest compounding monthly to be repaid at \$2365.00 per month over 20 years.

The recurrence relation for this loan, using the same definitions as in the previous example would be: $B_0 = \$330000, \quad B_{n+1} = 1.005B_n - \$2365$

An amortisation table for this loan would have to show 240 repayments, creating a document that would run over a large number of pages. Even just looking at the first few repayments on the calculator (see Figure 2) shows that the proportion of each repayment paying down the principal is about 30% (715/2365), compared to over 90% for the much smaller personal loan.

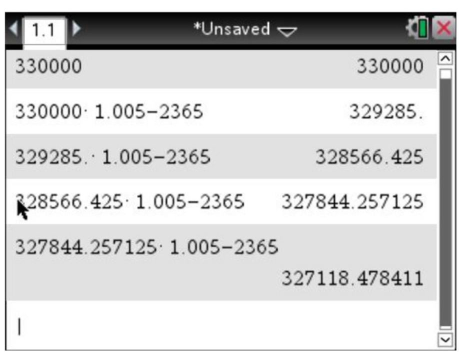

Figure 2. Calculator display of loan calculations

Calculations involving such a high value financial situation become the provenance of the Finance Solver (or similar tool/app in the graphic calculator). This would be used to determine how much was still owed halfway through the loan period, how long would it take to cut the amount owing to half of what was borrowed, or, shades of Examination 1 Q9 in 2014, determine the amount of the final repayment. (In this case, \$1995.78 is still owing after the 239th payment, meaning that the final payment will be \$2005.76 after interest is added.)

The typical annuity would generate a recurrence relation similar to that of the mortgage, as the investor has interest credited to their account before the annuity amount is paid out. So for an investor who puts $350 000 into an annuity paying 6% p.a. compounding monthly and receives $2500 per month, the recurrence relation for the balance of the investment would be: $B_0 = \$350000, \quad B_{n+1} = 1.005B_n - \$2500$

An annuity investment, where regular payments are added to an amount already present in an account would have a plus sign for the amount regularly deposited. So if $400 is added each quarter to an initial investment of $5000, while earning interest at 4 % p.a. compounded quarterly, the recurrence relation for the balance of the investment would be:

$B_0 = \$50000, \quad B_{n+1} = 1.005B_n + \$400$

In an interest-only loan situation where the borrower is paying just the interest charged each period of time without paying down the amount borrowed, the periodic payment required or the amount that could be borrowed for a particular repayment limit, could be calculated using either a calculation based on simple interest or a recursive relation, since the current and future balance of the loan will always be the same.

Similarly with perpetuities, where only the interest is paid out and the amount invested is preserved, a calculation based on simple interest or a recursive relationship could be used to determine either the investment required or the amount paid out each period.

The suggested teaching outline for Year 12 as shown in Table 3 is also quite generous with time allowed, both for student-centred classwork and teacher familiarisation with the material.

Early indications from teaching this topic at Year 11 are that the assembly of the recurrence relation is the most difficult part. How the Year 12's will cope, together with using the Finance Solver for the difficult calculations remains to be seen. I look forward to the challenge!

This paper has been developed from the VCAA/MAV Professional Development Sessions given around Victoria in May and June 2015, produced in association with Peter Jones from Swinburne University.

Table 3 *Suggested Teaching Outline for Year 12*

Lesson(s)	Topic(s)
1	Recursive relationships Revisit linear, geometric, Introduce combination
2	Depreciation modelling
3	Set up recurrence relation, nth term, graphing
4	Flat Rate, Reducing Balance, Unit Cost

5	Interest
6	Simple/compound interest in recursive relations Effective interest rates in compounding situations Graphing
7	Reducing Balance Loans
8	Step-by-step, recursive relationships
9	Calculator solution for the *n*th term (large), graphing Amortisation Tables
10	Annuities
11	Step-by-step, recursive relationships Calculator solution for the *n*th term (large), graphing
12	Interest Only Loans and Perpetuities
13	Step-by-step, recursive relationships Calculator solution for the *n*th term (large), graphing Calculator limitations
14	Annuity Investments
15	Step-by-step, recursive relationships Calculator solution for the *n*th term (large), graphing
16 – 18	Assessment

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HREF1:http://www.vcaa.vic.edu.au/Documents/vce/mathematicsMathematicsSD-2016.pdf (last visited 16/09/2015)

MATHEMATICS ON ONENOTE

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Microsoft OneNote is a program that has great benefits for students and teachers of mathematics. It helps organise mathematics notes for a class and for individual students. Using OneNote, mathematics can be handwritten on a tablet computer. OneNote can be used for learning and teaching mathematics online. Potentially, OneNote provides a mechanism which allows the teacher to adapt learning materials to suit the cognitive needs of individual students.

Mathematics on OneNote

Microsoft OneNote is a note-taking program that allows the user to collate information in a variety of forms including text, images, spreadsheets, audio commentaries and videos. OneNote provides a structure for the organization of notes and it supports multi-user collaboration. OneNote is also suitable for taking handwritten notes on devices that are operable with a stylus. These features make OneNote practical and beneficial for use in mathematics education.

OneNote provides an organisational structure

OneNote supports learning organization by providing a structure consisting of sections, pages and sub-pages as illustrated in *Figure 1*.

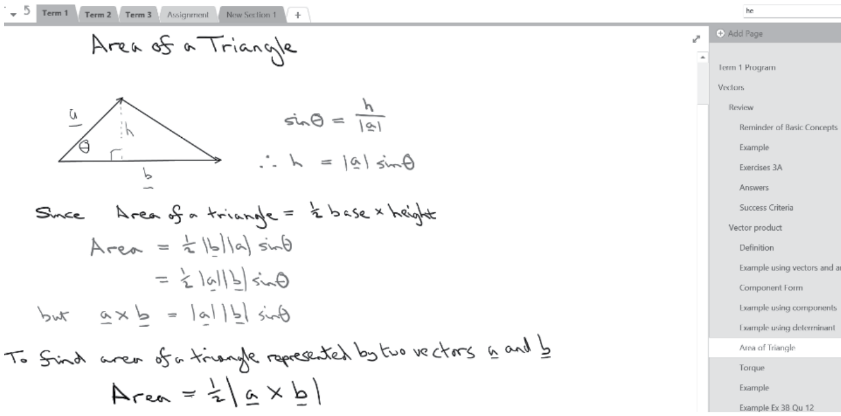

Figure 1. OneNote provides an organized structure.

The organisational power of OneNote has been elevated even further through the development of the OneNote Class Notebook application. This application, has expanded the functionality of OneNote by providing three spaces to work in. The first is a Collaboration Space in which students can input together at the same time. This allows students to collaborate and give one another feedback. The second space is the Content Library. This space is controlled by the teacher and students have read only access to it. In the third space, each student has a notebook of their own with sections and pages. The individual student controls and organises their own notebook. No-one else has access to it apart from the teacher. This arrangement encourages the students to be self-regulated learners, an important skill for the 21st century (Microsoft Partners in Learning, 2012), and makes it easier for the teacher to give differentiated support and feedback.

OneNote provides a natural interface

Mathematics is a language. It has its own syntax, grammar and vocabulary. It is therefore a language in its own right, and not just an extended version of an existing language (Krussel, 1998). The language of mathematics is represented by a diverse array of symbols, diagrams, and characters borrowed from other languages. This brings many challenges to the process of communicating in mathematics in digital form. As far as text is concerned, the use of a keyboard, arguably, has the advantage of speed, leaving more time for thinking (Chemmin, 2014). However, whilst it is relatively easy to communicate with text using a keyboard, some more sophisticated tools are required to support digital communication in mathematical language (Charles & Gaill, 2011).

Instead of trying to communicate in mathematical language with a keyboard, a simple alternative would be to use pen and paper. Another alternative is to use a digital pen (stylus) and write on the computer screen. Research into students' work on hypothesis-generating tasks has revealed differences in communicative fluency according to the type of interface that the students used (Oviatt, 2013). The use of a computer keyboard was found to elevate linguistic fluency, whereas non digital pen and paper elevated non-linguistic fluency (sketching, diagramming and annotating). The use of a digital pen interface was found to elevate non-linguistic fluency even more than non-digital pen and paper tools. These results were replicated in problem solving tasks, strengthening the finding that people communicate more non-linguistic content when using a digital pen interface such as a stylus on a tablet computer. Further, it was found that keyboard-induced linguistic fluency actually suppresses the generation of ideas, whereas the use of a digital pen interface increases the generation of ideas (Oviatt, 2013). In relation to hypothesis generation and problem solving – important skills in STEM disciplines – it appears that the pen is mightier than the keyboard, but, in turn, the stylus is mightier than the pen.

Using OneNote with a stylus on a tablet computer helps students deal with the intricacies of mathematical language by simply handwriting it rather than having to depress a complex sequence of keys. It also affords the user the ability to gesture, annotate and diagram, thereby supporting non-linguistic thinking. An example of non-linguistic thinking communicated through the use of OneNote with a stylus on a tablet computer is shown below.

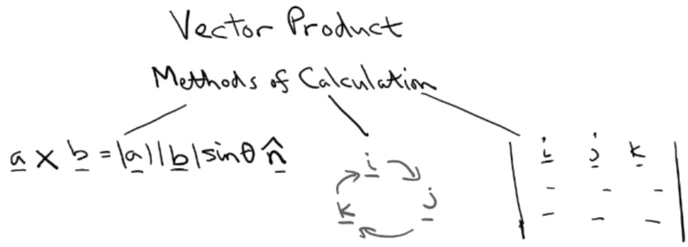

Figure 2. Non-linguistic thinking on OneNote.

OneNote can be used for online learning

Mathematics courses can be delivered online effectively using a blend of synchronous and asynchronous methods. Some of the learning takes place with the teacher and students communicating together online at the same time, and other learning is supported by resources made available over the Internet. In conjunction with other software, OneNote can be used to support both the synchronous and the asynchronous components of online learning.

In 2014, a class of Year 11 students from Ormiston College in Brisbane Australia were enrolled in a Year 12 course in Probability and Statistics. The students completed the course entirely online, and then sat the examinations at school under supervised conditions, and in accordance with the assessment criteria of the Queensland Curriculum Assessment Authorities. OneNote was used in the synchronous and asynchronous aspects of the course. The use of OneNote online is depicted below. Using the show desktop feature of the software Skype for Business, online lessons were conducted in which OneNote was used to handwrite mathematics on a tablet screen. OneNote was also used in screencast recordings made with an add-in feature of PowerPoint called Office Mix. In the recordings of these examples, OneNote is an integral part of a multimodal experience which is spontaneous, visual and has a friendly human voice.

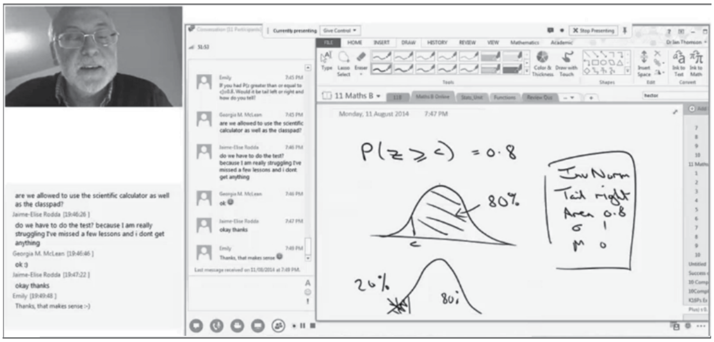

Figure 3. Using OneNote for online learning.

Matching materials to the cognitive needs of individual students

When students are learning mathematics they rely on their working memory to cope with new information. This places a burden on their thinking. This burden is known as cognitive load. Cognitive load exists in three forms (de Jong, 2010). First, intrinsic cognitive load is that associated with the actual subject matter in hand. Second, extraneous cognitive load is caused by information in the instructional material that is not relevant to the learning. Third, germane cognitive load arises in the construction of new mental schema, and is typified by learning processes such as interpreting, inferring, exemplifying and organising. Each of the three types of cognitive load- intrinsic, extraneous and germane - have unique features. These features may be examined in more detail, in order to identify ways that technology, in this case OneNote, may possibly provide support associated with each type.

Intrinsic cognitive load is to a large extent dependent on the subject matter, which in the subject of mathematics may be, for example, right-angled trigonometry, simultaneous equations or statistics. Intrinsic cognitive load is also affected, however, by the prior knowledge of the student. According to their learning backgrounds, some students will experience higher intrinsic cognitive loads than others.

Extraneous cognitive load comes from aspects of the instructional material that do not contribute to knowledge construction. The classification of cognitive load as extraneous may be dependent on the prior knowledge of the student. Some materials may be necessary for students with limited prior knowledge of a topic, whereas other students may find this same material to be unnecessary, and therefore extraneous. For the latter group of students, the nature of the extraneous material may result in a learning reversal effect (Kalyuga *et al.*, 2010). They find themselves unable to avoid the distraction of the aspects of the material that, to them, are extraneous. This results in an inefficient use of their working memory, and actually retards their learning.

Germane cognitive load can be associated with the complex reasoning processes that students employ when extending their knowledge. It relates to ways of thinking such as classifying, abstracting, induction, deduction and constructing support (Marzano, 2007). Unlike extraneous cognitive load, germane cognitive load is desirable and necessary. It stimulates the thinking processes and promotes the generation of new mental schema. As with other aspects of cognitive load, however, the distinction between extraneous and germane cognitive load is influenced by the background of the student. When acquiring knowledge, for example, it may be helpful to have multiple representations of a new concept. To one student, a graphical representation of an algebraic concept may be germane because the student constructs new knowledge by thinking though the connection between one representation and the other. To another student with more prior knowledge, however, the graphical representation may be superfluous and, to them, extraneous.

The OneNote Class Notebook, as described earlier, allows each student to access a common library of content whilst at the same time having an individual private section which they control and to which no one else has access apart from the teacher. This means that the student can customise the content, and thereby attune the intrinsic cognitive load according to their individual needs. The teacher can also assist in matching the intrinsic cognitive load to the needs of the individual student by adding or cutting information in the student section. This provides a mechanism which allows cognitive load to be classified as intrinsic or extraneous relative to the individual student rather than on a "one size fits all" premise. In a similar way, the classification of cognitive load as being germane rather than extraneous can be made in relation to the student, and materials can be customised accordingly.

"Back to the Future" of handwriting mathematics

Cognitive load can be affected by the type of computer interface that the students use. Research by Oviatt et al. (2006) which compared the effects of different types of educational interfaces found that students' performance on mathematics word problems was constricted by the use of a keyboard. The use of a keyboard to produce mathematical notation induces extraneous cognitive load that would not be experienced when writing in a natural way by hand. Pen and paper would be less problematic in this regard. A tablet computer with a stylus, however, offers an alternative that takes us "Back to the Future" of handwriting mathematics. Writing mathematics by hand on a tablet computer avoids the extraneous cognitive load associated with a keyboard. It has the advantage, however, of producing a digital artefact which, although handwritten, may be stored, copied, transmitted, edited and annotated. The fact that OneNote facilitates writing by hand on the computer screen is arguably the most significant benefit of OneNote in relation to mathematics education since it combines the benefits of a natural interface with a digital product.

Benefits of sound and vision

There are instructional benefits to be accrued when a teacher models a think-aloud process when working through the reasoning of a problem. The students cope better when the explanation and the visualisation are presented in different modalities (Marzano, 2007). This modality principle also applies in the context of multimedia presentations, where it is found that an auditory explanation accompanying a visual description is more manageable than having both the explanation and the description in a visual format (De Oliveira et al., 2015) In other words, the combination of sound and vision has a lighter extraneous cognitive load than the purely visual combination of text and diagrams.

The combination of auditory and visual presentation is achievable using OneNote. This facility was utilised in the example of online learning described earlier. The students could experience this in a synchronous way (students and teacher online at the same time) and in an asynchronous way (students watching a recording of the presentation). Either way the extraneous cognitive load was mitigated by the combination of auditory and visual explanations.

Summary

OneNote is a note-taking application that is very suitable for use in mathematics education. If offers an interface between students and computers that supports the organisation of their learning. OneNote can be used to make handwritten notes on a tablet computer using a stylus. This helps the students cope with mathematical notation in a digital environment, since it is easier for them to handwrite rather than type the details. It also elevates non-linguistic thinking which improves problem solving. The OneNote Class Notebook application has a collaboration space, a content library and individual notebooks for each student. This supports multi-user collaboration, self-regulated learning on the part of the individual student, and differentiated support for the class from the teacher. OneNote can also be used effectively to deliver online mathematics courses using a blend of synchronous and asynchronous methods. Potentially, OneNote provides a mechanism which facilitates the fine-tuning of intrinsic, extraneous and germane cognitive loads on students' working memory.

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EVALUATION OF PI AND OTHER MATHEMATICAL CONSTANTS AND FUNCTIONS

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An understanding of evaluations of constants pi, e, and the functions cos, sin, $\log_e$ is accessible to most senior students.

The Circle

A **circle** is a line drawn on a plane that is the same distance from a point called the **centre**. The width of a circle is called the **diameter (d)**. The distance around a circle (its **perimeter**) is called the **circumference (C)**. A line drawn from the centre to the circle is called the **radius (r)**

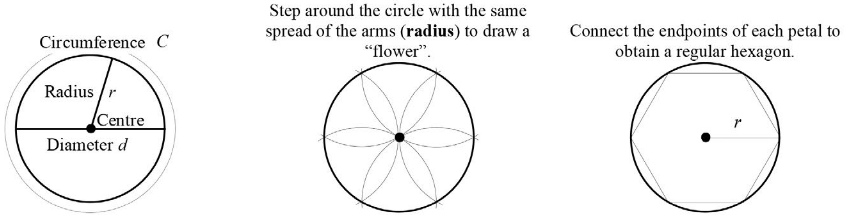

Figure 1. Inscribing a hexagon in a circle.

Circumference of a Circle

As the distance around the hexagon is $6r$, the circumference is greater than $6r$ i.e. $C > 6r$ or $C > 3 \times 2r$. As $d = 2r$ then $C > 3d$. The exact number of diameters in the circumference is pi (π), so $C = \pi d$ or $C = 2\pi r$ from which it follows $\pi = \frac{C}{d}$. π. can be approximately evaluated by measuring the circumference and diameter of circular objects and then dividing the circumference by the diameter.

π is approximately 3·14 as a decimal or approximately $\frac{22}{7}$ as a fraction. The exact value of π cannot be written as a fraction or a decimal. For most practical purposes the decimal 3·1415926536 is "exact" as is the fraction $\frac{355}{113}$.

Area of a Circle

Figure 2 shows a circle divided up into sectors. These sectors are re-arranged into a 'rectangle' (one white sector is divided in half and each half placed on either end). The distance around the curved part of the grey sectors (the 'width' of the 'rectangle') is half the circle's circumference i.e. πr. The height of the 'rectangle' is r so its area is $\pi r \times r = \pi r^2$ which must also be the area of the circle.

Perceptive students will realize that the width is less than πr and will rightly be sceptical that the area of the 'rectangle' is $\pi r \times r = \pi r^2$. This provides an ideal opportunity to talk of limits by stating that if the circle were divided into more and more sectors, the 'bumpy' top and bottom of the 'rectangle' becomes smoother and in the limit becomes indistinguishable from a straight line.

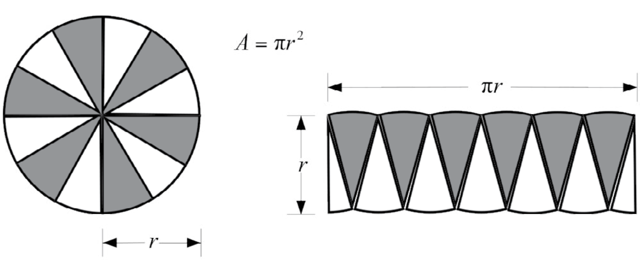

Figure 2. Circle area is same as 'rectangle' area showing $A = \pi r^2$.

Squaring the Circle

An enlightening activity for younger secondary students is to get them to cut a circle and a square of the same area from the same uniform material (usually paper). Students need to know that area is a measure of how much surface. It may be helpful to show the first two diagrams in *Figures 3* and *4* (without dimensions) and discuss the relative areas of the square and circle. This activity can be assessed by weighing the square and circle (using the electronic scales found in most secondary schools) and expressing the smaller weight as a percentage of the larger. The percentages shown in *Figures 3* and *4* give the smaller or equal shape's area as a percentage of the larger or equal shape's area.

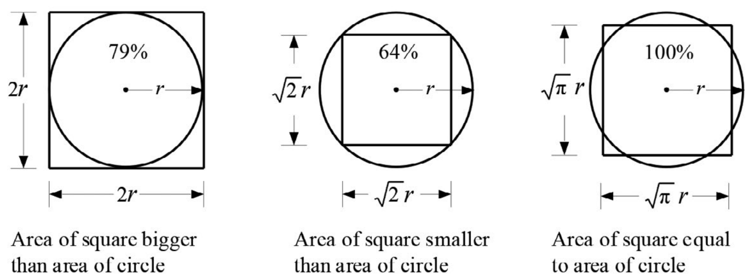

Figure 3. Various squares with the same size circle.

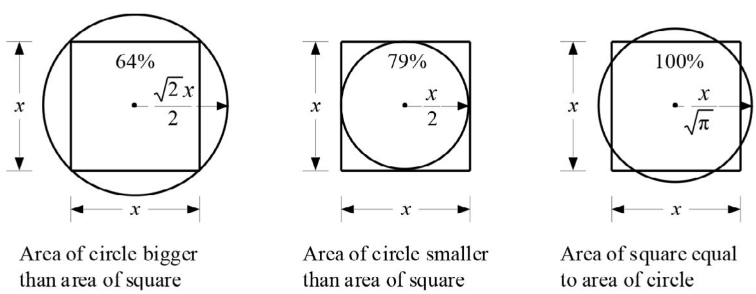

Figure 4. Various circles with the same size square.

Unit Circle

Plotting the twelve points (1, 0), (0, 1), (-1, 0), (0, -1), (0.6, 0.8), (0.8, 0.6), (-0.6, 0.8), (-0.8, 0.6), (-0.6, -0.8), (-0.8 -0.6), (0.6, -0.8) and (0.8, -0.6) which satisfy the relation $x^2 + y^2 = 1$ leads students to appreciate this is a unit circle centred on the origin. θ is the distance from (1, 0) anti-clockwise around the circle. $\theta = \pi$ halfway around the unit circle. $\cos(\theta) = x$ and $\sin(\theta) = y$. The distance from (1, 0) along the tangent on the right of the circle, where the straight line passing through θ and the origin meets is $\tan(\theta)$.

For the point (-0.8, -0.6), $\theta \approx \frac{100\pi}{83}$ so $\cos\left(\frac{100\pi}{83}\right) \approx 0.8$, $\sin\left(\frac{100\pi}{83}\right) \approx 0.6$ and $\tan\left(\frac{100\pi}{83}\right) \approx 0.75$. *Figure 5* shows this plot.

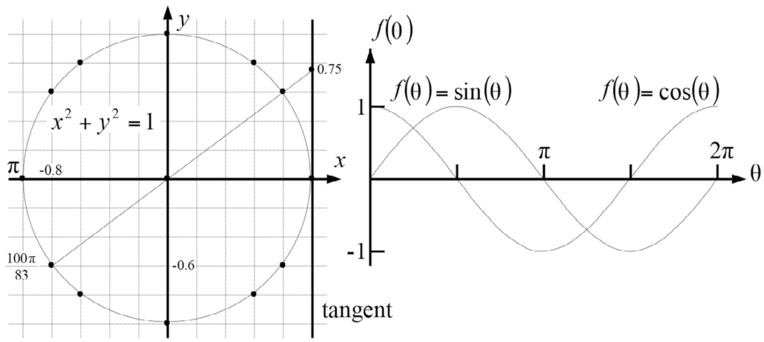

Figure 5. Unit circle with tangent. Graphs of $f(\theta) = \sin(\theta)$ and $f(\theta) = \cos(\theta)$.

As $x = \cos(\theta)$ and $y = \sin(\theta)$, $x^2 + y^2 = 1$ can be written as $\cos^2(\theta) + \sin^2(\theta) = 1$. Dividing by $\cos^2(\theta)$ changes this to $1 + \tan^2(\theta) = \sec^2(\theta)$. Perusing the gradient of $f(\theta) = \sin(\theta)$, shows it is credible that the derivative of $\sin(\theta)$ is $\cos(\theta)$. It can be shown $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. $0.75 = \frac{-0.6}{-0.8}$ in *Figure 5*, illustrates this. This knowledge with the quotient rule can be used to show that the derivative of $\tan(\theta)$ is $\sec^2(\theta)$.

i.e. $\frac{d\tan(\theta)}{d\theta} = \frac{d}{d\theta}\left(\frac{\sin(\theta)}{\cos(\theta)}\right) = \frac{\cos(\theta)\times\cos(\theta) - \sin(\theta)\times -\sin(\theta)}{\cos^2(\theta)} = \frac{\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} = \sec^2(\theta)$.

If $r = \tan(\theta)$, $\frac{dr}{d\theta} = \sec^2(\theta)$ so $dr = \sec^2(\theta)d\theta$. If $r = 0$, $\theta = 0$ and if $r = x$, $\theta = \tan^{-1}(x)$.

$$\int_0^x \frac{1}{1+r^2}dr = \int_0^{\tan^{-1}(x)} \frac{\sec^2(\theta)}{1+\tan^2(\theta)}d\theta = \int_0^{\tan^{-1}(x)} \frac{\sec^2(\theta)}{\sec^2(\theta)}d\theta.$$

$$= \int_0^{\tan^{-1}(x)} d\theta = [\theta]_0^{\tan^{-1}(x)} = \tan^{-1}(x)$$

Evaluation of π

The exact value of pi is given by the endless series $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right)$. This series can be evaluated by filling down cells A2, B2 and C2 in the spreadsheet shown in *Figure 6*. A more accurate evaluation of pi can be obtained by averaging the

values in c … e formula
=(C1+C…

Formula view

	A	B	C
1	1	1	4
2	=-A1	=B1+2	=C1+4*A2/B2

Normal view

	A	B	C
1	1	1	4
2	-1	3	2.6667

Figure 6. Spreadsheet to evaluate pi (π).

The following calculations briefly show how this series may be derived.

$$
\begin{array}{r l}
 & 1-r^2+r^4-r^6+\ldots \\
1+r^2 \,\big)\!\! & 1 \\
 & \underline{1+r^2} \\
 & -r^2 \\
 & \underline{-r^2-r^4} \\
 & r^4 \\
 & \underline{r^4+r^6} \\
 & -r^6
\end{array}
$$

$$
\begin{aligned}
\int_0^x \frac{1}{1+r^2}\,dr &= \int_0^x \left(1-r^2+r^4-r^6+\ldots\right)dr \\
\left[\tan^{-1}(r)\right]_0^x &= \left[r-\tfrac{r^3}{3}+\tfrac{r^5}{5}-\tfrac{r^7}{7}+\ldots\right]_0^x \\
\tan^{-1}(x) &= x-\tfrac{x^3}{3}+\tfrac{x^5}{5}-\tfrac{x^7}{7}+\ldots \\
\text{when } x=1,\ \tan^{-1}(x) &= \frac{\pi}{4} \\
\text{so } \tfrac{\pi}{4} &= 1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots \\
\pi &= 4\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots\right)
\end{aligned}
$$

Figure 7. Evaluation of pi (π).

Machin series for π

The Machin series, $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{4}{5^{2n+1}}-\frac{1}{239^{2n+1}}\right)$, requires some lengthy algebra and arithmetic, use of $\tan(\alpha\pm\beta)=\dfrac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$ and finally use of the series for $\tan^{-1}(x)$ (derived in *Figure 7*) i.e. $\tan^{-1}(x) = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$.

Formula view

	A	B	C
1	1	1	=4*A1*(1/B1*(4/5^B1-1/239^B1))
2	=-A1	=B1+2	=C1+4*A2*(1/B2*(4/5^B2-1/239^B2))

Normal view

	A	B	C
1	1	1	3.1833
2	-1	3	3.1406

Figure 8. Spreadsheet to evaluate pi (π) using Machin series.

This series converges rapidly with only ten terms required to obtain π correct to fourteen decimal places.

$$
\begin{aligned}
\tan(4\theta) &= \tan(2(2\theta))\\
&= \frac{2\tan(2\theta)}{1-\tan^2(2\theta)}\\
&= \frac{\left(\dfrac{4\tan\theta}{1-\tan^2\theta}\right)}{\left(1-\dfrac{4\tan^2(\theta)}{(1-\tan^2(\theta))^2}\right)}
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
&\frac{\left(\dfrac{4\tan(\theta)}{1-\tan^2(\theta)}\right)}{\left(\dfrac{1-6\tan^2(\theta)+\tan^4(\theta)}{(1-\tan^2(\theta))^2}\right)}\\
&= \frac{4\tan(\theta)}{1-\tan^2(\theta)}\times\frac{(1-\tan^2(\theta))^2}{1-6\tan^2(\theta)+\tan^4(\theta)}\\
\tan(4\theta) &= \frac{4\tan(\theta)(1-\tan^2(\theta))}{1-6\tan^2(\theta)+\tan^4(\theta)}
\end{aligned}
$$

Figure 9. Derivation of formula for tan(4θ).

$$
\begin{aligned}
\tan\left(4\tan^{-1}\left(\tfrac{1}{5}\right)-\tan^{-1}\left(\tfrac{1}{239}\right)\right) &= \frac{\tan\left(4\tan^{-1}\left(\frac{1}{5}\right)\right)-\tan\left(\tan^{-1}\left(\frac{1}{239}\right)\right)}{1+\tan\left(4\tan^{-1}\left(\frac{1}{5}\right)\right)\tan\left(\tan^{-1}\left(\frac{1}{239}\right)\right)}\\
&= \frac{\dfrac{4\tan\left(\tan^{-1}\left(\frac{1}{5}\right)\right)\left(1-\tan^2\left(\tan^{-1}\left(\frac{1}{5}\right)\right)\right)}{1-6\tan^2\left(\tan^{-1}\left(\frac{1}{5}\right)\right)+\tan^4\left(\tan^{-1}\left(\frac{1}{5}\right)\right)}-\frac{1}{239}}{1+\dfrac{4\tan\left(\tan^{-1}\left(\frac{1}{5}\right)\right)\left(1-\tan^2\left(\tan^{-1}\left(\frac{1}{5}\right)\right)\right)}{1-6\tan^2\left(\tan^{-1}\left(\frac{1}{5}\right)\right)+\tan^4\left(\tan^{-1}\left(\frac{1}{5}\right)\right)}\times\frac{1}{239}}\\
&= \left(\frac{\frac{4}{5}\left(1-\frac{1}{25}\right)}{1-\frac{6}{25}+\frac{1}{625}}-\frac{1}{239}\right)\times\left(1+\frac{\frac{4}{5}\left(1-\frac{1}{25}\right)}{1-\frac{6}{25}+\frac{1}{625}}\times\frac{1}{239}\right)^{-1}\\
&= \left(\frac{\frac{96}{125}}{\frac{476}{625}}-\frac{1}{239}\right)\times\left(1+\frac{\frac{96}{125}}{\frac{476}{625}}\times\frac{1}{239}\right)^{-1}\\
&= \left(\tfrac{120}{119}-\tfrac{1}{239}\right)\times\left(\tfrac{119\times239+120}{120\times239}\right)^{-1}\\
&= \tfrac{28\,561}{119\times239}\times\tfrac{119\times239}{28\,561}\\
&= 1\\
\tan\left(4\tan^{-1}\left(\tfrac{1}{5}\right)-\tan^{-1}\left(\tfrac{1}{239}\right)\right) &= \tan\left(\tfrac{\pi}{4}\right)\\
4\tan^{-1}\left(\tfrac{1}{5}\right)-\tan^{-1}\left(\tfrac{1}{239}\right) &= \tfrac{\pi}{4}\\
\pi &= 4\left(4\tan^{-1}\left(\tfrac{1}{5}\right)-\tan^{-1}\left(\tfrac{1}{239}\right)\right)\\
&= 4\left(\tfrac{4}{5}-\tfrac{1}{239}-\tfrac{4}{3}\left(\tfrac{1}{5}\right)^3+\tfrac{1}{3}\left(\tfrac{1}{239}\right)^3+\tfrac{4}{5}\left(\tfrac{1}{5}\right)^5-\tfrac{1}{5}\left(\tfrac{1}{239}\right)^5-\ldots\right)\\
\pi &= 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{4}{5^{2n+1}}-\frac{1}{239^{2n+1}}\right)
\end{aligned}
$$

Figure 10. Derivation of Machin series for pi (π).

e - Euler's Number

Consider \$1 000 invested for 10 years at 10% per annum. With simple interest the \$1 000 investment grows to \$2 000. With interest compounded annually the \$1 000 investment grows to \$2 5937.42; monthly to \$2,707.04; weekly to \$2,715.68; daily to \$2,717.91; hourly to \$2,718.27. If interest is compounded every minute, second or instant the investment grows to \$2,718.28. The maximum growth factor when the interest is compounded every instant is known as $e \approx 2.718281828459045...$

The Binomial Theorem can be used to evaluate the limit of $\left(1+\frac{x}{n}\right)^n$ as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\tfrac{x}{n}\right)^n &= \lim_{n\to\infty}\left[1^n + {}^nC_1 1^{n-1}\left(\tfrac{x}{n}\right) + {}^nC_2 1^{n-2}\left(\tfrac{x}{n}\right)^2 + {}^nC_3 1^{n-3}\left(\tfrac{x}{n}\right)^3 + \ldots\right] \\
&= \lim_{n\to\infty}\left[1 + \tfrac{nx}{1!}\times\tfrac{1}{n} + \tfrac{n(n-1)}{2!}\times\tfrac{x^2}{n^2} + \tfrac{n(n-1)(n-2)}{3!}\times\tfrac{x^3}{n^3} + \ldots\right] \\
&= \lim_{n\to\infty}\left[1 + x + \frac{(n^2-n)x^2}{2!n^2} + \frac{(n^3-3n^2+2n)x^3}{3!n^3} + \ldots\right] \\
&= \lim_{n\to\infty}\left[1 + x + \left(1-\tfrac{1}{n}\right)\tfrac{x^2}{2!} + \left(1-\tfrac{3}{n}+\tfrac{2}{n^2}\right)\tfrac{x^3}{3!} + \ldots\right] \\
\lim_{n\to\infty}\left(1+\tfrac{x}{n}\right)^n &= 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \ldots \\
\text{In general}\quad e^x &= 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \tfrac{x^5}{5!} + \tfrac{x^6}{6!} + \tfrac{x^7}{7!} + \tfrac{x^8}{8!} + \tfrac{x^9}{9!} + \ldots \\
&= \sum_{n=0}^{\infty}\frac{x^n}{n!}
\end{aligned}$$

Figure 11. Derivation of series for Euler's number, *e*.

Derivation and Evaluation of cos, sin, $\log_e$ and π

Following calculations show how cos, sin, $\log_e$ and π are derived and evaluated.

$$\begin{aligned}
\text{If } z = \cos(\theta) + i\sin(\theta) \text{ then } \frac{dz}{d\theta} &= -\sin(\theta) + i\cos(\theta) \\
&= i^2\sin(\theta) + i\cos(\theta) \\
&= i(\cos(\theta) + i\sin(\theta)) \\
&= iz
\end{aligned}$$

Figure 12. Derivative of $\cos(\theta) + i\sin(\theta)$ where $i = \sqrt{-1}$.

$$\begin{array}{r|r}
\frac{dz}{d\theta} = iz & \log_e(z) = i\theta - c \\
\frac{d\theta}{dz} = \frac{1}{iz} & z = e^{i\theta - c} \\
& z = e^{-c}e^{i\theta} \\
& z = Ae^{i\theta} \quad \text{where } A = e^{-c} \\
\theta = \frac{1}{i}\int \frac{1}{z}\,dz & \cos(\theta) + i\sin(\theta) = Ae^{i\theta} \\
i\theta = \log_e(z) + c & \text{if } \theta = 0, \cos 0 + i\sin 0 = Ae^{i0} \\
\text{where } c \text{ is constant} & 1 = A \\
& \text{so } \cos(\theta) + i\sin(\theta) = e^{i\theta}
\end{array}$$

Figure 13. Derivation of Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Series for cos, sin and Euler's Formula

$$\text{Now } e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \frac{(i\theta)^8}{8!} + \frac{(i\theta)^9}{9!} + \ldots$$

$$\text{since } e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\cos(\theta) + i\sin(\theta) = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \frac{i^7\theta^7}{7!} + \frac{i^8\theta^8}{8!} + \frac{i^9\theta^9}{9!} + \ldots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \frac{\theta^8}{8!} + \frac{i\theta^9}{9!} - \ldots$$

Figure 14. Series for Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

$$\begin{array}{c|c}
\text{Equating real parts give} & \text{Equating imaginary parts give} \\
\text{a series for } \cos(\theta) & \text{a series for } \sin(\theta) \\
\text{i.e. } \cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \ldots & \text{i.e. } \sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \ldots \\
\cos(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} & \sin(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}
\end{array}$$

$$\text{If } \theta = \pi \text{ then } e^{i\theta} = \cos(\theta) + i\sin(\theta) \text{ becomes } e^{i\pi} = 1$$

Figure 15. Series for cos(θ) and sin(θ) where (θ) is in radians.

If $\theta<0$ or $\theta>2\pi$ then multiples of 2π can be added or subtracted until the resulting value θ_1 satisfies the condition $0\le\theta_1 <2\pi$. If $\frac{\pi}{2} <\theta_1 <2\pi$ then a value θ_2 with a possible function sign change can be found such that $0\le\theta_2 \le \frac{\pi}{2}$ using the identities $\cos(\pi \pm \theta) = -\cos(\theta)$, $\sin(\pi \pm \theta) = \mp\sin(\theta)$, $\cos(2\pi - \theta) = \cos(\theta)$ and $\sin(2\pi - \theta) = -\sin(\theta)$. If $\frac{\pi}{4} < \theta_2 \le \frac{\pi}{2}$ then a value θ_3 with a function change can be found such that $0 \le \theta_3 \le \frac{\pi}{4}$

using cos($\frac{\pi}{2}$ –θ)= sin(θ) and sin($\frac{\pi}{2}$ –θ)= cos(θ). This means that the largest value of θ used in the series for cos(θ) and sin(θ) is $\frac{\pi}{4}$.

