MATHS ROCKS

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Editors: Jill Vincent, Gail FitzSimons and Joanna Steinle

THE MATHEMATICAL ASSOCIATION OF VICTORIA

Maths Rocks

Editors

Jill Vincent, Gail FitzSimons and Joanna Steinle

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Should readers wish to contact any of the authors, full addresses are available from the Professional Officer, the Mathematical Association of Victoria.

FOREWORD

The 51st conference of the Mathematical Association of Victoria has the theme Maths Rocks — a theme that is intended to inspire teachers and to encourage them to pass on their enthusiasm for mathematics to their students. Several authors have used the theme in the titles of their papers. The theme is open to numerous interpretations, including having fun with mathematics, and this comes through in several contributions. Another interpretation is of teachers forging closer relations with their students, trying to optimise student engagement and understanding, making a difference in students' current lives and encouraging them to continue their studies in mathematics. Regardless of how authors have chosen to interpret the theme, all papers portray a sense of purpose in teaching mathematics.

As editors we have enjoyed reading the many submissions to the proceedings. The papers cover diverse interest areas, ranging from effective methods for teaching topics that students traditionally find difficult to enriching the mathematical experiences of able students through mathematics competitions. Although not all of us will have the opportunity to visit the Museum of Mathematics in New York, we are treated to some fascinating examples of the mathematically rich displays.

Papers submitted for Double blind review and Peer review have undergone a rigorous review process. Summary submissions have accommodated those presenters who wish to share key points related to their presentation in a shortened format. We hope that readers will find the papers enjoyable and valuable. We also thank the reviewers of the blind and peer reviewed papers for their willingness to review and for their carefully considered constructive comments. Their contribution to the continued success of the MAV conference proceedings is greatly appreciated.

We close by thanking the MAV conference staff for their professionalism and for the support provided to us in our role as editors.

Jill Vincent, Gail FitzSimons and Joanna Steinle, The University of Melbourne (Editors)

The Review Process for the Mathematical Association of Victoria 51st Annual Conference Proceedings

Papers were submitted for double-blind review, peer review or as summaries. The Editors received 11 full papers for the double blind review process, for which the identities of author and reviewer were concealed from each other. Details in the papers that identified the authors were removed to protect the review process from any potential bias, and the reviewers' reports were anonymous. Two reviewers reviewed each of the 11 blind review papers and if they had a differing outcome a third reviewer was required. Ten of the 11 papers were accepted for publication. In addition, we received 14 full papers for the peer review process, where the names of the authors were identified to reviewers; 11 were accepted for publication as peer-reviewed papers and two were accepted as summary papers. Eight papers submitted as summary papers were reviewed by a combination of external reviewers and the editorial team. Seven of these were accepted for publication (one as a peer reviewed paper).

In the Conference Proceedings, double-blind and peer reviewed papers are grouped together and arranged in alphabetical order of author names. Double-blind reviewed papers and peer reviewed papers are indicated by ** and * respectively following the paper title. Summary papers follow the double blind and peer reviewed papers.

Of the total of 33 papers received, 30 papers are published: ten double-blind reviewed papers, twelve peer reviewed papers and eight summary papers. Altogether, 21 reviewers assisted in the review process, all of whom provided thoughtful feedback and were outstanding in responding quickly to our invitations.

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**** Double blind reviewed paper**
*** Peer reviewed paper**

DEVELOPING ALGEBRAIC THINKING: PROVIDING NEW TOOLS TO UNDERSTAND MATHEMATICAL RELATIONSHIPS *

George Booker

Educational Consultant: george.booker@optusnet.com.au

Problem tasks can be used as a basis for developing algebraic thinking through the use of materials that allow insight into the underlying algebraic ideas, thus providing a bridge to more formal algebra that will be developed later. An approach via problem solving reflects both recent research (Lins, 1992; Radford & Puig, 2007; Windsor, 2010) and the way in which algebra developed historically (Katz, 2007). As solutions are obtained and described, generalisations can be formed among related problems, and ways of thinking developed that focus on the relationships within problems.

Algebraic Thinking

Notions of algebraic thinking have been discussed for many years as a forerunner of formal algebra as well as a guide to building fundamental algebraic processes. The essence of algebraic thinking appears with the move from particular numbers and measures towards relations among numbers and measures (Carraher, Schliemann, Brizuela, & Earnest, 2006). In this way, mathematical relationships rather than mathematical objects become the objects of study with generalising, inverting and reversing operations, treating computational processes in general ways and reasoning about patterns the means to build this way of thinking (Lee, 2003; Sfard, 1995). Thus, algebraic thinking can be summarised as a conception that recognises general mathematical relationships and expresses these in increasingly sophisticated ways, from seeing patterns, to describing them with words or diagrams, before leading to the use of symbols that can express generalities concisely and carry meaning independently of the activity with which they were established.

> Algebraic thinking … should go beyond mastery of arithmetic and computational fluency to attend to the deeper underlying structure of mathematics. [This] requires the development of particular ways of thinking including analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting. That is, early algebra develops not only new tools to understand mathematical relationships, but also new habits of mind. (Cai & Knuth, 2011, p. ix)

Building an Ability to Generalise

The sequence of problems shown below introduces the use of additive thinking to represent and coordinate relationships within problems. The problems highlight the use that can be made of coloured counters, building on the earlier use of materials to develop number concepts and processes. In particular, the counters can be used to reveal relationships within the problems that lead to solutions (see Windsor & Norton, 2011, 2012). These problems can also be used to show the value of tables and graphs to reveal thinking, but students often find these difficult to set up until they determine the relationships needed to provide solutions. An initial use of counters can assist in understanding what needs to be displayed in the tables or graphs.

1. Caitlin's sister is 7 years older than her. When they added their ages together they found that they had lived a total of 25 years. How old is Caitlin?
2. When Roger and Susan went to the airport they had to weigh their backpacks at the Check in counter. The two backpacks weighed 24 kilograms. They noticed that one backpack weighed 6 kg more than the other. How much does each backpack weigh?
3. When Emma and her younger brother Damien went to buy birthday presents for their father, they found that Emma spent $35.00 more than Damien while together they spent a total of $95.00.
   How much did each spend?

Building Relational Thinking

Relationships within a problem can also be represented with coloured counters that can then be manipulated in order to give rise to a solution. However, this thinking needs to be developed through a careful sequence of problems that builds an understanding of equivalent expressions in order to find the form that gives a solution.

1. Brett and Peter got a job to paint the outside of a small kennel using three colours. When Brett checked the prices for the paint and brushes, he found that 5 paintbrushes and 3 cans of paint cost $55.60. Peter also checked the prices and he found that 3 paintbrushes and the 3 cans of paint cost $49.80. What is the cost of a paintbrush? How much does a can of paint cost?
2. When at the surf shop, 5 T-shirts and 3 caps cost $153.00 while 2 T-shirts and 3 caps sell for $81.00.
   What is the cost of a cap? How much does one T-shirt cost?
3. At the newsagent, 1 comic and 1 jigsaw puzzle cost $45.00 while 2 comics and 3 jigsaw puzzles cost $120.00.
   What is the cost of a comic? How much does a jigsaw puzzle cost?
4. At the market, Karen bought some potatoes, onions and mushrooms.
   1 sack of onions, 1 bag of potatoes and 1 box of mushrooms weighed 17kg.
   2 boxes of mushrooms and 2 bags of potatoes weighed 14kg.
   2 sacks of onions and 1 bag of potatoes weighed 22kg.
   What is the weight of each item?

Building to Symbolic Representations

This use of coloured counters both aligns with and extends these suggestions to provide a concrete introduction to algebraic thinking that can underpin any introduction to symbols to represent the generalisations that have been discovered.

Figure 1. Try several numbers. What happens?

The use of symbolic statements, though, is not readily accepted and used by learners—whilst algebraic symbols are a source of power because they can signify anything, at the same time they are a source of difficulties because they seem to stand for nothing. Arcavi (1995, p. 147) analysed this obstacle in a more detailed manner, noting how the acceptance of algebraic symbolism actually called for a "meaningful detachment of meaning," a task that is very difficult for both the learner and the teacher, requiring activities and examples that generate this detachment while still maintaining symbolic power. Long before this, when symbolic algebra was in its infancy, Hobbes (1588–1679), a philosopher and mathematician, drew attention to the problems the new symbols created, a view that is echoed by many school algebra learners:

> Symbols, though they shorten the writing do not make the reader understand it sooner than if it were written in words ... there is a double labour of the mind, one to reduce the symbols to words, another to attend to the ideas they signify. (Hobbes, cited in Mancosu, 1995, p. 87)

Conclusion

In time, initial verbal descriptions can give way to more mathematically based explanations, preparing for the more concise, symbolic arguments that will eventually develop into algebra as it is used in further mathematics.

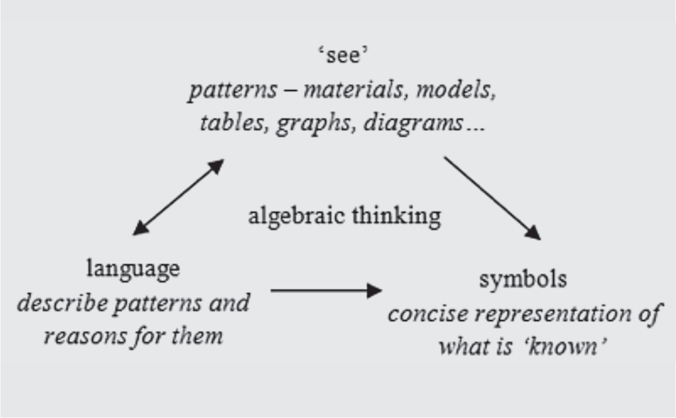

Figure 2. Model for introducing algebraic thinking.

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WHEN ARE WE GOING TO USE THIS? *

Mike Clapper

Australian Mathematics Trust

This paper explores ways in which we might answer this question in a practical way, emphasising the dangers of mathematics teaching becoming an exercise in technical skill development at the expense of problem solving and problem formulation. Although there is a philosophical element to this discussion, the emphasis is on practical pedagogies which engage students in the mathematical narrative.

When are we going to use this? (Dedicated to the boy in the back row)

Hands up those who have had this question? In my experience, this question is most often asked by the boy in the back row whose attention to proceedings may be marginal, but there would be very few mathematics teachers who have not had to field this enquiry from time to time. Indeed, statistics suggest that it may be girls who are really asking it, albeit silently, given the alarming reduction in numbers doing senior mathematics (Wilson, 2014; Mack & Walsh, 2013) so I think it is a question, annoying though it may be at times, which we have to take seriously.

Is this a reasonable question? (And if so, what answers can we give?)

We can always say – 'shut up and get on with your work'. But in a spirit of greater reflection perhaps we should ask ourselves, "why are they saying this?" Would they ask their English teacher the same question? And do we need to have an answer for those who are not very good at mathematics as well as those who are?

One common approach to dealing with the issue is to make everything *relevant*. However, I have seen some very distorted mathematics units, which place a premium on fitting into a topic or theme at the expense of good mathematical pedagogy, and I don't believe that there are always practical ways of making a topic appear immediately relevant. Often this ends with a kind of artificial or contrived relevance and the students will be the first to see through this.

I believe that, to some degree, we have created a rod for our own backs by giving primacy to technical development rather than to problem formulation. It is all too easy to start a new topic with an exposition on technique followed by Exercise 5.1, LHS, (with the added bonus of RHS if you finish quickly!) rather than starting with a problem which connects the topic to the real world in a manner which students might find engaging and out of which the desire to acquire techniques to solve the problem might arise. In short, I believe that students need to understand that Mathematics is as much, if not more, about asking questions than answering them.