Series for log$_e$

$$\int_0^r \frac{1}{1+x}\,dx = \left[\log_e(1+x)\right]_0^r = \log_e|1+r| - \log_e(1)$$

$$\int_0^r \frac{1}{1+x}\,dx = \log_e|1+r|$$

$$\begin{array}{r|l}
\begin{array}{l}
\quad\ \ \underline{1 - x + x^2 - x^3 + \ldots} \\
1+x\,\overline{)\,1} \\
\quad\ \ \underline{1+x} \\
\qquad -x \\
\qquad \underline{-x - x^2} \\
\qquad\qquad x^2 \\
\qquad\qquad \underline{x^2 + x^3} \\
\qquad\qquad\quad -x^3 \\
\qquad\qquad\quad -\underline{x^3 - x^4} \\
\qquad\qquad\qquad\quad x^4
\end{array}
&
\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \ldots \\
\int_0^r \frac{1}{1+x}\,dx &= \int_0^r 1 - x + x^2 - x^3 + \ldots\,dx \\
&= \left[x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \ldots\right]_0^r \\
&= r - \tfrac{r^2}{2} + \tfrac{r^3}{3} - \tfrac{r^4}{4} + \ldots \\
\log_e|1+r| &= r - \tfrac{r^2}{2} + \tfrac{r^3}{3} - \tfrac{r^4}{4} + \ldots \\
&= \sum_{n=1}^{\infty} \tfrac{(-1)^{n+1} x^n}{n} \quad \text{for } -1 < r \le 1
\end{aligned}
\end{array}$$

Figure 16. Series for log$_e$ $|1 + r|$.

$$\begin{aligned}
\log_e(756) &= \log_e\left(0.756 \times 10^3\right) \\
&= \log_e(0.756) + 3\log_e(10) \\
\log_e(756) &= \log_e(0.756) - 3\log_e(0.1) \\
r_1 &= 0.756 - 1 = -0.244 \\
\log_e(1 + -0.244) &= (-0.244) - \tfrac{(-0.244)^2}{2} + \tfrac{(-0.244)^3}{3} - \tfrac{(-0.244)^4}{4} + \cdots \\
\log_e(0.756) &= -0.2797\ldots \\
r_2 &= 0.1 - 1 = -0.9 \\
\log_e(1 + -0.9) &= (-0.9) - \tfrac{(-0.9)^2}{2} + \tfrac{(-0.9)^3}{3} - \tfrac{(-0.9)^4}{4} + \cdots \\
\log_e(0.1) &= -2.3025\ldots \\
\log_e(0.756) - 3\log_e(0.1) &= 6.628\ldots
\end{aligned}$$

Figure 17. Using series to evaluate log$_e$ (756).

Pi - π

$$
\begin{aligned}
\log_e\left(\tfrac{e^{i\theta}}{\cos(\theta)}\right) &= \log_e\left(\tfrac{\cos(\theta)}{\cos(\theta)} + \tfrac{i\sin(\theta)}{\cos(\theta)}\right) \\
&\log_e\left(1 + i\tan(\theta)\right) \\
&= i\tan(\theta) - \tfrac{(i\tan(\theta))^2}{2} + \tfrac{(i\tan(\theta))^3}{3} - \tfrac{(i\tan(\theta))^4}{4} + \ldots \\
\log_e\left(\tfrac{e^{i\theta}}{\cos(\theta)}\right) &= i\tan(\theta) + \tfrac{\tan^2(\theta)}{2} - i\tfrac{\tan^3(\theta)}{3} - \tfrac{\tan^4(\theta)}{4} + \ldots \\
\text{also } \log_e\left(\tfrac{e^{i\theta}}{\cos(\theta)}\right) &= \log_e\left(e^{i\theta}\right) - \log_e\left(\cos(\theta)\right) \\
&= i\theta - \log_e\left(\cos(\theta)\right) \\
\text{so } i\theta - \log_e\left(\cos(\theta)\right) &= i\tan(\theta) + \tfrac{\tan^2(\theta)}{2} - i\tfrac{\tan^3(\theta)}{3} - \tfrac{\tan^4(\theta)}{4} + \ldots
\end{aligned}
$$

Equating imaginary parts gives

$$\theta = \tan(\theta) - \frac{\tan^3(\theta)}{3} + \frac{\tan^5(\theta)}{5} - \frac{\tan^7(\theta)}{7} + \ldots$$

$$\text{when } \theta = \frac{\pi}{4}, \tan\left(\frac{\pi}{4}\right) = 1$$

$$\text{so } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Figure 18. Derivation of series for pi (π).

DEVELOPING A SENSE OF SPACE: WORKING WITH THREE-DIMENSIONAL OBJECTS

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The ability to make sense of the three-dimensional world we live in is crucial for human existence and technological advancement. Spatial thinking and reasoning is found to play a crucial role in boosting science, technology, engineering and mathematics. This paper discusses key elements in the development of spatial thinking and outlines ways in which teachers can help nurture children's ability to work with and reason about three-dimensional space.

Introduction

Since the publication of Howard Gardner's first book, *Frames of mind: The theory of multiple intelligences* (Gardner, 1984), there is a growing assumption that students have distinctive learning styles. It is believed that some students are verbal learners and others visual learners. It is an ill informed belief that spatial ability is fixed, that there is a gender differences in spatial thinking and this differences is hardwired biologically (Newcombe & Stieff, 2011). Recent research has shown that spatial ability is a trainable skill and that there is no genetic cause in the way male and female thinks spatially (Kozhevnikov, Hegarty, & Mayer, 2002). Importantly, the ability to reason about spatial relationships is crucial for human existence and technological advancement. In the primary years, spatial reasoning has a direct effect on children's mathematical performance (Cheng & Mix, 2012). In higher education, spatial thinking expands students' problem solving repertoire and enable them to switch strategies when solving scientific problems such as investigating molecules that have unique spatial structures (Stieff, 2011). Indeed, individuals who are proficient in spatial reasoning can conduct different mental operations, rotating, reflecting, folding and unfolding complex figures to produce accurate and elegant solution to spatial problem. Large-scale longitudinal research has consistently showed that spatial ability plays a unique role in excelling in science, technology, engineering and mathematical domains (Wai, Lubinski, & Benbow, 2009).

The learning of geometry is well placed in promoting spatial reasoning and helps develop ways of thinking mathematically. Geometry provides opportunities for nurturing mathematical 'habits-of-mind' – the ability to perform experiments, develop visually based reasoning styles, cultivate an eye for invariants, and use these and other reasoning styles to generate constructive mathematical arguments (Goldenberg, Cuoco, & Mark, 1998). It develops the skills of visualisation, intuition, perspective, problem-solving, deductive reasoning, and logical argument (Jones, 2002). Specifically, knowledge of three-dimensional (3D) objects assists day-to-day activities such as packing, designing garden beds, and assembling flat pack furniture. The ability to represent and interpret 3D objects is particularly valuable in many technical occupations. It allows designers and architects to create blueprint based on what they have visualised in their mind, and medical professions to make diagnosis through interpreting plane scans. The usefulness of geometric ideas such as parallel and perpendicular lines, and angles can be better appreciated in 3D spaces. Accordingly, helping children develop ways of working with 3D objects is of considerable importance.

Unlike number concepts, the learning of geometry does not follow a linear path. Children do not learn all about three sided shapes before moving on to learn four sided shapes. Instead, their initial geometric ideas are largely based on prototype models of shapes such as triangles, squares and circles they experienced through play. Over time, their knowledge become increasingly integrated and synthesised. A hexagon is no longer the regular hexagon they found in pattern blocks but any two-dimensional (2D) shape that has six angles. A rectangle no longer refers exclusively to an oblong but a right-angled four-sided shape. The learning of 3D objects follows a similar path but with additional challenges. Children's encounter with mathematical objects must proceed from tangible experience to interpreting and producing visual and spatial representations of 3D objects. The transition from reasoning based purely on physical appearance to reasoning base on abstract properties is highly dependent on good teaching, teaching that is centred on helping children 'see' and construct geometric knowledge. It is the connection between visualisation and language that allows geometric relations to be conceptualised.

Spatial Reasoning and Visualisation

Spatial reasoning is the ability to make sense of spatial relationships between objects. This thinking encompasses an understanding of the feature, size, orientation, location, direction or trajectory of geometric objects and being able to make spatial transformations.

To reason spatially, one has to 'see' and extract information from concrete materials, diagrams, figures or images you are presented with irrespective of one's orientation. Visualisation then, is a form of cognitive activity that enables individuals to study the movements of spatial relationships, create mental and physical images, and to manipulate them in solving various practical and theoretical problems. It is an essential skill in the development of spatial reasoning ability and building blocks for understanding 3D objects.

There are two distinct types of visual thinking, reflecting different ways of generating mental images and processing visual-spatial information (Kozhevnikov, Kosslyn, & Shephard, 2005). The first type of visualiser, object visualiser, tends to encode images globally as a single perceptual unit based on actual appearances. They generate detailed pictorial images of individual objects and process the information holistically (Kozhevnikov et al., 2002).

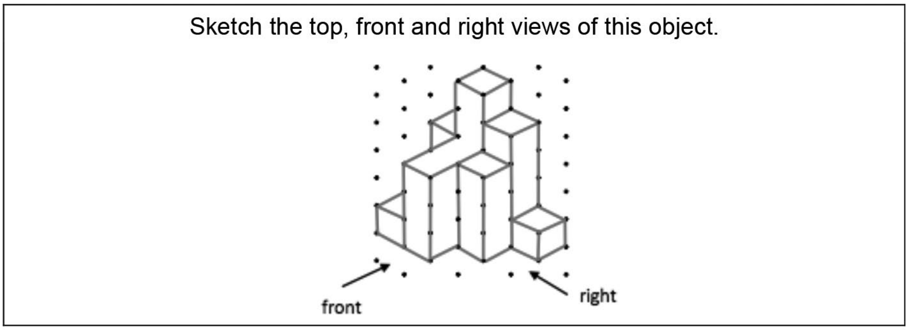


Figure 1. A visualising task on interpreting 3D object from 2D format

For example, when asked to interpret and reconstruct 3D objects using 2D format (Figure 1), object visualisers often reproduce images that resembles the actual object (Figure 2).

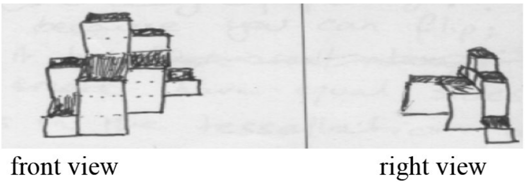
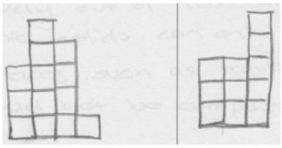


Figure 2. An object visualiser's orthographic projection of the isometric drawing

Conversely, spatial visualisers tend to encode and process images analytically, part by minute part, using spatial relations to generate schematic and abstract images from what they see. They are better able to interpret and analyse abstract representations and produce technical drawing demanded by engineers and architects (see Figure 3).



front view right view

Figure 3. A spatial visualiser's orthographic projection of the isometric drawing.

While object visualisers are faster and more accurate when performing recognition and memory tasks (Kozhevnikov et al., 2005), spatial visualisers outperform object visualisers in mathematical tasks that require analytical, creative and practical intelligence (Pitta-Pantazi & Christou, 2010). Analytical intelligence refers to one's ability to analyse, evaluate, judge, or compare and contrast information when faced with problems where the judgements to be made are of a fairly abstract nature; creative intelligence measures how well an individual can cope with relative novelties and practical intelligence involves applying skills to problems faced in daily life (Sternberg, 2005). The collective research findings to date show that students with spatial and visualising ability have a wider range of problem solving strategies, which they could flexibly employ to suit the situation.

Reasoning with 3D Objects

There are four essential spatial abilities students need to develop when working with 3D objects, namely spatial representation, spatial structuring, conceptualising mathematical properties and measuring reasoning (Pittalis & Christou, 2010). Spatial representation relates to the handling of different representational modes of 3D objects. Much of the learning of 3D objects is done through visualising plane representations. To reason spatially, students need to be able to interpret what they saw into some form of internalised mental images and reproduce the interpretation of that image. Since no single type of representation can depict and convey all the information of a real-world object, the convention used by engineers, designers and architects often involves a combination of perspective and orthogonal drawings. There is a need to explicitly teach students how to represent and interpret 3D objects from 2D format to support their spatial reasoning ability.

The second ability, spatial structuring, relates to the mental act of understanding and constructing an object, being able to identify and establish its spatial components and composites. For example, in Battista & Clements' (1998) packaging problem (Figure 4), students must appreciate issues relating to spatial structuring and be able to coordinate the spatial structure of a situation with an appropriate numerical scheme.

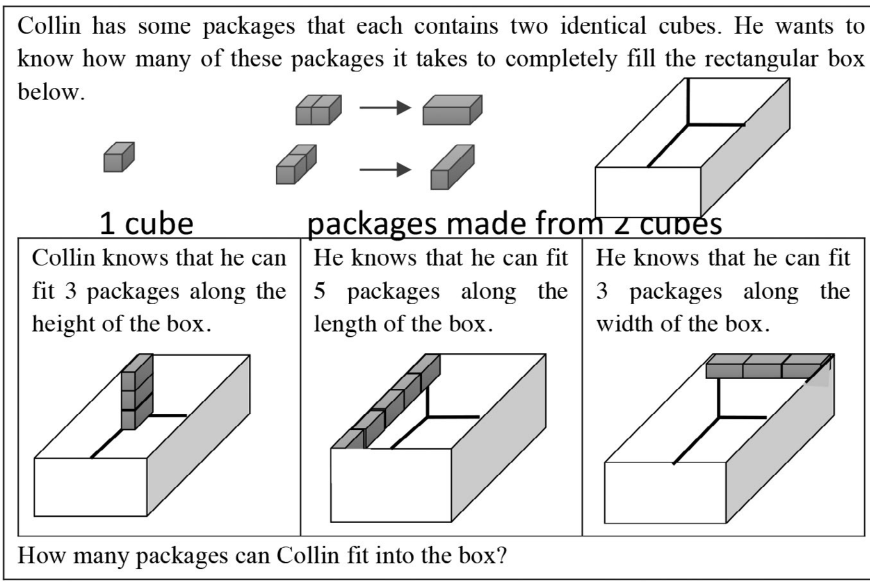


Figure 4. Battista & Clement's package problem.

The third ability deals with one's knowledge about invariant, symmetry, and transformations (Jones, 2002). In 3D space, this knowledge helps students recognise and compare the properties and classes of 3D objects. The etymology of geometry, earth measure, reflects geometry's connection with measurement. Finally, the fourth spatial ability concerns students' ability to calculate the volume and surface area of 3D objects. It links numerical operations through the use of formulas with visualisation and knowledge about the properties of 3D objects. To enrich the learning of geometry, instructions must focus on ways to nurture and promote these four spatial abilities.

Promoting Reasoning with 3D objects

The learning of 3D objects begins concurrently with learning about the names of 2D shapes. Young children learn about cones, cubes, pyramids, prisms, spheres and cylinders as they build, stack and roll different objects. Three-dimensional shapes are called solids even though they may be hollowed in the middle (Booker, Bond, Sparrow, & Swan, 2014). Both 2D and 3D shapes include those bounded only by straight lines/edges, curves or a combination of both. While the names of polygons indicate the number of angles, the names of 3D objects with all its faces or surfaces flat are named according to the number of faces. For example, an octahedron is a compound word consists of 'octa' meaning eight and 'hedron' meaning faces. One can extrapolate this naming system to refer to any object that has eight faces an octahedron. However, solids are better known according to their family groups with shared features.

Prisms are solids with two identical, parallel faces joined to one another by rectangles. Pyramids have one face as its base and other triangular faces joined together meeting at one common point. Prisms are named according to the type of 2D shapes that the parallel faces have whereas pyramids can be constructed through using different 2D shapes as the base and are named accordingly (Figure 5). Such naming system accentuates the conceptual properties between 2D shapes and 3D objects.

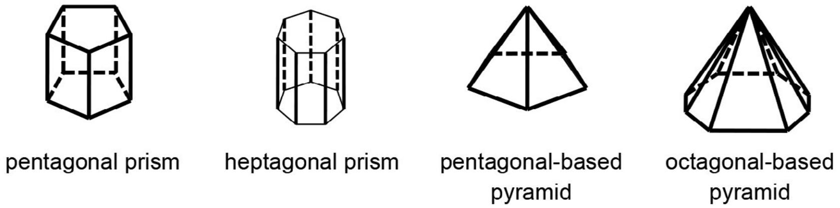


Figure 5. Types of prisms and pyramids

When reasoning about spatial relationships, visualised information needs to be interpreted and reconstructed through a logical and deductive process. The accuracy of interpretation is highly dependent on individuals' knowledge of geometric properties (Seah, 2015). For example, it is not uncommon for children to assume that Figure 6 is a pyramid instead of a triangular prism. They may associate pyramid as being made up of triangles with a pointy top and fail to see that the two end triangles are parallel to each other and joined by three rectangular faces.

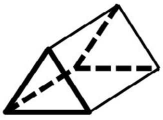


Figure 6. A triangular prism

Similarly, students also have difficulties determining whether a cylinder is a prism. At first glace, the ''face' of a cylinder is tube shaped rather than rectangular so it cannot be a prism. Closer investigation to the meaning of prism shows otherwise. The word 'prism' came from the Greek to mean 'cut', depicting the smaller versions of the same object when it is dissected parallel to its base. From this perspective, one would argue that a cylinder is a prism since the result of cutting produces two smaller versions of the same cylinder.

Knowing the origin of geometric terminologies places students' learning in context and supports the conceptualising and visualising of 3D objects. Although the origin of the Egyptian word for 'pyramid' is unknown, the Greek word 'puramis' can be referred to a 'wheat cake' because it resembles a pointy-topped cake or a structure built of stone with a square base. Reference for a monumental building is also used by the Arabs, and the Chinese's hieroglyph 金字塔 (pronouns as jin zi ta) depict the pyramid as a golden tower with an apex (Yakovenko, 2008).

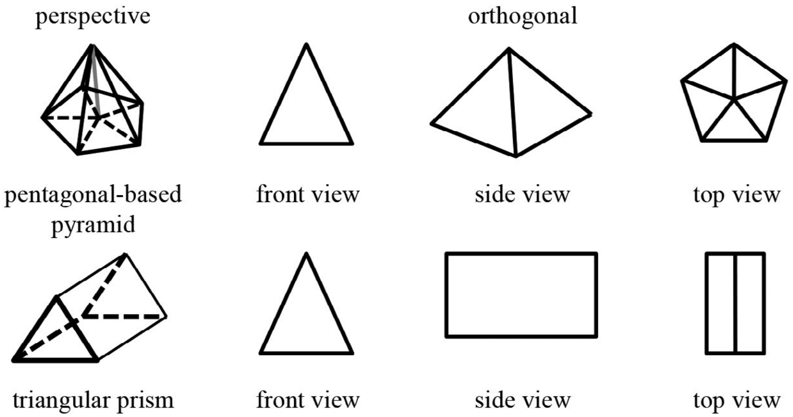


Figure 7. Perspective and orthogonal view of a pentagonal-based pyramid and a triangular prism.

Using cardboard, straws or Geofix to construct prisms and pyramids with different bases allows students to conceptualise the properties of solids (Figure 7). To ensure that the pyramid is not slanted and at risk of collapsing, the point where all edges meet must be above the centre of its base. This means the edges forming the slope must be of equal length. Hence, only equilateral and isosceles triangles can be used to construct pyramids. While the height of the pyramid is dependent on the gradient of each triangular face, equilateral triangles can only construct triangle-, square- and pentagonal-based pyramids. On a hexagonal based pyramid, the equilateral triangles will collapse and tessellate.

The same reasoning applies to platonic solids. Platonic solids are polyhedrons that are constructed with only regular polygons. These solids have faces that are identical in shape and size, and edges that have the same length. Their perfect symmetry caught Plato's imagination and he wrote extensively about them, hence the name platonic solids (Figure 8). There are only five platonic solids due principally because the faces must meet at one point, the vertex, with angles less than 360°. Otherwise, the surface will either tessellate or overlap and not able to meet at the vertex.

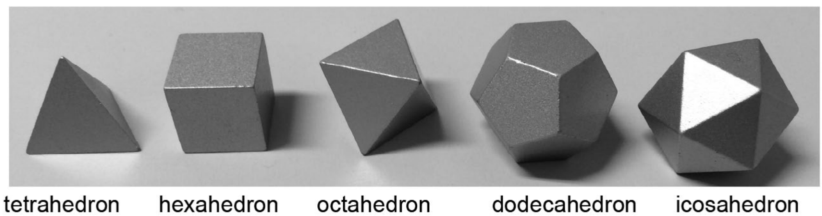


Figure 8. Platonic Solids.

The use of Nets is particularly useful in enhancing students' visualisation and helps students conceptualise geometrical properties between parts of the object. Nets allow students to comprehend spatial relations inside an object, and analyse visual images from different orientations, thereby developing their visualisation and understanding of the properties of solids (Pittalis & Christou, 2013). Similarly, learning about orthographic and isometric projection sharpens students' ability to visualise and make deduction from different perspectives (Figure 9). The term isometric is from Greek meaning 'equal measure'. Since the angles where the three faces meet must be 360 °, a 30° angle format preserves the dimension of objects on 2D format (Figure 10).

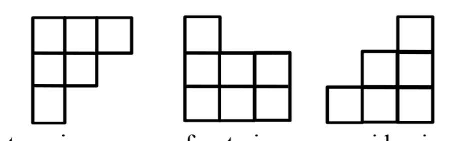


Figure 9. Orthographic projection of a 3D object.

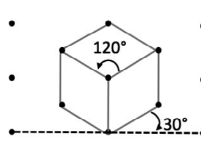


Figure 10. Isometric drawing of a cube.

Creating orthographic and isometric drawings cultivates students' ability to encode and process images analytically, using spatial relations to generate schematic and abstract images that is the hallmark of spatial visualisers.

Conclusion

This paper highlights the importance of visualisation in promoting spatial reasoning. While visualisation is needed when learning about geometric properties of 2D and 3D shapes, secure knowledge can only be constructed conjointly in language rich instruction. Moreover, students must be encouraged to create their own visual representations of 3D objects. In this way, learning becomes personally meaningful and relevant to their understanding. It is through the drawing, making and analysing of 3D objects that students learn to become spatial visualisers to a substantial degree that can be transferred across situations.

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ILEARN:WELEARN: USING A PRODUCTIVE SCHOOL PARTNERSHIP TO ENHANCE MATHEMATICS SKILLS OF ID STUDENTS

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It has long been known that for students with intellectual disabilities (ID), technologies, both hardware and software, can assist learning and access to communication skills (for example, Wilkinson & Hennig (2007)). While assistive hardware (switches, voice controls and many others) has allowed severely physically disabled individuals to participate as students in education and as adults in the world, software for ID students has had a more chequered history. The iLearn:weLearn project has sought to take advantage of the portability and interactivity of tablet technologies to provide applications (apps) in two mathematical areas (fractions and money) for a class of ten year 8/9 students in a Melbourne State school for students with ID. The delivery mechanism was apps specially written by a voluntary team of talented year 9/10 students at a Melbourne independent boys' school, and targeted at the particular needs of the ID students. The resulting enhancements of mathematical understanding and confidence were obviously visible outcomes, but unplanned outcomes were the impacts on team building, empathies and creative talents of the app coders.

Introduction: Tablets as Facilitators of Learning for Learning Disabled Students

While there is much debate about the final educational value of using simple computer technologies with all students, there has been a growing acknowledgement that for ID students the technologies can be highly advantageous. Educational strategies for learning outcomes with such students are not the same as those with mainstream classes. There are many ways of defining the student cohort that we are considering here, but one that may be useful here is "those, who, because of a disability, require special education and related services to achieve their fullest potential" (Hasselbring & Williams, 2000)

Technological support for foundational mathematics needs of ID students has received little attention compared with literacy needs. Some key mathematical skills are as essential as literacy skills for these students to survive in society, and yet few researchers have paid heed to this area. Brown, Ley, Evett & Standen (2011), working with students with intellectual disabilities to improve fractions skills, remark that mathematics plays a major, and often hidden role in society and the day-to-day activities of all individuals. It is a 'functional skill' that is crucial to ID students because it allows them to lead their adult lives as independently as possible. It is with mathematics that the iLearn:weLearn project described here engaged, working with a year 7, going on to a year 8 class of ID students with widely heterogeneous needs in a special school in Melbourne.

Mobile technologies and potential

Mobile technologies have the ability to address issues of "fullest potential" through providing educational access and equity. There is a need to examine how mobile technologies, and in particular multiple platform mobile devices, can be utilised to differentiate learning for students with special needs in the classroom and to create more personalised learning experiences. How teachers can utilise mobile technologies to differentiate learning for students with intellectual disabilities at present 'a best guess' scenario as little research exists on how mobile technologies can, and should, be used holistically in the classroom. Whilst many Apps designed to 'include' special needs students exist, they tend to be device-specific and focus on assistive technologies to support communication and/or digitised materials to increase access. Support for the implementation of these apps is disparate and often limited to advice on how individual Apps can be used to address specific learning difficulties. iLearn:weLearn, a collaboration between a State School for students with intellectual disabilities, gifted students from an Independent School, and a University, addressed issues of access and participation by involving students in the development and evaluation of technological innovation relevant to their own learning needs and for broader social benefit.

Tablet technologies and the needs of ID students

Students with intellectual disabilities have particular pedagogical and organisational needs. Well before the advent of technology to support ID students, Deshler, Alley, Warner and Schumaker (1981) wrote: "Results of research conducted with a wide range of ID adolescents have shown that severely learning disabled students need very stringent and systematic instructional procedures in order to acquire and apply learning strategies." This maxim applies equally to technology-mediated learning and learning environments without technology. Fernandez-Lopez et al (2013) assert that students with special education requirements (intellectual disabilities) are characterised by their widely differing needs and learning characteristics. This heterogeneity exists to a much greater degree in ID students than in students in conventional settings. They claim that applications for tablets such as iPads can be of great assistance to ID students, but only if they contribute to the individualisation of the learning process to meet the individual needs of the students. Thus "the development of customizable and adaptable applications tailored to users with special education needs brings many benefits as it helps to mold the learning process to different cognitive, sensorial or mobility impairments" (p78).

There are a few studies that are relevant to this area. The impact of iPads on learning disabled and emotionally disturbed students led Shah (2011) to describe and cite instances of students with mathematical fears and disabilities, being drawn into numeracy through iPad apps. For example, Shah cites Chance, who arrived with a fear of mathematics, an inability to sit still, and a deficit of patience, who became happy to spend hours working on math problems on an iPad using an app called Math Ninja. Shah identifies a key element of the use of tablets as their simplicity and the ease with which tablets can be customized. He claims that it is this aspect that aids ID students.. "The touch screens offer instant gratification for students with limited patience or those who can't understand the connection between a mouse and computer screen." It is this very need for individual customised applications that the current project set out to address. Evmenova and King-Sears (2013) also suggest that it is the transparency of personal digital assistants, smart phones and tablet computers for users with disabilities that contributes to the inclusion of individuals with various abilities and needs into the society. They assert that for learning disability students, opening doors to literacy and numeracy is a critical need, and technology can achieve this.

Tablet Technologies, needs of ID Students and Mathematics

There is only limited literature that can help guide or provide comparisons to the iLearn: weLearn project. In particular there is hardly any recent literature that relates iPads, or indeed any tablet technologies, to mathematics education with ID students. O'Malley et al (2013) found the same problem in one of the few mathematics studies conducted in this area: "The number of technology based math intervention studies is limited". They also state: "Lacking in the research is the use of instructional technology for academic interventions for students with moderate to severe disabilities in special education settings". Abbott et al (2011), in a comprehensive meta-review of all research studies relating assistive technologies to learning difficulties, looked at 135 papers, and found three dealing with mathematics instruction, and one dealing with mathematics assessment. Kagohara et al (2013) looked for all peer reviewed studies between 2008 and 2012 that involved iPods, iPads and related devices in teaching programs for individuals with developmental disabilities, and found zero studies that addressed the teaching of mathematics. One study that is available, albeit as a conference paper, is that of Brown et al (2011), who report on important research that they carried out with students with intellectual disabilities where they used a game on the computer to improve specific mathematical skills with fractions. Eight matched pairs took part, providing 8 students for the study and eight controls. Brown et al report that the intervention group significantly improved their understanding of fractions compared with the control group over a short term. The study is a useful indicator of ways to proceed and of aspects of what to expect from the study reported here, but its use is somewhat limited, because it did not use tablets with their interactive affordances, and it used a single commercially available game. There is a problem with one-for-all applications available through application stores on-line because they are not tailored to the wide differences in ID students' needs (Fernandez-Lopez et al, (2013)).

In a study of ten year 7 and 8 educationally disabled students to determine if iPads could improve basic mathematical fluency, O'Malley et al (2013) conducted a short 4 week study. Their findings indicated considerable growth in fluency during the intervention weeks, and a rapid loss of fluency in the 'B' weeks. The usefulness of such a study is unfortunately limited, like other studies mentioned, by the poor choice of a commercially available application that is 'one application fits all' and not adapted for the special needs of these students. The study was also constrained by its short duration when the learning disabilities involved require extended involvement.

Project Goals

With so little evidence of similar school partnerships to draw on, and so little evidence of similar initiatives, projected outcomes were somewhat tentative, but included:

- To provide app resources for the ID students to enhance two mathematics skills crucial to their future as citizens.
- For the coding group, to establish and foster an experience of working towards a goal in a team.
- For the coding students to learn and apply the very useful life skill for today's world of coding applications.

Additional Behavioural Goals

- To develop awareness by the coding students of the needs and difficulties of students less fortunately intellectually endowed than themselves.
- To develop the ability of the ID students to work cooperatively with people they would not normally get the chance to know but whom they will meet in the real world.
- For the coding students to offer genuine service to those needing it.

The iLearn:weLearn Project. Organisation and Implementation

The setting

The current study took place in a year 8 mathematics classroom of a school for children with moderate to severe learning disabilities. iPad tablets were available at all times for the class, but the use in mathematics was previously restricted to some mainstream published games applications. There is widespread scepticism about random use of such technologies in any classroom. Peluso (2012) comments: "simply allowing them to use their iPads, or providing them with classroom sets of iPods, does not implicitly mean they will be learning educationally beneficial material." This comment applies even more to ID situations, because as class 8B's teacher remarked, "every student has different special needs and has individual requirements".

It was this realisation that became the genesis of an idea to build on an existing relationship between the school for intellectual disabilities and a mainstream private school. In that existing relationship, year 9 and 10 students were involved with hosting an annual camp. This study is unusual, possibly unique, in that it has sought to invite the intellectually talented students to write the applications specific to the requirements of the special school students in aspects of mathematics and to interact with the ID students before and during the development of the apps. The teachers of the ID school are not strictly constrained by the standard curriculum, but are particularly concerned with adequate mathematical skills that will help their students survive within the adult world they will enter. They suggested that the coders work within the two key areas of practical understanding of money, and simple conceptual understanding of fractions. It is flexibility in app design that sets the iLearn:weLearn project apart from other studies in this area.

The year nine and ten students involved in the independent school were invited from a strongly performing mathematical group, but every member of the group was a volunteer. The project was planned to operate over one year but organisational difficulties led to its extension to a second year. This meant that some of the volunteers ended up in a high stakes year eleven level, yet so committed did they become that no-one dropped out after the initial few weeks. The time demand for the app writing team of 13 students was not insignificant: weekly team app writing sessions; one or two afternoons per term working with the ID students in their classroom; a number of weekend and school holiday sessions.

Development of the project

Following self-selection of the coders from the independent school, the group spent their first afternoon with the ID team, during which they shared experiences while the ID students worked with several commercial mathematics apps. The coders had been trained by their coordinating teacher to look for learning impacts and outcomes, and although they had discussed the mathematics pedagogies required at the very fundamental levels at which the ID students were working, we have evidence from early interviews on how surprised they were at the understandings required, and how far they themselves were removed from this level of mathematical reasoning.

The coordinating teacher at the independent school was also the mentor for helping the students to develop coding skills. The existing experience levels in coding ranged from theoretically highly skilled to almost zero skill, but none of the group had ever taken an app to completion. Some of the students claimed in early interviews that a major reason for their participation was to learn to code, for others it was to be able to use existing skills to support others with particular needs, and for two students, it was the opportunity to apply skills to a finished real project. For most of the students these reasons changed radically by the end of the project, with team learning support for less fortunate individuals prominent.

Assessment

Any project that studies the impact of a programme of activities on mathematical performance needs some form of assessment by which to judge the mathematical outcomes. There are difficulties when working with ID students for several reasons. It is almost impossible to use standardised tests because the individual needs of the students are so severe and widely different. In addition, most tests need a basic ability to read and reason questions through words,and for some ID students this is not possible. Yet some form of assessment would be required. The researchers devised a test for each topic using the app Explain Everything which was selected for its multimedia capabilities. Questions were selected from existing tests, but mainly from Schleiger Diagnostic Mathematics Tasks (Schleiger H & Gough, J. (2002), and adapted by making all the questions able to be listened to, allowing students to record responses on screen, either as speech-to-text or direct recording file, and offering a digital pen to draw or write answers on-screen if preferred to keyboard use. These adaptations allowed all the ID participants to successfully use the tests, and they were administered at the start of the project, and one year later.

Development of teamwork

Although teamwork was not an initial aim of the project, the coders developed into a cooperative team. The implementation developed a recursive nature, in that once initial apps had been coded the coders spent an afternoon working with the ID class, using the apps and observing the interactions with the iPads. As the session progressed, problems and errors appeared, and the coders started talking with the ID students about how they felt it could be improved. At the end of each session, a focus group with the coders was held, at which findings were discussed and possible solutions brainstormed. The coders then took these ideas back to their own school and began to find solutions as a team. This happened in all four times.

One outcome of this recursive procedure was that team members realised that they had different talents to bring to the coding of the apps. In a survey at the end of the project, it was clear that they had recognised their own areas of contribution. Some realised that their main strengths were in creative thinking through problems. Some accepted that they were the implementers: when presented with a problem brainstormed by the creative ones, they set to work to code the ideas. Still others were the calmers, students who had the ability to help others and bring the team together when frustrations threatened harmony and productivity.

A notable example of this team process and the growing relationships between the two diverse school groups came early in the project, towards the end of the first afternoon working together. The coders' coordinating teacher had a gaming mindset towards app coding, and the original app included buttons to make selections, like most commercial education apps, and also included scoring. One of the ID students discovered that in the fractions app, by pressing buttons at random, he could accidentally find the correct response, and then press that multiple times to accumulate a large score. His discovery rapidly spread round the room. Much of the focus group following the session was taken up with this problem and someone suggested that a way was needed to slow the thinking process down and hence avoid rapid pressing of buttons. About a week later, back at their own school and in the next team coding session, the creative idea emerged of abandoning buttons, but replacing them with sliders. One of the coders again verbalised this as "slowing them down so they have time to think". This proved to be a turning point in the effectiveness of the apps and created a thinking environment in the ID classroom. In the next joint session, it also gave the coding team a chance to discuss on-screen actions with the ID students.

Outcomes

Mathematical Outcomes

Mathematical outcomes for the ID students were impressive (Table 1), but need to be treated with caution.

Table 1: *Mean scores for the two mathematics topics, money and fractions*

Money		Fractions	
Pre	Post	Pre	Post
37%	72%	46%	71%

There is a twelve month gap between administration of the identical tests, and under normal circumstances we need to take into account natural progression to partly account for the score increases. However, natural progression is not guaranteed, or even expected, with ID students, and in interviews, the teachers of the students expressed surprise not only at the improvement but that it was maintained. When asked to what she attributed the improvements, their class teacher commented on the social cohesion in the group that developed during the project, and the fact that the students felt valued and "special".

Other Outcomes

While the support of mathematics understanding by the ID students was the overt aim of the project, other outcomes assumed equal or greater significance.

The experience of working towards a common goal through teamwork developed into a highly efficient and powerful learning experience, one that is usually only found in the adult world of business and is not usually compatible with working within a curriculum framework. For the coding group, the development of empathy and respect for the needs and also the strengths of the ID students were observed strongly by the researchers. For the ID students, there appears to have been a strong correlation between their acceptance by the coders as partners, their pride and pleasure in the engagement with the mathematical concepts through the app development process, and their successful learning. A significant observation has been that several of the ID students, in interviews with the coders, commented on how they were listened to when making suggesting improvements to the apps, and that they were able to make a valuable contribution themselves.

Conclusions

Any project working with ID students to enhance learning is fraught with research difficulties because of the strongly individualised needs of the students. However, in this project, it has been shown that it is possible to engage with and energise a group of such students through working cooperatively with non-ID students. The project also offered pointers to the power of using non-curriculum framework team-based goals to challenge students in the middle years of secondary school in creative ways using real-life goals. With the growing development of coding in the school curriculum, making mathematics apps specifically targetted to the needs of others is clearly a valid activity that can produce multiple outcomes beyond simply curriculum ones. What it will take is a willingness to break away occasionally from the assessment-based rigid curriculum framework, and to provide an organisational flexibility within which team-based learning situations can be developed.

Acknowledgments

This project was assisted by a grant provided by the Telematics Trust

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FORMATIVE ASSESSMENT – CLASSROOM CHALLENGES AND TECHNOLOGICAL SOLUTIONS

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Significant research evidence in combination with practical teaching experience has highlighted the difficulties and complexities of implementing formative assessment in classrooms. Technology offers the potential to assist with formative assessment however in practice, the information provided may not be well suited to informing learning. Drawing on research and my personal teaching experiences, this paper provides insights to help teachers successfully implement formative assessment in their classrooms. It also explores the benefits technology can bring to this process and the key requirements for easier, faster and more effective implementation.