Four Possible (and Reasonable) Answers

1. Give an immediate example (of course, it helps if you go to the lesson armed with some).
2. Buy time – this usually consists of – it's difficult to explain how you are going to use this until we have done a bit more work (but often we're a long way away from that point).
3. You might not use it yourself, but others will (and we can give examples where technology is based on mathematics).
4. You may not use this specifically, but it is all helping you to think logically. This should be a good answer but of questionable veracity, as I will discuss later.

The Reality of the Classroom

Most classrooms have a very great mix of ability levels. Teachers are often under time pressure to 'get through the syllabus'. There are behavioural issues which impact on our capacity to teach creatively. Younger teachers, in particular, need to focus on classroom management. These are all parts of our classroom reality and any answers to our difficult question need to respect this reality.

The Value of Context

Students often panic when they get to worded problems, because they have been socialised to think they are harder. This socialisation arises from every chapter starting with drill and practice, with only the students who finish all of these questions getting on to the worded questions. Given that the drill and practice questions get harder as you work through the chapter it follows that the worded problems at the end are the hardest of all.

But they're not! The double whammy here is that the weaker students who most need a purpose for working through the boring stuff never get to the worded problems.

In my experience, students are engaged by stories, so start with the story and let the mathematics follow. I will give a few examples of this below. Good teachers will have their own stories (indeed, from my observation, this is the key difference between good and not so good teachers!)

Fraction Example – Which Is Easier?

1. At Jim's birthday party, he thinks everybody will eat a third of a pizza. He asks his dad to order 4 pizzas. How many people are coming to his party?

   or

2. $4 \div \frac{1}{3}$

I would argue that most students will be able to use their intuition to work out the first, but in the second (which is the decontextualized version of the first) they will often enter into a mode of thought which is more about 'now what do I do in this situation' than reflecting on the meaning of the question. They are just trying to remember a technical skill at the expense of their own intuition.

Ratios and Dog Food

Dog food usually comes in cans that are either 700g or 1200g. One or the other is usually on special. How do I find which is better value? Of course there are easier examples where the ratios are 2:1, but there is never a difficulty in finding practical examples which students should be able to relate to. We have to let students get a feel for ratios and scaling before we go into the technical stuff. That is to say, students have to discover the importance of ratios in practical contexts before they are introduced to formal ratio notation and its associated rules.

Indices and Piano Tuning

If you have some musicians in your class, understanding the difference between Pythagorean tuning and well-tempered tuning is a rich example involving indices. Working through the major thirds or the cycle of fifths shows why tuning was such a difficult problem for musicians. Briefly, Pythagoras showed that simple pleasing harmonies were formed from simple ratios of string lengths or tube lengths (e.g., halving the string length gives the note one octave higher). The perfect fifth is in the ratio 3:2, the major third in the ratio 3:4 and so on. The problem occurs when you want to incorporate all of these intervals in a keyboard instrument such as a piano. If each perfect fifth is to be in the ratio 3:2 over the preceding fifth, then after 12 applications of this ratio, we should arrive at the note 7 octaves above (see diagram). However, this would require that $\left(\frac{3}{2}\right)^{12} = 2^7$ which, unfortunately it does not (though it is quite close). Of course, you would perhaps not use this example where you didn't have some musicians who would find it interesting! (HREF1; HREF2)

Measurement

Have you ever taught a unit on measurement without measuring anything? I have, but I never will again. Measuring real things (volume of wood in a tree, height of a building, amount of paint required to paint a room, anything but the simplest areas) is quite hard and students should experience these difficulties and not just learn about conversion of units. They need to develop estimation skills that can only happen if they have 'played' with the units involved. In particular, students have great difficulty with the idea of areas and volumes being proportional to the squares and cubes of linear dimensions, but this difficulty is much reduced when you have packed half of your class into a cubic metre. Try a measurement test in which you ask them to give the distance from their classroom to the Gym, or the area of their desk (or the school grounds), or the volume of a container by estimation. This will give you an idea of what units they use (without being told) and whether they have any intuitive understanding of the size of these units.

Trigonometry

The above can be extended to trigonometry. This should start with the need to measure awkward things such as the height of a tree. Once they have grasped the similar triangle concept and the idea of scaling they can go and measure some trees or buildings. Ideally, they should make their own tables of height to base ratios for given angles (I wouldn't call them tangents yet) and use their tables to calculate the heights of some objects. This will give them the understanding that a decimal value can be a ratio, something which is quite hard to grasp unless you have used it as such.

Generalisation

One of the most powerful aspects of mathematics is its capacity to generalise. Some students might go right through their schooling and never see this power at work. They need to be exposed to this at some point if they are to appreciate the power and importance of mathematics. The fundamental tool for expressing mathematical generalisations is, of course, algebra. However, for students who may lack the confidence or technical skill to deal with algebraic generalisations, a good example is to use one of the pictorial proofs of Pythagoras' Theorem (HREF3) for many good examples.

Algebra as Story Telling

I believe that Algebra teaching needs to have a strong narrative flavour. Algebra expressions and equations are essentially stories. With each element of algebra that students learn, they need to be able to solve some practical problems to appreciate the power of the method. I have dealt with this at length in a paper in the 2013 MAV Conference proceedings under the paragraph title *Algebra as story telling* (Clapper, 2013). An example of a question which I do not like is a worded problem in which the algebraic elements are contrived, e.g. a rectangle of given area with sides ($x + 3$) and ($2x - 10$). An example of a question which I prefer is one in which the algebra arises naturally in solving a problem, e.g., a rectangle of given area and perimeter and the requirement to find its dimensions.

What is the Point of Mathematics?

To return to the original question, we need to be able to articulate the purpose of mathematics, both for them and for others, including professional mathematicians. (HREF4, HREF5, HREF6) I would suggest the following:

- Learning some every day practical skills (such as counting)
- Learning to analyse a problem and break it down and formulate it mathematically
- Learning to be logical
- Understanding the power of generalisation
- Appreciating complexity

If students are taught creatively, with a strong sense of story and an approach that embraces the identification of problems and their solution, students will see its purpose and understand why it is such a vital component of the curriculum. I am not against the learning of technical skills. But they have to be grounded in conceptual understanding and the capacity to formulate problems. Otherwise:

- if students don't learn to be able to formulate problems mathematically, it is a waste of time them learning the technical skills;
- they may be able to use them to pass tests, but they will not retain them and they will say later on;
- they claim I never use any of that stuff I learnt (or didn't learn) in maths, I don't know why we had to do it.

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IMPLEMENTING PROBLEM-SOLVING AT EAGLE POINT PRIMARY SCHOOL: A WHOLE SCHOOL APPROACH *

Natalie Clarke and Anna Duncan

Eagle Point Primary School

Gaye Williams

Deakin University

This paper captures Anna and Natalie's reconstruction of the process of change to the teaching and learning of mathematics at Eagle Point Primary School, and Gaye's reflections on this process as critical friend. The paper is structured by questions Natalie and Anna asked themselves as they interrogated the process of changing to a problem-solving approach. Initially they asked: Why change? What types of structures should we put in place to increase our chances of success? On looking back: What conscious steps did we take that were useful? What strategies helped? Evaluating progress: What have we achieved so far? And, looking forward: What do we see for the future? Gaye reflects on characteristics of the school and staff, and what has contributed to successes achieved.

Introduction

Problem-Solving as a Vehicle for Deep Learning

It is more than thirty years since research findings showed students can develop deep and connected understandings of mathematics when they have opportunity to struggle to come to new understandings during mathematical problem-solving. Knowing what works does not mean this can be easily implemented. Even teachers who want to implement a problem-solving approach have difficulty doing so (in ways that lead to deep learning) initially, without support. Finding ways to support schools as they introduce a problem-solving approach is Gaye's focus of interest, and Natalie and Anna's focus is on bringing about such changes at Eagle Point Primary School. This paper captures that endeavour.

Introducing the Authors

Natalie Clarke and Anna Duncan are the Mathematics Drivers at Eagle Point Primary School. Natalie is the Grade 6 teacher and Anna the Foundation level teacher. Eagle Point Primary School in the Bairnsdale Region in Gippsland is a five-classroom school, with a current enrolment of 113 students.

Concerns That Stimulated the Change Process

In 2011, together with their principal Jenny Leggatt, Natalie and Anna assessed learning at Eagle Point Primary, and identified a number of concerns including: (a) many students perceived mathematics as a set of procedures that they required teacher help with (whenever they 'got stuck'), (b) NAPLAN results left much room for improvement, (c) there were inconsistencies in mathematics teaching and learning approaches across year levels, (d) increased teacher awareness of strategies to enhance learning was needed, (e) consistency in teacher judgments about the quality of student work was required, (f) there were teachers' with limited confidence around teaching mathematics, and (g) increased teacher mathematical content knowledge was needed. Something needed to be done! The school required support during the process of responding to these concerns. Gaye Williams was invited to work with Eagle Point Primary School as part of an initiative to support teachers in the Bairnsdale Region as they developed expertise in implementing unfamiliar challenging problems to increase student participation in learning.

Structures Developed to Stimulate Change

Cluster initiatives, Eagle Point Primary School's (PS) attention to strong leadership in the area of mathematics, and commitment to release as many staff as possible (for ongoing professional learning) provided initial structures to support change.

The Bairnsdale Network had previously solicited coaching in literacy, drawing on the expertise of John Munro (University of Melbourne). Coaching in numeracy (mathematics) became the next focus. This was sourced from Gaye Williams (Deakin University) and Mike Askew (Monash University), who independently offered different but complementary professional learning to Bairnsdale Region schools. The strong professional relationship between these schools was crucial to increased opportunities for small schools to access extensive professional learning.

Eagle Point Primary School's principal Jenny Leggatt considered leadership strength a priority if changes to the teaching and learning of mathematics were to occur across the school. She described what she valued in the selection of Mathematics Drivers: "I looked for staff who were solid teachers who had a good grasp of content knowledge with a passion for learning and preparedness to take risks with their own learning".

Gaye considered Jenny, Anna, and Natalie all possessed a passion for learning, and a preparedness to take risks, *and* that Jenny has faith in her staff to undertake their roles. These infectious characteristics contributed greatly to the learning culture developing in the school. Natalie and Anna's solid knowledge of mathematics and realistic confidence in their ability to teach mathematics, in conjunction with their passion for learning, immediately provided a professional learning resource. Anna and Natalie could model 'changes on-the-run in class' and respond to brief discussions with Gaye *during* the lesson. For teachers in the professional learning program to possess these abilities from the start, increased the likelihood of other teachers within the program seeing this as something that was possible. This progressed the professional learning for other teachers considerably and meant Anna and Natalie progressed at a fast rate because they could pinpoint new ideas they could integrate into what they already knew. This contributed greatly to the richness of task implementation and student learning. Thus, the attention of Eagle Point PS to strong leadership progressed professional learning across the region.

Eagle Point PS invested heavily in Professional Development making decisions about how to target it effectively to bring about changes. The school developed strategies to release staff: Natalie and Anna for leadership sessions in the Bairnsdale Region, and for the problem-solving implementation at their school and other schools each time Gaye visited (6–9 days a year). Other teachers who were enthusiastic about the approach were released for problem-solving planning, implementing or observing, and reflecting, half days with Gaye each time she visited. Over time other teachers were released more frequently as they became more responsive to problem-solving as a learning tool.

Releasing almost every staff member in such a small school was not easy. Jenny employed relief staff and covered classes herself. She was prepared to stay behind to oversee the school while numerous staff were away observing and participating in lessons and discussions at other schools (even though she would have loved to attend all the professional learning sessions herself).

Professional Learning is integrated into planning and development at Eagle Point PS. Staff attending professional learning are expected to engage in peer conversations and observations in their own time, and report back to Professional Learning Meetings on what they had learnt through these observations. Thus, change over time is planned, recorded and measured.

From Gaye's Perspective

Gaye agrees with the teacher release strategy to resource professional learning: engaging those who are most responsive for a start. This means there are successes that are excitedly recounted to others and, over time, those who may not initially have been as interested begin to want to learn more.

The leadership team at Eagle Point Primary School was crucial to the changes achieved. Jenny Leggatt has a passion for increasing opportunities for students to learn, and a deep understanding of how group work can promote such learning. She appointed passionate school leaders (drivers), Anna Duncan and Natalie Clarke: expert teachers who approach new situations with an open mind, a willingness to learn more, and insights into what could increase learning opportunities. Jenny's faith in her drivers enabled their autonomous exploration of new ideas, both in their classrooms and in finding ways to encourage and support other teachers developing new understandings.

One of the key ingredients Gaye has found makes a difference to the change process is teacher faith in children's ability to think. She considers that successful professional learning requires a similar faith of school leaders and professional learning providers in teacher ability to puzzle things out as a team and find what is going to work for that team. This puzzling includes experimentation, reflection, and ongoing changes based on these reflections. It appears to Gaye that this faith in others' (children and teachers) ability to experiment and explore is an essence of teaching and learning at Eagle Point PS.

'Engaged to Learn' Professional Learning Program

Gaye's Engaged to Learn Approach to problem-solving involves students working in small 'same-pace-of thinking' groups (3–4 students) with problem-solving tasks that are accessible through either simple or more complex mathematics, and multiple representations. More about this approach can be found in Williams, Harrington, and Goldfinch (2012) where teachers interrogate the approach. In this approach, teachers compose groups using criteria Gaye developed as a teacher. Groups decide how to approach the problem, what to use, and how to use it, so the usefulness (or not) of materials available depends on the solution pathway selected. By not telling students what materials to use or how to use them, the problem is left very open-ended with many opportunities for groups to pursue pathways *they* consider useful. Each group reports to the class at regular intervals with a different reporter each time. The teacher does not give hints, agree or disagree with mathematics developed but rather asks questions to elicit further student thinking.

Gaye works with staff in Bairnsdale Network schools (principals, mathematics drivers, early years, middle years, and senior years teams, and individual teachers) in various contexts including Curriculum Days, planning meetings with principals and drivers, and in planning, implementing, and reflecting on problem-solving with staff at different levels in the school (Early Years, Middle Years, Upper Years). Gaye sometimes modelled the implementing of a task, and over time, teachers have undertaken this modelling. Natalie and Anna were immediately able to model the Engaged to Learn Approach and flexibly make 'changes on-the-run'. Gaye's intention was to build and strengthen teacher team-work through a *problem-solving approach to teaching* to implement *mathematical problem-solving in class*. Between Gaye's visits, teachers who had observed lessons implemented the task in their classes (with at least one other teacher observing) to stimulate post-lesson reflections. Through this process tasks were progressively refined.

What Have Eagle Point Achieved So Far? What Helped?

Natalie and Anna:

Although there are differences in the degree of uptake of the Engaged to Learn Approach by different teachers, they have all progressed in various ways during this professional learning program.

We have developed a lesson structure based on Gaye's Engaged to Learn Model, and used this structure to develop detailed *lesson scripts*. We have increased our ability to break tasks into parts to analyse them in detail and plan their implementation. We now identify the foci of cycles of group work and reporting. We also record questions students ask to further their understanding, and questions we might ask as a result. We have improved in our ability to identify: (a) mathematics that may emerge at different points in a task, (b) types of difficulties students might encounter (where they may become 'stuck'), and (c) types of questions we might ask at different stages of the lesson. We have modelled these questions on the style of questions we have heard Gaye ask in tasks she has implemented in classes, and when working with teachers on curriculum days. We have begun to develop the skills to recognize the order in which to place reports so that: (a) every group has a chance to report something new, and (b) mathematical ideas build as the reports progress.

Roles of team members in groups have been defined in all classes (reporter, recorder, encourager, and timer). Children are at various stages along the way to developing collaborative skills:

- thinking out loud, listening to each other;
- building upon each others' ideas;
- asking focused questions of the group when they do not understand;
- making sure all group members understand;
- explaining again a different way; and
- including everyone in group conversations.

Student willingness to problem-solve is increasing. Although students are learning in these groups, their learning will be further enhanced when *all* begin to *really listen* to group reports, and ask reporters questions to extend understanding of report content.

Teacher confidence in teaching mathematics is 'on the rise', and there are more professional discussions around mathematics and strategies used. All teachers willingly provide assistance to other staff as needed/requested. Teachers are valuing student successes both in class and in discussions with other teachers. We set aside a lot of time for conversation and celebration. We listen and celebrate alongside someone who has a success story to tell. This occurs in the staff room, during meetings, recess, and lunchtimes. We schedule visits to all classrooms to see work displays and work in progress. This keeps enthusiasm high and timelines on track because staff know we are coming to look. In class we are working toward *children* identifying 'Quality Work' through self-evaluation, and we are experimenting with goal setting based on these conferences with students. This is also intended to increase staff consistency in judging quality work.

Teacher inclination to write lesson scripts that break down the task, identify the lesson cycles, and select appropriate questions to ask along the way is growing. This is catalyzed by the reported successes of those staff utilizing this strategy. Teacher impetus to learn more about the approach and how to implement it increases as teachers see their activity leading to positive changes in students.

What Helped? As a result of engaging with the problem-solving tasks, and seeing how these could differentiate learning in the classroom, Anna and Natalie have worked intensively with staff on yearly, term, and weekly planning documents which become more targeted over time. They describe strategies they are trialling to cater for different students during this approach: "We have created a whole school Maths Document in number which focuses not only on the core skills that need to be achieved by students at each level but also a bank of strategies that could be used to achieve these skills. We tried to ensure that fluency, explicit teaching, and problem-solving are all catered for in these strategies. This document is in the trial phase and will be reviewed in Term 4 and in 2015".

In addition, planned fortnightly Professional Learning Meetings to catalyse change (with maths, writing or spelling foci) have been arranged, and actions set with clear time frames (which are then advertised). Timelines are strictly adhered to as the action often relies on completion of the previous action first. The content, and leadership, of these meetings has changed over time. Initially, Natalie and Jenny (with support from Anna), focused each meeting around 'Gaye Williams' tasks': what we had seen and trialled. As teachers became more fluent with the language and the structure, staff began to contribute more and Natalie took on more of a 'facilitator of discussions' role, asking questions that staff considered then discussed, sharing their ideas and experiences. Lately, the structure has changed again. Less confident staff are now bringing concerns and problems to the table for input and ideas on how to move the students forward in their thinking. Our list of 'thinking questions' in our lesson scripts came from these meetings.

We are presently focusing on the task we are implementing. We brought our lesson scripts, shared them, and each checked to ensure the tasks increased in complexity from Prep through to Year 6. Changes were made to tasks and expectations of students highlighted. At this meeting we decided that the tasks needed to cater for the children we have *now* (we delayed planning concepts for the future as needs may change in years to come). This type of collegiate sharing is working at a much deeper level that the original meetings where Natalie, Jen and Anna were seen as the experts. Now participation is equal and strong collegial conversations occur.

Gaye's Perspective on What Has Been Achieved by Eagle Point PS

There is no doubt that Eagle Point PS have come a long way towards developing a culture of learning through staff collaboration. This has been fuelled by the passion of the leaders and those staff who immediately wanted to know more, and an increased inclination to change by other staff working within this environment. Crucial to the progress made was the staff breaking down the Engaged to Learn process into the aspects that needed to be studied in detail, and identifying strategies to enable the development of more detailed understanding of what was involved: communicated in lesson scripts. Leaders and other teachers knew exactly what they wanted to confer about with *me* on each visit.

By experimenting with ideas, and reflecting on what occurs, Eagle Point PS teachers have developed an extremely useful way to document lessons within a task to support others who may undertake the task in the future, and provide a tool through which to add ongoing refinements. These lesson scripts break the lesson into cycles (each including group work and reporting) and identify what might emerge (e.g., types of student experimentation, different representations, mathematical ideas, places where students might experience difficulties, types of reports student might prepare). The scripts also include possible teacher actions, such as questions teachers might ask (with appropriate times) and the order of reporting (to progressively report new ideas. They also give all groups opportunity to provide something new), as well as mathematical ideas the teacher could draw out. To be sure this is done during busy school life, Natalie collects, collates, and files these scripts, samples, and reflections.

Documentation at all levels of planning (from whole year to one cycle of group work and reporting in class), and continual refinements through staff collaboration (within levels, whole school, region level) have all contributed to what has been achieved, and will continue to be achieved in the future. Observing or taking classes at Eagle Point PS is a continual reminder to me of how open many students are to problem-solving, and how this is encouraging others to problem-solve too.

What Do We See for the Future?

Anna and Natalie:

Looking ahead, we see that learning from each other will be crucial to sustaining and enhancing what we have achieved, and essential to introducing new staff to this mathematics problem-solving culture. There is still work to do to increase teacher confidence, and expertise in *timely* asking of questions to elicit reasoning, deeper thinking, and the development of new mathematical ideas (in groups and whole class). Intended student outcomes from these changes to mathematics teaching and learning include changes to classroom culture that we have already started to see. We expect increases in student results on external assessments, and further consistency between teacher judgments about the quality of student work to follow. The next big step will be raising our awareness of how to select 'good problems', and design our own. We intend to make problem-solving a weekly activity, including a longer more complex problem at least once a term. We are currently trialling this in Eagle Point PS classrooms. We are also focusing on how information and data collected from problem-solving tasks can be incorporated into assessments, and future planning.

Gaye:

Natalie and Anna's reflections on Eagle Point PS's needs for the future illustrate their strength in driving this process. Look back at the original concerns, to see how much has been achieved, and how well they have pinpointed what is still needed. Mathematics is no longer perceived as only rules and procedures, and there is more consistency to teachers' approaches. There is a more collaborative teaching culture with sharing of strategies to enhance learning. A structure to increase teacher consistency in judging quality work is in place, and deeper teacher understandings of mathematics are developing task by task. Teachers are now sufficiently confident to ask to find out more, and focus is shifting to task design. Developing 'good problems' for the Engaged to Learn Approach is not easy but I am certain that the grit and determination of those at Eagle Point PS will prevail and this expertise will be developed and shared. Changes to grades should soon begin to show with sustained use and development of the Engaged to Learn Model.

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SMORGASBORD OF MATHS – WHAT'S ON THE MENU TODAY? DIFFERENTIATION IN THE PRIMARY CLASSROOM *

Cathy Davidson and June Penney

Bacchus Marsh Primary School

Jenny Dockeary

Melton South Primary School

Students come to school with complex learning needs. Teachers need to tap into student interest by presenting problems that are engaging. There is a growing expectation that teachers are aware of individual students' needs and take those needs into account in planning instruction in mathematics.

Your favourite restaurant set menu has *two* starters, *three* main courses and *two* desserts. How many different three-course dinners could you order?

Differentiated instruction involves providing students with different avenues to acquiring content; and to processing, constructing, or making sense of ideas; and developing teaching materials so that all students within a classroom can learn to their potential, regardless of differences in ability. Students come to school with complex learning needs. Teachers need to tap into student interest by presenting problems that are engaging. To open up the above example, students could be asked to nominate the type of restaurant, the number of starters, main courses and desserts, and to discuss the different ways they can come up with their three-course combinations.