Introduction

Formative assessment (FA) is widely recognised as a high impact teaching practice that improves student learning (Black & Wiliam, 1998; Hattie, 2009; Marzano, 2010; Popham, 2008). In 2012 the Victorian Government emphasised the importance of FA, highlighting high quality evidence and feedback as a key driver to improve student learning (Department of Education and Early Childhood Development, 2012, p. 9). An emphasis on both evidence and feedback matches the definition of FA where, "information from the assessment is fed back within the system and actually used to improve performance….." (Wiliam & Leahy, 2007, p. 31).

This paper provides a brief introduction to FA and investigates the potential for technology to reduce the challenges and constraints associated with its classroom implementation.

What is Formative Assessment?

Assessments are often identified as either formative or summative, although depending on how the information is used they can fulfil both functions (Gardner & Gardner, 2012). Formative assessment (also known as assessment for learning) is ongoing assessment during a lesson or unit of study which provides insights about student learning so timely adjustments can be made, while summative assessment (also known as assessment of learning) provides evaluative information at the end of a unit as a final measure of learning (Tuttle, 2014). Black and Wiliam's (1998, p. 5) seminal work defines FA as all "activities undertaken by teachers, and/or their students, which provide information to be used as feedback to modify the teaching and learning activities". This paper adopts Popham's definition of FA as "a process, not any particular test", with an emphasis on using the information during instruction to provide feedback and adjustments to improve students' learning (2008, p. 5).

The FA process as presented by Greenstein (2010) is provided in Figure 1 and includes 5 stages where teaching and assessing are part of a cyclical process for continuous improvement.

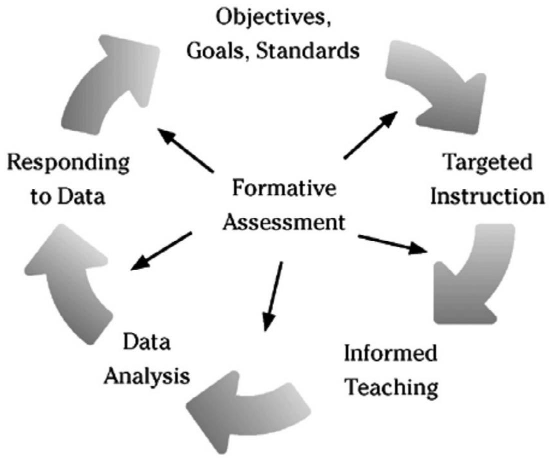

Figure 1. Summarised stages of the formative assessment process (Greenstein, 2010).

What are the Benefits and Challenges of Formative Assessment?

Research widely supports the educational benefits of FA to improve student learning (Black & Wiliam, 1998; Hattie, 2009; Marzano, 2010; Popham, 2008) with Marzano (2006, p. 10) concluding "formative classroom assessment is one of the most powerful tools a classroom teacher might use". A 2005 Organisation for Economic Co-operation and Development study found FA is "highly effective in raising the level of student attainment" (Organisation for Economic Co-operation and Development 2005, p. 91) while (Black & Wiliam, 1998, p. 61) found achievement gains "among the largest ever reported for educational interventions". However, in spite of these potential benefits, in practice many challenges have also been identified.

A 2005 evaluation of the Scottish 'Assessment for Learning' program found the complexity of FA and managing the collection of evidence were key challenges to implementation (Hutchinson & Hayward, 2005, p. 231). Other studies have also observed challenges with implementation including an emphasis on specific assessment techniques and marking at the expense of effective feedback and improvements to instruction (De Lisle, 2015; Hayward, 2015; Hume & Coll, 2009). Reviews by Hayward (2015) and Petour (2015) recognise while many countries have adopted FA, the complexities of achieving practice change in classrooms have remained. This highlights the current challenges teachers around the world face in bringing this effective and valuable teaching practice into their classrooms.

Technology and the Challenges of the Classroom

In my experience teaching involves many challenges with classrooms being busy, complex and intense environments in which to work. Teachers are continuously observing, integrating, managing, assessing and responding to behaviours and communications from all students in the class each and every minute of the class (Green, 2015). The 2015 Australian State of our Schools Survey showed many teachers face increasing workloads. Results showed 73% of teachers said their workload had increased in the past year, with high workloads identified as the most significant consideration for teachers thinking about leaving the profession ("Australian Education Union: Teachers need extra resources to cope with rising workloads," 2015).

A 1995 study of teacher perceptions of assessment showed that a significant concern for most teachers was "knowing how to assess, or even talk about" student learning with teachers often asking for "better guidelines to track kids' reading" and guidance on "what we should be looking for" (Johnston, Guice, Baker, Malone, & Michelson 1995, p. 362). They also observed assessment was "associated with very powerful feelings of being overwhelmed, and of insecurity, guilt, frustration, and anger" ((Johnston et al., 1995, p. 362)).

The time constraints and assessment challenges faced by teachers can be further exacerbated by the implementation requirements for effective FA which include data collection and analysis followed by response to the data which requires decision making and feedback (Figure 1). This is supported by the observation that teachers are often faced with time and resource constraints to implementing FA (Organisation for Economic Co-operation and Development, 2005). Therefore successful implementation of FA requires mechanisms and tools to help overcome the additional time and resource demands teachers face.

Technology has the potential to address many of these challenges (Khirwadkar, 2005) and billions of dollars have been spent on the purchase of technology for schools (Bennett & Oliver, 2011; Christensen, Johnson, & Horn, 2010). While technology allows "freedom to access information in a flexible environment" and "customized instruction in a more cost effective and efficient manner" (Leer & Ivanov, 2013, p. 19), Kopcha (2012, p. 1110) observes "teachers continue to report that they lack the time, resources, and training to use classroom technology for instructional purposes". Therefore while technology may be a potential solution it can also introduce additional challenges and difficulties to the classroom.

Common barriers to technology use in classrooms have been well reviewed (Ertmer, 1999; Hew & Brush, 2007) with key factors for teacher's use of technology including access to resources, quality of software, ease of use, and level of computer training (Mumtaz, 2000) as well as time constraints and the reliability of technology (Kopcha, 2012).

Therefore any technological solution to assist with FA must reduce the classroom challenges teachers face rather than increasing them as well as addressing the common barriers to technology adoption. If the technology solution does not address these issues it is likely that it will either not be used or only be used to a limited extent.

How Can Technology Help With Formative Assessment Implementation?

It is clear that successful FA requires effective feedback that provides sufficient information to help students and teachers make decisions that will improve learning (Black & Wiliam, 1998). In order for assessment, evidence or teacher insight to improve learning, the data and its implications for learning must first be understood and then used as the basis for teaching decisions regarding learning needs. Popham (2008, p. 23) highlights that "FA is all about decision making" about whether an adjustment is needed and what that adjustment should be. In other words it is not the *collection* of the information that matters it is the use of that information to inform teaching practice and student knowledge.

From my experience, understanding and interpreting the information from FA, deciding what this means for learning *and* communicating it back to the student in a way that helps them improve learning is the most challenging part of successful FA. This is supported by De Lisle's (2015) evaluation of FA implementation which showed effective feedback was a difficult and poorly executed step in the FA process.

In light of the challenges with implementing FA, the general challenges of classroom teaching (Green, 2015) and the integration of technology into the classroom (Kopcha, 2012) it is imperative any technological aid to implementing FA must be simple to implement, easy to use while still reducing the burden of data collection and analysis.

Many existing technology products collect information about student learning allowing students to practice mathematics and directly enter answers into the computer. These products reduce data entry time however the functionality of these products may not align well with the key elements of FA. This is supported by Popham (2008, p. 9) who raises concerns that many products and methods of FA are "not actually consonant with the research that validates formative assessment". Wiliam (2006) also highlights that many of the products that describe themselves as FA do not reflect the FA principles that research has shown to be effective.

Drawing on my own experience, if a technology solution is strong in one area, for example providing informative help to students trying to master a skill, it is often less effective in another area such as ease of use, reporting or allowing teachers flexibility to set defined tasks. These trade-offs often mean the benefits of using a given product can be negated by the additional burdens they introduce. In many cases the complexity of the user interface and the skills and knowledge required to use the technology in a way that aligns with the teaching curriculum and the stages of FA can often be overwhelming, further contributing to the challenges.

When considering a technological solution to support FA in the classroom I have found it is important to consider many elements including:

- Consistency of question types and complexity with the curriculum and learning standards
- Ability to modify/adjust the questions, skills and emphasis to match the unit content
- Ease of use (for both students in terms of practice and learning and teachers in terms of data clarity, access and analysis)

- How well the information informs the understanding of student learning and subsequent instruction needs
- Ability to identify students who require intervention (either remedial or extension)
- Ability to summarise and present information back to students that explains their learning progress and what if any changes are required
- Ability of the software to allow students to problem solve their own learning issues and to develop higher order thinking skills that will support life long learning

In summary, while technology clearly has an important role in implementing FA in schools, its application and suitability needs to be carefully considered to ensure it compliments and enhances teaching and learning in classrooms rather than adding to the burden, complexity and challenges teachers already face.

Conclusion

While FA is clearly a valuable tool to improve student learning, research has identified challenges and limits to its current implementation. These are often closely linked with the complexity of the teaching and classroom environment and have been shown to compound the time and resource limitations experienced by teachers.

While technology offers many solutions to assist with implementing FA in classrooms it is vital that any technology solution is easy to use and easy to implement while still providing the flexibility, information access and data analysis capabilities to effectively inform classroom teaching and learning.

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TASKS AND RESOURCES FOR DEVELOPING CHILDREN'S MULTIPLICATIVE THINKING

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The development of multiplicative thinking determines largely the extent of the mathematics that a person learns beyond middle primary school. Our current research project has so far revealed that many primary children have a procedural view of aspects of multiplicative thinking that we believe inhibits their progress. This workshop focuses on some of the teaching resources and tasks that have been developed from our research. The purpose of these tasks is to promote the development of conceptual understanding of 'the multiplicative situation' and the many connections within it and with other big ideas such as proportional reasoning and algebraic thinking.

Multiplicative Thinking

In their discussion paper "School mathematics for the 21st century: What should school mathematics of the 21st Century be like?" (2009), the peak body for mathematics education in Australia, the Australian Association of Mathematics Teachers (AAMT) stated: "The so-called Big Ideas in Mathematics are key to connecting other aspects of mathematics, both between and within the Mathematical Concepts and Mathematical Actions. They are overarching ideas that are neither 'concepts' nor 'actions'. Most of the "Big Ideas" pervade a number of conceptual areas of mathematics, and provide connections between them" (p.4). Basically the big ideas are those central ideas fundamental to mathematical success. According to research (AAMT, 2015; Charles, 2005; Siemon, Bleckly and Neal, 2012), one of the big ideas is multiplicative thinking. It is therefore imperative that for continued success students become 'solid' multiplicative thinkers (Siemon, Breed, Dole, Izard, & Virgona, 2006).

Multiplicative thinking is vitally important in the development of significant mathematical concepts and understandings such as algebraic reasoning (Brown & Quinn, 2006), place value, proportional reasoning, rates and ratios, measurement, and statistical sampling (Mulligan & Watson, 1998; Siemon, Izard, Breed & Virgona, 2006). Further, Siegler et al. (2012) advocate that knowledge of division and of fractions (another part of mathematics very much reliant on multiplicative thinking) are unique predictors of later mathematical achievement.

The Difficulties of Multiplicative Thinking

Unfortunately, research (Clark & Kamii, 1996; Siemon, Breed, Dole, Izard, & Virgona, 2006) has found that the label of being 'solid' multiplicative thinkers cannot be applied to most students. Clarke and Kamii (1996) found that 52% of fifth grade students were not 'solid' multiplicative thinkers, and Siemon, Breed, Dole, Izard, and Virgona (2006) established that up to 40% of Year 7 and 8 students performed below curriculum expectations in multiplicative thinking, with at least 25% well below the expected level.

Whereas most students enter school with informal knowledge that supports both counting and early additive thinking (Sophian & Madrid, 2003) students need to re-conceptualise their understanding about number to understand multiplicative relationships (Wright, 2011). Multiplicative thinking is distinctly different from additive thinking even though it is constructed by children from their additive thinking processes (Clark & Kamii, 1996). The difference between additive thinking and multiplicative thinking has been characterised by Confrey and Smith (1995) as the difference between a "counting world" and a "Splitting world". Essentially a "splitting world" is the ability to share (split) and is an idea to which many students are very sensitive from their earliest experiences (Confrey, 1994). Therefore, this makes splitting part of the multiplicative situation. The counting world however, identifies additive increments and often 'interferes' with the splitting concept. The counting world does not lead students' thinking into the world of rational numbers (fractions, percentages ratios etc.) in the same way as the Splitting world does (Confrey & Smith, 1995).

Multiplicative Thinking. What Helps Students?

If, as the research tells us, multiplicative thinking is vital for further success in mathematics, but difficult to learn, then teachers need the content and pedagogical knowledge to succeed in their endeavours to effectively teach it. Carroll (2007) has constructed a list of strategies and ideas that support multiplicative thinking.

- Allow children to work out their own ways to solve problems involving multiplicative thinking.
- Compare additive and multiplicative thinking approaches.
- Use models that clearly illustrate the idea/s.
- Sometimes students are introduced to the ideas symbolically before the groundwork has been done to establish meaning and become comfortable in working with them.
- Make and discuss the links between fraction ideas, rates, ratios and proportion.
- Use authentic contexts and models to exemplify situations.
- Estimation is really important as it demonstrates understanding of the concepts involved.
- Engage in conversations about the ideas and talk about the links, discuss the similarities and differences between the ideas.
- It is development of fuller, deeper and more connected understandings of the number system that makes a difference.

Carroll (2007, pp. 41 - 42)

In the remainder of this article we would like to pursue dot point three of Carroll's (2007) list; "Use models that clearly illustrate the idea/s". Although this will be the focus, it should be noted that by carefully considering the model used to 'carry' the understanding, at one time or another, all of the other dot points could and definitely should be exercised.

A Model for the Development of Multiplicative Thinking

One model for trying to build a conceptual understanding of multiplication is the multiplicative array. Multiplicative arrays refer to representations of rectangular arrays in which the multiplier and the multiplicand are exchangeable (Figure 1). These arrays are seen by some as powerful ways in which to represent multiplication (Barmby, Harries, Higgins & Suggate, 2009; Young-Loveridge & Mills, 2009). Young-Loveridge (2005) asserts that multiplicative arrays have the potential to allow students to visualise commutativity, associativity and distributivity. Further, Wright (2011) states that multiplicative arrays embody the binary nature of multiplication, and that as a representation, they have value, as they also connect to ideas of measurement of area and volume and Cartesian products.

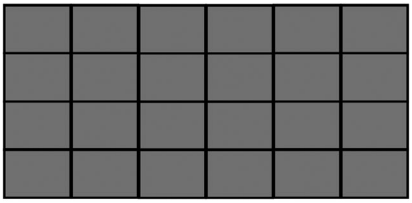

Figure 1: Multiplicative array

We will visit the use of multiplicative arrays in the activity called "A bag of tiles."

Task - A bag of Tiles

This task can be varied to suit the teaching and learning of several aspects of multiplicative thinking. Essentially it is based on students working with a set of 2cm x 2cm plastic tiles. These can be given out in a plastic snap-lock bag and can vary in number, depending on the task and the targeted aspect of multiplicative thinking. However for the following activity each pair of students is given 24 tiles. Although each student could be given their own set of tiles, if we want the students to engage in meaningful conversations about the task and about the multiplicative thinking behind the task, then having the students in pairs is actually a more productive setting.

The basic task is for children to make an array with the tiles so that rows and columns contain the same number of tiles with none left over. In this case it is an array of 24 tiles in a 6 x 4 configuration. The different arrays A and B (Figure 2) provide an interesting discussion point in leading children to a realisation that although the two arrays arrive at the same total, the manner in which they are constructed is important in certain contexts. For instance, there may be very big ramifications in not understanding that, although in a day you would end up taking twelve tablets, taking two tablets, six times a day may have very different effects from taking six tablets twice a day. This idea can be further developed by asking children to 'tell a story' about each number fact to show that 4 x 6 (four rows of six) is indeed different to 6 x 4 (six rows of four).

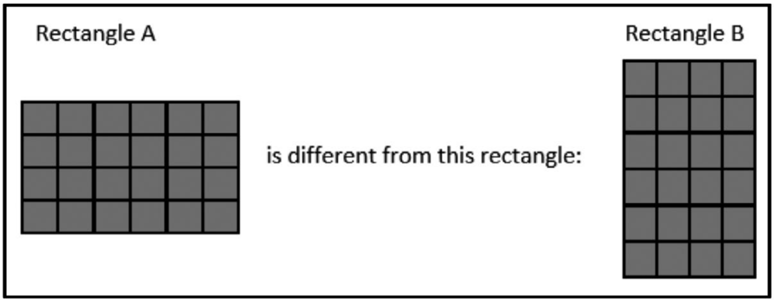

Figure 2: Two "different" rectangles, 6 x 4 and 4 x 6

As well, the two arrays offer a good opportunity to develop an understanding of the commutative property in a deeper way that by simply rotating the array. Different coloured strips of four and six squares can be laid over the array to show that while the two arrays represent the same product, they are in fact different (Figure 3).

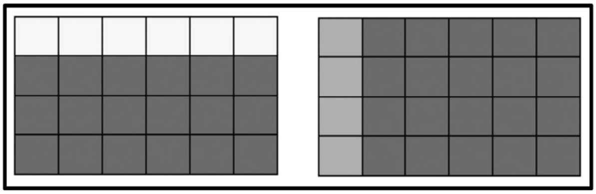

Figure 3: Overlaying Rectangle A with different strips

The students are then asked to find all of the different rectangles that can be made from 24 tiles. The bag of tiles activity works very well as a physical representation to develop an understanding of factors and multiples as well as the commutative property of multiplication, both very powerful understandings which will be often called upon in mathematics. This is also a good opportunity to make connections between the representations of the arrays and the way we symbolically record them. Further we can exploit the opportunity to discuss and show the links between the multiplicative situations of multiplication and division, that is; 6 x 4 = 24, 4 x 6 = 24, 24 ÷ 6 = 4 and 24 ÷ 4 = 6. Over and above the mathematical content that this activity contains, it also embodies the four proficiency strands (Problem Solving, Understanding, Reasoning and Fluency) as articulated in the Australian Curriculum: Mathematics (ACARA, 2015).

Further to the richness that can be gleaned from using 24 tiles and asking the questions that are articulated above, the same activities can be entered into with arrays of other sizes to further develop and re-inforce the understandings. The tiles can then be employed to investigate prime, composite, square and triangular numbers.

For this activity the students are given more tiles to work with, and are instructed that they cannot rotate the tiles (employ the commutative property) and still consider them to be 'different'. Therefore a 6 x 4 configuration is considered to be the same as a 4 x 6 configuration. They are then asked see if they can make a rectangle with two tiles in more than one 'different' way? They build the array and record the finding that the only configuration for two tiles is a 2 x 1 array. The students then investigate three tiles and continue their investigations as required, but probably for not less than the 24 tiles with which they started (Figure 4). There is a conversation to be had with first of all, four tiles, and then with nine tiles, and then 16 tiles (possibly even 25 tiles) about whether a square is a rectangle.

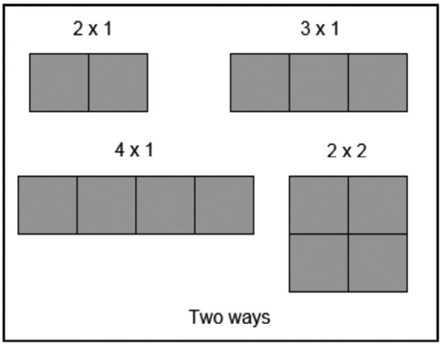

Figure 4: Arrays for 2 tiles, three tiles and four tiles

Initially what the students are making and recording are the factors for whole numbers between two and 24. What is also occurring, is an opportunity to talk about the numbers for which two or more sets of factors cannot be found (prime numbers) and the numbers which have multiple sets of factors (composite numbers). The exploration here of course is what makes some numbers prime numbers and others, composite numbers. Also, by previously considering arrays for the numbers four, nine, 16 and perhaps 25, an exploration of square numbers and why we call them square numbers can be undertaken.

Conclusion

In this article we have only just begun to scratch the surface of the opportunities afforded by multiplicative arrays to support students in understanding the multiplicative situation. Arrays can also be used to link to the division construct for fractions – e.g., a group of twenty four tiles can be split into quarters so that one quarter of 24 is 6, two quarters of 24 is 12, three quarters of 24 is 18. The same can be said about sixths. Further the concept of why we calculate area as we do can be explored by overlaying the tile array with a clear grid of the same number of squares (i.e., for a 4 x 6 array, use a clear grid of 2 cm squares in a 4 x 6 pattern). Students are asked to describe the area of the grid.

What the array offers is a powerful illustration of multiplicative situations and rich opportunities to use this construct to problem solve, develop understanding, practice fluency and cultivate reasoning.

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A WOMAN IN STEM

Jude Alexander

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Most people grow up with a basic understanding of traditional roles – doctor, blacksmith, candlestick maker, builder. But does the average student know what's involved in becoming a process operator, fly-in fly-out geologist or research project officer? There is broad agreement that we need more people, particularly women, working in science, technology, education and maths (STEM). However 50% of women terminate their STEM careers early, with a sense of frustration at the waste of time, energy and money. In order to address this problem, we need to gain a deeper understanding of the diverse range of STEM industries, including what the roles entail and what the culture is like, in order to prepare young people to engage fully with these industries.

Science, Technology, Engineering and Maths - Why Women Leave, and How We Can Help

When women leave STEM careers, it is a large personal and career hurdle. Some retrain in other fields, others find work in related areas and all of us abandon our original career plan. When I joined the education sector a year ago, I heard the phrase 'women in STEM' in my workplace, and decided to write about my experience – twenty years working in the area.

The first barrier to writing about STEM is that it is incredibly hard to make any generalisations about a concept that spans four very diverse areas. Science, technology, engineering and maths are four enormous fields, and the jobs within them are hugely varied. The reality is nuanced. There is probably no single initiative that will get more women, or more people, into STEM. Workers include nurses, offshore oil rig operators, lab technicians and computer programmers. These jobs suit different people, with different strengths, goals and skill sets.

Over twenty years, I spent

- two years in factories,
- three years in mining and cattle stations (the most extreme environments I worked in),
- five years in soil testing, working for small companies near cities,
- five years in bushfire science in a city office.

So I went from extreme workplaces to very mainstream ones. The observations in this paper apply strongly to the early, remote workplaces. The last of these fields was by far the least extreme, and all of the female colleagues in that team (except me) still work for the same employer.

The Early Cheer Squad

I studied maths and science in a very congenial environment. It was quiet, and the teachers were knowledgeable. Everyone agreed that placing women in STEM was a great idea, yet no one seemed to have any experience working in STEM fields. I spoke to an uninspiring geologist once at a careers day, and decided nevertheless to go into geology. University was similar, with excursions to look at rocks.

I worked in factories in my summer holidays, my first STEM jobs, and experienced some of the reality of this particular workforce. The work was hard, and I struggled with the practical aspects of it. I was repeatedly lifting more than I could manage, and working in hot environments.

For example, one task: you open the furnace doors, insert a little steel saucepan with a very long handle, dip the saucepan in the molten aluminium, pull it out and pour the aluminium into a sample mould. That process takes maybe three seconds. Afterwards, the chocolate bar in your chest pocket is entirely melted, front to back. The heat is like a physical blow. I loved it. It was tough.

The Impossible Goal of Thinking Practically

In that factory environment, nearly all the thinking was practical. We had never studied this. I had done well at school, but, like many high-achieving students, simple thinking tasks like remembering my lunch AND my locker key seemed to evade me. In many of the STEM environments that I worked in, there was a great deal of that type of practical 'remember your lunch and your locker key' thinking, for example, you need four metal tabs in your pocket when you go to put strapping tape around a tonne of aluminium ingots, otherwise you'll waste time when you have to go and get them. I would relentlessly forget the metal tabs. Science and engineering are often highly practical fields. People with extremely low literacy were often better at the practical aspects of our work than any of the educated professionals.

After graduating from university, I flew out to my first minesite in Western Australia. Like all new graduates, I was seen as quite bright, highly impractical, and a liability to myself and those around me. Like many women, I didn't know the difference between an excavator and a bulldozer. I was surrounded by men, and many of them seemed to have deep seated problems with women, which may explain why they found work in the outback where there were hardly any of us. Sexual harassment was a daily reality for all of the women, and we developed a tough 'us and them' mentality. On the other hand, STEM workplaces are amazing playgrounds of physics, chemistry and biology. I got to do some exciting work, make some great friends and have some wonderful times. Where else do you give an eighteen year old a saucepan and let her handle molten metal at 660°C? Or let her drive an excavator, or spend a whole day in the bush with a pile of science equipment and a clipboard?

Why Women Leave

When discussing women leaving STEM, there are a few things that it is useful to understand:

We have choices.

A young woman with a university degree can work elsewhere. The choice is not STEM or nothing. We have already proved we are trainable and employable. STEM workforces do not need to be vastly more suited to young, female employees than the alternatives, but they do need to avoid being dramatically worse.

I left because of safety.

It seems odd to reduce a complex, nuanced, life-changing choice to that one sentence, however I left extreme environments and my original plan because I had been subjected to too many life-threatening environments. I realised that there was a small, but realistic chance of serious injury, exposure to dangerous toxins (a daily reality) or death. Those risks would not be as prevalent in a more mainstream environment (it's important to note here that the death rate in the mining industry has dropped substantially since that time).

Many women leave to have babies and don't return.

This can seem normal until you work in an industry where they vast majority of women who have babies stay at work. We stay because we like or at least tolerate our jobs, we have a friendship network, we definitely need the money and our work is part of our identity. From that perspective, the leave-and-don't-return workplaces start to look like they are probably doing something wrong.

What Doesn't Help

There are a number of approaches to women in male-dominated workplaces that set us up for failure. I had hoped this had changed for the next generation of women, however the attitudes that didn't help me are still fairly prevalent.

Women are going to be wonderful.

We're not. Breaking into a new field is difficult. Imagine a troupe of rugby-playing men moving into nursing in the 1970s: the existing women wouldn't be able to have their normal conversations in the tearoom and might get disgruntled. The uniforms wouldn't fit the new men, and their fingers would be too large for some of the tasks. They may not have the emotional capacity that has been a standard part of the job for many years. Existing employees might question whether the new employees were able to do the job. It may be the case that tasks would need to be adapted to fit new workers.

Similar challenges affect traditionally male workplaces adapting to new, female employees.

You're a pioneer feminist adventurer!

It sounds great in theory, but we are often grumpy and exhausted, we make mistakes and sometimes we slack off. This makes us a lot like the other humans in the workplace. Women in STEM environments are not exalted heroines. We do our jobs. Now and again we might publish a great piece of research or set a production record. That is exciting. After that, we go back to drinking coffee and waiting for lunch.

If only you'd just be a little bit nicer.

Sometimes you need to make room for the things you cannot see. The man that is sexually harassing the woman you are expecting niceness from is doing it when you are not there. If she says anything, he will deny it and ridicule her. He started off being nice, too. Sometimes trust needs to be built. You may be a kind and decent person who listens and understands, but the man that's sexually harassing the women in your workplace is not. Women can get hard and bitter for a number of reasons. There is often a greater expectation of niceness from women, while they are working in a nastier environment than the men around them.

I have introduced the brown note of sexual harassment.

Sorry about that. There does seem to be a taboo about talking realistically about the type of problems we're cheerfully sending young women off to face, alone, in isolated environments. This taboo is the final element in this list of unhelpful attitudes.

What Does Help

Conversely, there are a number of helpful attitudes; too many to name here.

I'm here for you.

Maintaining networks with people outside of the new environment is often the only way many women stay in harsh environments for any length of time. This is still important if a woman is working in the city in a workplace with a strong gender divide and no role models. There are, of course, limits to the caring and sharing anyone might want to do with their high school maths teacher, and you probably don't have time to keep up with everyone, but being available to discuss technical problems can be helpful.

How are you going to handle that?

One of the most useful and supportive things anyone did was to challenge me to develop a rape strategy before I left for the mines. I devised a strategy that I wouldn't recommend to anyone, but it seemed to work. Safety is an imperative in the workplace. This specific example may be inappropriate for you, but developing some realistic ideas and strategies around likely problems is a good idea. For example: you're going into the mining industry. Do you know the difference between a bulldozer and an excavator?

Nice one.

Celebrating achievements rather than pipe dreams is an excellent way of supporting anyone's career. My mum used to relate back the strangely garbled accounts she gave her friends about what I was doing, and it was obvious she had no idea what that actually was. But her pride was also obvious, which helped. Of course, I have no idea about what she does in her career either (she's a historian). 'It never gets easier,' she says, cheerfully. 'You're always on a steep learning curve. I thought I would know what I was doing by now, but I still don't.'

So, Where Does Maths Come Into All This

When working in STEM I was frequently using maths, especially techniques I had thought would never, ever be useful. Quadratic equations. Trig. Lots and lots of volumes. The thing I used the most was estimation. It frequently underpins decisions worth thousands or hundreds of thousands of dollars. For example:

Here is another calculation geologists and major contractors do every day on a minesite:

The digger loads ore into the trucks in the mine. The trucks drive up to the ore pad and dump their ore on the ground. The front end loader then loads the ore into the crusher, where it is processed.
There have been 110 truckloads of ore carted to the ore pad during this dayshift.
Each truck contains about 120 tonnes of ore.
The loader driver has loaded 367 buckets of ore into the crusher.
One loader bucket contains about 35 tonnes of ore.
How many tonnes of ore has gone into the crusher, according to:
The digger driver?
The loader driver?
What is the difference, in tonnes, between the two sources?

You are counting piles of rock on the ore pad. Each pile is about 100 tonnes. There is a row of thirteen piles, and a patch that is three by seven piles.
How many rock piles are there?
How much ore is there?
The loader puts about 1000 tonnes of rock per hour into the crusher. How long have you got before the ore pad will be empty?
You call the digger driver on the two way radio. He says it will be four hours before he's digging ore again. What do you say to him?
0 Hurry up.
0 Take your time.
0 Why not have a longer lunch break?
0 I give up.

To really train people, or women, to do this in a realistic environment, you would need to take them to a dusty, hot minesite and get them to do six hours of work, then pose this question to them while standing next to noisy equipment, without a calculator, while someone is asking her to go home with him after work. We can never do this at school, of course, but the women who would do well in that kind of challenge might be a different cohort to the ones who do well given the same problem in a clean, neat, quiet environment.

I also did quite a lot of environmental work with the environmental scientist on one minesite. Her maths was like this:

> You have a borehole with a diameter of 8cm. It's 20m deep. In order to test the water in the borehole, you need to pump out a volume of water equivalent to three times the volume of the borehole, and then start measuring the water quality. This ensures an uncontaminated sample is taken.
> You measure the water depth by dropping a sensor attached to a measuring tape from the top. The measuring tape reads 5.3m when the sensor touches the water.
> What is the depth of water in the borehole?
> What is the volume of water in the borehole?
> How much water do you need to pump out before you can take a sample?

I then went and worked on cattle stations, which are not STEM workplaces. I did absolutely no maths, only revolting maths, like 'you've got twenty rotting birds in a water tank. How long will it take you to get them all out, and will you be able to eat lunch afterwards?'

> Geologists earn, on average, $89 000 per year. Environmental scientists earn $67, 000 per year. Cattle station hands earn $36 000 per year.
> How much money am I losing per day if I don't study maths before starting my career?
> [Ballpark figures are OK]

When I later pondered the lack of maths in this field, I realised that the maths that applied to the situation was of a more overarching nature:

Some Conclusions

In short, the training we give people entering STEM is not always relevant to the work we do in the field. The only time I really used my science training was when I worked in research. My advice would be that when you are looking at young women and men and deciding whether to encourage them to look at the thousands of careers available in STEM, it might be a good idea to broaden your outlook from the students who are passionate about science, maths or tech, to the girls who are good at endurance running, or the students who can deal with behavioural problems. Unless you are recommending a career in research or pure maths, you might look closely at the organised young woman who always remembers her lunch AND her locker key.

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A PEER MENTOR NUMERACY PROJECT

Tina Fitzpatrick, Dona Martin, Simon Turnbull, Brent Ritchie, Karen Smith, Elle Livingston, Jennifer Curtis, Adam Clusker, Vicki Mitchell, Ariana Te Arihi, Jasmin O'Sullivan.

La Trobe University

In an inclusive, judgment free environment, which promotes open dialogue, work undertaken in numeracy workshops at La Trobe university capitalizes on third year pre-service teacher practice. The work discussed here demonstrates how we work to extend students' ability by assisting them to make meaningful connections to prior learning. Having gained proficiency with number, many third-year students act as peer mentors to demonstrate to first-year students what it takes to develop competency with number. Their voluntary involvement in a series of peer-mentored lessons adds legitimacy to a need to build understanding of number for all teachers, enables genuine discourse, and creates a non-threatening learning environment where all class members feel comfortable in being open and honest - all of which serves as a catalyst for deep learning. The peer mentors, first-year students and their teacher work together as a team. This team approach reinforces the importance of content and pedagogy in a way that more traditional classroom practice cannot. Indeed, 'This camaraderie has been the foundation for achieving huge success.' (Elle)

Purpose

Across the four years of our undergraduate teacher education mathematics classes, at La Trobe University, we expect pre-service teachers [herein PSTs] to engage with the fabric of both thought and action. Importance is placed on each learner's preparedness to engage with the mathematics and time is devoted to addressing student anxiety by strengthening levels of self-belief. Our work integrates learning, development, and performance by taking account of things like a learner's mindset and past mathematical experiences.

We highlight early in our classes, how many learners perceive the teacher as owning the material, as it is the teacher that publically takes responsibility for its validity. Then we work to gain an early consensus that teaching and learning are inextricably linked. To this end we engage our PSTs in deep learning via peer mentoring opportunities.

It is important to note that to the uninitiated, peer mentoring simply provides struggling learners an opportunity to access more support than one teacher in a classroom could hope to offer. This paper is purposefully designed to dispel that idea. Peer mentoring, as conducted at La Trobe, is much more than that and it is explored here in a direct attempt to demonstrate the benefits it has to every member of the class.

Introduction

In a direct attempt to have PSTs take responsibility for their own learning, we introduced a purely voluntary, peer-mentoring program, outside of all set classes, between third and first year PSTs. One lecturer and an average of ten first year PSTs together with two third or fourth year PSTs made up a weekly class. Within these classes all members shared, as a prized focus, opportunity to develop *deep learning* in mathematics education.

The classes developed after discussions across first and third year levels about the value of setting up peer mentoring opportunities and after a general consensus by all involved that there was genuine benefit in peer teaching. That it offers 'both tutor and tutee benefit academically, [and that it is often] the tutor more than the tutee' that benefits, as discussed by Biggs and Tang (2007, p. 118). These sentiments are also endorsed by Topping (2005) who states that 'peer learning can be defined as the acquisition of knowledge and skill through active helping and supporting among status equals or matched companions. It involves people from similar social groupings who are not professional teachers helping each other to learn and learning themselves by so doing.'

From their first year, the PSTs in a four-year teaching degree, are encouraged to value mastering mathematics. The basis of this work builds from research on human development (Perry 1970, Belenky, Clinchy, Goldberger & Tarule 1986), which reinforces the notion that some learning has transformative results. Work in mathematics at La Trobe also draws heavily from research findings and recommendations for teaching actions, as discussed in the Australian Education Review on Teaching Mathematics: using research-informed strategies (Sullivan, 2011), work which was further enhanced at a 2014 presentation to our La Trobe PSTs by Professor Peter Sullivan. Peter encouraged us all to consider a set of six principles from the review, designed to guide teaching practice and provide key principles for effective mathematics teaching:

1. Articulate goals - identify key ideas that underpin the concepts you are seeking to teach
2. Make connections - build on what students know, mathematically and experientially
3. Foster Engagement – utilise a variety of rich and challenging tasks
4. Differentiate challenges – have all classroom participants interact as they engage in the experiences
5. Structure lessons – adopt pedagogies that foster communication and share responsibilities
6. Promote fluency and transfer – practice, reinforcement and prompting transfer of learnt skills

Context

For 8 years now, PSTs have had the opportunity to take voluntary mathematics classes in addition to their regular lessons, in order to improve their competence and confidence with mathematics. These classes, referred to as 'numeracy workshops' run for an hour per week during the same semester in which the PSTs undertake formal mathematics subjects. PSTs have the option to participate in this series of workshops twice; once in second semester of their first year and/or in first semester of their third year. The aim of these workshops is to consolidate PSTs' mathematical and pedagogical content knowledge.

Within these classes PSTs work through a Numeracy Workbook, which covers areas of most concern (e.g. fractions and long division). All topics are presented in a way that allows for deep learning of the material. Traditional algorithms, rules etc. are generally the final step and work develops logically during the process. The emphasis of these workshops is on deepening PSTs conceptual understanding.

In 2015, for the first time, some third and fourth year PSTs became Peer Mentors [herein PMs] to the current first year PSTs. Most of the PMs had done the Numeracy workshops in the second semester of their first year (2013) and in the first semester of this year (2015). In effect, these mentors would now be assisting the lecturer, Tina, to teach within the numeracy workshop program after having already undertaken the classes twice themselves as students.

Each week the PMs attended a session with Tina before the workshop classes for that week. Together they discussed the content to be delivered and the best ways to teach the weekly topic. We regarded these sessions as 'mathematics staff meetings', not unlike team meetings that actually occur in schools. During these meetings the group worked to ensure that the teaching was in line with current research and best practice.

The PMs attended one or more numeracy workshops during the remainder of the week and became mentors to the first year PSTs. Many PMs discovered such value in outcomes that it was not uncommon for them to attend up to 4 lessons per week.

Whilst Tina remained ultimately responsible for the class, at times PMs took leading roles during the lesson. The mentors worked with students one to one or in small groups. PMs often instructed the whole class using the whiteboard and other classroom aids.