Kennedy (2011) notes that:

> the reality of modern classroom teaching is that no teacher has a class of students who are all working at one level. Teachers who aim their maths lesson at only one or two groups of learners are choosing to believe in a myth that this type of class still exists, if in fact it ever did. Within every single class, even if students are streamed, we have a diversity of learners to cater for. For a single teacher with 25 or so different students to teach, this can be very difficult to handle.

Tomlinson (1999) highlights the differences between the typical traditional classroom and a differentiated classroom. The difference between students is used as a basis for all planning. Diagnostic assessment is used to design curriculum that is targeting student needs; individual goal-setting becomes an established part of the differentiated classroom. Teachers no longer use a single form of assessment but assess in multiple ways.

The National Council of Teachers of Mathematics (NCTM), the United States professional organisation whose mission it is to promote, articulate, and support the best possible teaching and learning in mathematics, recognises the need for differentiation. The first principle of the NCTM Principles and Standards for School Mathematics reads: "Excellence in mathematics education requires equity—high expectations and strong support for all students." In particular, NCTM recognises the need for accommodating differences among students, taking into account their readiness, their level of mathematical talent, their interest and confidence, to ensure that each student can learn important mathematics. "Equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students" (NCTM, 2000, p. 12).

While some students are still struggling with their multiplication facts or addition and subtraction with decimals, others are comfortable with complex reasoning and problem solving involving fractions, decimals and percents. The differences between students' mathematical levels begin from when students begin kindergarten.

Teaching to the 'Big Ideas'

Differentiating instruction is not a new idea, but the issue has been gaining an ever higher profile for mathematics teachers in recent years. More and more, educational systems and parents are expecting the teacher to be aware of what each individual student—whether struggling, average or gifted—needs and to plan instruction to take those needs into account. In the past this was less the case in mathematics than in other subject areas, but now the expectation is common in mathematics as well. Many teachers believe that curriculum requirements limit them to fairly narrow learning goals and feel that they must focus instruction on meeting those specific student outcomes. Differentiation requires a different approach, one that is facilitated by teaching to the big ideas. It is impossible to differentiate too narrow an idea, but it is always possible to differentiate instruction focused on a bigger idea.

Some of the strategies that have been suggested for differentiating practice include:

- the use of menus from which students choose from an array of tasks;
- tiered lessons in which teachers teach to the whole group and vary the follow-up for different students;
- learning stations where different students attempt different tasks; and
- other approaches that allow for student choice, usually in pursuit of the same basic overall lesson goal.

A big idea at Year 1 is that students understand and represent commonly used fractions such as 1/2 and 1/4 and must be able to illustrate how fractions represent part of a whole. Can students explain how many equal parts there are and demonstrate how they know the parts are equal? A Year 1 lesson where content is tiered according to readiness around this big idea could be tiered as follows (Adams & Pierce, 2004):

- Tier 1: Using paper circles (pizza) and squares (sandwich), students in pairs demonstrate how to share food equally and illustrate by folding the paper. Have two pairs determine how much they can share equally with four people. They can cut the parts and stack them to see if they match.
- Tier 2: Using paper circles (pizza) and squares (sandwich), have students determine how to share the food equally and illustrate by folding the paper. How can they share equally with six people? Repeat the process for sharing a birthday cake with 12 people. In each case, students can cut out the parts and stack to match. Have them start with half a cake and divide equally for 3, 6, and 12 people.
- Tier 3: Using paper rectangles (sandwiches) and triangles (slices of pie), have students in pairs determine how to share the food in three different ways to get two equal parts. Have them illustrate by folding the paper. Are there other different ways to divide each shape equally? How many ways are there? Have students determine which shapes – circles, squares, rectangles, triangles, - are easier to divide evenly and illustrate why with a particular food of their choice.

When a teacher tries to teach something to the entire class at the same time, "chances are, one-third of the kids already know it; one-third will get it; and the remaining third won't. So two-thirds of the children are wasting their time" (Lilia Katz, quoted in Youd, 2013).

The fact is "teaching is hard, and teaching well is fiercely so" (Tomlinson, 2000, p. 6). Effective teaching requires multiple approaches and adaptations in the areas of *content* (what students learn), *process* (the ways students learn and how content is taught), and *product* (how students present or demonstrate their learning) (Tomlinson, 2000). Teachers therefore need to have a clear understanding of what it is they are trying to teach. Pre-assessment, formative and summative assessment are all regular parts of the teaching and learning cycle.

Assessment Drives Instruction

In differentiated teaching, assessment drives instruction. Assessment information helps the teacher map the next steps for different learners and the class as a whole. Assessment occurs consistently as the unit begins, throughout the unit and as the unit ends.

> Never say anything a kid can say! This one goal keeps me focused. Although I do not think that I have ever met this goal completely in any one day or even in a given class period, it has forced me to develop and improve my questioning skills. It also sends a message to students that their participation is essential. Every time I am tempted to tell students something, I try to ask a question instead. (Reinhart, 2000, p. 480)

As well as flexible grouping, differentiation is also achieved through careful question and task design. By designing and using effective questions and tasks, teachers are able to differentiate content, process, and product, as well as cater for the readiness, interests, and learning preferences of students.

There is a fine line between a question that encourages the student to think and one that provides the student with too much information or inadvertently solves the problem for the student. Being able to straddle this fine line comes with reflective practice. (Ontario Ministry of Education, 2006).

Bloom's Taxonomy

There are a number of different models of thinking that can assist teachers to ask appropriate questions, such as Bloom's Taxonomy.

Table 1. *Bloom's Levels of Mathematical Thinking*

Levels of thinking	**Guide questions**
Memory: *recalls or memorises information*	What have we been working on that might help with this problem?
Translation: *changes information into another form*	How could you write/draw what you are doing? Is there a way to record what you've found that might help us see more patterns?
Interpretation: *discovers relationships*	What's the same? What's different? Can you group these in some way? Can you see a pattern?
Application: *solves a problem - use of appropriate generalisations and skills*	How can this pattern help you find an answer? What do think comes next? Why?
Analysis: *solves a problem - conscious knowledge of the thinking*	What have you discovered? How did you find that out? Why do you think that? What made you decide to do it that way?
Synthesis: *solves a problem that requires original, creative thinking*	Who has a different solution? Are everybody's results the same? Why/why not? What would happen if....?
Evaluation: *makes a value judgement*	Have we found all the possibilities? How do we know? Have you thought of another way this could be done? Do you think we have found the best solution?

Learning goals stem from curriculum expectations. Overall expectations (or a cluster of specific expectations) inform teachers about the questions to ask and the problems to pose.

By asking questions that connect back to the curriculum, the teacher helps students centre on these key principles and big ideas. Students are then better able to make generalisations and to apply their learning to new problems. Here is a 'big idea' example: the same object can be described by using different measurements.

Teacher's learning goal: to make a connection between length, width, area and multiplication.

Problem: a rectangle has an area of $36cm^2$. Draw possible rectangles.

Possible questions:

- As you consider the shapes you made, what are the connections of the length of the sides to the total area?
- If you know the shape is a rectangle, and you know the total area and the length of one side, what ways can you think of to figure out the length of the other three sides?

Asking Open Questions

Effective questions provide a manageable challenge to students—one that is at their stage of development. Generally, open questions are effective in supporting learning. An open question is one that encourages a variety of approaches and responses. Consider 'What is 4 + 6 ?' (closed question) versus 'Is there another way to make 10?' (open question). Or, 'How many sides does a quadrilateral figure have?' (closed question) versus 'What do you notice about these figures?' (open question). Open questions help teachers build student self-confidence as they allow learners to respond at their own stage of development. Open questions intrinsically allow for differentiation. Responses will reveal individual differences, which may be due to different levels of understanding or readiness, or to the strategies to which the students have been exposed, or to how each student approaches problems in general (or to a combination of these things). Open questions signal to students that a range of responses are expected and, more importantly, valued. By contrast, yes/no questions tend to hinder communication and do not provide us with useful information. A student may respond correctly but without understanding (HREF1: Asking Effective Questions, p. 2).

In planning for differentiation, then, teachers need to ask themselves:

- what do I want my students to know, understand, and be able to do?
- what do I need to do instructionally to get my students to learn these?
- how will my students show their learning?

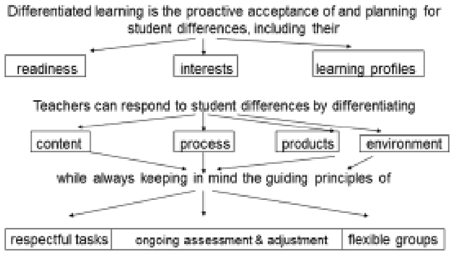

Figure 1. Differentiated learning [From Irwin, HREF2].

It is a fundamental premise of differentiating instruction that teachers know exactly what to expect from students. They need to have an understanding of what students need to learn and how they can show their learning. Failure to do this is a major barrier to effective differentiation.

Differentiation is responsive teaching rather than one-size-fits-all teaching. *One size does not fit all!*

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TEACHER QUESTIONING IN MATHEMATICS CLASSES IN CHINA AND AUSTRALIA: A CASE STUDY **

Lianchun Dong and Wee Tiong Seah

Monash University

David Clarke

The University of Melbourne

Asking questions is an important instructional activity frequently employed by teachers in mathematics classrooms worldwide. This paper will compare teachers' questioning behaviour in Australian and Chinese classrooms by examining mathematics lessons taught by four experienced junior secondary teachers. Its cross-cultural approach allows researchers and practitioners to examine teaching practices within a much broader context, and thus contributes to a deeper and more explicit understanding of teaching behaviour in the two countries. It employs new ways of analysing and coding teacher questions and prompts in cross-cultural settings, and some of these will be summarised and discussed in this paper.

Introduction

Asking questions is one classroom instructional strategy which is used so frequently that it could be observed and identified in almost every mathematics classroom. Teachers not only need to be well prepared in how to ask a mathematical question that could facilitate students' understanding of mathematics, but they should also provide effective follow-up actions to those students who respond to the questions. Due to the diversity in terms of students' potential responses to one question, it is usually challenging for the teachers to cope with students' responses in ways that will facilitate students' mathematics learning (Franke et al., 2009).

Investigating experienced teachers' questioning practices could provide us with valuable perspectives and ideas in regard to a variety of potential ways of employing questioning strategies. Exploring cases from different cultural settings, such as Chinese and Australian mathematics classrooms, has the potential to provide sufficient variation in terms of the classroom interaction through which teacher questioning behaviour functions. A high percentage (approximately 50%) of lesson time in Australian mathematics classrooms is spent on private teacher-student interaction when students work individually, in pairs, or in groups (Hiebert et al, 2003; Hollingsworth, Lokan, & McCrae, 2003). In contrast, in China 75% of mathematics lesson time is spent in public interaction whilst private interaction consisted predominantly of the occasions when students studied individually (Hiebert et al, 2003).

This study was designed to examine teachers' questioning strategies in mathematics lessons taught by four competent junior secondary teachers from mainland China and Australia. It intended to analyse what kinds of verbal questions and prompts were initiated by the teachers and in what ways the teachers took students' verbal contributions into consideration so as to facilitate their construction and acquisition of mathematical knowledge.

Setting and Participants

There are four participants in this study: two are from Beijing City and from Nantong City in China and another two from Melbourne. All the teachers are recognised as competent according to local criteria. Video cameras were employed to record the participants' instructions in their classrooms. For each teacher, one lesson was analysed and the detailed information about the teachers and the lessons is listed in Table 1.