Acquiring Peer Mentoring Skills

There is an old saying that 'to teach is to learn twice'. Throughout their course, PSTs learn to plan and prepare well for the classes they teach. However, most teachers have experienced at least once, that no matter how well they have prepared for a lesson, their actual lesson delivery can be less than what they desired. Improvement comes with experience. The PSTs in this project acquired more teaching skills as a result of their efforts. As Topping (2005) states '[mentoring] … is characterised by positive role modelling, promotion of raised aspirations, positive reinforcement, open-ended counselling, and joint problem-solving.' As a means of demonstration of clear outcomes, following is a series of quotes directly from the PMs:

- As a third year student, I witnessed the importance that mentoring had on both the mentor and the mentee. The mentor is able to share their knowledge and experiences, which allows them to gain more knowledge, as well as promoting deep understanding for the mentee. The first year students benefit from working with peers who were not long ago in their very position. Each one of the first years has spoken about how grateful they are for our help, which highlights how important this program has been. (Jasmin)
- I greatly valued the weekly mentor meeting where the discussion focused on the structure of the next mentoring session. The language, the explanations, the use of materials, and the manner of demonstrations were discussed and refined. All

mentors had input and gave suggestions based on personal experiences and current understandings. All the mentors came away with new ideas and approaches to the subject matter. I found myself looking forward to these sessions to share techniques I had seen in the classroom that might aid in clarifying an issue we had discussed just a few weeks prior. (Jenny)

- Each session was successful in demonstrating what it is like to plan with other teachers; how to best teach maths using both higher order thinking skills and cooperative learning approaches. Within the classes mathematical literacy is being developed as are ways to improve existing mathematical methods. We now encourage discussion and debate in positive, fun and interactive ways. As a result, the mentees and mentors alike are engaging in maths in meaningful and relevant ways. These discussions allow room for growth mindsets, a re-examining of current beliefs and provide ways to add depth to our current knowledge and skills. (Ariana)
- This entire mentoring experience has had a positive influence on my confidence in teaching mathematics. I view the workshops as a seed for further nurturing and developing the mathematics teacher that lies within. Indeed, as a future teacher I believe it is not only my responsibility to encourage learners to become proactive with their learning but also to act as a role model, sharing my experiences with future teachers. By actively partaking in opportunities to enhance the overall learning experience, I can provide some insight into what lies ahead. Overall, I enhance my own learning and teaching experience through the opportunity to team-teach with like-minded peers and future teachers currently at the beginning of their studies. (Vicki)
- The peer-mentoring program gives us an opportunity to consolidate and deepen our learning as we are challenged to explain and share our thinking with others. Being able to reflect on our learning, to paraphrase and communicate, has really ordered my learning. I am comfortable to say that this program has allowed me to develop a deep and much stronger conceptual understanding of mathematics. It has been brilliant in terms of providing me opportunity to work on a range of teaching strategies and techniques for whole group, small group and one on one teaching. The program has also shown me the importance of planning together and has provided me with the opportunity to practise the use of correct mathematical vocabulary. Mentors need to be well prepared by making sure they understand the content themselves at a deep level. This is of utmost importance in order to ensure that the first year students do not lose confidence in the mentors as a whole. (Brent)

Outcome

Over the years, students have expressed that one of the most significant aspects of these workshops was the opportunity given to them to practise teaching. Teaching mathematics 'up at the board' can be very stressful. In allowing students to come 'up to the board' in a safe, nurturing environment we have created opportunity for students to practise how best to use correct mathematical language, how to stand at the board, ways to write on the board to communicate well, how to recover from mistakes. The mentors teach by example all these various aspects, and encourage the first years to do this also. Students in these classes have many opportunities to teach each other. For as Webb (1991, p. 210) discusses, 'it is also important to acknowledge that both the peer tutor and tutee can both benefit from the interaction. Students who give elaborate explanations typically learn more than those who receive them'.

The peer mentoring of first-year PSTs provides a context to see and to articulate connections between learning as a process and learning as an outcome. Through being closer to that process, PMs are more aware of the difficulties of teaching mathematics for understanding, which of course also builds opportunity to teach more empathetically.

Conclusion

The Peer Mentoring Program developed a collaborative environment in which both the first and third year PSTs were active learners. The long-term benefits for all are as follows: PSTs that are more competent at mathematics, are more confident in their approach, have a much better disposition towards mathematics, display improved teaching practices and a greater university performance. The most important aspect is the positive outlook and growth mindset being built towards mathematics. Hopefully this attitude further develops a group of highly skilled mathematics teachers; teachers who will have a positive impact on the mathematical journeys of the next generation and can make maths a subject that is celebrated rather than dreaded. (Brent)

In a direct effort to highlight the PSTs' perspectives of the benefit of peer mentoring, the authors draw attention to a further paper within these MAV proceedings 'Learning and teaching together – peer mentor numeracy project'.

Teaching mathematics in a way that develops students' conceptual (deep) understanding, that enables students to experience what it is like to have constant positive maths experiences, that instills a passion for teaching and learning and that ultimately benefits the community, are the reasons why we undertake projects such as this.

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LEARNING AND TEACHING TOGETHER - PEER MENTOR NUMERACY PROJECT

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Pre-service teachers at La Trobe University consolidate their understanding of learning and teaching mathematics while undertaking peer mentoring within numeracy workshops. This work aims to extend educators' understanding of the value of peer mentoring, through sharing reflections on making meaningful connections and through the experience of involvement in a voluntary peer-mentoring program.

Introduction

From the beginning of a four-year undergraduate teaching degree, pre-service teachers [herein PSTs] at La Trobe University are encouraged to value both conceptual understanding as well as procedural fluency in mathematics. Many take responsibility for their own learning via voluntary involvement in a peer-mentoring program between third and first year PSTs. Within these classes, PSTs and their teacher, Tina, work to develop *deep learning* in mathematics education along with an increased confidence and skill in teaching mathematics.

This paper relates to another paper presented within these MAV proceedings, 'A peer mentor numeracy project', which sets the scene for the following reflections. These reflections provide readers with a series of first-hand accounts on the value to learners of becoming peer mentors [herein PMs]. The program places high importance on learner preparedness to engage with mathematics. Time is devoted to addressing student anxiety and self-belief, to developing a growth mindset and a conceptual understanding of mathematics.

Structure

Following is a series of reflections that demonstrate how peer mentoring of first-year PSTs provides third-year PSTs with a context to see and articulate connections between learning as a process and learning as an outcome. Being involved in this process enables third year PSTs to better understand the traps and difficulties of teaching mathematics for understanding and builds opportunity to teach more empathetically. The reflections are presented using the following headings:

- Attitudes, beliefs, anxiety
- Building number sense
- Effective reasoning
- Developing a growth mind set and self-directed learning skills
- Increased motivation and self-belief for learning

Attitudes, Beliefs, Anxiety

Many first year PSTs face a number of issues detrimental to their success in mathematics; feelings of anxiety at having to learn and teach mathematics, feelings of failure from past negative experiences, a lack of motivation, a belief that they will never learn mathematics ('learned helplessness'), a belief that they are not 'maths minded'. Well aware that these issues could lead to an attitude of avoidance and continued negativity, the PMs wanted to help change poor attitudes and beliefs and to set peers on another path – that of enjoying the learning and teaching of mathematics.

- I hated Mathematics at school and was absolutely terrified of doing further classes at university. I believed I knew enough mathematical content. But after attending these workshops in third year I realise how much I still need to learn. (Jasmin)
- During different practicums I've had teachers tell me I was welcome to teach all the maths because they didn't really feel comfortable teaching it themselves. If this was their attitude, how could they teach it effectively? Maths always made sense to me, I was surprised to find 'maths phobia' so prevalent in our schools. Toward the end of first year, I realised many peers shared this phobia. So I decided to become a primary school mathematics teacher, believing this is where the fear of mathematics begins. When I tell people I want to teach maths, they are usually

surprised. "Oh why would anybody want to teach maths? I hate maths; it doesn't make sense". I have a huge job ahead of me. (Simon)

- You either love or hate maths. After much research and participation in university tutorials, I realise that attitudes and current perceptions about maths often relate to past learning experiences. I once loathed mathematics myself so I understand that to give students positive, engaging, meaningful and relevant learning experiences I had to overcome my fears and anxieties. (Ariana)
- As mentors, we show how maths can be fun and engaging. We demonstrate flexible ways of working with number. Using conceptual methods and concrete materials, I'm now part of a positive change. We want peers to enter future classrooms confident and excited about teaching maths, to be part of what Swan (2004) calls a positive 'flow-on'. We're part of a group consolidating skills and knowledge and helping to build expert learners. (Ariana)
- As third year PMs we're well placed to share with the first years. They soon realise that these workshops allow them to develop metacognitive skills. This understanding is gained from positive experiences that can change attitudes to mathematics. (Brent)
- Previous maths experiences left me with a complete lack of confidence. In high school I was treated as if I was not making an effort, I ended up left behind. So I understand what it's like to feel overwhelmed and the anxiety evoked from that. These feelings easily transform into negative thinking and low self-confidence. The PM program has helped me change from having a *fixed* to a *growth* mindset. I now believe I can *do* maths, and more importantly I can teach it. (Adam)

The initial aim of the PM program was for the PMs to gain more teaching experience, however all the PSTs discovered many more benefits from participating in this project. PMs helped to positively influence first year PSTs in many ways, while opportunities to practise teaching mathematics in a supported environment had a significant impact on their own teaching approaches. In acknowledging past difficulties, negative attitudes and beliefs, PMs were able to 'lead by example'; to show the first year PSTs that there is much to gain by working at mathematics in nurturing classroom environments.

Building Number Sense

PSTs were exposed to the ideas of 'number sense' as they worked at a conceptual level. The work was in line with that of Boaler (2015a, p. 2) who in her paper *Fluency without fear* discusses 'the fact that students achieve more highly in mathematics if they have number sense [and that] … number sense is the foundation for all higher-level mathematics'.

Following is an example of a typical numeracy workshop lesson which explored multiplication in depth. We began by asking students how they would solve 18×5 without writing or using the traditional algorithm (Boaler, 2008, p. 156). To highlight the importance of learning to use each other's methods, we discussed different ways students thought about the problem. For as Boaler (2015a, p. 1) explains, 'People with number sense … use numbers flexibly'. Some responses follow:

$10\times5=50$ then $8\times5=40$ then $50+40=90$

$20\times5=100$ then $2\times5=10$ and finally $100-10=90$

$9\times5=45$ and $9\times5=45$ then $45+45=90$

Students were then taken through the different stages of understanding the multiplication concept, starting with arrays. 'The use of arrays … reveals the connections between division and multiplication that are at the heart of multiplicative thinking' (Booker, Bond, Sparrow, & Swan, 2014). Using counters, students built arrays, discussed the correct recording of multiplication problems and the appropriate language to use. The commutative property was discussed as in the following example:

3 rows of five items or '3 fives' (Figure 1) has the same number of items as the 5 rows of three items or '5 threes' (Figure 2).

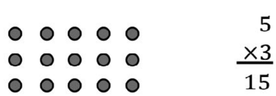

Figure 1

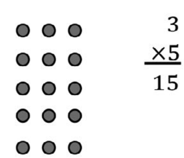

Figure 2

The distributive property of multiplication was explored by sectioning parts of the arrays.

E.g. 3×5 was represented as $3\times2+3\times3$

Using region models with grid paper followed and lead into the more efficient way of representing arrays for larger numbers. E.g. 123×3 would require drawing 3 rows of 123 items. The 'area model' or multiplication diagram (Figure 3) was the next step in studying the multiplication concept.

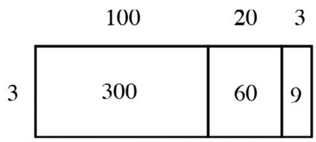

Figure 3

$$\begin{array}{r} 123 \\ \times \quad 3 \\ \hline 9 \\ 60 \\ 300 \\ \hline 369 \end{array}$$

Figure 4

These diagrams clearly show the different 'parts' to a multiplication problem and lead into the preliminary vertical recording 'by parts' (Figure 4) and finally the traditional recording of the multiplication algorithm. Place value mats and base 10 materials were used to visually represent the multiplication. Further methods such as lattice multiplication and multiplication with straight lines was also investigated.

- In first year I attended the mathematics workshops. I had no idea then that they would transform my understanding. Now as a third-year PM, blending a learner's perspective with a teaching perspective, I have further broadened my understandings. (Vicki)
- I grew up through the procedural style of mathematics instruction, so engaging with mathematics in a conceptual way was a revelation to me. I now see how ideas relate. For example, what occurs when the '1' is 'carried', how multiplication and division relate to each other as well as to fractions. Mathematics is constantly revealing new ways of thinking and calculating. There were so many simple details I'd missed or misunderstood. I enjoy mathematics now. (Adam)
- This program helped the PMs review, revise, and refresh our maths understanding. We de/re-constructed mathematical equations and connected our thinking with different teaching methods. We had opportunity to practise appropriate language and methods, gaining more than just deeper understandings. Practising newer methods of teaching and learning is so relevant and appropriate, not just for us but also for our future students. (Karen)
- Using appropriate language and demonstrations of techniques helped us deepen understanding, knowledge, and teaching practices. Our explanations use correct

maths language, which is often difficult to master, but it really assists everyone to understand procedures and reduces misunderstandings. The material is delivered in a manner that emphasises understanding of underlying concepts, by using concrete materials and pictorial representation. Many times I heard 'this just makes so much sense' which indicates that the first years are not just gaining greater understandings but also an awareness that teaching maths is both possible and enjoyable. (Jenny)

- Number sense is where I see all the wonder in mathematics. I've had more 'Aha!' moments from sharing different ways to solve problems than in any other part of my maths experiences. Number sense is integral to my success and to a student's success. I need to support my students to recognize and understand the differences in their ways of thinking. (Adam)

Students were constantly surprised to learn of the many varied approaches to solving problems. When encouraged to think deeply, to represent solutions in as many ways as possible, to discuss the mathematics with their peers and to reason through a process, students developed a deep conceptual understanding.

The importance of assisting students to make meaningful connections between procedures and concepts is imperative in developing proficient mathematical learners (Reys et al., 2012). By taking students step by step through the development of a concept (such as in the multiplication example above) and to explore it deeply, students had the opportunity to feel inspired and are now better equipped to research future topics.

Effective Reasoning

Boaler (2015b) states that 'the core of mathematics is reasoning - thinking through why methods make sense and talking about reasons for the use of different methods'. This focus was reflected in PMs explaining, demonstrating and questioning the mentees' level of understanding. The explicit nature of this style of teaching involves watching the PMs demonstrate the concept using materials, 'then everyone working together and finally independently in a manner which demonstrates understanding' (Jenny). 'Reasoning mathematically involves observing patterns, thinking about them and justifying why they should be true in more than just individual instances' Reys (2012 p. 99).

- The peer-mentoring program assisted first year PSTs to learn about teaching mathematical operations. It's hands-on learning that provides a critical review of teaching processes. As PMs we gathered weekly to discuss not only the mathematical operation but also the different ways it could be taught. All operations were modeled, discussed, explained and refined to allow continuity in transfer of learning between mentors, lecturer and students. (Karen)

- In one session, a jubilant learner finally understood a multiplication algorithm once demonstrated as in the example above. In another, a learner questioned the use of the factor tree, seeing it as time consuming. We discussed how an understanding of factors is useful when simplifying fractions. He later said, "I see the value now; it's good to understand terms such as factors, prime numbers, composite numbers and multiples". In another session, learners were asked how they would demonstrate the problem 42-13 using a place value mat and base 10 materials. One student placed both numbers (42 and 13) on the place value mat (as would be done if adding), then struggled to explain how this subtraction could be demonstrated. 'I know the answer is 29; I don't know how to show it'. We modeled the problem through to completion. (Ariana)

In talking and *doing maths* in interactive, relevant and engaging ways, in being part of a positive environment, learners were peer-mentored through the demonstrating processes. Many significant milestones were evident for some of our very negative learners. The social aspect - working together to construct knowledge and justify solutions, was key in helping them adopt more positive attitudes to maths. 'Being a part of this unraveling was a positive experience, as learners worked together to reach mutual understandings and learned to value the work of establishing positive norms in the classroom' (Ariana).

Developing a Growth Mind Set and Self-directed Learning Skills

From the beginning, we discussed the best learning environment for mathematics. We drew much of our research from the 'Positive Classroom Norms' work of Boaler (2015b) and worked to encourage a growth mindset in each student. We promoted effort, rather than performance as a means to success, and valued mistakes as learning opportunities. We encouraged deep thinking and mathematical discourse over speed and rote learning styles.

- I was extremely grateful for the support I received in first year and felt that involvement in these workshops would allow me to give something back. After four weeks I realised this program was still giving me more than I was giving out. Being in a real team teaching environment and reflecting on current teaching methods allowed me to engage in meaningful discussions. I've been involved in creating some amazing dialogue. As a direct result, I am determined to become the best maths teacher I can and I am going to do this by integrating maths right across the curriculum. (Simon)

- These numeracy classes have strengthened my mathematical knowledge. I continue to gain rich understandings as I explore, teach and explain the problems with my peers. Tina's workshops allowed me to become confident with content. Now I love maths! The workshops dramatically increased my confidence. Many first years had the same terrified attitude I had. Through involvement in the numeracy classes we've all had the opportunity to develop stronger understandings and are all inspired to encourage others. (Jasmin)
- We connect with first year PSTs, we relate to their experience, share familiar stresses and easily track a deep dislike for maths back to misunderstandings, negative beliefs and lack of confidence. As PMs we reassure that mindsets can change. Our work is friendly and supportive, rewarding and wholesome, it reflects an authentic team-teaching experience in a versatile, supportive and inclusive environment. First and third year students and the lecturer share high quality teaching/learning outcomes. (Vicki)
- The environment of safety and trust builds rapport. We get to know learners on a personal and professional level. We've been able to find learners' current perceptions of maths and where they'd like to be. Teachers are accountable for their students' academic, personal, and social success. Learners need to face any fears and improve any areas of weakness. By engaging in these extra classes, researching, and examining proven approaches on how to set up positive norms, I now know about addressing student needs, through understanding learning styles and preferences. Valuing mistakes is a key component in learning and teaching for both the teacher and the student as it underlines the fact that none of us are perfect and allows us room for growth. (Ariana)
- During the workshops, working in intense mentoring sessions, we discovered we can fully support, guide and encourage and we learnt the importance of not moving on to the next step unless everyone is ready. The mentoring experience reached everyone. Every day there was another enlightening experience. As future teachers, we share an ambition of providing mathematical learning environments that foster a love for learning and doing. We're enriching experiences and positively influencing everyone's mindset. (Vicki)
- First years willingly shared ideas through small group and whole class discussion and by answering questions on the whiteboard, demonstrating a real growth mindset. They shared ideas with confidence and came to understand that mistakes are useful in building deep learning. Rarely are ideas completely incorrect and

students now take comfort in finding what is right, value working things through and using the knowledge of others. They now work together, think critically and are deep, confident learners. We have created a sense of ease. Students are confident to answer questions and to pose problems. (Brent)

- We value mistakes and the process of solving problems. In authentic situations we overcome issues and challenges, such as admitting the mistake, documenting them on a 'mistakes chart' and re-assessing the problem, to find where things went wrong. Unpacking work always exposes the foundation of a problem, helping those struggling to see what's required. Our work encourages positive mathematical discourse, which helps to remove negative attitudes, myths, stigmas, and barriers that restrict learners from seeing the beauty of maths. (Ariana)

In the workshops, PSTs began to believe in themselves and their mathematical ability. They became comfortable in acknowledging that mistakes can be opportunities for learning. PMs modeled how mistakes could be addressed. We valued every question asked and made sure students justified their solutions. We stressed that all students should 'refuse to go on' if they did not understand. We tried to find as many different ways as possible to represent solutions to problems, thereby hoping to understand the many and varied needs of learners. We valued effort in our work and depth in understanding over speed and the outcome of tests. PSTs acknowledged the importance of reflecting on their own learning and teaching practices.

Increased Motivation and Self-belief for Learning

Setting up and modeling positive norms in the classroom, using positive messages and approaches toward maths (Boaler, 2015) has seen learners develop confidence, become willing to take risks, and rise to challenges outside their expected level of comfort. 'For some learners, this has been a first time experience and has elicited a sense of social and emotional safety' (Ariana).

- We are comfortable asking questions and sharing misconceptions. We see how content taught well can clarify understanding. (Elle)
- Over the last few years, maths has turned into constant 'Aha!' moments. I've overcome long-existing negative beliefs. The challenge was at times frustrating but encouragement and persistence prevailed. I'm now confident and eager to teach mathematics. (Ariana)
- As a PM, I'm able to provide reassurance that it takes time to become efficient i.e. keeping up with modern and often unfamiliar mathematical language. I have clarified my learning and understanding, which has had a really positive influence

on my confidence. I'm now better able to provide enriched learning experiences and I understand why we need to continue with current research. (Vicki)

- Mentoring for me is applied learning. The opportunity to practise, to make mistakes and to learn in a much deeper way, can only be recommended. Be open, be flexible, listen and enjoy the social atmosphere of mutual learning in a safe environment. (Adam)

In these workshops PMs value the act of carefully listening to the mentees. PMs question PSTs to find out their misconceptions and current knowledge of a topic. PMs give positive feedback on persistence and as a consequence, noticed an increased motivation in themselves and in the mentees. Attending these workshops had no grade value for either group. They attended because they are self-regulated learners who will decide what and how they will learn.

Conclusion

The PMs gained experience in team-teaching and in being part of a program which developed a collaborative environment for active learning. The long-term benefits relate to improved conceptual understanding, teaching practices, and greater performance. However, the most important aspect has been the positive outlook and growth mindset that both mentors and mentees have developed.

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CAS OR PEN-AND-PAPER: DECISIONS FOR SELECTED YEAR 11 PROBLEMS

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Teachers and students working in a classroom where CAS and pen-and-paper (p&p) is expected need to make choices about use of CAS or p&p for teaching, learning and doing mathematics. This paper analyses six Year 11 problems providing possible CAS and p&p solutions and some considerations for teachers when using similar problems. Results from a study investigating how a class of Year 11 students solved these problems showed that students' choices about the use of CAS or p&p varied with each problem.

Background

A Computer Algebra System (CAS) can automate many routine procedures in mathematics. For example, a CAS can automatically solve a quadratic equation or factorise an expression without any intermediate p&p work. Teachers and students working with CAS have to work with the brand of CAS they are using, learning appropriate syntax and developing the ability to interpret given displays. Teachers need to decide when to use CAS for demonstration purposes and students need to decide when to use CAS or p&p for solving problems. These decisions need to be made problem-by-problem and within a problem.

Access to CAS has been assumed in VCE mathematics for many years in Victoria, starting with a pilot study in the early 2000's. Curriculum documents and a CAS-assumed examination in Year 12 mathematics provided systemic support for the introduction of CAS, following on from a long tradition of integration of graphics calculators in senior school mathematics. The study design and examinations for VCE mathematics can be accessed from the Victorian Curriculum and Assessment Authority (VCAA) website (www.vcaa.vic.edu.au).

Problems, Pen-and-paper and CAS Solutions and Considerations for the Classroom

The following problems (refer to Tables 1, 3, 5, 7, 9, 11) are from a research study reported previously in Cameron and Ball (2014, 2015). The participants were a class of seven Year 11 Mathematics Methods (CAS) students and their teacher. The study investigated the influence of students' attitude towards CAS on use of CAS and compared students' use with their teacher's expectations. Data was collected via a calculus worksheet completed by students at the end of the topic. The worksheet was designed with the classroom teacher to ensure problems were appropriate for students. All students had their own CAS, except Emily (who borrowed the teacher's CAS). Students completed the problems under normal classroom conditions. After completing the worksheet, students were interviewed to gain insight into their attitude and use of CAS. Further information can be found in Cameron and Ball (2015).

Depending on the time of the year particular functions will be familiar or unfamiliar (for the students in this study) as topics are covered in the curriculum. The teacher classified problems as containing familiar or unfamiliar functions and we used the same classification. For each problem a CAS and p&p solution is presented and we discuss possible considerations for teachers, as well as potential for students to have a choice of approach when solving each problem.

Following discussion of each problem is a summary of techniques students used in their solution (refer to Tables 2, 4, 6, 8, 10, 12). If students reported using either CAS or p&p exclusively then this is recorded as CAS or p&p respectively; if students reported using a combination of CAS and p&p then this is referred to as 'mixed'.

Problem 1

Students were required to differentiate a polynomial function; the teacher identified this function as familiar. This is not surprising as polynomial functions of this type feature in the Mathematical Methods (CAS) curriculum (VCAA, 2010). Differentiation can be carried out using CAS, however Flynn, Berenson and Stacey (2001) reported that students would be able to quickly answer this problem using p&p. Students need to make a choice about the use of CAS or p&p.

Table 1 *Problem 1: Differentiation of a Polynomial Function in Expanded Form*

Problem 1	Find f'(x) if $f(x) = 95 + 2.7x + 4x^2 - 0.1x^3$
Possible p&p solution expected at Year 11	$f(x) = 95 + 2.7x + 4x^2 - 0.1x^3$ $f'(x) = 2.7 + 8x - 0.3x^2$
Possible CAS solution using TI NSpire	$\frac{d}{dx}(95+2.7\cdot x-4\cdot x^2-0.1\cdot x^3)$ $\quad$ $-0.3\cdot x^2-8\cdot x+2.7$
Considerations for teaching	• Syntax may be time consuming • Need to recognise when problems can be answered quickly with p&p

Note – Problem 1 from Flynn, Berenson and Stacey (2001, p. 12)

The teacher expected students to use p&p as "it would just be quicker to do it with p&p than type it into the calculator" (Cameron and Ball, 2015, p. 144). The teacher suggested p&p as a good choice as it was believed to be faster than CAS, and it was reported that students also considered relative speed of CAS and p&p when solving. Three students used p&p and four used CAS for this problem. Kate used p&p noting that her writing was quicker than calculator entry, whilst Sam chose CAS for speed and precision - "I knew that I'd probably get a more precise answer if I used my calculator and it's quicker" (p. 144). (Cameron and Ball, 2015). We expect a teacher would anticipate students to use p&p for routine problems (such as problem 1), but it is interesting to consider why some students might choose CAS for a problem easily solved using p&p. There could be a variety of reasons, for example, students may use CAS to compensate for limited p&p skills. For students growing up in a technological age, where handheld personal technology is commonplace, it may be that syntax entry is seen to be quicker than p&p work and this may also drive CAS use.

Table 2 *Problem 1: Teacher Expectation and Students' Choices of CAS, p&p or Mixed Techniques for Entire Problem*

Name	Teacher	Sam	James	Emily	Simon	Jessica	Amy	Kate
Technique	p&p	CAS	CAS	p&p	p&p	CAS	CAS	p&p

Problem 2

Students had to differentiate a polynomial function using first principles; this is traditionally demonstrated using p&p techniques. Whilst calculus is the core focus here, lengthy algebraic manipulation is required. Algebraic manipulation may seem to be the mathematical focus, rather than calculus.

Students with limited p&p skills may use CAS to supplement their p&p skills making problems of this type accessible. Students who have facility with both CAS and p&p need to make a choice about how to solve.

The teacher expected students to use p&p as he had demonstrated problems using p&p and anticipated that students would replicate his approach. Five students used p&p and two students used a combination of CAS and p&p (refer to Table 4). Sam used p&p because "we would always do it [i.e., differentiation from first principles] with p&p" (p. 144). Jessica used p&p for algebraic manipulation and CAS to calculate the limit, commenting that she felt p&p was required for this problem (Cameron and Ball, 2015).

> You can't just put it in [to the calculator] and go well there's your answer, because then they'll [the teacher] go, "well where's your working out?" (p. 144)

Cameron and Ball (2015) suggested that students felt compelled to replicate techniques demonstrated in class. Teachers must consider how they balance use of CAS and p&p in class, as their preferences may influence students. For students with limited p&p skills, CAS use may enable a focus on the mathematical topic at hand.

Table 3 *Problem 2: Differentiation Using First Principles*

Problem	Use first principles to find the derivative of the function $f(x)=x^3-4x+1$
Possible p&p solution expected at Year 11	$f(x) = x^3 - 4x + 1$ Calculate $f(x+h)$ $f(x+h) = (x+h)^3 - 4(x+h) + 1$ $f(x+h) = (x+h)(x+h)(x+h) - 4(x+h) + 1$ $= (x+h)(x^2 + 2hx + h^2) - 4(x+h) + 1$ $= x^3 + 2hx^2 + xh^2 + hx^2 + 2h^2x + h^3 - 4x - 4h + 1$ $= x^3 + 3hx^2 + 3xh^2 + h^3 - 4x - 4h + 1$ Calculate the limit $\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$ $= \lim_{h\to 0}\frac{(x^3 + 3hx^2 + 3xh^2 + h^3 - 4x - 4h + 1) - (x^3 - 4x + 1)}{h}$ $= \lim_{h\to 0}\frac{3hx^2 + 3xh^2 + h^3 - 4h}{h}$ $= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 4$ $= 3x^2 - 4$ $\therefore f'(x) = 3x^2 - 4$
Possible CAS solution using TI NSpire	Define $f(x)=x^3-4\cdot x+1$ *Done* $\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$ $3\cdot x^2-4$
Considerations for teaching	• CAS can reduce algebraic manipulation required • CAS may reduce the possibility of errors • Syntax may be time consuming - Defining functions useful - Limit syntax non-trivial

Table 4 Problem 2: *Teacher Expectation and Students' Choices of CAS, p&p or Mixed Techniques for Entire Problem*

Name	Teacher	Sam	James	Emily	Simon	Jessica	Amy	Kate
Technique	p&p	p&p	p&p	p&p	p&p	Mixed	p&p	Mixed

Problem 3

Students anti-differentiate a polynomial-like function after extracting appropriate information from the problem statement. This problem may present difficulties for students used to working with whole-number exponents, as one exponent is given as a fraction.

Problem can be solved using CAS or p&p so students need to make a choice.

Table 5 *Problem 3: Integration of a Function that is Familiar to Students*

Problem	An art collector purchased a painting for $500 from an artist. After being purchased the value of this artist's paintings increase, with respect to time, according to the formula $\frac{dP}{dt} = \frac{5}{2}t^{\frac{3}{2}} + 20t + 100$, where P is the anticipated value of the painting t years after it is purchased. Find an equation that will give the price of the painting at a given time. Consequently, find the price of the painting 4 years after it was purchased.
Possible p&p solution expected at Year 11	Anti-differentiate to find price at any time, t. $\frac{dP}{dt} = \frac{5}{2}t^{\frac{3}{2}} + 20t + 100$ $P = t^{\frac{5}{2}} + 10t^2 + 100t + c$ Solve for c, given that $P = 500$ when $t = 0$ $500 = 0^{\frac{5}{2}} + 10\times0^2 + 100\times0 + c$ $c = 500$ $\therefore P(t) = t^{\frac{5}{2}} + 10t^2 + 100t + 500$ Evaluate at $t = 4$ $P(4) = (4)^{\frac{5}{2}} + 10(4)^2 + 100(4) + 500$ $P(4) = 1092$ After 4 years, the painting will have a value of $1092.

Possible CAS solution using TI NSpire	$\int\left(\frac{5}{2}\cdot t^{\frac{3}{2}}+20\cdot t+100\right)dt$ $t^{\frac{5}{2}}+10\cdot t^{2}+100\cdot t$ $t^{\frac{5}{2}}+10\cdot t^{2}+100\cdot t+c$ $t^{\frac{5}{2}}+10\cdot t^{2}+100\cdot t+c$ $\text{solve}\left(500=0^{\frac{5}{2}}+10\cdot 0^{2}+100\cdot 0+c,c\right)$ $c=500$ $4^{\frac{5}{2}}+10\cdot 4^{2}+100\cdot 4+500$ 1092
Considerations for teaching	• Students with limited p&p skills may have difficulty solving - CAS can supplement p&p skills • Syntax may be time consuming - Integral non-trivial to enter

Note – Problem 1 from Flynn (2001)

Cameron and Ball (2015) found that the teacher wanted students to use p&p, however noted that CAS use may minimise the potential for errors associated with p&p working, in particular errors associated with finding an anti-derivative involving fractions in the exponent. Although the teacher expected p&p he was cognizant of the fact that some students might use CAS and anticipated why this might be the case. Only three students in the study completed this problem (refer to table 6). Jessica used a combination of p&p and CAS, whilst Amy and Kate used CAS exclusively. Jessica used p&p to complete “the easy stuff” (p. 145) and CAS for “the more complex things” (p. 145). It is possible that, for the students in this study, performing anti-differentiation with powers involving fractions may have been considered non-routine – even though the teacher considered the problem to be accessible to students. Whilst CAS can extend the range of problems accessible to students, explicit consideration and teaching may be required for these problems to be accessible to all students, rather than only some.

Table 6 *Problem 3: Teacher Expectation and Students’ Choices of CAS, p&p or Mixed Techniques for Entire Problem*

Name	Teacher	Sam	James	Emily	Simon	Jessica	Amy	Kate
Technique	p&p	No Evidence	No Evidence	No Evidence	No Evidence	Mixed	CAS	CAS

Problem 4

Students were required to anti-differentiate a trigonometric function, an unfamiliar function based on the curriculum (VCAA, 2010). Students had to apply their conceptual understanding of calculus, using CAS to solve. Whilst the function is unfamiliar, anti-differentiation was not, so CAS was the only option.

Table 7 *Problem 4: Integration of an Unfamiliar Function*

Problem	Determine the antiderivative of the function f'(x) = sin(2x + 1) +7 cos(x)
Possible p&p solution expected at Year 11	No solution expected
Possible CAS solution using TI NSpire	$\int(\sin(2\cdot x+1)+7\cdot\cos(x))\,dx \qquad 7\cdot\sin(x)-\frac{\cos(2\cdot x+1)}{2}$
Considerations for teaching	• CAS can be used to solve problems beyond students' p&p skills • Syntax may be time consuming

The teacher considered the problem difficult due to the unfamiliar function, but anticipated that his confident students would use CAS. Five students completed this problem using CAS. James, who did not complete the problem, stated "the whole sine and cosine ... confuses me" (p. 146). Of the five students who completed the problem, many were perturbed by the unfamiliar function. For example, Simon stated "I don't think I'd done these problems before" (p. 146) but solved the problem using CAS. Kate knew she could use CAS to solve problems of this type (Cameron and Ball, 2015). This suggests that conceptual understanding, and understanding how to use CAS, can enable solution of problems beyond p&p skills.

Table 8 *Problem 4: Teacher Expectation and Students' Choices of CAS, p&p or Mixed Techniques for Entire Problem*

Name	Teacher	Sam	James	Emily	Simon	Jessica	Amy	Kate
Technique	CAS	CAS	No evidence	No evidence	CAS	CAS	CAS	CAS

Problem 5

Students had to differentiate a polynomial function given in factorised form. Whilst the function is familiar, students may not regularly deal with functions in factorised form or a number of parameters. As students had not been taught the product rule, students needed to expand brackets prior to differentiation; thus meaning that lengthy algebraic manipulation was required for a p&p solution. This problem could be solved using CAS or p&p, so students had to make a choice.

Table 9 *Problem 5: Differentiation of a Polynomial Function in Factorised Form*

Problem	If f(x) = k(x–a)(x–b)(x–c) what is the simplified form of the derivative f'(x)?
Possible p&p solution expected at Year 11	Expand original function $f(x) = k(x-a)(x-b)(x-c)$ $= k(x-a)(x^2 - xc - xb + bc)$ $= k(x^3 - x^2c - x^2b + xbc - ax^2 + xac + xab - abc)$ $= kx^3 - kx^2(a + b + c) + kx(ab + ac + bc) - kabc$ Derive expanded function and simplify $f'(x) = 3kx^2 - 2kx(a + b + c) + k(ab + ac + bc)$ $f'(x) = k(3x^2 - 2x(a + b + c) + a(b + c) + bc)$
Possible CAS solution using TI NSpire	$\frac{d}{dx}(k\cdot(x-a)\cdot(x-b)\cdot(x-c))$ $\quad k\cdot(3\cdot x^2-2\cdot(a+b+c)\cdot x+a\cdot(b+c)+b\cdot c)$ $\text{factor}(k\cdot(3\cdot x^2-2\cdot(a+b+c)\cdot x+a\cdot(b+c)+b\cdot c))$ $\quad k\cdot(3\cdot x^2-2\cdot(a+b+c)\cdot x+a\cdot(b+c)+b\cdot c)$
Considerations for teaching	• CAS can find solutions for general equations quickly • Teachers need to help students develop familiarity with syntax

Note – Problem 5 from Flynn, et al. (2001, p. 12)

The teacher anticipated that students would use p&p (Table 10), but commented "CAS would expand it [i.e., the function] for you and then find the derivative too" (p. 146), suggesting that both CAS and p&p were viable. The teacher anticipated some student difficulty as the function was factorised.

Three students completed this problem; all used CAS. Kate discussed using CAS to explore problems stating "if it looks a bit weird [i.e., the function] then I'll stick it on the calculator and see what happens" (p. 147). However, the unfamiliar function did not stop some students using p&p. Simon expanded the brackets before reaching a point where "I have no idea what I am doing" (p. 146). This illustrates the importance of students developing CAS and p&p techniques, and being able to choose the most effective technique for a given problem and also when working through intermediate steps.

Table 10 *Problem 5: Teacher Expectation and Students' Choices of CAS, p&p or Mixed Techniques for Entire Problem*

Name	Teacher	Sam	James	Emily	Simon	Jessica	Amy	Kate
Technique	p&p	No evidence	No evidence	No evidence	No evidence	CAS	CAS	CAS

Problem 6

Students had to locate the stationary points of a function. The function was unfamiliar based on the curriculum (VCAA, 2010), however the first few steps of the problem could be completed using p&p (refer to Table 6). Once students find the derivative, it is unlikely they will solve the equation $f'(x)=0$ using p&p, so students had to use CAS. As students can start with p&p it may not be immediately obvious that CAS is required. This highlights the importance of students evaluating the potential for CAS or p&p use on a step-by-step basis.