Table 1. *Lesson topics delivered by the four participating teachers*

Teacher	**Gender**	**Year**	**Lesson content**
AUS1	Male	9	Trigonometry: Introducing ratios in right-angled triangles
AUS2	Female	9	Pythagoras Theorem and Trigonometry: 1) Finding the distance between any two points in the Cartesian plane by using Pythagoras's theorem; 2) Application of Pythagoras's theorem in real life situations
CHN1	Female	9	Trigonometry: Finding the lengths of sides and the measures of angles in right-angled triangles.
CHN2	Male	8	Quadratic function: Investigating the graph of $y = ax^2$

Data Analysis

The terms 'question' and 'prompt' refer to what the teacher says to elicit students' verbal responses related to mathematical content. Questions or prompts that were not mathematical were excluded unless they were associated with other mathematical 'talk'. Questions immediately repeated using the same wording were counted only once. In the analysis of teacher questions and prompts, a distinction was highlighted between initiation questions and follow-up questions. Initiation questions are those questions asked by teachers for purposes such as to start a conversation or discussion. In contrast, follow-up questions are those questions asked for purposes such as in response to students' answers or their contributions to teachers' previous questions. Where the student/s replied to teacher questions and the teacher did not to respond, these interactions were categorised as Question-Answer (Q&A) pairs. Teacher Initiation-Student Response-Teacher Follow up (IRF) sequences (Mortimer & Scott, 2003) were those where the teacher responded to students' answers that were triggered by the previous teacher question.

A coding system was developed to categorise the initiation questions and follow-up questions. Instead of inventing a name for each category in advance, the questions and prompts documented in our data were analysed first and then given descriptive names. The development of the coding system in this study was informed by previous research (Hunkins, 1995; Hiebert & Wearne, 1993; Oliveira, 2010). The coding systems are presented in Table 2 and Table 3.

Table 2. *Sub-categories for Initiation Questions*

Category	**Description**	**Example**
Understanding checking	Questions used to check whether students can follow or agree with teacher's teaching or other students' opinions	"He [some student] said for this parabola, it is a mirror image across the y-axis , [do you agree]?"
Review	Questions used to elicit previously learnt concepts, propositions, formulas, procedures from students	"Where did I get the eight from?"
Information extraction	Questions requiring students to identify and select information from text descriptions, graphs, tables, diagrams so as to elicit students' interpretation of texts, graphs, tables, diagrams	"What is b, what's the mathematical word for what b is asking you to find?"

Exemplification	Questions requiring students to provide examples	"Could you list some examples?"
Result/ product	Questions used to elicit results of mathematical operation, or the final answer of the problem solving from students	"What is the square root of 80?"; "Were your measurements the same as Kevin's?"
Strategy/ procedure	Questions used to elicit the procedures or strategies of problem solving	"How can we solve this problem?"
Explanation	Questions requiring students to provide explanations	"How would this be perceived from the perspective of a function?"
Cognitive Regulation	Questions requiring students to monitor and regulate the process of reasoning and problem solving	"But have you answered the question?" "Now can we [on the basis of the information that we have already found] work out the length of *BC*?"
Comparison	Questions requiring students to compare mathematical identities, e.g., mathematical descriptions, or graphs.	"Now let us have a look at how Wang and Liu described the graph of $y = x^2$. Which of the descriptions do you prefer to agree with?"
Reflection	Questions requiring students to have a reflection after the completion of mathematical activities, e.g., problem solving or geometric drawing.	"What kind of knowledge have we already used to solve one triangle?"
Variation	Questions requiring students to consider the variations of the previous mathematical tasks.	"So what if I got a hundred and twenty seven in that answer?"

Table 3. *Sub-categories for Follow-up Questions*

Category	Description	Example
Clarification	Questions requiring a student to show more details about his/her answers or solutions	"How did you get this 16?"
Seeking confirmation	Questions used to confirm understanding of students' intended meanings in their responses	"So what you're saying is 'go down'?" "Any right angle triangle?"
Justification	Questions requiring students to justify their answers	"Why did you choose this method to solve this problem?"
Elaboration	Questions asking for additional information under the circumstances, when the students' previous responses fail to provide sufficient mathematical information to fully answer teacher's questions	"So then the green line becomes the what?" "In other words, toward which direction should it [the graph of one parabola] extend?"
Extension	Questions used to extend the knowledge under discussion to other situations or to establish the relationship between the knowledge under discussion and previous knowledge.	"Would this work with other numbers?"
Supplement	Questions used to request for supplement.	"Did anyone do this problem in a different way?" "What else? What else is important?"
Cueing	Questions used to direct students to focus on key elements or aspects of situation in order to enable problem-solving, especially when the students fail to make progress or when students give the incorrect responses.	"What is the problem asking you to find?"
Repeat	The teacher repeats or rephrases question.	

Agreement request	After a student provides answer to teacher's question, the teacher asks this type of question to check whether the rest of the class agree.	"So would you agree that the height of this one is going to be a hundred and forty nine?"

Results

1. Proportion of initiation questions and follow-up questions

Figure 1 shows the number and proportion of initiation questions and follow-up questions asked by the four participating teachers.

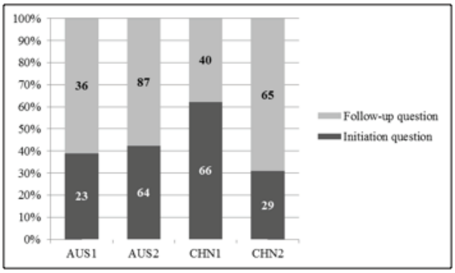

Figure 1. Proportion of initiation questions and follow-up questions.

The four teachers show different characteristics in the number of questions posed: CHN1 asked the largest number of initiation questions, while AUS2 asked the largest number of follow-up questions. AUS1 posed the least number of both types of questions.

2. Nature of initiation questions

Table 4 presents further details about the kinds of initiation questions the participating teachers asked in classrooms. Three types of questions were found to be the common initiation questions used in all four teachers' classrooms: Understanding checking, Review, and Information extraction. Of these three, Review questions take up a considerable proportion of the initiation questions for all four teachers.

Table 4. *Breakdown of Initiation Questions Asked by the Participating Teachers*

	UND	REV	INF	RSL	STG	EXPL	COG	COMP	REF	VAR	EXEP	Sum
AUS1	1	10	1	5	1	0	0	0	5	0	0	23
AUS2	8	10	8	11	8	6	9	2	0	2	0	64
CHN1	1	14	6	22	20	0	2	0	1	0	0	66
CHN2	6	5	5	0	0	7	0	4	0	1	1	29

Note: UND=Understanding Check; REV=Review; INF=Information extraction; RSL=Result; STG=Strategy; EXPL=Explanation; COG=Cognitive regulation; COMP=Comparison; REF=Reflection; VAR=Variation; EXEP=Exemplification.

In terms of the diversity in asking initiation questions, AUS2 teacher's questioning strategies include nine out of eleven kinds of identified initiation questions, presenting more variations than the other three teachers. Moreover, AUS2 teacher also demonstrated a more balanced way of asking questions. The initiation questions asked by AUS1 teacher mainly consist of seven different types of questions, namely Understanding check, Review, Information extraction, Result, Strategy, Explanation, and Cognitive regulation. CHN2 teacher also showed relatively diverse and balanced characteristics in terms of asking initiation questions. In total, he asked seven types of initiation questions, mainly consisting of five: Understanding check, Review, Information extraction, Explanation, and Comparison. Compared with AUS2 teacher and CHN2 teacher, the other two teachers employed questioning strategies in a less diverse and balanced way. While there are six types of initiation questions in AUS1 teacher's classroom, his questions predominantly consists of three types, Review, Result and Reflection which altogether takes up around 87% of all initiation questions. Similarly, in CHN1 teacher's classroom, three types of initiation questions (Review, Result, and Strategy) compose the most part (about 85%) of all the seven initiation questions asked in her classroom.

3. Nature of follow-up questions asked by the participating teachers

Table 5 shows the breakdown of follow-up questions asked by the four participating teachers. Five kinds of follow-up questions were used by all four teachers: Clarification, Seeking confirmation, Elaboration, Cueing and Repeat. Moreover, out of these five types of commonly-used follow-up questions, Cueing questions occupy a considerable proportion in the follow-up questions for all four teachers. The approximate statistics are separately 28% (the initiation question type with the second highest proportion in AUS1 teacher's class), 31% (the initiation question type with the highest proportion in AUS2), 20% (the initiation question type with the third highest proportion in CHN1), and 25% (the question type with the second highest proportion in CHN2).

The data reflects consistent features of the teaching practices in a particular cultural system. It is evident that three types of follow-up questions (Cueing, Seeking confirmation, Repeat) were employed more often by AUS1 and AUS2 teacher than the two Chinese teachers. Compared with the two Australian teachers, CHN1 and CHN2 teachers asked more Elaboration and Supplement questions.

For AUS1, around 94% of all follow-up questions are of four types: Clarification, Cueing, Seeking confirmation and Repeat questions. For AUS2, 69% of follow-up questions are of three types: Cueing, Seeking confirmation, and Repeat. For CHN1, the majority of follow-up questions are of four types: Clarification, Cueing, Elaboration, and Supplement (68%). For CHN2, three types of follow-up questions predominated: Cueing, Elaboration, and Supplement (75%).

Table 5. *Breakdown of Follow-up Questions Asked by the Participating Teachers*

	CLA	CUE	CON	ELA	REP	SUP	JUS	AGG	EXT	Sum
AUS1	14	10	5	2	5	0	0	0	0	36
AUS2	8	27	19	10	14	2	5	1	1	87
CHN1	6	8	2	9	2	10	3	0	0	40
CHN2	4	16	3	25	1	8	3	5	0	65

Note: CLA=Clarification; CUE=Cueing; CON=Seeking confirmation; ELA=Elaboration; REP=Repeat; SUP=Supplement; JUS=Justification; AGG=Agreement request; EXT=Extension

Discussion

This study examined and identified the diversity of questioning strategies employed by four mathematics teachers in two different cultural settings. Through the development of separate coding systems for initiation questions and follow-up questions, the study attempted to reveal, in a more explicit way, the features of questioning strategies used by the participating teachers. The study identified eleven types of initiation questions and nine types of follow-up questions used by the four participating teachers, reflecting the variety of options for mathematics teachers for asking questions. Nevertheless, further examination of the proportion of each question type, shows that all four teachers employed only a small number of question types for both initiation questions and follow-up questions.

In terms of initiation questions, all four teachers' choices of asking questions show different features, except that they all asked a considerable proportion of Review questions. Of the four teachers, AUS2 and CHN2 asked initiation questions in a more diverse and balanced way than AUS1 and CHN1 teachers. This might relate to the lesson content. For AUS2 and CHN2 teachers, the lessons involve a variety of mathematical knowledge, such as Pythagoras' Theorem and its applications in geometry and real life, as well as different kinds of properties with regard to the graph of quadratic function. In contrast, AUS1 and AUS2 teachers' lessons mainly contain mathematical procedures and rules. The examination of follow-up questions reflects some culturally-specific features in terms of dealing with students' responses and contributions in classroom instructions. Although cueing questions were commonly asked with a high proportion by all the four teachers, the two Chinese teachers employed more elaboration and supplement questions, whereas the two Australian teachers asked more seeking confirmation and repeat questions. To some extent, this reflects different principles underlying questioning strategies in China and Australia.

In summary, the employment of cross-cultural research design and the development of coding systems focusing separately on initiation questions and follow-up questions could help to understand better the characteristics of teacher question strategies. It helps to identify a variety of options that could be used in mathematics classrooms to facilitate students' construction and transformation of mathematical knowledge. Besides, it can also help in-service teachers to reflect on their own teaching practices through using the coding systems developed in this study and thereby improve mathematics classroom teaching.

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ASSESSMENT FOR LEARNING IN SECONDARY MATHEMATICS *

Heather Ernst

Kurnai College and Federation University

The strategies described in this paper could help teachers and students identify the students' mathematical strengths and weakness before the formal assessment. Using mini-whiteboards, flash cards and a variety of other methods, with a range of closed and open-ended questions, teachers can discover which students know what they are doing. The students' focus changes from finishing the work, to understanding the mathematical concepts. These methods involve little preparation, increase students' interest and increase overall test results.

Introduction

Several worldwide studies (Jablonka, 2012; Murray, 2011; Povey, 2010) have reported that students find secondary school mathematics boring and difficult. Consistent feedback by way of *Assessment For Learning* is one way to add interest and challenge to our classes (Black & Wiliam 1998; Hattie & Timperley, 2007; Wiliam, 2011, 2012).

Assessment For Learning (AFL)

Assessment for Learning (AFL) or Formative Assessment is assessment before or during a topic. Assessment for Learning is used by teachers who would like to know their students' mathematical strengths and weakness before the formal assessment tasks. Students would also benefit, especially if the students acted on this information to improve their skills and understanding.

Assessment for Learning can take many forms: pre-tests, quizzes, mini-whiteboards, matching cards, clickers, random name generators, and class discussions. Five mathematics teachers in a regional secondary school trialled these AFL methods as part of an Action Research Project.

Pre-test and Quizzes

Pre-tests and quizzes highlighted students' strengths and weaknesses; however, the time delay for marking and the students' interest in the numerical grade, rather than self-improvement, made their use less effective. Comment-only feedback did encourage the students to analyse the comments on their work, although this reporting process was time consuming. Self or peer corrected quizzes were the most effective as this reciprocal evaluation meant students received instant feedback and teachers gained insights into students' knowledge, which could subsequently be acted on before the formal assessment. Students also gained insight into the marking process. It should be noted that a positive collaborative class atmosphere is necessary for this to be successful.

Mini-whiteboards

The teacher asked the class mathematical questions which the students responded to on individual mini-whiteboards. The students then displayed their responses to the class. The teacher was able to identify strengths and weaknesses, and class discussions developed from the shared learning activity. Introductory closed questions were asked, but the most useful were the open-ended instructions or questions.

An example of an open-ended instruction for Mathematical Methods is: "Write an expression which can be differentiated by the product rule." Some students chose to use a text book question, while others would challenge themselves and choose a question which needed the product rule used twice. For some sample responses see Figure 1.

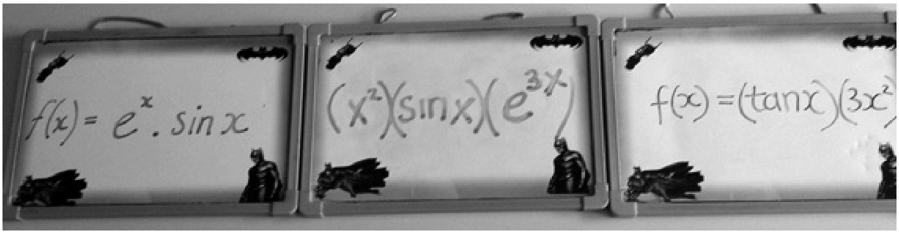


Figure 1. Mathematical Methods students were asked to write a function which could be differentiated by the Product Rule.

Another example is from General Mathematics where the students were asked to write 1/3 as a decimal. The students responded with 0.3, 0.3333, and 0.3333333333333333333.

This led to a lively discussion of the degree of accuracy required. In the topic of Networks, the students were asked to draw a Network which had an Euler Circuit but not a Hamilton Path.

Students at times worked in pairs so the pair negotiated a solution, supporting mixed ability learning. Students copied each other, but this was observable, and added to the teacher's knowledge.

Matching Cards

Questions and answers were written on cards (see Figure 2) to be matched by small groups of strategically chosen students. The students would check each other's work and finally check against answers keys. Discussions evolved and lively debates occurred, which led to interested, self-motivated students. Extra preparation was involved in making the cards as the students requested to keep the cards for extra practice.

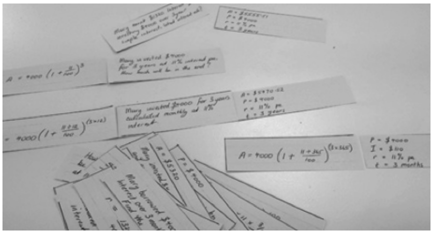


Figure 2. Matching cards, interest rate questions and answers.

Clickers

Quizdom handheld devices are a brand of *Clickers* which were used in some classes. Students could input responses to text book questions, quizzes or questions from a *PowerPoint* and the devices instantly communicated with the teachers' laptops. Which questions the students were doing well and common mistakes were immediately evident for the teacher. This system did involve preparation, but the students appreciated the instant responses and reported they felt more interested and focused in class.

Random Name Generators and Class Discussion

In traditional *chalk 'n talk* lessons, the teacher asks the class a question and some students put their hands up to respond. Unfortunately, some students volunteer and some students put their eyes down and doze. With random name generators any student may be asked, so all students are more alert. Random name generators are names on either icy-pole sticks or computer programs. Wait time or check-with-a-friend time is important so all students have a chance of giving an acceptable and appropriate response, and the pressure is on the partnership not the individual. Once one student has replied, another student is called on to respond, which results in interactive class discussion rather than isolated questions and answers.

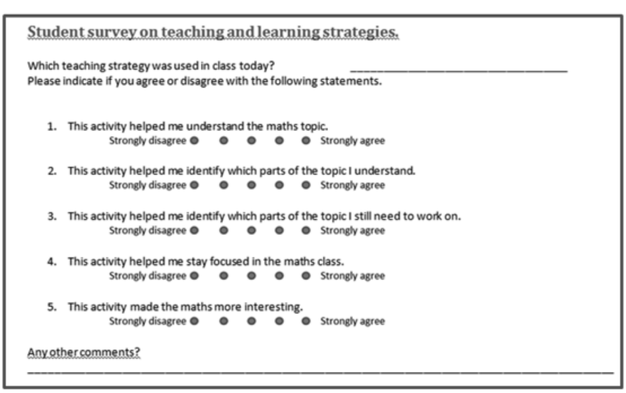
Action Research Project

The AFL strategies described here were trialled as a result of a whole school focus initiated by the Leadership Team, and they were also the centre of an Action Research Project. A mixed method was employed with interviews and surveys of teachers and students.

The Action Research Project formally involved 107 General Mathematics students in five classes. After 14 of the AFL lessons, surveys were completed by the students. Interviews were conducted with 20 students who were chosen for their mixed ability. The survey questions are shown in Figure 3, and the interviews were based on the same questions.

Students' Responses

Students completed the short survey after the AFL lesson. The surveys appear to have been taken very seriously as there were few responses which were straight down the page. Students would volunteer to stay after the class to complete their surveys in more detail. Students' survey results showed they found the lessons added interest and focus to the class. Most students felt the AFL strategies helped them identify their strengths and weaknesses, and it changed their focus from completing the work to understanding the work. Discussions became more on-task. Even asking the students to complete the surveys was seen as a positive activity as the students seemed to consider that the teachers cared about their learning and were trying to cater to the students' needs.


Student survey on teaching and learning strategies.

Which teaching strategy was used in class today? ____________________
Please indicate if you agree or disagree with the following statements.

1. This activity helped me understand the maths topic.
   Strongly disagree ● ● ● ● ● Strongly agree
2. This activity helped me identify which parts of the topic I understand.
   Strongly disagree ● ● ● ● ● Strongly agree
3. This activity helped me identify which parts of the topic I still need to work on.
   Strongly disagree ● ● ● ● ● Strongly agree
4. This activity helped me stay focused in the maths class.
   Strongly disagree ● ● ● ● ● Strongly agree
5. This activity made the maths more interesting.
   Strongly disagree ● ● ● ● ● Strongly agree

Any other comments?
____

Figure 3. Student survey on AFL strategies.

As expected, some students did not welcome the change in routine and the pressure of demonstrating their knowledge to the teacher, but they were fewer than two or three students per class. Some students appeared to resent the group work, but these were students who had previously completed very little mathematics work, so their low work ethic became more apparent.

Some quotes from the students' surveys and interviews include:

- *It helped give everyone a chance to answer a question and talk about the workings out.*
- *It should not be used all the time.*
- *It was an overall fun activity, can we do it again?*
- *Gets you thinking.*
- *I liked working with partners.*
- *I didn't like it but I see that it was good for other students.*
- *It was good because we could see if we were correct straight away.*

Teacher Responses

The teachers chose the AFL activities they would trial themselves, and reported back to the group. Occasionally the teachers observed each other's classes. Teachers commented on the extra preparation, especially for the Clickers classes, but the teachers also appreciated that the material could be used again the following year. Teachers also shared *Good Questions* (Cline, Zullo, & VonEpps, 2012) and lesson ideas. One teacher was concerned with the increased noise level in the classroom with the group activity, but other teachers commented on the improved quality of the conversations between the students. Oral language exchanges heightened the students' understanding of the mathematical concepts being learned.

Summative Results

The results for the previous three years tests and examinations are shown in Table 1. It can be seen that 2013 test and examination results have improved over the previous two years by between 7% and 14%. The tests, examination and curriculum were consistent in this three year period. The same staff were involved and the student profile remained constant. The only change was with the implementation of AFL practices.

Table 1. *Summative Assessment using Topic Tests for Year 11 General Mathematics (Approximately 100 Students each Year).*

Year	**Networks**	**Arithmetic**	**Linear equations**	**Finance**	**Exam**
	Mean%(SD)	Mean%(SD)	Mean%(SD)	Mean%(SD)	Mean%(SD)
2011	61 (21)	54 (27)	63 (19)	51 (24)	52 (26)
2012	55 (23)	53 (19)	64 (18)	48 (21)	52 (18)
2013	69 (20)	65 (21)	72 (18)	57 (22)	60 (19)

Action Research

Action research is a reflective process whereby the teachers set goals and trial practices to improve their own teaching (Thomas, 2011). The process is cyclical, with goals being set, trials carried out, results considered, and then goals being set again. Each month the teachers met to discuss and evaluate the lessons. Teachers encouraged and supported each other. The colleagues of the research practitioner (the author), all found the process worthwhile, especially as it was a component of the professional review process, so it was an excellent way to collect data on teacher improvement.

Conclusions

Whilst secondary school mathematics has been reported as boring and difficult, the feedback from Formative Assessment was found to add interest and challenge to our classes. With a variety of lesson styles and teachers' collaboration on lesson planning it also supported the students' learning – a win-win for teaching and learning.

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VALUES ALIGNMENT STRATEGIES FOR MATHEMATICS TEACHERS **

Darren Fitzpatrick

Hillcrest Christian College

> *Understanding student and teacher values can sustain and enrich teaching, broaden student values and create an avenue of cross-cultural bridge building to improve the communication and learning of mathematics.*

Introduction

Compared with teaching other subjects, there are specific challenges that come with being a teacher of mathematics. We teach a subject that generations of people have found difficult. Some of our students may have been conditioned by their parents and by other teachers to dislike mathematics. I have heard teaching colleagues speak in a derogatory way about mathematics, and even boast about how bad they were at mathematics. These are negative aspects in educational culture which face teachers of mathematics when they face a classroom of students. We need to find a way to overcome these factors to successfully equip students to learn and excel in mathematics.