Table 11 *Problem 6: Finding the Stationary Points of a Function that was Unfamiliar to Students*

Problem	What are the coordinates of the stationary points for the function $f(x) = (4x + x^{-2})^2$?
Possible p&p solution expected at Year 11	$f(x)=(4x + x^{-2})^2$ $= (4x + x^{-2})(4x + x^{-2})$ $= 16 + 8x^{-1} + x^{-4}$ $f'(x) = 32x - 8x^{-2} - 4x^{-5}$ Solve $f'(x) = 0$ $f'(x) = 0 = 32x - 8x^{-2} - 4x^{-5}$ Expected that Year 11 students cannot solve

Possible CAS solution using TI NSpire	$\frac{d}{dx}\left(\left(4\cdot x+x^{-2}\right)^2\right) \quad \frac{4\cdot\left(2\cdot x^3-1\right)\cdot\left(4\cdot x^3+1\right)}{x^5}$ $\text{solve}\left(\frac{4\cdot\left(2\cdot x^3-1\right)\cdot\left(4\cdot x^3+1\right)}{x^5}=0,x\right) \quad x=\frac{-2^{\frac{1}{3}}}{2} \text{ or } x=\frac{2^{\frac{2}{3}}}{2}$ $\left(4\cdot\frac{-2^{\frac{1}{3}}}{2}+\left(\frac{-2^{\frac{1}{3}}}{2}\right)^{-2}\right)^2 \quad 0$ $\left(4\cdot\frac{2^{\frac{2}{3}}}{2}+\left(\frac{2^{\frac{2}{3}}}{2}\right)^{-2}\right)^2 \quad 18\cdot 2^{\frac{1}{3}}$
Considerations for teaching	• Teachers can introduce more difficult functions • Students need competence in entering correct syntax for multi-step problems

Cameron and Ball (2015) investigated teacher expectations and student solutions for problem 6. A summary is provided in Table 12. The teacher noted that this problem would be within the range of p&p skills of Year 12 students, rather than Year 11: "I suppose in Year 12 you could use the rule to find the derivative; some p&p work could be used to find when the derivative is equal to zero" (p. 147). The teacher expected his students to use CAS as the problem was outside students' expected range of p&p skills.

Three students used CAS to complete this problem. Kate, who used CAS, "stuck it [i.e., the function] into the calculator because it knows how to do it for me" (p. 147). For Kate, her conceptual understanding and knowledge of CAS allowed her to complete problems (i.e. problems 4&6) that lie outside the range of her p&p skills. Conversely, Jessica considered whether p&p was a viable option to solve the problem before choosing CAS "I thought I could do it by hand, but I didn't really know" (p. 147). For teachers, it is important to consider whether students have the conceptual understanding of a topic as well as knowledge of CAS syntax and conventions in order to be able to solve unfamiliar problems. Teachers working in CAS classrooms will need to consider ways in which they can help students make effective decisions about the use of CAS or p&p.

Table 12

Problem 6: *Teacher Expectation and Students' Choices of CAS, p&p or Mixed Techniques for Entire Problem*

Name	Teacher	Sam	James	Emily	Simon	Jessica	Amy	Kate
Technique	CAS	No evidence	No evidence	No evidence	No evidence	CAS	CAS	CAS

Concluding remarks

For teachers and students working in CAS classrooms, there are many considerations for teaching, learning and doing mathematics. Some considerations include the possibility of using unfamiliar functions that extend students and the need to balance demonstration of CAS and p&p. It is important that students make decisions about appropriate use of CAS or p&p when solving problems. In a curriculum where CAS is assumed it is reasonable to expect that students learn to work effectively with CAS and p&p, not just one or the other exclusively.

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MATHEMATICAL VALUATIONS

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This paper explores some aspects of the use of mathematics in the property industry in determining valuations. The principal author completed VCE, Further Mathematics and Mathematical Methods several years ago, and is now an assistant valuer with a company that provides services in property consultancy, valuations and asset management for a municipal council. The effective use of mathematics in a ratings and taxation department supports good quality valuations that are acceptable to property owners as part of valuation in a municipality. We consider several scenarios based on realistic data to illustrate how sales ratios, graphs and simple geometry are part of these processes. Spread sheets are used as enabling technology for data and related computations.

Introduction

Educators often provide, with varying degrees of impact and success, rationale for the study of mathematics in terms of 'it's useful for' or 'it's importance for further study in' while some use a more aesthetic rationale in terms of 'austere elegance and beauty' for a cultivated mind.

In this paper we approach this matter from the obverse direction, providing an account of how a student found they used mathematics in the transition beyond school.

The principal author completed VCE in 2012, including study of both Further Mathematics and Mathematical Methods. While Further Mathematics came naturally, with the applications making sense, Mathematical Methods was more challenging in the application of concepts and techniques.

After completing VCE she enrolled in a nursing degree at the Australian Catholic University which required basic calculations for medication (see Tyreman, 2013). Students were required to solve equations without a calculator to ensure they had a sound, basic understanding of mathematics. However, as medication equations became more difficult to solve, calculators were also used before administering medication. While completing practicum placement at a hospital, the ability to solve these equations became second nature – as evidenced by the ability of more experienced nurses to readily solve medication equations with or without technology as applicable.

Following holiday employment at two real estate agencies, the principal author changed courses and is currently enrolled in a Bachelor of Property and Real Estate degree at Deakin University whilst concurrently employed 4 days per week at a valuation firm – Westlink Consulting. The author was somewhat surprised at the use of higher level mathematics in some of her subjects such as Property Development where she conducted financial feasibility studies to determine whether a given development would produce an acceptable profit margin.

Similarly, the subject of Property Investment has a substantial mathematical aspect, including the application of both the Direct Capitalization Approach and the Discounted Cash Flow (DCF) Analysis. The DCF utilizes the current CPI index across predetermined in-goings and out-goings across a 10 year period. Additionally, the Property Economics subject explores various investment options such as shares, bonds and property investment through companies such as Australian Real Estate Investment Trusts or A-REITs (HREF1). This includes calculation of projected profit for a 10 year period and analysis of results to determine the outcome that best met the needs of the investor. Both VCE Mathematics studies have been helpful in this regard. The use of mathematics within property valuations is prominent and leans more toward the concepts and skills learnt in Further Mathematics. However, some of the more challenging aspects of valuations utilize concepts and skills learnt in Mathematical Methods. Both studies are valuable and together provide a strong background for working confidently in this area. While the usefulness of Further Mathematics is readily apparent, the study of functions in Mathematical Methods provides additional conceptual depth. In this paper we provide three application case studies of mathematics in this field.

Context

Westlink Consulting is a valuations company predominantly focused on the provision of statutory taxing and rating valuations for local councils in Victoria. It also provides services in other areas such as finance valuations, private client and institutional advice and valuations, specialized property valuations and advisory and specialist services.

Having worked at Westlink for a year the author has learnt how to carry out a variety of different tasks that together lead to the valuations that occur in a particular council. The author's initial task was the completion of supplementary valuations which involved simply measuring building and garage areas from plans and entering them into a database. This task encompassed simple methods such as finding the area of square and using Pythagoras's theorem to determine the area of dwellings. These methods were also used on plans of subdivision to help determine lot sizes.

While continuing to complete supplementary valuations, the author moved onto work associated with the 2016 General Valuation – a biennial process that requires all Victorian rateable and non-rateable leviable land to be valued for the purposes of a general valuation. These values are then used in the calculation of council rates and taxes. This work involves the use of various ratios to help determine whether the values a council places on properties are within a pre-determined deviation from the overall sales median. The sales median is calculated from all the sales within the specified data group, referred to as the Sub-Market Group (SMG) that the dwelling is located in, for a 6 month time period starting from the date the analysis begins. All of the sale ratios from each property are calculated and compared against the overall sales median to determine the deviation.

Case Study 1 – Data analysis

The gathering and analysis of data is one of the most commonly used tools in property valuations. Three categories of data have been identified for use in the determination of values: general data; specific data; and competitive supply and demand data. The property market itself is one of the largest sources of information which falls into the category of supply and demand data. From this data we note that the residential property market has experienced significant movement over the past 18 months leaving the previous council valuations undertaken for the 2014 General Valuation from the 1st of January 2014 lagging behind the current market. The analysis of sales over the last 6 months provides valuers with the information needed to re-value dwellings in each municipality. Table 1 indicates a rapid increase of value in comparable properties in the market.

Address	Land Area	Sale Date	Sale Price
3 Farnham Road Bayswater	1014m2	7th March 2013	$405,000
2 Farnham Road Bayswater	1022m2	19th May 2015	$700,000
Percentage Change	72.84%		

Table 1 – *Comparison of two comparable sales (Realestate.com, 2015)*

The values of two properties that have comparable land sizes (and therefore development potential) and located opposite each other have increased by a 72.84% in a little over 2 years.

From the sales data gathered for a specific region, valuers determine the sales ratio of each sale. This indicates whether the current Capital Improved Value (CIV) placed on the property for the revaluation period of 2014 - 2016 is within the margins set by the Valuer General Victoria (0.900 – 1.00) or whether it is over, or under, valued. The sales ratio is calculated with the following formula:

Sales Ratio = Pending CIV/Sale Price

Table 2 shows the three different scenarios that emerge from determining the sales ratio. The sales 'A' and 'C' are undervalued while sales 'D' and 'E' are considered to be overvalued:

CIV	Sale Price	Pending CIV	Sales Ratio (Decimal form)	Sales Ratio (Percentage Form)
Sale A	$742,000	$605,000	0.8154	82%
Sale B	$556,000	$530,000	0.9532	95%
Sale C	$510,000	$415,000	0.8137	81%
Sale D	$415,000	$490,000	1.1807	118%
Sale E	$460,000	$510,000	1.1087	110%

Table 2 – *Sales Ratios*

In some cases sales fall into the undervalued or overvalued categories due to incorrect data in the system; perhaps the dwelling for sale 'A' is an extensively renovated 1960s brick veneer dwelling, however this renovation has not been recorded in the council database. Once this correction has been made (by including an estimate of 10% increase) to the data and the sales ratio is recalculated, the chance of it falling within the correct range is highly likely as shown in Table 3:

Sale A	
Pending CIV	$605,000
Renovation Value	$40,000
Increase of building code to (+10%)	$60,500
Sale Price	$742,000
New CIV	$705,500
Old Sale Ratio	0.8154
New Sale Ratio	0.9508

Table 3 – *Adjustment of sales data*

The analyses of the sale data not only shows where the current market is headed but also indicates where there are errors in the collected data.

Case study 2 – Direct Capitalization

As defined by the Australian Property Institute (API), direct capitalization is '...used in the income capitalization approach to convert a single year's income expectancy into a valuation amount'. In valuations, the direct capitalization approach is one of the most common when it comes to determining the value of a property. This approach works well with both commercial and residential investment properties but is mostly used for commercial properties.

Suppose we are going to be analyzing a commercial property, the subject property (which is categorized as a second grade office building) has a building area of 1663 square meters and a site area of 1900.74 square meters. Five comparable sale properties have been identified and are shown below in Table 4 with their projected annual rental income:

Comparable Property Net Rental Income	
Sale A	$782,000
Sale B	$352,800
Sale C	$475600
Sale D	$1,365,714
Sale E	$614,625

Table 4 – *Identified sale properties*

Each of the sale properties are then capitalized to produce a capitalization rate. The capitalization rate is calculated using the following formula:

Capitalization rate = Projected annual net rental income/Sale price

The corresponding spreadsheet calculation is shown in Figure 1:

	Sale A	Sale B	Sale C	Sale D	Sale E
Sale Price	$9,750,000	$4,000,000	$5,800,000	$13,800,000	$8,250,000
Net Operating Income	$782,000	$352,800	$475,600	$1,365,714	$614,625
Capitalization Rate	0.0802051	0.0882	0.082	0.09896478	0.0745

Figure 1 – Capitalization rate of sale properties

No two properties are exactly the same – as such, valuers try to find sale properties that are most comparable to the subject property. Sometimes there is a paucity of comparable sales and in this case valuers generally add weightings to the comparable sale/s to determine which properties have the highest comparability and which properties have a lower comparability. This weighting will determine how much each sale affects the end value as can be seen in the spreadsheet output in Figure 2 which assigns weighting to the comparable sales based on comparability;

Comparable	Capitalization Rate	Weight	Weighted Capitalization Rate
Sale A	0.080205128	10%	0.008020513
Sale B	0.0882	35%	0.03087
Sale C	0.082	30%	0.0246
Sale D	0.098964783	5%	0.004948239
Sale E	0.0745	20%	0.0149
Subject Properties Cap Rate			**0.083338752**

Figure 2 – Weighted capitalization rates

The weightings are expressed in percentage terms; this percentage is then multiplied by the original capitalization rate to create the weighted capitalization rate for each of the comparable sales. The weighted capitalization rates are then added together to create the capitalization rate that will be used on the subject property.

Generally, as the subject property has not recently sold, valuers will need to calculate the subject property's Net Operating Income (NOI) to be able to determine the value:

Net Operating Income = Potential gross income – Operating expenses

The following is a list of general calculations that are performed to obtain the NOI of the subject property:

Potential Gross Income	$597,106
Less: Operating expenses	
Land Tax	$5,800
Municipal Rates	$8,500
Water Rates	$4,100
Air Conditioning	$15,000
Building Liability Insurance	$10,300
Common Area Cleaning	$17,400
Electricity	$17,600
Fire Services/Protection	$6,900
Garden Maintenance	$4,230
Lifts	$6,000
Management/Administration	$13,500
Repairs & Maintenance	$16,000
	($125,330)
Net Operating Income	**$471,776**

Once the NOI has been determined the value is then capitalized to create the overall value for the subject property as can be seen below;

Net Operating Income	**$471,776**
Capitalized at 8.33%	**$5,663,577.43**
Say	**$5,660,000**

The relevant calculation is (100/8.33) × 471,776.21= 5,663,577.43. Therefore, in this example, we would determine the value of the property to be around $5,660,000 (5 Million Six Hundred and Sixty Thousand Dollars).

The spreadsheet output in Figure 3 indicates how the capitalization rate affects the value of the subject property;

Net Opertating Income	Capitalized at	End Value
$471,776	9.24%	$5,105,800.87
$471,776	6.56%	$7,191,707.32
$471,776	8.33%	$5,663,577.43

Figure 3 – Capitalization rates affecting the end value

As can be seen, the rate at which the property is capitalized has the biggest influence on the end value placed on the subject property.

Case study 3 – Statistics

Statistics are vital to rating and taxation valuations. As there are significant numbers of residential dwellings in each municipality, for instance approximately 65,000 in the City of Knox, it is impractical to perform individual valuations on each dwelling during the revaluation period. Each council therefore uses mass valuation techniques to ascribe values to all dwellings by using statistics from the last six months of sales.

To assist in completing this mass valuation, the overall median sales ratio is calculated, the upper and lower quartile sales ratios are calculated and the absolute deviation of each of the sales properties is calculated from the overall sales median. By utilizing the absolute deviation, valuers are able to identify if their data set is in the acceptable range from the median sales ratio or if the properties are overvalued or undervalued. Further, valuers use a variety of statistics and measures as can be seen in Figure 4 to assist in the valuation of each specific SMG.

SMG	ABC
Number of properties in the SMG	1,670
Number of Sales Analysed	33
% Number of Sales	1.98%
Number of Sales Excluded	6
Median Sales Ratio	0.9677
Minumum Sales Ratio	0.7636
Maximum Sales Ratio	1.2124
Range Sales Ratio	0.45
COD	8.12
COV	10.54
PRD	1.00
Average SV Change	19.74%
Average SV Change Non-Sale	18.62%
Difference in SV Change	1.12%
Average CIV Change	25.50%
Average CIV Change Non-Sale	16.98%
Difference in CIV Change	8.52%
Median CIV (sold props)	[redacted]
Median Sale Price	[redacted]
Median CIV (unsold props)	[redacted]
CIV Q1	[redacted]
CIV Q3	[redacted]
Median Sales Ratio Lower Q	0.9247
Median Sales Ratio Upper Q	0.9917
Minumum Sales Date	6/01/2015
Maximum Sales Date	24/06/2015
Sale Date Range	6 months

Figure 4 – Statistics used for revaluation purposes

This shows the range of statistics and measures that are considered to determine whether the specific SMG is in line with, behind, or ahead of the market. Properties which are over one standard deviation from the overall sales median (as can be seen in Table 4) are considered to be outliers and require further investigation. These properties are generally incorrectly recorded in the database, or a change in the property (eg alterations and additions to the dwelling) has not been addressed. In some cases these properties are considered true outliers to the data set and are therefore excluded from further analysis. However, sales evidence must be provided against properties to demonstrate that they are outliers before they are allowed to be excluded.

Overall Sales Median	Sales	Sale Ratio	Absolute Deviation	Included/ Excluded
0.9677	Sale A	0.8769	0.0908	Included
	Sale B	1.0811	0.1133	Excluded
	Sale C	0.8558	0.1119	Excluded
	Sale D	0.7636	0.2041	Excluded
	Sale E	0.8587	0.1090	Excluded

Table 4 – *Absolute deviation of sale properties*

The use of these statistics in combination with the data analysis in case study 1 is the main component of the general valuation process. By utilizing these techniques valuers are able to track where the market has been and create accurate values for all dwellings within the specified municipality.

Horses for courses

VCE mathematics studies are popular choices for students, with discussions about 'suitable' and or 'desirable' selections the topic of robust discussion between teachers, students, parents and the media. Over the past five years the number of Unit 3 enrolments in English has been fairly stable at around roughly 50 – 51,000 students. The corresponding figures for Further Mathematics are roughly 30 – 31,000 students, with a slight increase over the years, and for Mathematical Methods are around 16,000 students, with this figure being fairly stable. Around 4,500 students have taken both Further Mathematics and Mathematical Methods, and this figure has also been fairly stable. Apart from meeting the English group requirements, Further Mathematics is by far and away the most popular study for student choice for inclusion in their VCE program, twice as popular at the next two (equally) most popular VCE subjects: Mathematical Methods and Psychology. Further Mathematics is clearly seen as an important and useful study by many students, and its role in engaging students the broad population of students in mathematics is acknowledged in the 2012 Victorian Auditor General's Report *Science and Mathematics Participation Rates and Initiatives* (HREF3). Mathematical Methods is also seen as an important and useful study by many students, in particular in its role as an enabler for future studies which require a calculus-based mathematics background, and to meet tertiary course pre-requisites.

Conversely, the area of valuations provides a context for realistic examples and data (see, for example, HREF4) which relate to the application of knowledge and skills for the Core area of study: *Data analysis and Recursion and financial modelling* of Further

Mathematics Unit 3. Mathematical Methods provides essential background for more in depth consideration of valuation functions in micro-economics.

Some reflections

The principle author was an enthusiastic student of Further Mathematics, less so for Mathematical Methods. In general she has found that Further Mathematics concepts and skills are broadly applicable, and Mathematical Methods concepts and skills provide additional depth and flexibility. Both have been useful in her post VCE study and employment.

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INVESTIGATING CHILDREN'S MULTIPLICATIVE THINKING

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Multiplicative thinking is a 'big idea' of mathematics that underpins much of the mathematics learned beyond the early primary school years. The conference presentation reports on a recent study that utilised an interview tool to gather data about children's multiplicative thinking. Using a workshop format, we present some of the interview tool and some of the findings, as well as demonstrate how the tool can be used for planning, teaching and assessment. The session also emphasises the importance of developing deep conceptual understanding as opposed to the teaching of procedures. This paper considers how evidence from the interview can be used to inform teaching.

Multiplicative Thinking

The importance of multiplicative thinking as a 'big idea' of mathematics has been well documented (Siemon, Bleckly, & Neal, 2012; Siemon, Breed, Dole, Izard, & Virgona, 2006), as has the importance of 'big ideas' in highlighting the myriad connections within and between them (Charles, 2005; Clarke, Clarke, & Sullivan, 2012). Charles (2005) asserted that 'big ideas' "link numerous mathematical understandings into a coherent whole", make connections, and that "good teaching should make those connections explicit" (p. 10). Multiplicative thinking is one such 'big idea'.

It therefore seems to follow, that to make these explicit links to develop multiplicative thinking, teachers should incorporate the Proficiency Strands (particularly Reasoning and Problem Solving) of the Australian Curriculum: Mathematics (Australian Curriculum, Assessment & Reporting Authority, 2015) rather than focus solely on the Content Strands (Number and Algebra, Measurement and Geometry, Statistics and Probability). For example, instead of teaching children a set of 'rules' for working with numbers, and teaching ideas like multiplication and division as separate entities, more effective teaching would focus on reasoning about why numbers behave as they do when operating, and understanding how multiplication and division are different ways of describing the same situation. This paper describes some research conducted with primary aged children to determine the extent of their multiplicative thinking. The results of that are very interesting in themselves. However, it is the inferences that can be drawn about teaching and the associated implications for teaching about multiplicative thinking that comprise the main thrust of this paper.

The Research

Semi-structured interviews were conducted with thirty eight children in Years Five and Six in two different schools (Schools A and B). Interviews lasted between 25 and 40 minutes. A questionnaire was developed from the interview format in order to generate a larger set of data in a shorter time. This was administered to nine whole class groups comprising 180 children in Years Four, Five and Six at a third school (School C) and the administration of the questionnaire took about 30 minutes per group. Both the questionnaire and interview were administered to the Year Five group at School A to establish the reliability of the results from the questionnaire. Whilst the data from the questionnaire was shown to be reliable in that the results from it were reflected in those from the interview, richer data were generated from the interview. Burns (2010) asserts that the power of the interview lies in the quality of the question posed by the interviewer or teacher. Examples include "Can you explain how you worked that out?" and "How did you get that answer?" irrespective of whether the child interviewee had the correct answer or not.

Typical questions from the interview and questionnaire included the following:

- In $7 \times 6 =$, what do the numbers 7 and 6 tell you?
- Do a drawing to show the number fact (or table) 4×3.

- Write as many multiplication facts (or tables) as you can that give an answer of 24. Circle all of the numbers that are factors and draw a square around numbers that are multiples. Explain what they are factors and multiples of, and how you know.
- My friend says that if you know the answer to 6 × 17, you must also know the answer to 17 × 6. Is he correct? Why/why not?
- My friend says that if you know that 6 × 17 = 102, you must also know the answer to 102 ÷ 6? Is he correct? Why/why not?

These questions are chosen because they explore key aspects of multiplicative thinking, the understanding (or otherwise) of which is likely to provide an indication of a student's level of thinking.

Overview of Results

The responses from the interviews and questionnaires made for some interesting overall observations. First, responses from the Year Five cohort at School A and the Year Six cohort at School B revealed a wide range of conceptual understanding. Second, responses to the questionnaire administered at School C revealed that the three class groups within each year level had varying levels of understanding. While there were large variations within each year level, a similar range was evident between year levels, and indeed, within each class group. This paper suggests that the differences may have resulted, at least to some extent, from different pedagogies, teaching styles, and/or may reflect different stages of development of children's understandings of multiplicative concepts. After all, Siemon et al. (2011) have noted that multiplicative thinking usually does not fully develop until the early secondary years.

Specific Results and Discussion

School A and School B

The purpose of this paper is not to compare performance of different school cohorts against one another or different sections of school cohorts against one another. Rather it seeks to identify aspects of multiplicative thinking that might be evident or otherwise in different children and to understand why that might be so. Hence interview results from Schools A and B are combined into one set. Table 1 presents a summary of responses to the five illustrative questions listed above for the Year Five cohort from School A and the Year Six cohort from School B (n = 38).

Table 1 - *Summary of Responses from Schools A and B*

Mathematical understanding demonstrated by responses to listed questions	School A Year Five School B Year Six (n=38)
Identifies numbers in multiplication fact as 'group size' and 'number of groups'.	39%
Represents given multiplication fact as an array.	34%
Defines 'factor' and 'multiple' and/or identifies factors and multiples in given number fact.	63%
Explains commutative property in a conceptual way and/or demonstrates it using an array.	29%
Explains inverse relationship in a conceptual way based on number of groups and group size	50%

It is also worth noting that of the 38 children in the combined sample, ten (26%) responded correctly to four or five of the above questions and a further seven (18%) responded correctly to three of the questions. This seems to indicate that approximately one quarter of the sample demonstrated a strong level of conceptual understanding of the selected aspects of multiplicative thinking and a smaller proportion showed a reasonable level of understanding. However, over half the children in the sample could only respond appropriately to two or less of the selected questions. This suggests that there is a wide range of understanding across the sample.

Some Typical Strong Responses

Typical responses demonstrating a strong level of conceptual understanding of the commutative property of multiplication include the following:

- Student Dylan – "It doesn't really matter which way it is – seventeen groups of six is the same as six groups of seventeen". He then used tiles to make three groups of five and five groups of three, and also rearranged twelve tiles saying "I just put them into a three by four grid – it's the same as a four by three".
- Student Dean – "It's just the same . . . you just flip it around". He then used tiles to make a three by five array and rotated the array to explain his point.
- Similarly, the following exchange during the interview with Student Jason shows some connection of ideas around the inverse relationship between multiplication and division, sharing into equal groups, and arrays.
- When discussing the division fact 24 ÷ 3, Jason showed it as an array and said, "Then I'm going to split it up into threes, because I'm going to see how many groups of three

> I can have in 24". Also said, when asked what the answer would be, "I started with knowing that how many threes go into 15 and that's five, then I counted by threes to get 18, 21, 24". He also said, "If I had 3 times 4 it would be 12. If I had 12 divided by 3 it would be 4". He also gave a similar example with "8 groups of 3 = 24, So 24÷8 = 3"

Such connected discussion seems to demonstrate a sound understanding of the concepts involved.

Responses Indicating Partial Understanding

The apparent lack of conceptual understanding in the responses of some children is of interest. It is difficult to draw conclusions about the depth of some children's conceptual understanding given the absence of links and connections between responses to different questions. That is, some children show some understanding of a particular idea which would lead one to reasonably expect they would show an understanding of related concepts. However, this was often not the case.

It is well accepted that the array is a powerful representation of the multiplicative situation (Jacob & Mulligan, 2014). However, while two of the children (Ellie and Tilly) drew an array to represent the given number fact, neither of them could explain why the commutative property works, in terms of the array. Rather, they said that the numbers were 'swapped around' (Tilly) or 'you've just swapped them around' (Ellie). Also, of the thirteen children who drew an array, only seven of them could describe factors and multiples.

Similarly, while 63% of the children (n = 24) could adequately describe factors and multiples and their roles in the multiplicative situation, only five of them talked about the 7×6 number fact in terms of group size and number of groups. Further to that, 39% of the children (n = 15) described the number fact in terms of group size and number of groups, yet only five of that group also drew an array. Some of the children (n = 11) adequately explained the commutative property and half (n = 19) explained the inverse relationship between multiplication and division. However, not all of the eleven children who explained the commutative property could also explain the inverse relationship. This is interesting because the ideas underpinning those interview tasks are inextricably linked – that is, group size/number, the factor-factor-multiple relationship, the representation as an array, the commutative property, and the inverse relationship. Hence, it might be reasonably expected that there would be more children who could perform well on all or most of the items, or on none (or very few) of them.

School A and B Implications

The inferences that can be drawn from the School A and B data suggest that the connections between those important ideas need to be made clear and more explicit so that a mutual understanding of them can be developed. This is supported by the observations that: some students drew an array; some others could explain factors and multiples; and some others could explain a multiplication fact in terms of group size and number. Perhaps it is because there has been passing mention made of these key ideas, rather than sustained and explicit teaching of them. For example, the fact that less than one third of the children could explain the commutative property in a conceptual way gives rise to questions about how the commutative property may have been taught. Perhaps it is also attributable to the fact that children's understanding of the multiplicative situation is developing and in a state of flux. After all, it has been noted (Siemon et al., 2011) that multiplicative thinking is a concept that does not fully develop until the secondary years around the age of fourteen and the students involved here are several years younger than that.

The lack of sustained teaching may also be because the teachers simply do not appreciate the critical importance of the ideas of factor/multiple, group size/number, and the use of the array. Hence they may have taught some of the ideas once, assuming that such exposure is adequate when that is clearly not the case.

School C

In School C, the questionnaire was administered to 180 children in Years Four, Five and Six. Table 2 represents the responses from the three year levels in School C to questions about the same concepts as shown in Table 1.

Table 2 - *Comparison of Responses from Different Year Levels at School C*

Mathematical understanding demonstrated by responses to listed questions	Year 4	Year 5	Year 6
Identifies numbers in multiplication fact as 'group size' and 'number of groups'.	14%	9%	23%
Represents given multiplication fact as an array.	28%	32%	39%
Defines 'factor' and 'multiple' and/or identifies factors and multiples in given number fact.	3%	15%	14%
Explains commutative property in a conceptual way and/or demonstrates it using an array.	0%	2%	5%
Explains inverse relationship in a conceptual way based on number of groups and group size	10%	1%	12%

In general it would probably be expected that the Year Six children would perform better than the Year Five children who would in turn perform better than the Year Four children. However, as can be seen, this is not always the case and even where it is, one would perhaps expect the comparative performance of the older children to be markedly better than it is.

Of more interest is the comparison within each year level in School C, as shown in Table 3 which shows responses from children in the three Year Four classes. Here, there are marked differences in the responses from different class groups, particularly in relation to the first two questions. It is indeed surprising that no child in Class 4A could identify 'group size' and 'number of groups' in multiplication facts when over a third (35%) of Class 4C could do so. Even more intriguing is that nearly two thirds (62%) of Class 4A drew an array to show a multiplication fact when not one child in Class 4C did that. As well, very few children in Class 4B responded correctly on any of the five questions. What does this indicate?

Table 3 - Comparison within Year Levels in School C

Mathematical understanding demonstrated by responses to listed questions	4A	4B	4C
Identifies numbers in multiplication fact as 'group size' and 'number of groups'.	0%	6%	35%
Represents given multiplication fact as an array.	62%	17%	0%
Defines 'factor' and 'multiple' and/or identifies factors and multiples in given number fact.	0%	6%	5%
Explains commutative property in a conceptual way and/or demonstrates it using an array.	0%	0%	0%
Explains inverse relationship in a conceptual way based on number of groups and group size	14%	0%	15%

School C Implications

Classes at School C are not streamed on ability. Hence, it seems reasonable to assume that the variation in responses may be due to different teaching occurring among the three Year Four classes. Perhaps there has been a clear emphasis in Class 4A on the use of arrays, rather than showing multiplication facts as separate groups. It is also worth noting that the responses from Class 4A (62%) to the array question were the highest of any class in the school – only one Year Six class (53%) and one Year Five class (50%) recorded a similar level of correct responses. Similarly, the teaching in Class 4C is likely to have emphasized the notion of 'group size' and 'number of groups' in the multiplicative situation. Again, Class 4C's response (35%) is the highest recorded of all classes with only one Year Six class (33%) recording a similar level of correct answers. However, it seems reasonable to imply that there is a need for explicit teaching of the connections between the five related ideas in the multiplicative situation, something which seems to be reflected in the responses from Schools A and B as well.

General Implications

The five selected questions from the interview and questionnaire represent less than a quarter of the full instrument yet the data generated from just three sets of children have provided plenty of food for thought. There are two main observations that can be made from the presented data. First, there are considerable differences in the levels of understanding of multiplicative concepts shown by two groups (Schools A and B) that were interviewed. Some children displayed more connected understanding than did others. Second, there is considerable difference in responses among classes in the same year level at the same school (School C) where the questionnaire was administered. In seeking reasons for this, it is reasonable to infer that the differences may be due to pedagogies.

The differences in responses are quite stark at times and the relative connectedness in the thinking of some children in the combined cohort from Schools A and B suggests that connections between ideas may have been made more explicit in some classes compared to others. At least, it is likely that some children from the School A/B group have been encouraged to justify, explain, and reason about their ideas, as well as interpret those of others. Apart from the differences in responses, some children were more forthcoming and articulate which suggests that they may have been more accustomed to discussing mathematical ideas than other children from the same combined group who were often unable to elaborate their answers. Also, in responding to questions other than those reported here, some children were reluctant and/or unable to depart from quite procedural responses which was not the case with other children.

Conclusions

In conclusion, there are implications for teaching in terms of what can be done to help children develop key multiplicative concepts in a connected way. The evidence presented here suggests that such pedagogical practices exist but may not be sufficiently widespread. Such teaching could include the following:

- Explicitly teach that the multiplicative situation is based on the number of equal groups and the size of each group.
- Develop an understanding of the terms factor and multiple through the use of arrays, and explicitly use them as 'mathematical language'.
- Teach multiplication and division simultaneously, not separately.
- Develop the commutative property through the use of arrays and physically show the 'x' rows of 'y' gives the same result as 'y' rows of 'x'.

If teachers view multiplication and division as different ways of representing 'the multiplicative situation', rather than as separate entities, the links and connections between the ideas discussed in this paper can be made explicit for children. When those connections are clearly understood, ideas such as the inverse relationship and the commutative property become much easier to grasp.

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USING STUDENT CENTRED APPROACHES IN SENIOR SECONDARY MATHEMATICS TO ASSESS LEARNING AND INFORM TEACHING

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This article reports on a lesson taught to a year 12 Mathematics Methods class (the middle level subject). The pedagogical approach has been widely and successfully adopted by teachers at upper primary and junior secondary levels. Teachers at those levels have found that students appreciate the opportunity to solve problems for themselves and to explain and justify their reasoning. Teachers have also found that students' responses to open-ended problems reveal rich and useful information about what students know and that those insights can be used in subsequent teaching. The proposition is that this approach may be able to be used productively by teachers in upper secondary mathematics classes as well.

Introduction

A common view is that upper secondary students learn mathematics best when teachers give clear explanations of mathematical concepts, usually in isolation from other concepts, and students are then given opportunities to practise what they have been shown. We have been working on a research project at primary and junior secondary levels that is exploring a different approach. Basically, this approach involves posing questions like the following and expecting (in this case, primary level) students to work out their own approaches to the task prior to any instruction from the teacher:

> The minute hand of a clock is on 2, and the hands make an acute angle.
> What might be the time?

There are three ways that this question is different from conventional questions. First, it focuses on two aspect of mathematics together, specifically time and angles. Contrasting two concepts helps students see connections and move beyond approaching mathematics as a collection of isolated facts. Second, the question has more than one correct answer. Having more than one correct answer means that students have opportunities to make decisions about their own answer and then have something unique to contribute to discussions with other students subsequently. Third, students can respond at different levels of sophistication: some students might find just one answer, while other students might find all of the possibilities and formulate generalisations.

The task is what is described as appropriately challenging. The solutions and solution pathways are not immediately obvious for middle primary students but the task draws on ideas with which they are familiar. An explicit advantage of posing such challenging tasks is that the need for students to apply themselves and persist is obvious to the students, even if the task seems daunting initially.

After the students have worked on the task for a time, the teacher manages a discussion in which students share their insights and solutions. This is an important opportunity for students to see what other students have found, and especially to realise that in many cases there are multiple ways of solving mathematics problems. Note that it is suggested to teachers that they use a document camera or some similar technology to project students' actual work. This has the advantages of saving time in comparison with rewriting the work, it presents the students' work authentically, and it illustrates to students the benefits of writing clearly and explaining thinking fully.

Subsequently, the teacher poses a further task in which some aspects are kept the same and some aspects changed, such as:

> The minute hand of a clock is on 8, and the hands make an obtuse angle.
> What might be the time?

The intention is that students learn from the thinking activated by working on the first task and from the class discussion, and then apply that learning to the second task. In other words, the students move from not knowing to knowing.

One of the foci of the research is to identify tasks that not only are appropriately challenging but also amenable to adaptation for the needs of particular students. For example, there may be some students for whom the first task is too difficult. Those students might be asked to work on a question like:

> What is a time at which the hands of a clock make an acute angle?

The intention is that those students then have more chance of engaging with the original task. Of course, there are also students who can find answers quickly and are then ready for further challenges. Those students might be posed questions like:

> Why are there six times for which the hands make an acute angle? Is there a number to which the minute hand might point for which there are not six possibilities?

There might even be advanced students who could be asked:

> What are some times for which the hands on a clock make a right angle?

The combination of the students' own engagement with the problem and the different levels of prompts means that the students' work products contain rich and useful information about what the students know that teachers can use not only to give the students feedback but also to plan subsequent teaching.

Among other things, the project has found that, contrary to the preconceptions of some teachers, many students do not fear challenges in mathematics but welcome them. Further, rather than preferring teachers to instruct them on solution methods, many students prefer to work out solutions and representations by themselves or by working with other students. The project has also established that students learn substantive mathematics content from working on challenging tasks and are willing and able to develop ways of articulating their reasoning.

In general, the tasks that work best are those in which students have the opportunities to:

- engage with important mathematical ideas that are central to the curriculum;
- plan their approach for themselves, especially sequencing more than one step;
- process multiple pieces of information, with an expectation that they build connections between those pieces, and see the concepts in new ways;

- choose their own strategies, goals, and level of accessing the task;
- spend time on the task, persisting if the task seems difficult, and record their thinking; and
- explain their strategies and justify their thinking to the teacher and other students.

For further details of the pedagogical approach, termed activating cognition, and project results, see Sullivan, Walker, Borcek, and Rennie (2015) and Sullivan, Askew, Cheeseman, Clarke, Mornane, Roche, and Walker (2014).

Exploring What this Might Look Like at Upper Secondary Levels

The purpose of this contribution is to explore whether the *activating cognition* approach may be useful for teachers at upper secondary levels. While there is an overall consensus within the mathematics education community that such approaches have potential to make school mathematics more engaging and support students in developing concepts that are more robust and useable (see Anthony & Walshaw, 2009; Kilpatrick, Swafford, & Findell, 2001), there has been limited research on the use of such approaches at upper secondary levels. This may be in part due to the limited interest in innovation by teachers at those levels. Part of the reticence of such teachers for innovation is that the syllabi are crowded, the assessments stakes are high and whatever appetite for change exists has been satisfied by rolling developments in hand held technologies.

The following is a lesson that was taught collaboratively by the authors, the intention of which was to explore the potential of more open-ended tasks. The teaching approach described above was used to engage students in formulating their own solutions and to contribute to and learn from discussions with each other on those solutions.

The lesson started with an explanation that it may be different in form from what students are used to but otherwise there was no discussion of solution pathways. The task (termed "Turning Point 1") was posed as follows:

> I was not paying attention in my mathematics class but I heard the teacher say "a turning point is at (2, -3)". What might have been the function? (give at least two different possibilities).

It was anticipated that the task would give students opportunities to use strategies ranging from the turning point form to approaches that use the gradient of the tangent being zero on a range of categories of functions. It is noted that the approach is different from many text book exercises that require the application of a pre-determined procedure. Rather than replicating a taught process, the students need to activate their thinking about different functions and their properties.

In anticipating some students might experience difficulty with the initial task, and to prompt a focus on the turning point form, a further task was prepared:

> I was not paying attention in my mathematics class. I heard the teacher say that "a turning point is at (0, -3)". What might have been the function?

This is one step simpler than the original task in that only translation of an appropriate function parallel to the y axis is required.

For those students who might find various approaches to Turning Point 1, the following was prepared:

> What are some further strategies for finding a function given a particular turning point?

This was intended to prompt those students to move towards considering a generalisation across approaches.

The plan was that, after working on the first task and having thoughtfully selected students explain the solutions they found, including projecting the solutions using a document camera, the following (termed "Turning Point 2") would be posed:

> I was not paying attention in my mathematics class. I heard the teacher say that "a turning point is at (2, 3)". What might have been the function? (give at least two different possibilities).

The purpose of the first problem was to activate the knowledge and thinking of the students on functions and turning points so that they could learn from each other. The purpose of the second problem is for students to apply and extend that thinking to a similar but different problem.

Learning From the Student Products

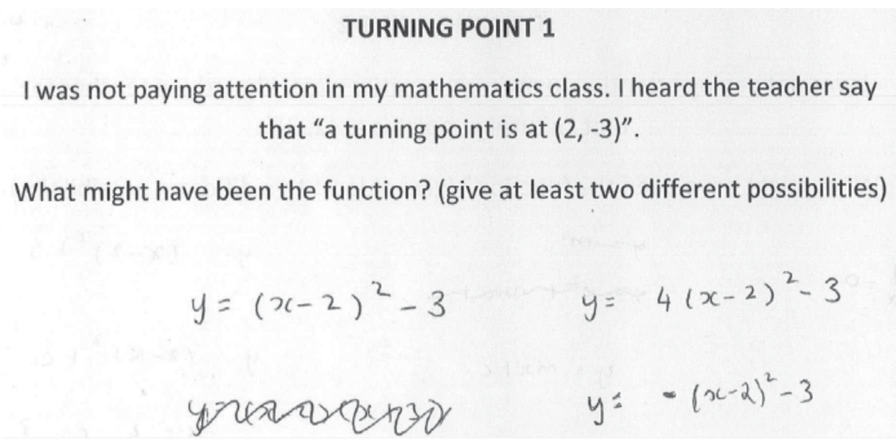
Part of the experiment was to evaluate how students responded to the lesson based on this plan. The main purpose though was to determine whether the information contained on the student worksheets offered opportunities for giving students feedback on their solutions and informing the teacher about emphases in subsequent teaching.

After the 50 minute lesson, the worksheets of the 16 students were collected and responses to both tasks inspected. There were three categories of responses.

First, there were six students who gave multiple more or less mathematically correct solutions using a variety of approaches to both tasks. It might be inferred that those students are familiar with a variety of approaches and can use those approaches readily if given the opportunity. Even still, there may be opportunities for providing feedback on the completeness of their explanations to such students. A key inference is that those students are ready for more significant challenges than offered by these tasks.

Second, there was one student who gave one response to Turning Point 1 and only one to Turning Point 2. It would be possible for the teacher to explore the reasons for this limited response further.

Third, and most importantly, there were nine students who used one approach to Turning Point 1 but incorporated multiple approaches in their responses to Turning Point 2. Figure 1 is an example of one such student's approach to "Turning Point 1".


TURNING POINT 1

I was not paying attention in my mathematics class. I heard the teacher say that "a turning point is at (2, -3)".

What might have been the function? (give at least two different possibilities)

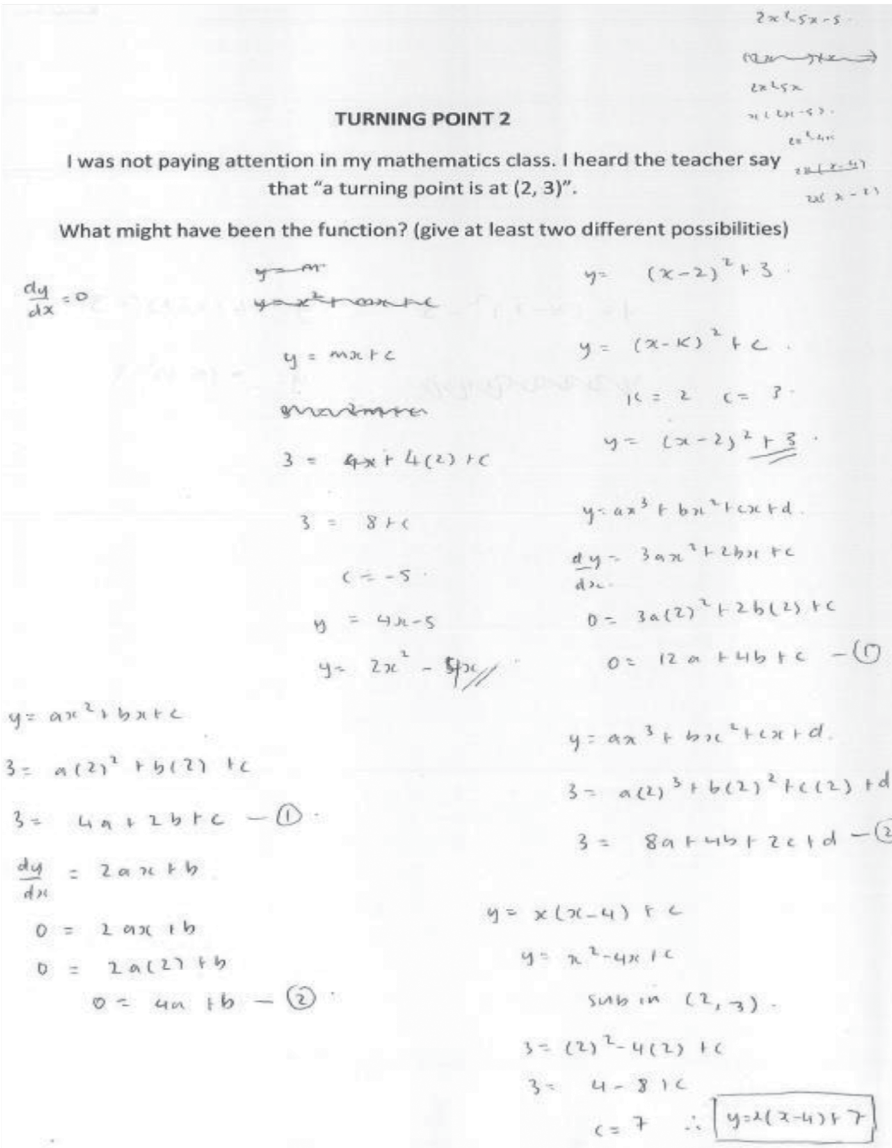
$y = (x-2)^2 - 3$ $\quad$ $y = 4(x-2)^2 - 3$

$y = -(x-2)^2 - 3$

Figure 1: A sample response to Turning Point 1.

The response indicated that the student used the generalised turning point form flexibly, giving three variations on the form with the co-efficient of the x term being 1, 4 and -1, respectively.

Figure 2 is that same students' response to Turning Point 2.



TURNING POINT 2

I was not paying attention in my mathematics class. I heard the teacher say that "a turning point is at (2, 3)".

What might have been the function? (give at least two different possibilities)

$2x^2-5x-5$

$2x^2-5x$

$x(2x-5)$

$2x^2-4x$

$2x(x-4)$

$2x(x-2)$

$\frac{dy}{dx}=0$

$y=mx+c$

$3=4x+4(2)+c$

$3=8+c$

$c=-5$

$y=4x-5$

$y=2x^2-5x$

$y=(x-2)^2+3$

$y=(x-k)^2+c$

$k=2 \quad c=3$

$y=(x-2)^2+3$

$y=ax^3+bx^2+cx+d$

$\frac{dy}{dx}=3ax^2+2bx+c$

$0=3a(2)^2+2b(2)+c$

$0=12a+4b+c$ — (1)

$y=ax^2+bx+c$

$3=a(2)^2+b(2)+c$

$3=4a+2b+c$ — (1)

$\frac{dy}{dx}=2ax+b$

$0=2ax+b$

$0=2a(2)+b$

$0=4a+b$ — (2)

$y=ax^3+bx^2+cx+d$

$3=a(2)^3+b(2)^2+c(2)+d$

$3=8a+4b+2c+d$ — (2)

$y=x(x-4)+c$

$y=x^2-4x+c$

sub in $(2,3)$

$3=(2)^2-4(2)+c$

$3=4-8+c$

$c=7 \quad \therefore \boxed{y=x(x-4)+7}$

Clearly this student had extended her thinking from Turning Point 1 to Turning Point 2, noting the range of responses, some of which were correct while others were only partial.

The solution on the right hand side at the top is the turning point form for this point. At the bottom of the page on the right hand side, the student used the x co-ordinate form based on the symmetry of the quadratic, and substituted to find values that translate the x axis. In between those two correct solutions, the students hypothesised a cubic, differentiated the general equation, and substituted values to seek to determine the coefficients. To complete this solution the student needed to assign a value for one of the coefficients after which the others could be found from the appropriately labelled equations. The student has also tried this approach for a quadratic, bottom left, which also need one of the coefficients to be assumed. The working at the middle of the page near the top is hard to follow and other than acknowledging the gradient at the turning point is not correct. All of the methods can be used in a subsequent conversation with the student.

It is stressed that all except one student had produced a range of self determined approaches to Turning Point 2 and the example above is representative of the others.

It is possible that students had more time to work on Turning Point 2, but a more likely explanation is that the class discussion of the Turning Point 1 prompted students to broaden their strategies for the second task. In terms of timing, the students spent just under 10 minutes working on Turning Point1 and 16 minutes on TP 2. The class spent about 8 minutes reviewing student solutions to TP 1 and 16 minutes reviewing solutions to TP 2.

There are three key aspects of the solution presented above (Figure 2) that emerge.

First, the correct responses can be affirmed for this student, as well addressing any issues of the representations of the solutions. This will be particularly important in preparation for external high stakes assessments in which marks are often awarded for the representation of the steps in solutions as well as the answers.

Second, the partial solutions can be addressed including suggesting possible ways of overcoming the barriers experienced. In the case of the student above, the possibility of assuming a value for one of the coefficients that allows values for the other coefficients to be found can be explained.

Third, the collected solution types can be discussed with the group. Specifically, in this case there were students who gave one or more examples of:

- The turning point form of a quadratic ($y = (x - h)^2 + k$)
- The turning point form of a quadratic generalised ($y = a(x - h)^2 + k$)
- The x co-ordinate form of a quadratic involving translation of the x axis (in the case of the above example, one approach is $f(x) = x(x - 4) + k$)
- The x co-ordinate form involving dilation (one student used $f(x) = \frac{1}{27}(x - 5)^2 (x + 1)^2$ producing a quartic)

- Choosing co-ordinates that gave a gradient value of 0, then anti- differentiating
- Calculating the period ($\pi/4$) of a trig function with amplitude of 3 (for the given turning point)
- Calculating the period ($\pi/4$) of a trig function with amplitude of 1 with a vertical translation.
- One student used absolute values for a quadratic, producing a double "u" shaped graph.

Since these various solutions were produced by the students, the teacher would be able to use these subsequently as part of teaching, perhaps for revision, as well as being aware of what those students are next ready to learn.

In other words, the diversity of strategies and the ways of representing solutions is a prompt for "conversations" on communicating mathematics and connecting representations. This is not a common outcome of working on textbook exercises.

Reflecting on the Teaching

The first author led the lesson but has not taught Year 12 mathematics for many years. This meant that there were aspects of this approach that operated differently at this level than in primary or junior secondary classes.

The examples and solution strategies produced by the students were diverse and it was hard to check their work for accuracy, or even be aware of the diversity in approaches, in the class in real time.

The students were not good at explaining their thinking. But neither was the first author clear in explicating unanticipated approaches. It is possible that when working at the limit of their mathematical understanding, teachers need to plan their explanations in advance, perhaps resulting in narrow approaches to solutions.

Student handwriting made it hard to read their work, and some wrote digits ambiguously. This did not help in the class but it is also likely to be a liability in examinations.

To gain a sense of the students' work, it was necessary to collect their work products and examine them later. This allowed more considered analysis of what the students had done. It did have the effect of allowing the first author to revise and consolidate his knowledge. On reflection, it is arguable that teaching such a lesson and being open to learning from the experience is an efficient way of extending mathematical knowledge of teachers, such as those teaching "out-of-field".

Conclusion

While the tasks and the *activating cognition* pedagogies operated differently from their use at other levels, it is arguable that posing students open-ended tasks and encouraging them to active their knowledge to find a range of solutions is an effective and efficient approach to teaching at senior secondary levels. Certainly this approach seems appropriate at the stage of revising units or the course.

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THE WISDOM OF THE CROWD

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The Wisdom of the Crowds has historical significance on estimation and averages. This workshop will look at a number of activities that will investigate the wisdom (or otherwise) of using central tendency to get the best estimate of a physical quantity. Casio Classpad Graphics calculators will be used for the activities and will be available for participants who do not bring one to the workshop.

Introduction and Background

The central tendency of data is a common theme throughout the study of statistics in high school and beyond. The introduction of the mean, median and mode would be a common element in many Australian classrooms where statistics is studied. What might vary is the way that central tendency is presented to a particular class.

One introduction that could be used is the example of the Wisdom of the Crowds. The Wisdom of the Crowds was attributed to Sir Francis Galton who used the estimates of the weight of an ox at a fair to calculate the median (and later the mean) to determine a 'best value' of the weight of the ox. He was 83 years old at the county livestock fair in England in 1906 when he observed nearly 800 villagers entering a contest to judge the weight of an ox (HREF1). It is reputed that he made the reasonable assumption that there would be as many overestimations as underestimations. In this manner he judged that he should be able to get a close estimate to the weight of the ox through the collected wisdom of the villagers who had made their individual estimate. This most suitable answer could be referred to as a Goldilocks estimate where it is not too high and not too low. The estimate by Sir Francis of 1207 pounds was found by calculating the median of all of the estimates. What was perhaps surprising was that his estimate was not only better than the estimates of the villagers but was also better than the estimates of the cattle experts that were present.

His estimate was the closest to the correct weight of 1198 pounds. Later that evening he supposedly calculated the mean (without the assistance of a modern calculator) to be 1197 pounds. This became known as the Wisdom of the Crowds and has implications in economics, mathematics and psychology (Surowiecki, 2004). This informative story led me to consider a classroom investigation that involved guessing the numbers of M&M's in a jar.

The Lesson Plan

The lesson plan involved students from two different classes being asked to estimate the number of M&M's in a jar that contained two 380g packets of M&M's. The results were tabulated on the whiteboard and the students asked to consider the estimate values and whether these values could be used to derive a better estimate. The students had access and familiarity to the Casio Classpad graphics calculator. The students tended to quickly think of using the mean and some suggested calculating the median to determine the 'best estimate'. The estimates are shown in Table 1. The students were also asked to represent the data graphically.

The Results

The low value of the guesses was surprising and the reasons were hard to ascertain with just a single lesson. The discrete nature of the data did vary from the continuous data that was used with the values found with the estimation of the weight of the ox. The students were initially not really sure why a variety of estimates were needed and presented the overall data in pie graphs, lines graphs and as stem and leaf plots. The 'wise choice' of the mean was calculated by the students and found to have a value of 332.5 in group A and 521.3 in group B. They further went on to also calculate the median with the use of technology. Group A found a median of 292 and group B found the median to be 520. The statistical values have been included in Figure 1.

There was a considerable discussion as to the accuracy of the values and what this meant in terms of the distributions. The students were very surprised with the actual number of M&M's that were in the jar. Even my 'wise choice' was considerably different to the actual value and I suggested to each class that I was not a good estimator. An astute student made the comment in my defence that my estimate was adversely affected by the estimates of the entire class.

Table 1 - *The number of M&M's in the jar as estimated by the students in Class A and Class B*

Class A estimates	Class B estimates
200	250
318	696
436	247
300	750
450	520
249	530
230	492
257	785
258	699
413	482
284	827
388	301
500	282
512	480
250	300
275	798
	330
	307
	576
	753
	405
	659

Conclusions and Comments

The actual number of M&M's in the jar was 804 and this was counted by an independent observer (before distributing the M&M's to the students). The 'wisdom of the crowds' was not really so wise in this instance but did assist both classes with the concepts of central tendency and variation. What may also of note was that the larger crowd was wiser than the smaller crowd. The variety of scores started some interesting discussions about what constituted a normal distribution in both of the classes. Using an item that is of a continuous nature (such as a large pumpkin) may allow better estimates from the wisdom of the very small crowd of a classroom.

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SMART TESTS FOR SMARTER TEACHING

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Springside P-9 College has been using Specific Mathematics Assessments that Reveal Thinking (SMART) tests since 2014 in Years 5-9. SMART tests are online diagnostic assessments developed by The University of Melbourne to assist teachers in identifying students' understanding and revealing students' misconceptions. Over the past year, 10 teachers have regularly used the SMART tests to complement their in-class assessments. Most importantly, SMART tests have enabled our teachers to be more confident about planning and teaching mathematical content appropriate to their year level, to identify student misconceptions and how to move students forward in regular class settings.

What are SMART Tests?

Teachers are often looking for quick and easy ways to identify the learning needs of their students to ensure they are targeting their specific needs. At Springside College, we have found a great way to do this, which isn't time consuming or labour intensive. Specific Mathematical Assessments that Reveal Thinking (SMART) tests provide teachers with an informative diagnosis of their students' conceptual understanding of most topics in senior primary and junior secondary school mathematics.

Price, Stacey, Steinle, Chick and Gvozdenko (2009) recognise that "It is often difficult to keep track of students' progress and identify exactly where they are struggling. It is often hard to identify when students are ready for the current topic, and exactly where they might be having difficulty" (p 2).Through SMART tests, their research-based diagnosis provides teachers with an immediate and efficient way to highlight the misconceptions of their students. This assists teachers to differentiate their lesson instruction in order to address the misconceptions identified by the results of the SMART tests. The same authors, Price et.al (2011), explain that "Understanding new mathematical concepts often relies on having good background knowledge and so, to avoid presenting some of the class with tasks they cannot do, we sometimes excessively revise earlier material and make sure that the tasks set for the bulk of the lesson are straightforward enough to be tackled by anyone." (p. 1)

SMART tests are relatively new and are available to all teachers in the Victorian government school system (see note 1). They are online assessments devised by researchers at The University of Melbourne which enable teachers to assess their students' understanding by using a range of assessment tasks such as diagnostic tests, short quizzes and short answer tests which are all marked by an assessment rubric. Steinle and Stacey (2012) found that the idea of formative assessment - where assessment directly influences lesson planning - was not well understood by teachers, who were more used to using assessment summatively to assist in writing reports, as distinct from driving their planning and teaching. Time constraints also restrict teachers in their ability to effectively utilise all pieces of assessment to inform their planning and teaching. Lack of time also affects how teachers moderate with their colleagues to discuss and analyse results. The SMART tests involve students logging on to a dedicated website, completing a teacher-selected assessment tool with multiple choice answers. Students' answers are collated online and are immediately made available to teachers, demonstrating which stage each student is at, as well as identifying key misconceptions. The other benefit of using the SMART tests is the 'Teaching Ideas' section of the website, where ideas are supplied for how to address specific misconceptions.

Research on Teacher Knowledge

Chick (2010) conducted research to examine aspects of teachers' knowledge for teaching the topic of ratio. Successful teaching of this topic requires background knowledge on the misconceptions students regularly face, and also how students' knowledge of fractions can be transferred to develop their understanding of ratios. This was part of a larger study, carried out by Chick, which aimed to explore teachers' Pedagogical Content

Knowledge (PCK) of mathematics. The questionnaire involved a task that was given to year eight students, and provides a sample answer given by one 'student'.

The questionnaire then asks teachers to identify why the student might have selected that answer, and what assistance or explanation a teacher might provide to assist this student. Chick (2010) was most concerned with the teachers' responses on the teaching suggestions to address this misconception. *"Very few of the suggestions proposed specifically addressed the additive error"* (p.7). In the Mr Short and Mr Tall problem, Mr Short was presented as being 6 paperclips tall, and then, when measured instead by matchsticks, was 4 match-sticks tall. Mr Tall, in contrast, was 6 matchsticks tall, and students were asked to determine Mr Tall's height in paperclips. Among the approximately two-thirds of students who could not answer this question correctly, many made the additive error of focussing on 6 – 4 difference in the two measurements for Mr Short, and then added this to Mr Tall's height of 6 to get his paperclip height as 8. This is an area in which teachers may lack confidence; they are usually confident in teaching the skill, however 'fixing' the misconception entails a different knowledge and understanding of the topic or task. Whilst Chick (2010) identified that teachers in the study had difficulty addressing misconceptions, she did not discuss how these teachers' capacities were developed as a result of the study. At Springside P-9 College, teachers are participating in an ongoing study as to how using SMART assessments in Mathematics assists their professional development and builds teacher capacity.

Steinle and Stacey (2012) conducted research regarding teachers' views of using an online formative assessment system for mathematics. Their research included interviewing ten focus groups with teachers at schools involved in the development of the SMART test system in its first two years. This included online surveys teachers completed after they had used a SMART test, and emails that teachers involved sent Steinle and Stacey on a random basis.

The online survey included two questions, and participants in the survey were invited to elaborate on their answers:

- As a result of using this quiz, have you learned something useful for you as a teacher?
- Did you adjust your teaching plan as a result of the diagnostic information?

They found that 82% of one hundred and forty three respondents replying to the first question said they had either found the SMART tests as "very valuable", or "useful" learning. Only 7% of respondents said that they had not gained new and useful knowledge. Steinle and Stacey also collected comments and anecdotal evidence from teachers involved in their study, and reported these comments in their research.

Responses to the second question, asking, "Did you adjust your teaching plan as a result of the diagnostic information?" provided further information on the usefulness of the SMART tests in terms of informing teachers planning and teaching. An encouraging 61% of respondents said that they had adjusted their teaching as a result of the SMART tests. The respondents who answered 'yes' were then asked 'In what way did you change your teaching plan?' The responses to this were varied. This research carried out by Steinle and Stacey (2012) showed that SMART tests were useful in improving teachers' mathematical content knowledge. They also found that, in general, teachers did adjust their teaching plan as a result of the SMART tests.

Using SMART Tests at Springside P-9 College

At Springside P-9 College, teachers in Years 5 to Year 9 use SMART tests as one form of formative assessment. Students are identified as a certain stage of development in regards to a specific topic, and any misconceptions a student might have is also identified. This assists teachers in planning lessons, or focus group activities, which cater for the individual needs of their students. During planning sessions, misconceptions are used to identify key teaching points or areas in which to be covered throughout the unit of work. Discussions also occur as to how best to address these misconceptions.

At Springside P-9 College, the SMART assessment tasks also assist in the planning of a unit of work. If a large majority of students are identified as having the same misconception, this will form the basis for an explicit teaching focus. Alternatively, if only a small number of students are identified as having a specific misconception, this will inform the planning of focus groups to target that specific misunderstanding.

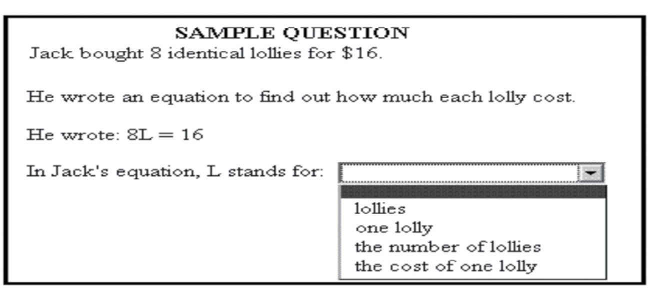

Figure 1: Sample item from a SMART test

Figure 1 shows a sample item from SMART tests on the sub-topic, *Letters for numbers or objects.* It is one of three different items for Quiz A. There is a second Quiz B which allows teachers to use two quizzes at the same time, or to use one quiz first and the results of the second as a measure of improved student understanding. Within the Algebra topic area, there are ten different themes, ranging across *Formulating expressions, Writing and solving equations, Representing linear functions, to Interpreting gradients of graphs.* For each sub-topic, the most important elements are illustrated by the following excerpt from this particular sub-topic. It is here that teachers are given a detailed explanation of stages students can be expected to transition through and the common misconceptions likely to be shown in students' responses to the given items. Then typical misconceptions are explained. And finally some teaching suggestions are offered:

> ### Detailed explanations of developmental stages and common misconceptions
>
> **Stage 0** students are below stage 1. They do not as yet interpret algebraic letters as standing for numbers.
>
> Stage 1 students rarely interpret algebraic letters as standing for numbers.
>
> **Stage 2** students sometimes interpret algebraic letters as standing for numbers.
>
> **Stage 3** students consistently interpret algebraic letters as standing for numbers. They have a correct understanding of the meaning of a pro-numeral.

Typical misconceptions are classified to assist teachers in recording the prevalent mistakes made by students. In this case, there are two main misconceptions.

> ### Misconceptions
>
> **LO** (letter as object) This group of students frequently uses algebraic letters to stand for objects rather than numbers. They have little understanding of the meaning of letters when used in algebra as pro-numerals. They usually think that algebraic letters are abbreviations for words. For example they may think that $2\boldsymbol{c} + 3\boldsymbol{d}$ "represents" 2 cats and 3 dogs. They may interpret $\boldsymbol{a} + \boldsymbol{b} = 90$ as "the apples and the bananas cost 90 cents".
>
> **SAC** (solution as coefficient) Like **LO** students, these students also interpret algebraic letters to stand for objects rather than numbers, but in a slightly more sophisticated way. These students are likely to translate the information "I bought $\boldsymbol{a}$ apples and $\boldsymbol{b}$ bananas for 90 cents. The apples cost 15 cents each and the bananas cost 20 cents each" into the equation $2\boldsymbol{a} + 3\boldsymbol{b} = 90$. They intend this equation to mean the true statement "2

apples and 3 bananas cost 90 cents", without realising this is not a valid equation. They have in effect found a solution to the equation first, then used the solution in the equation. They do not understand the usefulness of equations to solve problems.

The last section on this sub-topic provided some teaching suggestions which are related to typical misconceptions and the stages at which students are placed:

Teaching Suggestions

Both **LO** and **SAC** students suffer from the very common 'letter as object' misconception. **LO** (letter as object). This is misconception often arises from teaching. For example many teachers teach addition of 'like terms' by given examples like this: 2***a*** + 3***b*** represents 2 apples and 3 bananas and 6***a*** + 4***b*** represents 6 apples and 4 bananas and so (2***a*** + 3***b***) + (6***a*** + 4***b***) represents 8 apples and 7 bananas (i.e. 8***a*** + 7***b***). While this way of thinking will initially be helpful when gathering like terms, it is incorrect and leads to long term problems. This sort of thinking and teaching is sometimes called *fruit salad algebra,* because fruit is so often used as the example.

Stages 0, 1 and 2 The simplest teaching suggestion is not to teach students the misconception that letters in algebra stand for objects or abbreviated words. When teaching, do not use "fruit salad algebra". Algebraic expressions are always about numbers and number relationships. In addition, students need plenty of opportunities to write equations that describe situations in the real world, so they can see how to find the equation about the numbers involved and learn to avoid the easy false alternatives. Prefacing any formulated equation with the statement "Let ***n* be the number of** ......" is a very good habit, as is ensuring that students write this in their own algebraic work. It may also be useful to use variables that are not the initial letter of the name of the items involved. For example, "Let ***w*** be the cost of a banana". Time spent formulating expressions and equations before focusing on solving equations is also beneficial and will allow you to see errors as they occur.

There are also additional links to the Mathematics Developmental Continuum.

http://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/continuum/Pages/algebralett425.aspx

Stage 3 Students who use letters to represent numbers need many opportunities to formulate algebraic expressions and equations. Once students have completed their assessment, the data is instantly available for teachers to access. It can be exported into an excel document, or downloaded as a PDF file. At Springside P-9 College, we have taken the data analysis stage a step further.

Letters for Numbers or Objects	Stage 0	Stage 1	Stage 2	Stage 3
		These students generally do not interpret algebraic letters as standing for numbers.	These students sometimes interpret algebraic letters as standing for numbers	These students consistently interpret algebraic letters as standing for numbers.
		Student A Student B	Student C Student D	

Misconceptions	LO	SAC
	These students have a tendency to interpret algebraic letters as objects (instead of numbers)	These students also have a tendency to interpret algebraic letters as objects, but also when writing an equation, use the solution(s) as the coefficient(s) of the pronumeral(s). This is a type of interpreting algebraic letters as objects
	Student A Student B Student C	

Figure 2: SMART analysis at Springside

Figure 2 shows how we record SMART test data relating to whether students know that letters, when used in algebra, stand for numbers (in other words that they are 'pronumerals').

Figure 2 above demonstrates the way in which SMART test data is compiled for ease of planning. Teachers log on to the website, download their data and then place their students in the corresponding stage. This document gives teachers a snapshot of where their students are in terms of their understanding of the different stages of the SMART test. It also identifies how many students have a particular misconception, so that the best way to address this can be discussed and implemented.

Quiz: Letters for numbers or objects For years: 7,8,9

This quiz tests whether students know that letters, when used in algebra, stand for numbers (in other words, that they are 'pronumerals'). Many students interpret and use algebraic letters as abbreviations for words or to stand for things and they think that algebra is mainly just mathematical shorthand. This misconception is an underlying cause of many difficulties in writing equations.

Students are classified as having reached one of three stages in the development of their understanding of pronumerals. Those exhibiting one misconception have it noted

.Average time to do this quiz: 3 minutes. Range: 1 - 9 minutes.

Figure 3: Corresponding SMART test notes for the analysis in Figure 2

The SMART assessments are currently being used regularly in Years 5 to 9 at Springside College as both formative assessment and summative assessment. They have been a great resource for teachers to use when planning both a unit of work, and their individual teaching program based on the needs of their students.

Impact on Teachers: Planning of Instruction

Preliminary analysis of data collection shows a shift in teacher understanding and confidence in the areas of Knowledge of mathematics, Knowledge of content in the Australian Curriculum, Understanding of students thinking and Planning of instruction. Figure 4 shows the results for one item of a questionnaire given to members of one Year level teaching team.

Planning of Instruction

I am unsure of what my students need to learn and how to plan lessons and activities which suit their needs		I am confident in planning lessons and activities which cater to the needs of my students

	0	1	2	3	4	5	6	7	8	9	10
Pre					TeachA	TeachB	TeachC	TeachF	TeachE		
Post									TeachC	TeachF TeachA TeachE TeachB	

Figure 4: Teacher pre and post SMART comparisons on planning for instruction

In the questionnaire teachers were invited to place themselves on a ten-point scale showing where they were before using SMART tests and after one semester's use of SMART tests. From Figure 4, we can see that every teacher has reported a positive effect in Planning Instruction as a consequence of using SMART tests. Teacher A indicated the biggest shift (5 points), followed by Teacher B (4 points). Other teachers reported smaller shifts, but everyone saw SMART tests as having a beneficial effect on Planning of Instruction. These different reported effects can be explained partly in terms of teacher experience. These self-ratings can also be validated by reference to actual planning documents used by teachers.

The use of SMART test data has given teachers confidence when planning lessons, ensuring they cater to the needs of their students. SMART test data shows what stage of learning each of their students is working at, and also provides teachers with possible teaching suggestions to assist in their development towards the next stage.

Impact on Teachers: Understanding of Students' Thinking

Understanding of Students' Thinking

When looking at students work or responses, I understand some of the thought processes, explanations and possible misconceptions behind their answers		I have a reasonably strong understanding of most or all of the responses my students give when answering maths questions, and can explain the thought process and possible misconceptions behind their answers

	0	1	2	3	4	5	6	7	8	9	10
Pre			TeachB		TeachA	TeachE	TeachC		TeachF		
Post								TeachA	TeachC TeachE	TeachF TeachB	

Figure 5: Teacher pre and post SMART comparisons on Understanding students' thinking

Figure 5 demonstrates the effect SMART testing has had on the area of teachers understanding of students' thinking, especially when analyzing student work or responses. We can again see that every teacher has reported a positive effect in Understanding of Students' Thinking as a consequence of using SMART tests. Teacher B indicated the biggest shift (7 points) in confidence. Other teachers reported smaller shifts, but everyone saw SMART tests as improving their Understanding of Students' Thinking. It has allowed teachers to deepen their understanding of possible misconceptions, and to address these misconceptions, either through whole class introductions or targeted focus groups.

Other areas included in the teacher pre- and post-SMART comparisons were: Confidence when speaking to my colleagues about mathematics, Knowledge of mathematics, and Knowledge of content in the Australian Curriculum. These all showed positive impacts on the teachers involved in the study. In all cases, teachers' self-ratings were able to be triangulated with other available school data.

Teachers' Reflections on Their Experiences with SMART Tests

When completing the post survey on the use of SMART tests, teachers were also given the opportunity to comment on their experiences when using SMART tests. Here are comments from three teachers.

I feel that through the SMART tests, I now confidently know the misconceptions in each unit I teach, and can then create focus groups from that information. The students have enjoyed *the SMART tests. They (SMART tests) have helped with planning the units and focus groups-which has helped ensure I'm confident when introducing new topics. (Teacher G)*

SMART testing has brought me to appreciate perspectives on the same idea a little more. This has happened through the system of notation of common student error. (Teacher H)

Until using SMART tests I did(n't) feel that I had a good knowledge of the standards and what they require. SMART tests add a whole other dimension to mathematics teaching and learning and encourage a deeper and more thorough understanding of the mathematical content. After using SMART tests, I definitely feel more confident in this area (Understanding of Students' Thinking). Students give a myriad of responses to mathematics problems, and without knowing where they actually stem from, its near impossible to correct them. My understanding of potential misconceptions and their origins has already improved, and this will only continue. (Teacher I)

Teachers' Use of Knowledge About Students' Misconceptions

One teacher involved in the implementation of SMART testing at Springside P-9 College was asked how SMART testing is utilized at the college. They were asked 'What sort of things would the teachers do if they found there were misconceptions happening in their classes.'

An example of this would be in a class where we did the decimal comparison test (Understanding decimals). We did that at the start of the unit and we know that a lot of the students had the 'Longer is Larger' misconception, but only a few had the 'Money Thinking' and a few had the 'Shorter is Larger' misconception.

What we thought was if that many students had the 'Longer is Larger' misconception that informed an introduction and maybe a warm-up activity to try to address it and talk about it. Where if there were only a few students that had the 'Money Thinking', misconception, that would inform the forming of a focus group, and we thought of what activities we could do with those students to address that misconception. So we thought that if the whole class doesn't have it it's probably not worth spending a whole lesson on it with the whole class, but it's important for those students. Focus explicitly on this with those students while the other students are doing another task and address it then.

Conclusion

At Springside P-9 College, SMART testing has been a useful complement to other formative and summative testing. It has allowed teachers to explore the stages of development in particular topics, and also deepen their understanding of possible misconceptions that students might have, as well as suggestions on how to address these. Results are also shared with students, so they can set their own personal learning goals.

Note 1: SMART tests can be accessed by Victorian state school teachers at http://smart-test.edu.au/teacher/, teachers at schools affiliated with the Catholic Education Office, Melbourne can access SMART tests at http://www.smart-quiz.edu.au/teacher/.

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HOW TO STOP STUDENTS FORGETTING WHAT THEY LEARN

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Velvet Learning

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Long-term retention of learning is important particularly in mathematics because the subject is particularly sensitive to mastery of prerequisite material. Existing research offers direction for teachers to adapt their lessons so that students can retain more of their knowledge for a longer period of time. If successful this would allow teachers to spend less time revising material at the start of each unit.

The Significance of a Student Retaining What They Learn

Just as cramming right before a test can improve immediate performance but is inadequate for long-term retention, many classes today apply strategies leading to short-term gains in performance (like on a topic test) rather than retaining knowledge for future studies and later life. This practice contradicts a wide body of psychological research. This article offers suggestions for translating these long-term retention techniques to the mathematics classroom.

Out of all the subjects in school, mathematics perhaps requires the most amount of pre-requisite knowledge, this being the knowledge that students carry forward from previous years. A student in History may forget the date of the Battle of Hastings and its major actors – and, while lamentable, this has little effect on later studies of, say, World War II. By contrast, a mathematics student who is learning quadratic equations and does not remember the process for operating with negative numbers is at a severe disadvantage. In practice, gaps in pre-requisite knowledge result in mathematics units containing much revision of previous material that was never mastered or is now forgotten.

If the principles from the psychological research discussed below can be translated into the classroom, students will retain more from year to year. Furthermore, students will be better equipped to use what they have learned in later life, and teachers will be able to redirect time spent on revision towards mastery and enrichment of new content.

Principle 1: Use Distributed Instead of Massed Practice

The Research

Roediger and Pyc (2012) explain,

> *"if information is repeated back to back (massed or blocked presentation), it is often learned quickly but not very securely (i.e., the knowledge fades fast). If information is repeated in a distributed fashion or spaced over time, it is learned more slowly but is retained for much longer."*

This is "one of the oldest findings in experimental psychology (first reported by Ebbinghaus in 1885)".

The tendency in traditionally structured classes is to intensively learn one skill or idea in a single period, complete the corresponding questions in the textbook in a massed block. Indeed this is the design of all major textbooks. Likewise many schools isolate individual topics to only a few weeks of a year. For example, a student may only study probability for two weeks without seeing it again for another year.