The Two Phase Appraisal Model for Difficult Classroom Situations

There have been times when I have looked at a class with students who were disengaged and not learning and thought to myself, "Is this worth it? Do I really want to keep doing this?" I'm sure some other teachers have had moments like this. There is a need for teachers to be able to think effectively through these situations. A useful model for difficult situations in the classroom is the two phase appraisal model, developed by Den Brok, van der Want, Beijaard, & Wubbels (2013). In this model, the first evaluation concerns the meaning of what is occurring, and the second appraisal is the perceived response the teacher can take in the situation. So, for example, the first evaluation is a teacher's interpretation of the situation and what it means. Two possible, and unhelpful, teacher responses to difficult classroom situations could be to think "What terrible students" or "I am a terrible teacher". This is the first appraisal. The second appraisal leads to action. For the response "What terrible students," the teacher could plan how they could change these students or plan to teach different students; for "I am a terrible teacher," the teacher would either plan to leave the profession or to improve as a teacher. There are a number of more productive ways of appraising and responding to these situations.

Values and the Two Phase Appraisal Model

One model I have personally found to be useful in responding to difficult classroom situations involves values and the two phase appraisal model. This helped me understand that what had been happening in the classroom at those times were in contrast to what I value in the classroom. This realisation helps me to change the behaviour and attitudes in the class to be aligned with what I value. I value learning, students developing in skills and conceptual understanding of mathematics, respectful interactions, and helping people improve their approach to learning. If the classroom and school context don't enable these values to be honoured, I would not want, long term, to teach in that scenario. The realisation that this double appraisal model is in action helps frame first appraisals that lead to actionable second appraisals. Without the awareness and capacity to do this, teaching would be unsustainable. It is stressful to teach in a situation where our values aren't honoured, and emotional exhaustion is the key predictor of burnout in the teaching profession (Richardson, Watt, & Devos, 2013).

Teaching Mathematics as a Form of Cross-Cultural Communication

I regard teaching mathematics as cross-cultural communication. Culture can be defined as including shared values, attitudes, beliefs and behaviours (McDaniel, Samovar, & Porter, 2009). While this is an area that requires further research, I believe that mathematics teachers have different values, attitudes, beliefs, and behaviours concerning mathematics compared with many students. Certainly, this has been my experience and, anecdotally, is the experience of many mathematics teachers with whom I have spoken. Importantly, perhaps this is also the experience of some students, attempting to engage with a culture that is different and which they do not understand.

Three Categories of Values Relevant to Mathematics Learning and Teaching

An important aspect of culture is values, and in the past few decades a body of research has been built focusing on values in the mathematical classroom. Bishop (1996) analysed the types of values in three categories; mathematical, general education, and mathematical educational values. The first category, mathematical values, refers to what is important to researchers in the area of mathematics. They are subject specific values such as abstract symbolic rationalism. The second category, general education, refers to general life values such as honesty and hard work, which are not specific to mathematics but are still relevant within mathematics education (Commonwealth of Australia, 2005; Commonwealth of Australia, 2011). The third category, mathematical educational values, refers to values which are focused upon the mathematical classroom and learning within mathematics subjects at school such as problem solving, working out, fun and competition. At the same time, students each have their own values for mathematics, general educational values, and mathematics in the classroom, which may be different from each other and different from their teacher. I believe that a core challenge of a mathematics teacher is to bridge this gap through effective communication informed by cultural understanding.

Changing Our Approach to Teaching Mathematics

You may be wondering why it is our job to bridge the gap. Well, I caught up with a friend of mine for coffee recently, and she arrived looking drained. I asked her why she was so tired and she explained that a co-worker had an approach which was incompatible with her own, and the co-worker was not capable of working in a different way, so to make the situation feasible, she changed her own approach. As teachers we are adults and professionals, and we should be more capable of changing our approaches than the students who are children and, in general, less mature. The capacity to be flexible was described by Chen (2009) as a key characteristic of intercultural communication competence; the capacity to communicate with people who are from a different culture. According to Spitzberg (2000), this communication requires considerable motivation, efficacy and competence which are mutually reinforcing, that is, an increase in any of these three areas can lead to an increase in the other two. Following this, according to Spitzberg, a deeper understanding of the other culture can be developed which can lead to composure, other-centeredness, expressiveness, and an increase in adaptation. These are the characteristics and skills which are required to cross the gap between mathematics and students. As can be discerned from the description, it can require a considerable investment of time and effort to develop this capacity, but it is, in my view, an essential skill in teaching mathematics.

Values Alignment: Redefining, Reprioritising and Complementing

Seah (2014) used the phrase *values alignment* to describe this bridging of the cultural gap through redefining, reprioritising and complementing. Redefining involves the teacher finding an alternative way to express their own value which is acceptable to the students, while reprioritising involves the teacher changing their approach and complementing involves exploring to find ways of expressing both values in the same situation, according to Seah. These are three useful strategies which can be employed where the teacher holds to different values compared to the students.

My View of Values Alignment

The way I personally visualise the process of bridging of a cultural gap through values alignment involves a Venn diagram with two overlapping circles representing the values of the teacher and the values of the students. My key strategy is to use the shared values, the overlapping area, as much as possible, and use the values of the student-alone to promote the values which I hold. It is important to use shared values, as this means we are all working in our preferred approach. It is also important to use the student-alone values, because this can be used to motivate the students to complete actions that the teacher values. For example, if students value marks, marks can be allocated for review work, for homework and for other tasks, not just summative tests and examinations. This means both of our contrasting values can be satisfied to some degree. This approach can require creativity, for example, many students seem to value socialisation in the classroom. I do value this as well, but I also value focused concentration as being important for learning. My way of balancing this is to encourage students to have 80% of the focus on their work and 20% of their focus on their conversation. The hope would then be that students can develop and grow in their values. So, if they genuinely understand a mathematical concept and learn through superior concentration, this experience can help them value understanding mathematics and focused learning for their future mathematics classes.

Tool for Understanding Teacher and Student Values: WIFI

One way of coming to a deeper understanding of what students value in school mathematics learning is a survey developed by Seah (2013). The *What I Find Important* (WIFI) in mathematics learning questionnaire is an efficient way of mapping what students appreciate and value in mathematics learning. A scoping study by Seah and Peng (2012) investigated 19 values in the Australian and Swedish contexts. The values held in common by both nationalities included explanation, sharing and fun, but there were a number of other values which were held by students in Australia which were not held by students in Sweden. The other values in Australian mathematical classes were certainty, clarification, competition, efficiency, examples, multi-modal presentation, resources, and working (Seah & Peng, 2012). This survey is one tool which can be used to map teacher values and student values. These values can then be analysed for differences, between the student and teachers groups and within each group. A deeper understanding of what students' value can help within the model of cross-cultural communication, to understand the student-specific values. Teachers can then appreciate these student values and use them to promote some of the values which they hold and which the students may not hold.

Conclusion: Values Alignment Can be Fruitful

In conclusion, an explicit understanding of our own values and those of others can help us sustain our teaching and enrich our teaching. From this position of understanding, strategies can be employed to fruitfully utilise the values of the students to promote the learning of mathematics, and, hopefully, enrich and broaden the values of the students at the same time. The development and implementation of these strategies can require an investment of energy, because changing our own approaches can be difficult, but the benefits far outweigh the required investment.

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THE MATHS ONLINE INTERVIEW STILL ROCKS! *

Pam Hammond

ROPA Consultancy (ropa1@bigpond.com)

This interview, originally developed for the Early Numeracy Research Project (ENRP) incorporates the 'Domains' of Number, Space and Measurement, with many tasks including aspects of the Proficiency strands. It is available on the Victorian Department of Education and Early Childhood Development (DEECD) website. Many primary teachers have been using the Mathematics Interview for some years and know its value in planning for student needs. Is it still relevant? Are there connections to the Australian Curriculum: Mathematics? This paper will show that there are connections, and the relevance of the Interview now and into the future. It will also discuss an on-line teaching resource linked to tasks on the Mathematics Interview.

The Mathematics Interview

The Mathematics Interview was developed as part of the Early Numeracy Research Project (Clarke et al., 2002), funded by the Victorian Department of Education, Catholic Education Office (Melbourne) and the Association of Independent Schools Victoria. It is a task-based, one-on-one assessment, which focuses on students showing what they know, the mental strategies they choose to use and explaining their mathematical thinking.

The Interview includes domains in Number (counting, place value, addition and subtraction, multiplication and division); Measurement (time, length, mass); Space (properties of shape, visualisation and orientation), with aspects of the Proficiency strands embedded in many of the tasks. The interview was available as a booklet, distributed to all government schools in Term 4, 2001, which can be downloaded from the DEECD website. It is available as the Mathematics On-line Interview to all Victorian government school teachers.

There are around 60 tasks within this Interview (with several sub-tasks in many cases), although students do not move through all of these, as the pathway through the Interview is determined by a student's response to each task. If schools have access to the CDROM version or are able to log into the On-line Interview, profiles can be generated once the Interview has been used with a class.

- Group profiles show all points of growth in Number, Measurement or Space and highlight the students who have achieved each during the Interview, including the date at which this was achieved.
- The Point of Growth Summary profile gives data on the number of students in the group/class who have achieved each Point of Growth and the percentage that represents. This can be useful for schools wishing to track achievement over time.
- The Question Summary profile shows how many students in the group/class have successfully completed all aspects of each task and the percentage that represents.

The knowledge of students' mathematical strategies gained from the Interview can be analysed to determine the needs of the class overall and the areas to be targeted with specific students or groups of students.

Links Between the Interview and the Australian Curriculum: Mathematics

Most of the 60 tasks in the Interview have been linked to the Australian Curriculum: Mathematics to assist teachers to see the level that a student demonstrates when successfully completing these tasks (see HREF1).

This linking is organised under each Content Strand, showing links between the Interview tasks and the achievement standard, content description and levels Foundation to 5 of *Number and Algebra,* and *Measurement and Geometry.* Some of these connections are outlined in Figure 1. A list of the Mathematics content strands, sub strands and descriptions that do not directly link to the Mathematics Interview have also been included.

Section A: COUNTING		Interview Point of Growth	ENRP Growth Point		Achievement Standard	Content description
Task 1	**Teddy task**	PoG 12. Confidently count a collection of around 20 objects	GP 2. Counting collections	**Level F**	**Number and Algebra** Students connect number names and numerals with sets of up to 20 elements, estimate the size of these sets, and use counting strategies to solve problems that involve comparing, combining and separating these sets. Students order the first 10 elements of a set.	**Number and Place Value** Establish understanding of the language and processes of counting by naming numbers in sequences, initially to and from 20, moving from any starting point
1	**Teddy task (successfully counts 10 objects but unsuccessful beyond 10)**	PoG 1. Know some number names but have difficulty stating them in sequence above 10				
		PoG 2. Rote count the number sequence to 10 but are unable to reliably count a collection of that size				
		PoG 7. Count a collection of around 10 objects				
2a only	**Counting forwards, backwards, and breaking the sequence**	PoG 11. Rote count the number sequence to at least 20	GP 1. Rote counting	**F**		**Number and Place Value** Establish understanding of the language and processes of counting by naming numbers in sequences, initially to and from 20, moving from any starting point
2	**Counting forwards, backwards, and breaking sequence (a, b, c, d, e)**	PoG 18. Count by 1s forward/backward from various starting points between 1 and 100	GP 3. Counting by 1s (forward/backward, including variable starting points; before/after)			
3	**Before and after task (a, b)**	PoG 19. Know numbers before and after a given number up to 100		**1**	**Number and Algebra** Students count to and from 100 and locate these numbers on a number line. They partition numbers using place value and carry out simple additions and subtractions, using counting strategies.	**Number and place value** Develop confidence with number sequences to and from 100 by ones from any starting point. Skip count by twos, fives and tens starting from zero
4	**Counting from 0 by 10s, 5s, and 2s**	PoG 22. Count from 0 by 2s, 5s and 10s to a given target	GP 4. Counting from 0 by 10s, 5s, and 2s			

Figure 1. Mapping the Mathematics Online Interview to the Australian Curriculum: Mathematics
Retrieved 3rd October, 2014 from HREF1

The Mathematics Continuum

The Mathematics Continuum is a resource, available to all (Department, Catholic and Independent schools) on the DEECD website. The Continuum is organised into units of work that provide evidence-based indicators of progress, teaching strategies and activities that are illustrative of these.