Instead, spaced or distributed practice calls for the same amount of practice to be spread out over time. Booker et al. (2010) suggest organising whole units in a "spiral curriculum", where concepts are revisited over time at increasing levels of complexity.

Dunlosky et al. (2013) reviewed contemporary research in this area and found distributed practice had a large effect across a range of student ages, topics and forms of assessment. Several studies, notably Landauer and Bjork (1978), found that longer and increasing intervals between practice sessions have an even larger effect than short, consistent gaps.

Overlearning is where students complete additional practice in the same session after they are able to complete a problem correctly. Rohrer and Taylor (2006) ran two experiments to refute the idea that massed overlearning is superior for mathematics. Consistent with the studies on massed practice cited above, overlearning leads to lower performance in the long-term and even just a week later.

Some students find distributed practice more difficult or discouraging, as immediate performance gains are not as evident as with massed practice. Bjork (1994) introduced the concept of "desirable difficulties" to describe a class of learning techniques including spaced practice that are much better in the long-term despite upfront appearance of strain. For this reason, teachers should share the rationale and research as to why spaced practice is used and that a feeling of difficulty is normal rather than a reflection on the student.

Applying it to the Mathematics Classroom

When a new concept or skill is learned, practice should be spread over several sessions. A useful approach is to "front-load" teaching of several skills together, then drawing practice from multiple exercises or activities over the several days that would have been allocated to those multiple skills anyway.

To avoid the problem of students only studying a topic such as measurement once per year, schools could also break units up into a number of shorter units, which can be spread across the year. This has the added advantage of making all summative assessments formative, as students will return to the topic again with the same teacher.

Teachers should give students a small amount of practice on previously learned material throughout the year. This could be in homework or warm-up questions at the start of class. This can either be done with a formal schedule of spacing or ad hoc, from a teacher's judgement.

An objection that may be raised is that there is insufficient time because students must "get through" the curriculum. Note that the distributed practice approach trades off short-term performance for long-term retention. It is acceptable to do a smaller percentage of the activities usually done in a unit, leaving room for spaced practice of old material. This allocation of content can be done knowing that the remaining practice will take place in future sessions and thereby contributing to better long-term retention.

Principle 2: Clear Obstacles to Initially Storing Memory

The Research

Students will more easily retrieve a memory of something over a longer period when that memory is implanted well (Bjork 1994). Some obstacles in a maths classroom include: maths anxiety, divided attention and a lack of automaticity for prerequisite concepts.

How a student feels about themselves and mathematics is known to affect their learning (e.g. Martin and Marsh 2006; Brophy 1983). One reason is that students use avoidance strategies so they simply get less practice than their peers (Elliot 1999). Another is that anxiety interferes with working memory and retrieval from long-term memory storage (Ashcraft and Kirk 2001) such that tasks beyond a low level of complexity become much harder.

In a busy classroom, students' attentions are divided. Even if students are on task, it can be unclear what aspect of a task requires their attention (Hattie, 2009). For instance if a student thinks mathematics is about solving procedures accurately, they may choose to filter out any attempt by a teacher at introducing conceptual ideas, which means these ideas do not get stored with the memory.

When students in secondary school do not have basic multiplication or division facts memorised, their acquisition of future concepts is slowed because part of their working memory is taken up re-calculating these from scratch. There are similar fundamental facts or skills within every level and topic whose recall or execution should be automatic not just fluent. Something that is automatic does not take up time or cognitive load, enabling a student to either have more time or more mental space to practice and engage with the new, higher-level concept. Examples include: being able to relate the size of an angle to its drawn shape; solving linear equations; and, at the Year 12 level, knowing basic derivatives or trigonometric values.

Applying it to the Mathematics Classroom

Reducing maths anxiety is a long process. One strategy is to provide opportunities for success for the anxious student. Following Carol Dweck's (2000) work on growth mindset, teachers should emphasise continuous improvement as the success metric. Spaced practice allows students to re-attempt material, so that these improvements are visible.

To ensure that the subtleties of what is being taught are what the students focus on, teachers must be explicit about the intended learning outcomes. Note that an outcome written as "to add fractions" is too shallow and adds nothing beyond the obvious. It is important to specify the strategies used or level of understanding expected, such as "to explain how to add fractions using fraction strips".

Teachers should delineate for students what skills are required fluently versus automatically and set different targets in assessment. Specific practice for automatic skills should be given after students can perform the skill fluently. A focus on speed and less on articulating understanding is appropriate, unlike most practice.

Principle 3: Teach for Understanding

The Research

Conceptual understanding is already a goal of mathematics education. Thorough conceptual understanding that justifies and links facts and procedures also provides an excellent base for long-term recall, though research in this area is in its infancy in contrast to the principles discussed above.

Two closely related and easily implementable techniques are elaborative interrogation and self-explanation. Roediger and Pyc (2012) explain elaborative interrogation as "students generating plausible explanations to statements while they are studying (i.e., answering *why* some stated fact may be true)."

Self-explanation is where students answer specific metacognitive prompts that force them to connect ideas, notice what they know or don't know or justify statements. For example, Wong, Lawson, & Keeves (2002) studied self-explanation for two groups of ninth-grade students studying a new theorem in maths. Both groups were prompted to think aloud during their study of the new concept but one group was given specific prompts to respond to, specifically:

- What parts of this page are new to me?
- What does the statement mean?
- How does the new piece of information help with the proof of the theorem?
- Is there anything I still don't understand?

On a subsequent test of the material, given specific prompts, the group performed slightly better on material similar to what had been taught and much better on questions that required application to new circumstances.

Dunlosky et al. (2013) summarises research in this area: both techniques require little training to be effective. Students will complete less practice problems, as they slow down to explain their thinking, but this is compensated for by richer and transferable understanding. Studies show better long-term retention when these techniques are used though this effect may be confounded.

A caveat for these techniques is that they are less effective for students with low prior knowledge or skills, as they struggle to generate plausible explanations. Dunlosky et al. (2013) cite Didierjean and Cauzinille-Marmèche (1997) who studied "ninth graders with poor algebra skills [with] minimal training prior to engaging in self-explanation while solving algebra problems. ... Analysis revealed that students produced many more paraphrases than explanations".

Applying it to the Mathematics Classroom

Teachers should offer formal opportunities for students to explain their thinking or justify a step of working. As a point of convenience and frequent reference, a set of regular self-explaining questions could be made into a poster and displayed in the classroom.

Questions that specifically test these skills should be included in assessments i.e. 'explain why the formula for the volume of a cylinder is true' or 'For 2(x+3) = 5, justify whether another student should divide by two or expand the brackets as the first step in solving this equation'.

Conclusion

Students need to retain the mathematics they learn now for use in future years. Teachers can support this by incorporating the principles of distributed practice, removing obstacles to initial memory storage and teaching for understanding into their classroom practice.

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EMPOWERING CAS SKILLS IN SPECIALIST MATHS: VECTORS, CIRCULAR FUNCTIONS & COMPLEX NUMBERS

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Since 2006 students have been permitted to bring into the examination room their Computer Algebra Systems (CAS) in VCAA Specialist Mathematics Examination 2. There is certainly a great benefit in the effective and efficient use of CAS, particularly with multiple-choice questions. This paper will explore how CAS (both TI-Nspire CAS and ClassPad) can be used to solve some past multiple-choice questions from the VCAA Examination 2. The main focus will be on Vectors, Circular Functions and Complex Numbers.

Introduction

As technology advances, students continue to improve their ability to cope with these constant technological enhancements. With new apps and social media updates constantly being produced, it is clear that most students have the capability to quickly adapt to new technologies. Yet, despite their tech savviness, many students struggle when faced with CAS technology. Operating a device that is so rarely used in their everyday lives is most definitely a difficult task. CAS calculator or CAS software has been allowed in Examination 2 since 2006. However, students usually preferred to do multiple-choice questions by hand rather than making the most of the powerful technology they are allowed to have with them in the exam. Selected questions from past VCAA Exam 2 Specialist Mathematics are discussed in this paper and their solutions demonstrated by both TI-Nspire™ CX CAS Teacher Software and ClassPad Manager for ClassPad II. The purpose is to maximise students' ability to use their CAS calculator as an effective tool and to assist them in answering or checking questions in the multiple-choice section.

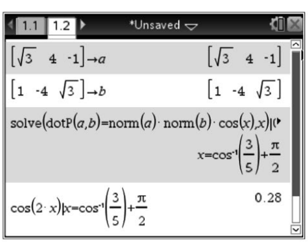
Vectors

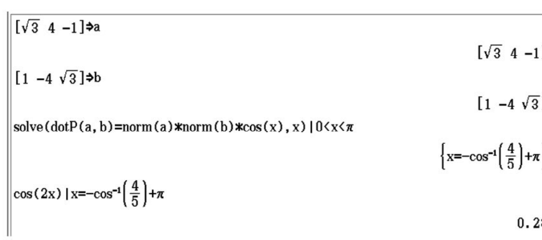
VCAA 2014 Exam 2, Question 15

If θ is the angle between $\underset{\sim}{a} = \sqrt{3}\,\underset{\sim}{i} + 4\,\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\,\underset{\sim}{j} + \sqrt{3}\,\underset{\sim}{k}$, then cos(2θ) is

A. $-\frac{4}{5}$ B. $\frac{7}{25}$ C. $-\frac{7}{25}$ D. $\frac{14}{25}$ E. $-\frac{24}{25}$

This question requires students to find the angle, *θ*, between the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and hence apply the double angle formula to find *cos*(2*θ*). CAS calculators would be quite useful in this situation as students can substitute the given vectors straight into the dot product formula and solve for the angle, *θ*. The solution is shown below. Note: storing the vectors into the CAS calculator initially and then recalling it in the *Solve* command would be less tedious.





VCAA 2014 Exam 2, Question 16

Two vectors are given by $\underset{\sim}{a} = 4\underset{\sim}{i} + m\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + n\underset{\sim}{j} - \underset{\sim}{k}$ where $m, n \in R^{+}$. If $|\underset{\sim}{a}| = 10$ and $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$, then *m* and *n* respectively are

A. $5\sqrt{3}, \frac{\sqrt{3}}{3}$ B. $5\sqrt{3}, \sqrt{3}$ C. $-5\sqrt{3}, \sqrt{3}$ D. $\sqrt{93}, \frac{5\sqrt{93}}{93}$ E. $5, 1$

Students did quite well on this question with 77% of students correctly answering (A). While the concept in this question is straight forward, students would have spent more time on this question had they not thought of incorporating the mathematical concept and the CAS device to solve for *m* and *n* in one step as shown below. It is more effective to include the values of *m* and *n* at the end of the *Solve* command with a restricted domain, ie *m*>0 and *n*>0 so that CAS only gives one set of solutions.

$$\text{solve}\left(\begin{cases}\text{norm}([4\ \ m\ \ -3])=10\\ \text{dotP}([4\ \ m\ \ -3],[-2\ \ n\ \ -1])=0\end{cases},m,n\right)$$

$$m=5\cdot\sqrt{3}\ \text{and}\ n=\frac{\sqrt{3}}{3}$$

$[4\ \ m\ \ -3]\rightarrow a$	$[4\ \ m\ \ -3]$
$[-2\ \ n\ \ -1]\rightarrow b$	$[-2\ \ n\ \ -1]$
$\begin{cases}\text{norm}(a)=10\\ \text{dotP}(a,b)=0\end{cases}\Bigg\vert_{m,n} \mid m>0\ \text{and}\ n>0$	$\left\{\left\{m=5\cdot\sqrt{3},n=\frac{\sqrt{3}}{3}\right\}\right\}$

VCAA 2013 Exam 2, Question 15

Let $\underset{\sim}{u}=4\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}$, $\underset{\sim}{v}=3\underset{\sim}{j}+3\underset{\sim}{k}$ and $\underset{\sim}{w}=-4\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}$.

Which one of the following statements is **not** true?

A. $|\underset{\sim}{u}|=|\underset{\sim}{v}|$

B. $|\underset{\sim}{u}|=|-\underset{\sim}{w}|$

C. $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are linearly independent

D. $\underset{\sim}{u}.\underset{\sim}{v}=0$

E. $(\underset{\sim}{u}+\underset{\sim}{w}).\underset{\sim}{v}=12$

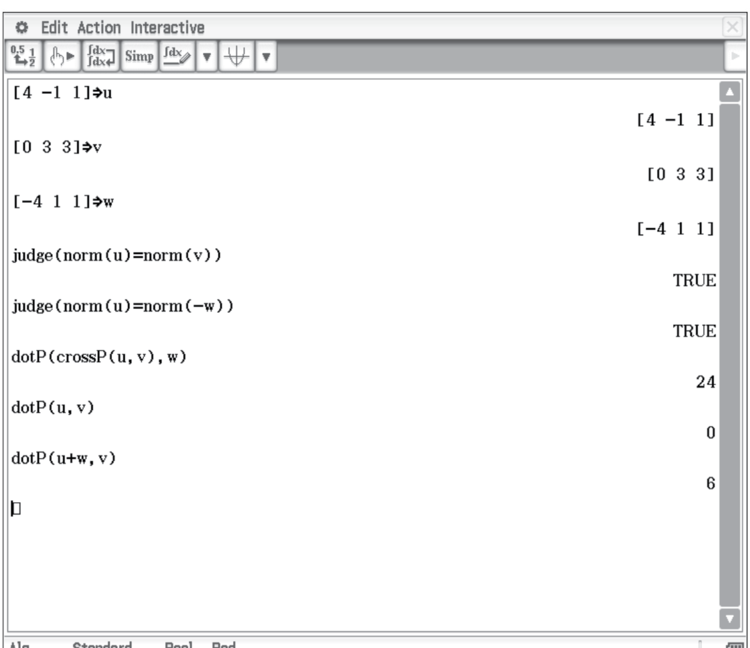
This question requires students to go through every option to find a correct answer. A by-hand approach can be quite time consuming and thus a CAS device would be a better option as shown below.

Note: the cross product requires both of the vectors to be three-dimensional vectors with the result of a cross product being a vector. The *Scalar Triple product* method has been used to determine whether the vectors lie in the same plane, ie coplanar.

Interpretation: If the scalar triple product of three vectors is zero then the vectors will lie in the same plane (ie *linearly dependent*) and if the the result is *NOT zero* then the vectors won't lie in the same plane (ie *linearly independent*).

TI-Nspire CAS:

ClassPad:



VCAA 2012 Examination 2, Question 15

The vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + m\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = m^2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ are perpendicular for

A. $m = -\frac{2}{3}$ and $m = 1$ B. $m = -\frac{3}{2}$ and $m = 1$ C. $m = \frac{2}{3}$ and $m = -1$

D. $m = \frac{2}{3}$ and $m = -1$ E. $m = 3$ and $m = -1$

Students did very well on this question with 85% of students correctly answering (D). Therefore, students demonstrated a solid understanding of the perpendicular vectors concept in this question. A dot product of the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ can be done by hand first and then use the *Solve* command from a CAS device to solve for *m*.

Alternatively, a more effective method would be to use the *Solve* and *dotP* commands at the same time from CAS to find the values of *m* and *n* in one step as shown.

$$\text{solve}\left(\text{dotP}\left(\begin{bmatrix}2 & m & -3\end{bmatrix},\begin{bmatrix}m^2 & -1 & 1\end{bmatrix}\right)=0,m\right)$$

$$m=-1 \text{ or } m=\frac{3}{2}$$

VCAA 2012 Examination 2, Question 17

If $\underset{\sim}{u} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = 3\underset{\sim}{i} - 6\underset{\sim}{j} + 2\underset{\sim}{k}$, the vector resolute of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{u}$ is

A. $\frac{20}{49}\left(3\underset{\sim}{i} - 6\underset{\sim}{j} + 2\underset{\sim}{k}\right)$ B. $\frac{20}{3}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$ C. $\frac{20}{7}\left(3\underset{\sim}{i} - 6\underset{\sim}{j} + 2\underset{\sim}{k}\right)$

D. $\frac{20}{9}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$ E. $\frac{1}{9}\left(-2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$

This question involves applying the *vector resolute* of $\underset{\sim}{v}$ in the direction $\underset{\sim}{u}$ formula and 60% of students correctly answering (D). The equivalent answer can be found in a few steps using store, *dotP* and *unitV* commands from CAS as shown below. Students are then required to take out a common factor of $\frac{20}{9}$ and thus obtain the answer as given in D.

$[2\ \ {-2}\ \ 1] \to u$	$[2\ \ {-2}\ \ 1]$
$[3\ \ {-6}\ \ 2] \to v$	$[3\ \ {-6}\ \ 2]$
$\text{dotP}(v,\text{unitV}(u))\cdot \text{unitV}(u)$	$\left[\frac{40}{9}\ \ \frac{-40}{9}\ \ \frac{20}{9}\right]$

Circular Functions

VCAA 2010 Examination 2, Question 5

For $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the graphs of the two curves given by $y = 2\sec^2(x)$ and $y = 5|\tan(x)|$ intersect

A. only at the one point $(\arctan(2), 10)$

B. only at the two points $(\pm\arctan(2), 10)$

C. only at the one point $\left(\arctan\left(\frac{1}{2}\right), \frac{5}{2}\right)$

D. only at the two points $\left(\pm\arctan\left(\frac{1}{2}\right), \frac{5}{2}\right)$

E. at the two points $\left(\pm\arctan\left(\frac{1}{2}\right), \frac{5}{2}\right)$, as well as at the two points $(\pm\arctan(2), 10)$

The only way to get the exact answers is to do this question by hand and it can be quite challenging for most students. A CAS calculator can be used to sketch both graphs within the given domain and then count the number of points of intersection which is probably a preferable method. Alternatively, using the *Solve* command to find the points of intersection within the restricted domain as shown on the screenshot below is also quite effective but moving across the screen to count all points of intersection can be frustrating. It would be much better to obtain the approximate answers from CAS as the exact answers can be quite daunting!

$$\text{solve}\left(\begin{cases} y=2\cdot(\sec(x))^2 \\ y=5\cdot|\tan(x)| \end{cases}, x, y\right) \Big| \frac{-\pi}{2} < x < \frac{\pi}{2}$$

$x=-1.10714871779$ and $y=10.$ or $x=-0.46364$▸

Complex Numbers

VCAA 2014 Examination 2, Question 5

If the complex number z has modulus $2\sqrt{2}$ and argument $\frac{3\pi}{4}$ then z^2 is equal to

A. $-8i$ B. $4i$ C. $-2\sqrt{2}i$ D. $2\sqrt{2}i$ E. $-4i$

Students with strong algebra skills may prefer to do this question by hand. However, it may be quicker to use CAS to find z^2 in one step as shown below.

$$\left(2\cdot\sqrt{2}\cdot e^{\frac{3\cdot\pi}{4}\cdot i}\right)^2 \qquad -8\cdot i$$

VCAA 2014 Examination 2, Question 7

The sum of the roots of $z^3 - 5z^2 + 11z - 7 = 0$, where $z \in C$, is

A. $1+2\sqrt{3}i$

B. $5i$

C. $4-2\sqrt{3}i$

D. $2\sqrt{3}i$

E. 5

If students know that, in general, the sum of the roots of $az^3 + bz^2 + cz + d$ is $\frac{-b}{a}$, then the answer to this question can basically be written down from mental computation, that is $-\ \frac{-5}{1}$, hence E is the correct response. A CAS device in this question may not be a much better approach but can be quite useful.

The commands *cZeros* and *sum* are used in *Ti-Nspire CAS* as shown below.

$$\text{cZeros}\left(z^3-5\cdot z^2+11\cdot z-7,z\right)$$
$$\left\{2-\sqrt{3}\cdot i,2+\sqrt{3}\cdot i,1\right\}$$
$$\text{sum}\left(\left\{2-\sqrt{3}\cdot i,2+\sqrt{3}\cdot i,1\right\}\right) \qquad 5$$

While for *ClassPad*, the commands *solve* and *sum* are used.

$$\text{solve}(z^3-5z^2+11z-7=0,z)$$

$$\{z=1, z=2-\sqrt{3}\cdot i, z=2+\sqrt{3}\cdot i\}$$

$$\text{sum}(\{1, 2-\sqrt{3}\cdot i, 2+\sqrt{3}\cdot i\})$$

$$5$$

VCAA 2014 Examination 2, Question 8

The principal argument of $\frac{-3\sqrt{2}-i\sqrt{6}}{2+2i}$ is

A. $-\frac{13\pi}{12}$ B. $\frac{7\pi}{12}$ C. $\frac{11\pi}{12}$ D. $\frac{13\pi}{12}$ E. $-\frac{11\pi}{12}$

In this case, a CAS device is proven to more effective than by-hand method as the answer can be found in one step using the *angle* or *arg* command.

$$\text{angle}\left(\frac{-3\cdot\sqrt{2}-i\cdot\sqrt{6}}{2+2\cdot i}\right) \qquad \frac{11\cdot\pi}{12}$$

VCAA 2014 Examination 2, Question 9

The circle $|z-3-2i|=2$ is intersected exactly twice by the line given by

A. $|z-i|=|z+1|$

B. $|z-3-2i|=|z-5|$

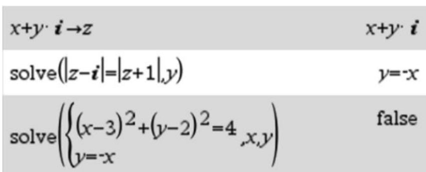
C. $|z-3-2i|=|z-10i|$

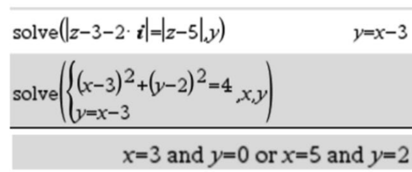
D. $\text{Im}(z)=0$

E. $\text{Re}(z)=5$

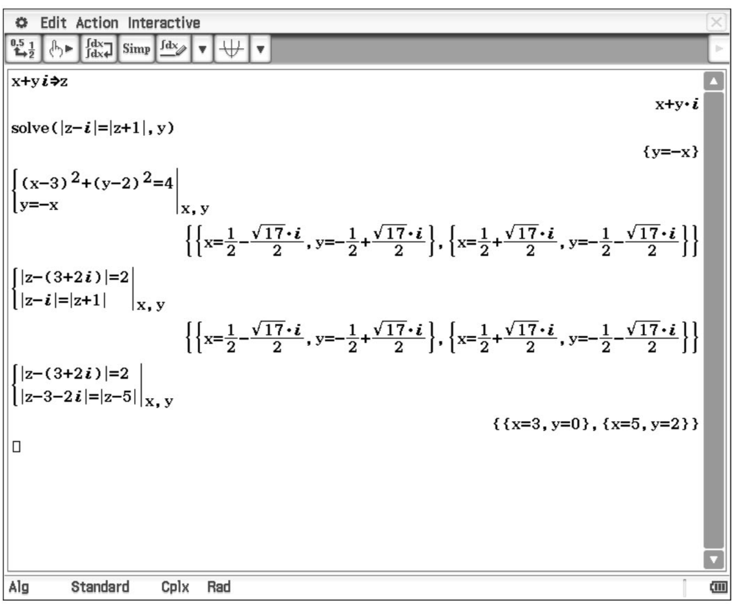
This question can be done using several different approaches. Students can immediately eliminate options D and E by drawing a quick circle with a centre of (3, 2) and radius of 2. Im(z) = 0 would give a line y = 0 while Re(z) = 5 would give a line x = 5 and both of these lines are tangential to the circle. However, doing other options by-hand can be quite tiresome and hence CAS may be useful in this question as shown below by both *Ti-Nspire CAS* and *ClassPad* devices, even it does take a number of steps to find the correct answer.

The command *Solve* was used in *Ti-Nspire CAS* as shown below.





And the solutions demonstrated by *ClassPad* is shown below.



Conclusion

The effective and efficient operation of CAS is certainly not easy, especially if the knowledge to provide the necessary understanding is not present. Often, CAS can be overlooked and disregarded, and rather seen as a helpful tool, it is perceived as a burden to some students. However, with hard work and practice, students would undoubtedly benefit in the use of their device in order to excel. Technology is universal, growing and thriving in an environment that will continue to embrace it. It is therefore imperative for us to keep up with this dynamic. We make the most of what has been given to us, letting technology be an instrument to greater success where possible.

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HREF: Selected multiple-choice questions from VCAA 2010 – 2014 Mathematics Examinations © VCAA; reproduced by permission. http://www.vcaa.vic.edu.au/Pages/vce/studies/mathematics/specialist/exams.aspx

Technologies

TI-Nspire™ CX CAS Teacher Software
ClassPad Manager for ClassPad II

UNIVERSITY MATHEMATICS AS SAC TOPICS

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This paper proposes a framework for evaluating the suitability of a university mathematics topic as the topic for a SAC. Some details of a Specialist Mathematics SAC, based on the approximation of functions by Taylor polynomials, will illustrate some of the lessons learned about the SAC-writing process.

Introduction

The motivation for developing the following framework came from the oft-repeated question, "When will we ever use this?" In the context of a Specialist Mathematics class, especially one in which three-quarters of students wished to study engineering in university, one answer is, "In two years you will use this to solve problems like ..." This throw-away answer prompted a review of several university mathematics textbooks for hard-science majors, in search of topics which could be adapted as SAC topics. The evaluative framework evolved during that process.

This paper is divided into three main sections. Firstly, a set of guiding questions is provided, together with some thoughts on using them to assess the suitability of a university topic for inclusion in a SAC. Secondly, some specific components from a SAC, based on the representation of functions by Taylor series, are presented with some discussion of the decisions for inclusion and/or exclusion of various questions and techniques. The student feedback came from unsolicited comments in the days following the SAC. The third section contains some reflection on the overall process.

The SAC in its final form requires approximately four hours of reading and writing time, and was administered under Exam 2 conditions (reference book with CAS calculator). The students were informed in the exam instructions that questions were to be answered algebraically unless specifically told to use technology.

Assessing a Topic

The following questions have helped in assessing the suitability of a university mathematics topic for use as the foundation of a SAC. Some thoughts are shared on how Taylor polynomials measure up against these questions.

a) Is the algebra accessible?

Some topics may have extremely convoluted algebra, for example, inverse Laplace transforms. Others, such as Taylor polynomials, have relatively straightforward algebra, involving differentiation and evaluations. However, this is a mechanical consideration. There may be a conceptual twist which keeps the topic out of reach.

b) Is there a conceptual twist?

Euler's method for solving Diophantine equations requires that a student repeatedly take a rational term from an equation and declare it to be an integer before continuing with the next step, a process which was trialled in a mathematics enrichment session, with limited success. If an expression looked like a fraction, many were incapable of treating is as an integer. By comparison, changing a tangent line to a "tangent parabola" was accepted readily. But even if the topic is easily understood, the topic might still require a theoretic framework which is beyond the students.

c) Can the mathematical rigor of the topic be sidestepped for the purposes of the SAC?

While it is true that a "proper" treatment of Taylor polynomials must address technical issues such as the interval of convergence, many of these concerns have no impact on a naïve treatment of the sorts of functions studied in a high school mathematics class. As will be noted, some substantial results may be explored based on a single assumption, namely, that a function can be approximated by a polynomial.

d) Is it interesting?

This is the "What if…" question. In the present SAC, the underlying question is, "What if tangent lines could be curved instead of straight?" This lead to a sense of discovery during the SAC, a much more powerful experience for students than merely calculating correct numerical answers. To conclude with a technique for calculating π, as in this SAC, or the gravitational constant g (as in a SAC on differential equations), can also give a rabbit-out-of-the-hat moment that can leave the teacher feeling as good as the students.

e) How much reading will be required?

Exploring a university topic may require a substantial amount of explanation. A lot of reading may have an unexpected time impact. This was partially side-stepped in the present

SAC by providing a take-home pre-SAC worksheet which explored the "tangent parabola" without the time constraint of a classroom setting.

This list of questions is not intended to be a definitive framework for evaluating topics for inclusion in SACs. Rather, it is intended to structure one's thinking about the evaluation. As such, it is merely a beginning.

Taylor Polynomials, Pre-SAC Worksheet.

A pre-SAC worksheet presented some revision and extension of the material MME1/2. A fully-worked example was provided, using the cubic function $f(x)=x^3-3x^2-2x+4$, in which the first three Taylor polynomials (the y-intercept treated as a constant function, the tangent line, and the "tangent parabola") were graphed on one set of axes. To avoid introducing the introducing the interval of convergence, the concept of an error function was used to demonstrate that "good" approximations could be made over larger intervals using higher-degree Taylor polynomials. The students then used the worked example as a template to determine the first three Taylor polynomials for a quartic.

The Pre-SAC Worksheet was created to address two main issues associated with Taylor polynomials.

Firstly, a major conceptual leap needed to be addressed, namely, what is meant by "approximation" when discussing the behaviour of a function. When tangent lines as a localised linear approximation are introduced in the Year 11 curriculum, we seldom refer to the constant function $y=0$ as an approximation of the function $y=\sin x$. Yet it is arguably a "good" approximation, as we know that it never deviates from the function's value by more than ± 1, the mathematical equivalent of "A stopped clock is correct twice a day." The author felt that students would benefit by considering this perspective on approximation, together with the attendant calculations, rather than dealing with any of the formalism normally associated with "convergence."

Secondly, the algorithm for building the Taylor polynomials is sufficiently different from the "tangent line recipe" of Year 11 that the author desired that students see the procedure in action. More specifically, this procedure makes use of an assumption that "the Taylor series representation of a function must have all its derivatives equal to the corresponding derivatives of the function," which was explored as an iterative process to improve the approximating power of the polynomials, rather than being stated as a precondition of a series.

The worksheet was made available to the students on the Friday before they sat Part 1, to be completed over the weekend. This work was included as an assessment component, albeit for merely ten percent of the aggregate SAC mark. The students signed a declaration that their submission was substantially their own work.

Student feedback included the following:

a) The fully-worked example was appreciated by most students.
b) A few students felt that the worksheet was too long for the benefit derived from it.

Taylor Polynomials, and Approximate Integration, Part 1.

The body of the SAC reviewed the usual formula for the sum of a geometric sequence, i.e. $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$. Using partial sums of this formula as Taylor polynomials showed that the topic is not as foreign as it first appears. The substitution of $-x$ to "derive" a different set of Taylor polynomials was intended to demonstrate an additional level of usefulness of the transformation-of-function rules. The anti-derivative of the formula provides the Taylor polynomials of a logarithmic function very tidily.

A "wrong answer" example was provided involving the evaluation of a Taylor polynomial outside of its interval of convergence to raise the students' awareness that this is not a fool-proof technique. It was hoped that they might connect the problem with the discontinuity in the function; none of them did. In future, the author would be more careful about using exceptions as examples.

A fair amount of redundancy was incorporated in this section of the SAC, namely, having the students generate the Taylor polynomials of sin x and cos x and e^x. This was done so that Part 1 could run in a double period, with the Taylor polynomials being used immediately to compare their behaviour with the derivatives and anti-derivatives of the functions themselves. In retrospect, this might have placed too great an emphasis on simpler skills, and not enough on more challenging techniques specific to Specialist Mathematics. However, this also gave the students a comfort level and familiarity with the topic that they would not have obtained through a single exercise.

Part 1 was worth forty percent of the aggregate SAC mark.

Student feedback on this part of the exam was non-committal.

Taylor Polynomials, and Approximate Integration, Part 2

The second half of the SAC included an additional resource, namely, a sheet of several Taylor polynomials for the most common functions being explored. This allowed for a ready comparison of the derivatives and anti-derivatives of these polynomials. It also allowed an easy comparison of some definite integrals of the functions with the definite integrals of their Taylor polynomials, i.e. $\int_0^1 e^x\, dx$ with $\int_0^1 (1+x^2)\, dx$.

Some techniques specific to Specialist Mathematics were now explored. For example, a common exam direction such as, "Take the derivative of $f(x)=xe^x$, and use to evaluate the definite integral $\int_0^1 xe^x\,dx$" now has the additional context of comparing with the Taylor polynomial of xe^x. The complexity of these questions can be increased to explore a definite integral which would normally require limits, such as, "If $f(x)$ is defined by the rule

$$f(x) = \begin{cases} \frac{1}{x} \times \sin x \ , & x \neq 0 \\ 1 & x = 0 \end{cases}$$

then approximate the definite integral $\int_0^1 f(x)dx$ by replacing the integrand by its second-degree Taylor polynomial."

Questions involving trigonometric identities may also be included. For example, a definite integral of $f(x)=2\times\sin x\times\cos x$ may be evaluated directly using either a double-angle identity or an appropriate u-substitution, followed by an approximation using Taylor polynomials.

No justification was provided for approximating definite integrals of the form $\int_b^a x\times f(x)\ dx$ by replacing the function by a Taylor polynomial, or for multiplying two Taylor polynomials together. No students questioned this "oversight."

This section of the SAC concluded by "calculating" e and π. The CAS calculator employed for this SAC (the TI-*n*spire) includes dedicated menu items for generating and evaluating Taylor polynomials. These were used to produce and evauluate 100th-degree Taylor polynomials for $f(x)=e^x$ and $f(x)=4\ ^{\tan^{-1}} x$ to yield decimal approximation to e and π.

Part 2 included the specific algebraic techniques learned in Specialist Mathematics, and were paired with Taylor polynomial approximations. Some error calculations were included to reinforce the students' confidence that Taylor polynomials behave as intended. Graphing a function together with several of its Taylor polynomials on the same set of axes may achieve the same end more simply. They also provided a self-check mechanism in the questions using specific integration techniques. No instances of students catching mistakes via this self-check were noted.

A certain sleight of hand occurred in this half of the SAC. The paper is purportedly about approximate integration, yet the techniques being developed also get turned to calculating individual values of transcendental functions. No students complained. The final section took advantage of the TI *n*spire's ability to perform a number of Taylor polynomial operations. The author has not investigated the ability of other calculators in this respect.

Part 2 was worth fifty percent of the aggregate SAC mark.

Student feedback included the following:

a) A few students commented that the SAC would have been easier if the Ancillary Resource were introduced earlier in the SAC. This will be considered further in the Concluding Thoughts.
b) The question, "Is this what mathematicians do?" was immensely satisfying.
c) Most students expressed amazement that they were able to calculate π so easily, but some of them then pointed out that the π button was easier still.

Concluding Thoughts

This SAC was supposed to eliminate the question "When will we ever use this?" While not successful in this respect, the SAC was well-received by the students. Calculating π was its own reward, even for students who found the work challenging.

One student opined that the paper required too much reading, which would disadvantage slower readers. This criticism added the fifth question to my guiding framework, complete with an admission that this should receive greater consideration in the drafting stage.

The scope of the SAC may be expanded to provide different emphases. For example, no graphing was required, as this was not considered an essential component within the context of Taylor polynomials as a computational device. The assessment across the year included graphing in other SACs. However, as stated earlier, graphing could be used to provide graphical evidence of convergence. The SAC could also be tailored to place greater emphasis on trigonometric identities. Doing so would require some careful consideration of the amount of work needed to generate Taylor polynomials.

As mentioned earlier, one student suggested that the Ancillary Resource should have been provided earlier in the SAC. A future incarnation of this SAC might be broken into hour-long segments, with an abbreviated Part 1 using graphing to accomplish the convergence exploration, and ending with the generation of the Taylor polynomials for $\sin x$ and e^x. Part 2 could commence with the Ancillary Resource for all the approximate integration. This would also allow greater emphasis to be placed on the material using new integration techniques.

One final thought. Three past students contacted the author, with news that they were acing the section on Taylor series.

RE-EXAMINING MIDDLE SCHOOL MATHEMATICS – LEARNING AND ASSESSMENT

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"Nothing can be so dampening on learning by middle school students as narrowly-construed assessment that serves only to reinforce a sense of failure and diminish self-esteem." (Pendergast & Bahr, 2005). Despite a substantial body of research and articles on assessment for, and of, learning in secondary mathematics, time-poor teachers may still be reliant on textbook bank tests generated by other staff and not necessarily used with discernment. The challenge for all mathematics teachers is to assess learning of more than just facts in a way that builds the students' understanding and confidence and includes the students themselves on their learning journey.

Re-Examining?

There is no doubt that teaching middle school mathematics can be challenging. But is the challenge just for the teacher, or for the student? The Australian Curriculum has provided a focus for curriculum content but the sheer volume of topics and content to be covered in approximately 250 minutes a week can be daunting to staff and students alike. Most secondary mathematics teachers have multiple classes, possibly in middle and senior school/VCE and may be teaching outside their main methods.. The range of experience and qualifications of teachers varies greatly within and between schools (Thomson & Fleming, 2004). And then there is the diversity of the students in the classroom. From national testing such as NAPLAN and other diagnostic testing such as PAT testing and OnDemand, teachers will observe that it is not unusual to have a Year 8 class, for example, with students that have mathematical abilities that range from a Grade 3 standard to a Year 10 standard. Siemon, Virgona and Corneille (2001) reported that within year levels "teachers can and should expect a range of up to 7 school years in numeracy-related performance" This poses a challenge for teachers to plan, deliver and assess learning that will meet the needs of all students in the classroom.

Recent years have seen a resurgence on the value of feedback given to students with (Hattie, 2012) highlighting the variability in feedback given by teachers and gives evidenced guidance on providing effective feedback that will enhance student learning. Many of the current trends have already been considered and, as one teacher said recently "done to death". However, is it time to stop and quietly reflect upon the current status of planning, learning and assessment in our own classrooms?

Teachers have a mighty toolkit of resources including textbooks, state government websites, professional organizations such as the Mathematical Association of Victoria and specialist websites such as the SAM Middle years modules (Supporting Australian Mathematics, 2015) to name a few. However, despite this plethora of information and support, teachers in secondary schools are still resorting to cutting and pasting from textbook generated tests or tests from previous years to assess understanding and application of mathematical skills, concepts and processes. It seems likely that most teachers would like to develop rich, problem-solving tasks or tests that cater to their particular cohort; but there is a lack of time and opportunity, and occasionally a lack of expertise or confidence in writing mathematical assessments. No time to start from scratch, to write tasks or questions that cover the seven-year numeracy gap, no time to write tasks that are relevant to the students' lives, that assess knowledge of facts, of concepts, of processes, tasks that provide opportunities for critical analysis or problem-solving and yet can be completed by students in less than one hour. A colleague recently commented that he spent seven hours working on developing one assessment task (in his own time) only to have it rejected by his peers because, although an excellent task, it would take too long to mark! There are still some hurdles in the development of quality assessment tasks to be faced, and overcome, in many schools.

With so many considerations to juggle, how do teachers make decisions about curriculum and assessment? Theoretically teachers are making decisions based on a wider perspective of curriculum planning (for example a backwards by design approach); ensuring that there is consistency of teaching, learning and assessment and, perhaps more importantly, consideration of the particular student cohort they are teaching. This may not be the case in reality.

In the TIMSS 2002 report (Thomson & Fleming, 2004), the authors report on the analysis which proposes that self-confidence in learning mathematics has the strongest association with mathematics achievement for Year 8 Australian students. What can Australian teachers change or implement, for their own classes, to bring about self-confidence in their students? It is unlikely that there is a one-size-fits-all solution to this question but it is imperative that every teacher asks this question and continues to be inspired to search for the answers. John Hattie ((Hattie, 2012) sees teachers as change agents, responsible for enhancing student learning.

The purpose of this article is to encourage all middle school mathematics teachers to ask the questions about pedagogy, learning, feedback and assessment; to look at their own expertise, their own practices, their professional teams and their students with a new perspective and a renewed inspiration and challenge to have all their students find their self-confidence in learning mathematics.

Progression vs Achievement to Raise Student Self-Confidence

In Victoria, mathematics curriculum is part of AusVELS. This is the integration of the relatively new Australian Curriculum subjects into the Victorian Essential Learning Standards. Although aligned to Year levels to some degree, AusVELS represent a *progression* of learning through the increasingly sophisticated development of mathematical knowledge, skills and numeracy capabilities during years of schooling. The important focus is in *progression*, recognizing that students do not start each year at a consistent level and may progress at varying rates in different content and proficiency strands. This focus is also reinforced in the development of the Mathematics Developmental Continuum (Siemon, Virgona, & Corneille, 2001)

Some national testing explores the progression and values growth rather than achievement of a particular level (within bands) however many schools operate an achievement-based reporting structure – one that can lead to a sense of failure for some students and one which may not give all students the important self-confidence in learning mathematics. For example, a student might start Year 8 with a diagnostic test indicating the student is at a Grade 4 standard. During the year the student works hard and progresses to a Year 7 standard but may still be scoring significantly lower than peers on tasks and topic tests. Although significant achievement has been made, the student may just see the percentage result as the summation of their efforts – still a fail. How might this mindset be changed?

One way to ensure students see their own growth is to provide feedback and processes that make the learning explicit to each student. This requires the student to have an understanding for each topic of what knowledge and skills they start and finish with.

This can be quite easily achieved by a combination of pre-testing, formative 'testing' and summative testing. It may appear like a lot of testing, but is not as onerous as it seems.

Planning for Learning and Assessment

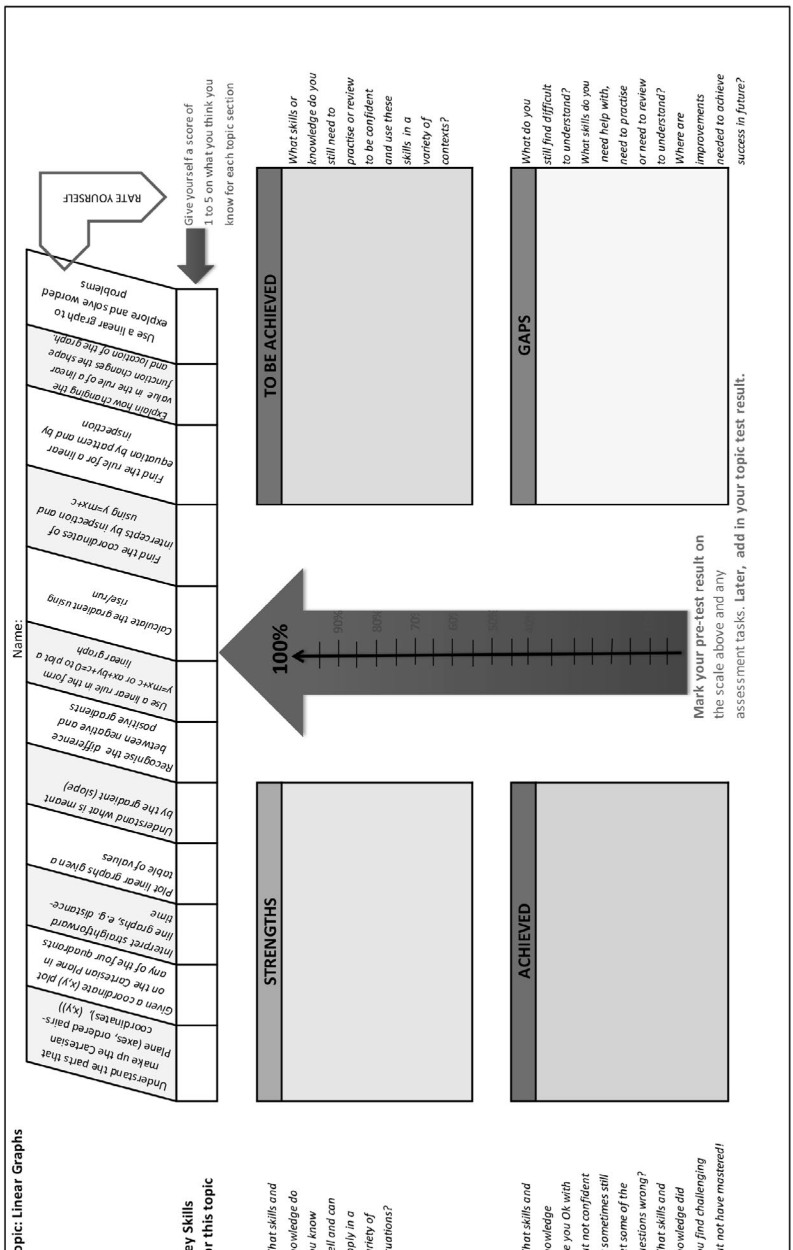
It will not be a surprise to teachers that curriculum, learning and assessment are not separate. However, what is surprising is that not all teachers see them as an integrated package and many continue to begin to develop assessments after the topic has started. Mike Ollerton (2003) suggests that planning a scheme of work has five key principles one of which is *creating a modular structure* (ideas and tasks connected by a common theme or concept). Using problem solving approaches, access and extension, providing opportunities to practise specific skills and pleasure are the other four. It is suggested that assessment of learning be a further part of writing a scheme of work and that the scheme can be allied with the student's individual learning by using a reflective learning grid (see *Table 1*). This should all be completed well ahead of delivery of a module/topic.

The approach outlined below originated from asking the question "how can I help students understand their own learning journey?" and was inspired by the Visible Learning Framework workshops. The reflective learning grid is based on the feedback grid developed within the Visible Learning Framework (with permission) (Hattie, 2012). The purpose of the grid is to make learning visible to the student, in particular to make the *progression of learning* visible.

The reflective learning grid works best as part of a total package or module for a topic. The following template is for a Linear Graphs topic and is a draft which is still being refined. It is important that students understand the language for the key skills and are able to match the skills against any pre-test, task or summative test.

Students are given a short pre-test as a diagnostic to show themselves (and the teacher) what they already know of the key knowledge and skills. They might reflect again later in the topic after a formative task or completion of work and then again after the summative task. Students identify their knowledge and skills and any areas for improvement a multiple points in the topic.

Table 1 *Example of Reflective Learning Grid for Linear Graphs (Draft)*



A Learning and Assessment Package

There are many ways that a scheme of work, module or topic package might be compiled. The following is a suggestion for one possible approach:

Create a Year Level Learning Team

- Allocate topics – ideally based on a passion of or expertise of the teacher
- The teacher produces the 'complete' package for the topic consisting of the AusVELS standards for the topic (with the curriculum standards for the topic of at least Years 5-10). The package should ideally consist of a pre-test or diagnostic task, a timeline, a formative task and a summative task as a minimum (catering for all abilities). Additional differentiated tasks, supporting material, multi-media resources, problem-solving activities, access and extension work, etc. are ideally provided to support students and teachers of the topic.
- Have the package fully completed well ahead of the planned delivery of the topic so that the learning team can have time to discuss the key skills and learning for the topic, teaching strategies, misconceptions, progression of the knowledge in later years, the assessments set, etc.)

Be Adaptive to the Needs of Your Students

- Look at the results of a pre-test or diagnostic task. (Note: it is important that any pre-test closely matches the final summative test, not in length but in how key skills are assessed. Question matching with changed numerical values is suggested)
- Ask your students to complete the reflective grid. Scan or take a copy of the grids so that you can find out the skills and knowledge your students already have and to have a back-up copy. Is there a way to reduce 'busy work' and build on existing learning? Does your teaching plan need to be revised to reflect your students' needs?
- What additional support do students with knowledge gaps or skills beyond the year level require?

Be Adaptive to Feedback from Tasks and Your Students

- Consider the results of any formative tasks – what feedback does this provide to the teacher about the student learning so far (or the teaching) and the confidence of the students? What changes need to be made to improve learning outcomes?
- Does the summative task need to be amended to take into account changing timelines or student ability levels? You may wish to use a textbook generated test as a base, but how can the questions be selected or amended to reflect the learning

in your classroom? Or how can questions be amended to be more relevant to your students? Does the summative task test the key skills that are covered in the topic? Is there significant repetitive testing of key skills?

Reflect on the Learning Progression of Students

- Ask students to reflect on their progression in knowledge and key skills and to identify where they have had success and where further improvement is still required. This is the most powerful part of the student reflection. Students are empowered by seeing progression rather than seeing just a percentage score on a summative task. Where students may have had very little knowledge in a pre-test, for example, they can see the achievement of skills over a short period of time – even if that progression was from a 5% result to a 45% result.
- Reflect on your students' achievements and your part in that process (as you would hopefully do anyway): what teaching strategies worked, what needs to be changed for next time, what assessment was effective and how could it be improved?
- Go back to the 'package' and write some comments and observations so that when you, or another teacher, need to use the topic/theme module again you can incorporate improvements and changes.

This process is empowering for both teachers and students as the learning is not just visible but obvious. Some students have a sense of achievement for the first time in completing the grids and you may be quite surprised by the change that this can trigger.

When attempting to implement this process, it may not be without some challenges along the way. Some teachers may refuse to take part as they believe they are already successful in their own approach, others will recoil at the volume of work seemingly required for a 'package' – without understanding that once the initial work is done, subsequent years are just spent improving and fine-tuning. It may be easier for teachers work across year levels on a single topic (this has an advantage of seeing how the knowledge and skills build towards senior mathematics).

Co-operative teamwork is the key to the success of this approach, but successfully preparing one or two topics a year may be all that is possible for an individual teacher– it is still a good start and positive feedback from students may inspire further developmental work.

Ideally aim to have a variety of formative tasks that provide problem-solving, analysis and can also highlight any gaps in key knowledge and skills. In his book "Developing Mathematical Thinking" (Katz, 2014), the author proposes a five dimension approach to mathematical teaching with a strong focus on inquiry-based learning and provides a detailed mathematics rubric to assist in developing tasks (along with a lot of other useful resources). Jonathan Katz provides a useful map for re-thinking learning in mathematics.

Share the workload and share ideas– if you are part of a network or regional group share tasks, curriculum documentation and teaching strategies. Speak up for a quality of learning experience and quality of assessment tasks. Have learning the impetus for writing assessments – not how long they take to mark!

A draft 'package' exemplar is available as a starting point and it is hoped that this material will provide further ideas for middle school mathematics' learning teams to discuss and improve upon. What questions will your team ask? What feedback will your students offer?

Further discussion and development of this topic would be beneficial and this article is offered as a springboard for further action.

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HANG ON. I'VE GOT IT!

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This paper reports on the findings of a literature review and a school-based peer research project prompted by the following scenario: a student asks a teacher for help, begins to explain their difficulty but then has an epiphany and announces that they have found the solution. What is it that enables students to solve problems simply by reading them out aloud and can we exploit this to improve student fluency?

Introduction

Throughout the years I have spent in mathematics classrooms I have intermittently experienced a phenomenon that, until recently, I paid little attention to. Every now and again a student would raise his or her hand to ask for my help and a dialogue similar to what follows would unfold.

TEACHER: Yes, Matthew. How can I help?

STUDENT: I'm stuck on question three.

TEACHER: What's the problem?

STUDENT: Well, it asks for the values of x for which the discriminant

... Hang on. I've got it! Don't worry. Thanks.

Although the dialogue above is apocryphal, it illustrates a relatively common occurrence: a student begins to explain what it is that they are stuck with, but as they think aloud, they have an epiphany and announce that they have solved the problem or at least now know how to proceed with a solution.

This paper is an attempt to determine the mechanism that enables students to solve problems simply by reading them out aloud and suggests ways in which this can be exploited to improve student fluency.

The theoretical underpinnings that follow in this paper come from a literature review I wrote for the fulfillment of a Negotiated Capstone Project in the Masters of Education course at the University of Melbourne (Schaffner, 2014). The review was far from conclusive. However, some tantalising clues came from a wide range of diverse perspectives, including sociocultural and information-processing learning theories, psycholinguistics and cognitive neuroscience. The practical research findings come from a peer project conducted at Penleigh and Essendon Grammar School (PEGS).

Thinking about Thinking

The information-processing theory of learning suggests that we learn by gathering, or encoding, information via our senses and storing it in our working and long-term memories (Banikowski & Mehring, 1999). Working memory is particularly important in the context of this paper because thinking out loud allows students to rehearse, a key process for the storage and future retrieval of memories (Schunk, 1986; Woolfolk & Margetts, 2010, p. 274). Indeed, studies of the production effect suggest that verbal rehearsal leads to better memory than silent rehearsal (Forrin, MacLeod, & Ozubko, 2012).

Thinking out loud is also important in the process of encoding, storage and retrieval because it helps students become aware of, evaluate and regulate their thinking (Holton & Clarke, 2006). In other words, thinking aloud mediates students' *metacognition* allowing them to improve their retrieval of information and therefore arrive at a solution or at least a pathway to a solution (Schunk, 1986).

It's on the Tip of My Tongue

If thinking aloud is a mediator for memory retrieval, perhaps my students are experiencing the mathematical equivalent of a tip-of-the-tongue lapse in memory, a phenomenon that has been studied in such depth that it is known by researchers simply by its acronym, TOT (Abrams, 2008; A. S. Brown, 2012; Schwartz & Brown, 2014). TOTs were first defined by Brown and McNeill (1966) as 'a state in which one cannot quite recall a familiar word but can recall words of similar form and meaning' (R. Brown & McNeill, 1966, p. 325). To understand how TOTs are caused, and resolved, we should first consider how speech is produced and how speech production is related to the cognitive processes controlling working memory.

Speech production is thought to begin with the formation of a plan of what to say, followed by a choice of words, which is then followed by a retrieval of the necessary sounds. These complex stages of conceptualisation, formulation and phonological encoding then direct the muscles of the mouth, tongue, lungs and diaphragm to produce speech (Abrams, 2008, p. 234; Harley, 2013, pp. 397-398). It is widely accepted that TOTs occur not because of a failure in conceptualisation or formulation, but in the phonological encoding of a word or idea (Abrams, 2008; A. S. Brown, 2012). This has interesting parallels to the earlier discussion about the importance of thinking aloud for memory since a key component of working memory maintenance is the phonological loop, a speech-based system of rehearsal (Harley, 2013, p. 472; Swanson, 2004, p. 649; Woolfolk & Margetts, 2010, p. 272).

Surprisingly, the effect of thinking aloud in resolving TOTs has been somewhat overlooked. Even Brown (2012) laments that 'given that the most influential theory on the etiology of TOTs involves a phonological system deficiency, comparisons of written versus oral presentation of the target word cue, and written versus oral responding, would seem relevant to better understand TOTs' (Brown, 2012, p. 203).

Suggestion 1: Reading Aloud

If the mechanism that allows my students to solve problems by reading them aloud is an alleviation of a 'phonological system deficiency' or a mediation of their ability to retrieve information from their memory, then the first suggestion for how to improve their fluency might be for them to simply continue to read problems out aloud. One method that began as a means for diagnosing reading errors, but became a very useful pedagogical tool, is Ann Newman's error analysis. This method uses the following prompts when students were solving problems

1. Please read the question to me. If you don't know a word, leave it out.
2. Tell me what the question is asking you to do.
3. Tell me how you are going to find the answer.
4. Show me what to do to get the answer. "Talk aloud" as you do it, so that I can understand how you are thinking.
5. Now, write down your answer to the question (White, 2009, p. 251).

Alternatively, students could use Explain Everything or similar mobile application software to screencast their solutions (Explain Everything, 2011-2015). Screencasting not only gives students the opportunity to explain their thinking out loud, it also makes their thinking 'visible', allowing teachers to give contextual and effective feedback (Richards, 2014).

Suggesting the use of screencasting merely to practice solving problems by thinking out loud, however, probably does not do justice to the potential of screencasting as a teaching and learning tool, one that deserves an entirely separate research project. In addition, as my literature review suggested, the relationship between spoken mathematics and problem solving is more complex than mere memory recall.

Thought Comes Into Existence Through Words

In his aphorism, 'thought is not merely expressed in words; it comes into existence through them' (Vygotsky, 1986, p. 218), Vygotsky set in train the sociocultural theory of learning. This theory differs markedly from the information-processing theory by suggesting that language and speech does more that merely mediate memory and thought, but actually *determines* thought. Indeed, Sfard et al (1998) go so far as suggesting that '*all* thinking is essentially discursive' (p. 50). More importantly, sociocultural theory stresses that this determination of thought does not happen in isolation, but 'requires intervention and support by others' (Alexander, 2006, p. 11). This 'support' is sometimes referred to as scaffolding, a concept that has now been expanded to include distinctions between, and within, 'expert' scaffolding, 'reciprocal' scaffolding (collaborative peers) and, indeed, self-scaffolding (Holton & Clarke, 2006). Self-scaffolding is of particular interest in this paper because Holton and Clarke (2006) contend that '*self*-scaffolding is essentially the same as metacognition' (Holton & Clarke, 2006, p. 138).

Suggestion 2: Self-scaffolding and Private Speech

Since thinking aloud has already been suggested as a mediator of metacognition, it must then follow that it would also mediate self-scaffolding. This idea, that 'the external dialogue of scaffolding becomes the inner dialogue of metacognition' (Holton & Clarke, 2006, p. 141), stresses the importance of the explicit instruction of metacognitive skills so that students can 'become better learners and better problem solvers in situations when they do not have expert assistance to hand' (Holton & Clarke, 2006, p. 142). Perhaps the explicit instruction of metacognitive skills, and placing importance on the development of private speech (Ostad, 2011, p. 10), is a way to improve the fluency of my students in secondary mathematics classes. The value of private speech is illustrated in Student A's experience discussed later in this paper.

Suggestion 3: Think Aloud Paired Problem Solving

It should be stressed, however, that sociocultural theories of learning emphasise that it is *social* interaction that promotes the internalisation of 'knowledge, ways of thinking and ways of doing' (Scott, 2009, p. 4) and the regulation of that thinking (metacognition). The role of spoken language in these social interactions is to mediate this internalisation. This suggests that *collaborative* problem solving involving discussion or thinking out loud is another way to improve student fluency.

In their investigation of collaborative problem solving using a Thinking Aloud Pair Problem Solving protocol, Goos and Galbraith (1996) report on a dialogue tantalisingly similar to the apocryphal one at the beginning of this paper. A student has a sudden flash of insight, triggered by a dialogue with a peer which the researchers identify as metacognitive (Goos & Galbraith, 1996). In another examination of students talking aloud during peer collaborations in mathematics, Kotsopoulos (2010) found that two of the functions of talking aloud were clarifying thinking and 'eliciting support from peers' (Kotsopoulos, 2010, p. 1065). These findings support the information-processing theory that thinking aloud is important for the regulation of cognition, but add that thinking aloud also mediates that cognition, particularly in social settings. In a school-based peer research, I also adopted a Think Aloud Paired Problem Solving (TAPPS) protocol in collaborative whiteboarding classrooms. Whiteboarding is a distinctive method of teaching and learning where students solve problems exclusively at large whiteboards that line the walls of a learning space. The results in terms of improvements in student fluency and resilience were significantly beneficial (Schaffner, Schaffner, & Seaton, 2015).

Peer Research Project

For my own peer project research I decided to use the mobile application software, Explain Everything (Explain Everything, 2011-2015) to capture my students' thinking and to see whether thinking out aloud made any difference to their ability to solve mathematical problems. I chose this application software because it not only records what the students say, but also the number, and duration, of pen strokes as they write their solutions.

In each case the students were presented with a problem similar to those they had recently experienced difficulty with. The 'difficulty' usually fell into one of two categories:

- being unable to perform appropriate algebraic manipulations
- making simple arithmetic or algebraic errors and not being able to recognise and correct those errors.

The students were asked to complete the problem as they normally would until they became stuck or had completed the problem. They were then asked to complete the problem again, but on this occasion to think aloud as they were solving the problem.

In the first example, Student A (Year 12 Mathematical Methods) was asked to attempt the problem shown in figure 1.

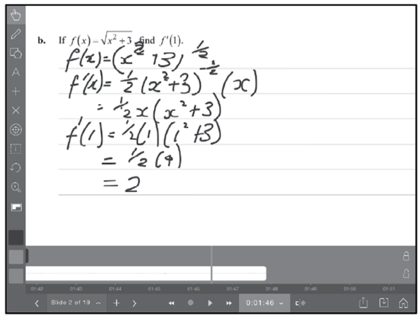

Figure 1.

Notice the small arithmetic mistake in writing the derivative of $x^2 + 3$ as x rather than $2x$. She would have lost a significant number of marks in an externally assessed Year 12 examination (it is likely that she would have scored zero out of three marks). The student was then invited to repeat the question, but this time to think aloud. Her solution is shown in figure 2.

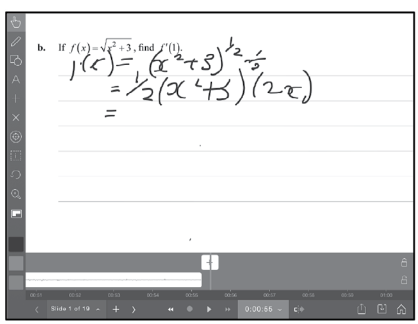

Figure 2.

> Student A: That's *f* of *x*. You rewrite it so that it is to the power of a half. That makes it easier. In the chain rule you put the power to the front. I think. Then you write what's in the brackets. And then you write the differentiation of what's in the bracket, which is 2*x*.
> [0:00:39] Not *x*.

At a time of [0:00:39] the student had the epiphany that the derivative was $2x$, not x, and went on to solve the problem successfully.

In the following example, Student A was also asked to attempt the problem shown in figure 3.

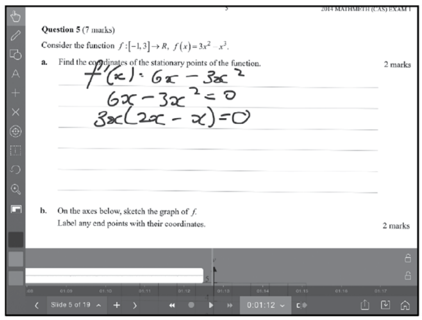

Figure 3.

Surprisingly, she stopped writing at [0:00:41] and announced at [0:01:09] that she was stuck. She was then invited to repeat the question, but this time asked by the teacher to think aloud. Her solution is shown in figure 4.

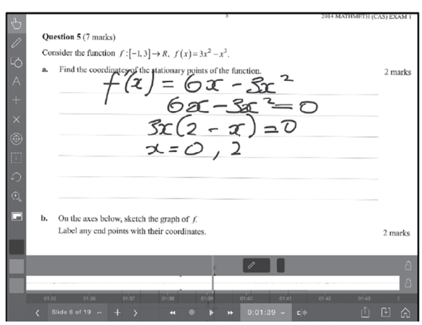

Figure 4.

> Student A: To find the stationary point, the derivative has to equal zero. Yes. So we do f dash x. So two times three is six x. Minus three x squared. This equals zero. Um, because there's a common factor we can divide by three x. So that then becomes 2 minus x.
> [0:00:55] Oh! Equals zero. So x equals zero or x equals 2.

Again, this student had an epiphany, exclaiming 'Oh!' at [0:00:55] and then continuing with her solution. Interestingly, she stopped at the point shown in figure 4 and declared that she had finished the question. The teacher then prompted her by saying, 'read the question aloud'. She began to read the question aloud and as soon as she read out the word 'coordinates', she had another epiphany and completed the question successfully as shown in figure 5.

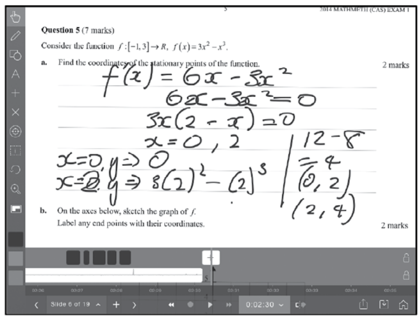

Figure 5.

Student A had a tendency in high-stakes, summative tasks to make the mistakes highlighted in the previous figures. When she thought out loud, however, she tended not to make these mistakes. In keeping with Holton and Clarke's (2006) and Ostad's (2011) suggestions, Student A began to use a form of private speech similar to the way she did when she thought out loud. Her scores on subsequent summative tasks improved and she remarked that she felt more confident about tackling routine algebraic manipulations.

Although thinking out loud gave Student A a metacognitive tool to mitigate the errors she tended to make, it is interesting to note the style of language she used. She suggested that she should '*put* the power to the front'. A more competent, or fluent, mathematician may have said, 'to differentiate we multiply the power by the coefficient and then subtract one from the power'. She also tended to use words like 'cancel', instead of division by the same number or pronumeral, and 'move over to the other side' or 'take to the other side', instead of 'add to both sides' or 'multiply by both sides'. Has this choice of language limited her mathematical fluency at a senior secondary level?

The next example highlights Student B's choice of mathematical language. Thinking aloud did not result in an epiphany for Student B, nor did it allow him to recognise and correct his mistakes. However, thinking aloud, with a particular choice of mathematical language, did change his reasoning and thereby revealed what I regard as a significant connection between language use and mathematical fluency.

The example reveals a common misconception that 'any term on the numerator can be cancelled with any term on the denominator' (Swedosh, 1996, p. 540). The problem, to simplify the expression $\frac{xa+xb}{x+xd}$, comes from Egodawatte's (2011) doctoral dissertation exploring secondary school students' misconceptions in algebra (Egodawatte, 2011, p. 123). The student on this occasion was asked to think aloud from the beginning. His result is shown in figure 6.

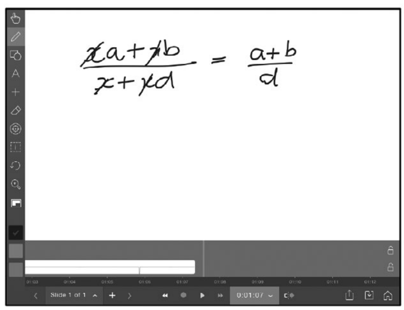

Figure 6.

> Student B: OK. I've got *xa* plus *xd* over *x* plus *xd*. So, I cancel the '*x*'s and that gives me *a* plus *b* over *d*. Pretty simple.

During a discussion with Student B he explained, incorrectly, that cancelling was possible simply because the numerator and denominator were the same. His conception of cancelling relied more on simply 'getting rid of' pronumerals, rather than the explicit use of mathematical operations. On the basis of further discussion, Student B was invited to repeat the problem, but to avoid using the word 'cancel' and to use 'divide by' instead. His result is shown in figure 7.

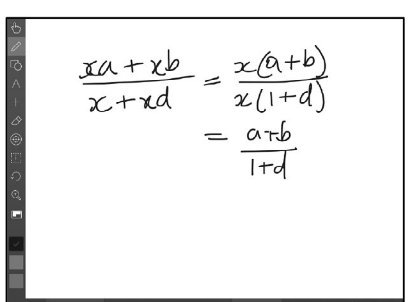

Figure 7.

> Student B: So, I can't use cancel. So, how am I supposed to get rid of the '*x*'s? Oh, hang on. Maybe I can take out a common factor. I can take out *x* at the top and the bottom. But now I can't cancel again. But I can do 'divide by', right? *x* divided by *x* is 1, so I can leave that out and I get *a* plus *b* over 1 plus *d*. Oh, now I get it.

The examples reported in this paper are preliminary and hardly conclusive. In particular, it isn't entirely clear whether simply repeating questions, with or without thinking aloud, is enough of an enabling prompt to allow students to identify their errors. However, these examples do invite further research in at least two areas. The first is that it seems that thinking aloud does indeed facilitate some sort of metacognitive self-scaffolding that allows students to

- retrieve their tip-of-the-tongue memory lapses
- recognise and correct algebraic and arithmetic mistakes
- read and answer questions correctly

The second is that, despite my literature review finding some convincing neuropsychological evidence to refute linguistic determinism in mathematical thinking (Schaffner, 2014, p. 5), the language that students in my study used did seem to influence the way they solved mathematical problems. In particular, algebraic misconceptions or misunderstandings appear to be ***reinforced*** by inappropriate, unsophisticated language, but ***alleviated*** by appropriate, sophisticated language.

The refutation of linguistic determinism and the notion that 'conceptual development drives the acquisition of counting words rather than the other way around' (Butterworth, Reeve, Reynolds, & Lloyd, 2008, p. 13182) is not the only counterpoint to my suggestion of the importance of thinking aloud as a means of improving mathematical fluency. The literature review also revealed that people who found speech difficult or even impossible, such as the deaf or those who suffer aphasia, are not precluded from solving even complex mathematical problems (Schaffner, 2014, p. 6). Similarly, the review revealed interesting cultural considerations. Despite having no opportunities to speak about mathematics, and very few opportunities to speak at all, students in many countries perform quite well on international assessments. Indeed, Clarke, Xu and Wan (2013) went so far as to question contemporary theories of learning that advocate the promotion of spoken mathematics in the classroom on the grounds that these theories of learning themselves are culturally specific.

Perhaps these counterpoints are best summarised by Anna Sfard, who has written extensively on discourse analyses in mathematics classrooms, when she reveals her uncertainty about 'the mechanism of mathematical thinking which makes verbalisation of mathematical ideas beneficial' and questions contemporary theories by suggesting that 'in spite of the long-standing interest in the complex relationship between thought and language, we still do not know enough to turn a *belief* (in the power of mathematical talk) into a theoretically sound *assertion*' (Sfard, Nesher, Streefland, Cobb, & Mason, 1998, p. 41) .

Closing Thoughts

In their attempts to reconcile sociocultural and neurological approaches in their research, Clarke and Hollingsworth (2013) suggest that 'the tensions and apparent contradictions that appear to pose the greatest challenge for useful interpretation and instructional advocacy also provide the greatest insight' (p. 38). Hopefully, the tensions in the competing and contrasting examples presented in this paper have also provided an insight into the beneficial role that thinking aloud can have in solving mathematical problems.

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WHAT TO DO WHEN KIDS ALREADY KNOW EVERYTHING - SERIOUS MATHS EXTENSION

Tierney Kennedy

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Sometimes when teaching, we come across a student who blows our mind. It seems like they either already know everything that we are trying to teach, or are so fast at making connections that we can't keep up. In this situation it can be really hard to know what to, especially if the student seems to understand more maths than we are comfortable with. This article provides four simple keys for extending student thinking and skills, without just making the numbers harder.

Key 1: Make the questions weirder rather than making the numbers bigger

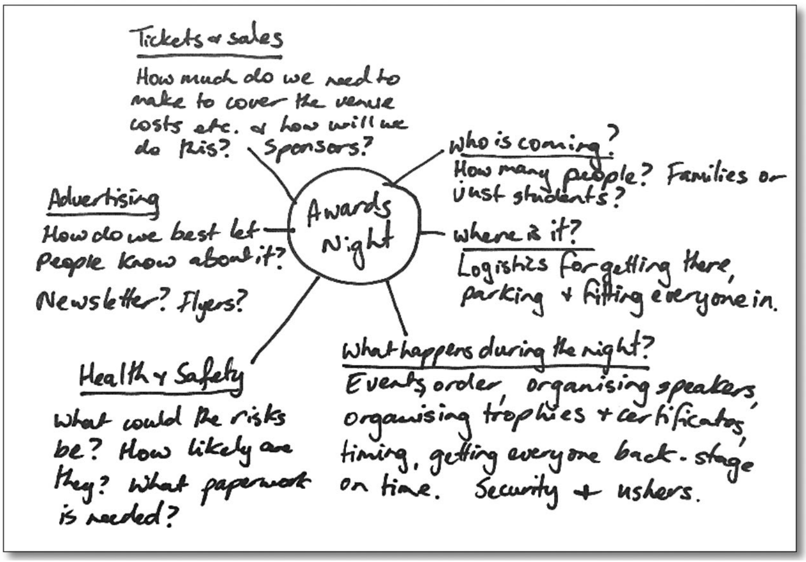
Students with high mathematical understanding tend to find routine-style questions very easy and often end up bored and frustrated. At times we are tempted to give "fast finishers" more and more routine questions to fill in time, effectively punishing them for having higher understanding than we were prepared for. In contrast to this approach, the Australian Curriculum recommends that students "make connections between related concepts and progressively apply the familiar to develop new ideas" (Australian curriculum website, 2015). To develop strong understanding in students we therefore need to consider how to use unfamiliar or unusual tasks to challenge their thinking.

One way to do this is to change the structure of routine questions rather than simply making the content harder. Here are a few ideas which can be easily prepared before-hand and adapted to any content area:

1. Think about using gaps, missing add-ends, or asking students to work backwards to a starting number rather than simply working forwards to an unknown answer. The picture below shows what happened when a teacher started with eight blocks and hid some.
   $\square + 3 = 8$ and $3 + \square = 8$ are both much harder for kids to solve than $3 + 5 = \square$
2. Give students the starting and ending numbers and ask them what happened to get that answer – get them to work out what operation/s were used.
3. Require students to adapt what they have already worked out to account for a change in circumstance or additional operations, for example: *What if I had also subtracted two to get the answer of eight? How would that change the number in the box?*
   How would your answer change if the three was a different number?
4. Add in an additional unknown using a pattern, for example: *What if we had three numbers that added to give eight, but two of the numbers were the same?*
5. Use a non-standard looking question, for example: present fractions in a 3D solid rather than an area model, use irregular rather than regular shapes, ask students to work out the time on an analogue clock without the minute hand.

Key 2: Make use of real-life events happening at your school

Consider what events you have coming up this term at school. Do you have an athletics carnival, concert, awards night, disco, swimming carnival, art show or excursion? These are fantastic sources of real-world mathematical thinking that are readily engaging for students. In order to make the best use of these situations, I find it is best to start out by thinking of what jobs/tasks need to be done to make the event successful rather than starting out by thinking about what mathematics is involved in the task. Check out the example below for organising an awards night:



Once you have teased out the task, choose one aspect and apply your "maths teacher" hat. What content areas best lend themselves to this job? What would be suitable for your students? What would they enjoy doing? For example: you might get students to organise the schedule for the night, including what time everyone needs to be backstage and ready to go on, when the lights/curtains/speakers/presentations happen etc. Another idea would be for students to work out the logistics and recommend a venue (how many people are we expecting and therefore how many seats/toilets/exits/drinks/tickets... are needed?), or work out how to get everyone from school to the venue for the practice run.

Key 3: Make them prove it

I find that often students with high mathematical understanding have difficulties explaining what they did to find the answer to a question. I expect that this is because they already knew the answer, or saw the solution so quickly that their thinking did not slow down enough to be able to identify the steps involved. Mathematical reasoning and communication are critical for these students to develop if they are to share their ideas, proofs or solutions with anyone beyond school. If we want students to explain their thinking, there are a number of strategies worth trying:

1. If students can do it in their heads, then the question is probably not hard enough. Make the maths harder so that students can't just see the answer by adding in enough different ideas, connections and steps that they need to write something down to keep track of everything that they are doing.
2. Instead of asking the student to explain what steps they went through, ask them to prove why a different answer is wrong. Personally, I've found that I get the best reasoning from students who are trying to prove that my answer is wrong!
3. Give the students a number of questions that all follow a similar structure and then ask them to explain the similarities and differences, as well as explaining how the questions are connected.

Key 4: Create an extension box

While it is necessary to include extension students in many of the normal activities that you do with your class, consider which lessons or parts of lessons they could use to do something more challenging. One great idea is to create a box in your classroom in which you place a challenging task for two to three students to work on together. This box can be accessed whenever you feel that it will be useful, for example: when students have finished their work, when you are explaining lower-level concepts, when the class is practicing skills that these students have already mastered. Each task should be written in the form of a student brief, with a simple description of what is required along with the key steps and/or requirements.

Here are a few simple examples from the *Back-to-Front Maths* website:

- You have been given a budget of $50. You need to plan a meal for ten people, make a shopping list, and work out the total cost for the meal. You will be provided with a recipe book and a catalogue from a local grocery store.
- Your school wants to apply for federal funding to build a new playground. As part of the grant application process the administration has decided to take suggestions from students on the design of the playground. Your task is to design a playground that will fit in the area of your school that your teacher indicates. You must include a map of your playground which shows a variety of shapes such as quadrilaterals and triangles on the ground. These can be garden beds, paths, or sections around

playground equipment. Your map must include the mapping conventions. A second map needs to contain details about the shapes within your playground (angles, sides). You must also hand in a 3D model of a climbing frame that you design.

- School planners and architects need to create models of rooms so that they can design spaces that work well. Your job is to use lego or other blocks to create a model of your classroom, including where the furniture is and where the door is. If you use lego, try to use 4 lego dots for each metre in length, but don't worry if it is a bit over or a bit under.

Whatever you do, remember that maths should be fun, challenging and should make kids think hard. Solving a new problem, adapting to a new situation, trying to find a pattern and modelling a real-life situation are a lot more engaging than answering routine questions. And that's true for all students, not just those at the top-end.

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