Indicators of progress are points on the learning continuum that highlight critical understandings required by students in order to progress, providing insight into the mathematical content within the unit and along with the illustrations and teaching strategies, can enhance teachers' *pedagogical content knowledge* (Shulman, 1987).

Once the analysis of student needs has occurred, decisions need to be made on the appropriate mathematical focus for the next sessions and the context and materials most suitable to move the students forward in their mathematical growth. There are many resources teachers can and do use (including their own proven activities), with the units of work on the Continuum offering excellent examples of tasks. Units of work have been developed in the strands – Number and Algebra, Measurement and Geometry, Statistics and Probability from Foundation to Level 10. (See HREF2)

Links Between the Interview and the Mathematics Continuum

Tasks in the Mathematics Interview have been linked to the activities within units of work on the Continuum (from Level Foundation up to Level 6) to indicate readiness for continuing the learning. (See HREF3)

An Example of a Unit of Work and the Links to the Interview

Each unit of work in the Mathematics Continuum includes indicators of progress related to the focus of the unit, illustrations of these, teaching strategies and activities. For Level 2 in the Australian Curriculum: Mathematics, in a unit 'Early division ideas', these elements are shown in Figure 2.

Indicators of progress: Success depends on students identifying that the operation of division applies in both partition and quotition situations.

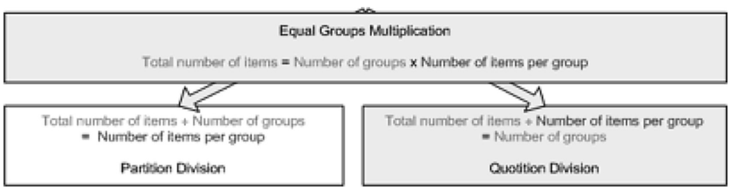

Illustration 1: Partition is the first division situation recognised by students. Students achieving Level 2 write number sentences such as: 20 ÷ 4 = 5 to describe partition (sharing). *Note:* In partition, total number of items is given, and number of groups. The answer is the number of *items per group.*

Illustration 2: Quotition division follows recognition of partition. A student who understands division applies to quotition situations can write number sentences such as:

20 ÷ 4 = 5 to describe e.g., *Shop sold apples in packs of 4. So 20 apples would fill 5 packs.* Quotition situations are often summarised as *"How many 4s in 20?"*

Note: In a quotition division, total number of items is given, and number of items per group. Answer is number of *groups.*

Illustration 4:

Examples of the types of the tasks that would be illustrative of division concepts, aligned from the Mathematics Online Interview:

- Question 28 – *Sharing teddies on mats*
- Question 31 – *Teddies at the movies*

Teaching strategies: Students working towards Level 3 extend their understanding of division to quotition. They will similarly model it with diagrams, learn to recognise that quotition problems can be expressed with division, and solve it (at this level) with materials or by knowledge of multiplication facts. Each of the four basic operations of arithmetic applies to real world situations that differ in context (whether the story is about lollies or pets, for example) and which also differ in structure. In preparing to use mathematics in their everyday life, students must encounter many situations with different structures.

Activity 1: Modelling quotition

Step 1: Select problem solved by quotition division e.g., Fruit shop sold apples in packs of 4. How many packs can be filled with 20 apples?

Step 2: Discuss how to solve the problem with concrete materials and how to record the process with a diagram, e.g., select 20 counters to represent the apples. Take out one group of 4 representing a pack, then take another group of 4 until there are none left.

Count the number of groups (5). Students interpret this in the context of the problem: 20 apples can fill 5 packs. Draw diagram and annotate with number sentence and a description.

Step 3: Ask what they notice about the result and how it links to what they know. Help them link with multiplication fact 5 × 4 = 20. Students point out the 5 groups of 4 apples, which makes 20 apples altogether.

Step 4: Choose quotition problems with different contexts and numbers for students to solve using materials and diagrams, and to make the link to known multiplication facts.

Step 5: As students consolidate this knowledge, they can omit the materials and the drawings, but they should still link their answers explicitly to the known multiplication fact.

Activity 3: Number line and repeated subtraction

Students illustrate quotition division on a number line, following the same general steps as in Activity 1. The links to repeated subtraction can also be seen this way. The number line can be abstract (numbers only) or in a context (e.g., 20 means 20 metres or 20 dollars).

How many 3s in 18? Start at 18 on the number line then move down in steps of 3, until you reach 0. There are 6 steps so there are 6 threes in 18.

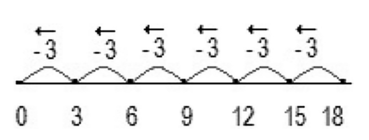

Figure 2. Early division ideas Level 2. [Adapted from HREF4]

Conclusion

This article set out to show the connections between the Australian Curriculum: Mathematics and the Mathematics Interview and hence its relevance to teachers in implementing the new curriculum. I believe, very strongly, that it most certainly is relevant into the future. As mentioned, the strength of the Interview is that it focuses on students' mathematical thinking and the strategies they choose to use, an integral component of the Australian Curriculum: Mathematics. Connections to the new curriculum are clear, so data from this interview will offer teachers support in their planning against the new curriculum and what should be included in their mathematics program for the mathematical growth of their students to continue.

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HREF3. Retrieved 3rd October 2014 from: http://www.education.vic.gov.au/studentlearning/teachingresources/maths/interview/default.htm

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THE IMPORTANCE OF FRACTIONS IN BEING A SUCCESSFUL MATHEMATICS STUDENT **

Derek Hurrell and Lorraine Day

University of Notre Dame Australia

Research and experience tells us that fractions are not easy to teach and learn and that a solid conceptual understanding of fractions has far reaching ramifications in the secondary mathematics classroom. A failure to acknowledge and remediate 'inherited' challenges to this understanding may result in the further success of students being severely impeded.

The Teaching and Learning of Fractions

The ideas associated with fractions are amongst some of the most complex, yet important, concepts that students encounter (Behr, Lesh, & Post, 1983). One of the reasons that fractions are so important is that they provide the foundations on which other number work, algebraic thinking, and proportional reasoning are built (Booth & Newton, 2012; Brown & Quinn, 2007; Chinnappan, 2005; Wu, 2001). Siegler, Fazio, Bailey, and Zhou (2012) stated:

> Poor fraction knowledge in elementary school predicts low mathematics achievement and algebra knowledge in high school, even after controlling for general cognitive abilities, knowledge of whole number arithmetic, and family education and income. High school algebra teachers recognize this relation; they rank students' fraction knowledge as among the largest impediments to success in their course. (p.18)

Apart from these number based aspects of mathematical understanding, the topic of fractions also supports students to make critical conceptual links in such strands as geometry and measurement (Pitkethly & Hunting 1996; Siemon, 2003). Other areas, such as probability, statistics, and rates of change require a solid conceptual understanding of fractions.

Fractions are Difficult

Fractions enjoy attention in curricula around the world but have long been documented to cause students difficulties (Anthony & Ding, 2011; Anthony & Walshaw, 2003; Capraro, 2005; Nunes & Bryant, 2009; Usiskin, 2007; Wu, 2005). Indeed, Smith (2002, p. 3) asserted; "No area of school mathematics is as mathematically rich, cognitively complicated, and difficult to teach as fractions, ratios and proportion." The National Assessment of Educational Progress Report, declared that fractions are "exceedingly difficult for children to master" (Braswell et al., 2001, p. 5). Chapin and Johnson (2000) concluded that "... this complex topic causes more trouble for elementary and middle school students than any other area of mathematics" (p. 73). Indeed Baba and Iwasaki (2003) found that some university students could not understand fractions.

Fractions and the Secondary School Student

In the content descriptors from The Australian Curriculum: Mathematics (ACARA, 2014) the writers stopped using the word fractions after the Year Seven entries. This provides a clear indication that the expectation is, that students on reaching secondary school, are already well on their way to a sound understanding of fractions. This expectation would perhaps be reasonable if not for the fact, as previously established, that fractions are hard to teach and to learn.

Data from a study by Clarke, Mitchell, and Roche (2005) determined that when asked to identify what fraction part D was of a circle (see Figure 1) only 43% of the Year Sixes asked (n=323) could do so. When asked to identify the size of Part B of the circle, although 86% of the same students responded with the correct answer, the remaining students, that is, 14% of Year Six students, could not identify a fraction they had been working with for most of their schooling years. The Australian Curriculum: Mathematics (2014) asks Year Twos to "Recognise and interpret common uses of halves, quarters and eighths of shapes and collections (ACMNA033)."

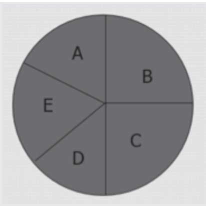

Figure 1. Fractions of a circle.

In 1998, Carpenter, Kepner, Corbitt, Lindquist, and Reys reported that when estimating the answer to a problem (see Figure 2), and being asked to select the correct response from a bank of four responses, only 24 percent of 13 year olds chose correctly. From the authors' own experience in working with adults in 2013, when asked to identify an equivalent improper fraction for a mixed numeral (see Figure 3) just over 12% of 224 people could not do so.

$$\frac{7}{8} + \frac{12}{13}$$

Which is the closest answer?
1, 2, 19, 21

Figure 2. Estimating a fraction sum.

Which fraction has the same value as $2\frac{3}{4}$

$\frac{8}{4}$ $\frac{9}{4}$ $\frac{11}{4}$ $\frac{14}{4}$

Figure 3. Equivalent improper fraction.

Fractions in the 'Real' World

Elizabeth Green (2014) wrote an article for the New York Times which was adapted from her soon to be published book. In this article she wrote:

> One of the most vivid arithmetic failings displayed by Americans occurred in the early 1980s, when the A&W restaurant chain released a new hamburger to rival the McDonald's Quarter Pounder. With a third-pound of beef, the A&W burger had more meat than the Quarter Pounder; in taste tests, customers preferred A&W's burger. And it was less expensive. A lavish A&W television and radio marketing campaign cited these benefits. Yet instead of leaping at the great value, customers snubbed it.
>
> Only when the company held customer focus groups did it become clear why. The Third Pounder presented the American public with a test in fractions. And we failed. Misunderstanding the value of one-third, customers believed they were being overcharged. Why, they asked the researchers, should they pay the same amount for a third of a pound of

> meat as they did for a quarter-pound of meat at McDonald's. The '4' in '¼,' larger than the '3' in '⅓,' led them astray. (p. 5)

Quite clearly then, many students are not arriving into the secondary mathematics classroom with a well-developed understanding of fractions, and neither are they leaving with one. If we want the students to be in a position to successfully work with fractions then an acceptance of the need to determine what understandings the students actually have, and then reach back into the curriculum and find the 'big' ideas which will help the students to progress are necessary. Once the 'big' ideas are identified it is also necessary to determine the pedagogies which are suitable for the secondary student.

Pedagogical Considerations

Walker (2004) wrote about understanding and recognising the nature of early adolescents and how this should inform the school curriculum. It is a recognition that although the mathematical content that needs to be covered is developmentally suitable for primary school aged students, the pedagogy adopted needs to be cognisant of the psychological needs of the adolescent. Carmichael and Hay (2008) argued that these psychological needs are competence, autonomy (being able to have some choice in what they do), and social-relatedness (personal involvement). Consequently, a guided discovery method (Dinham & Rowe, 2008) (which these authors might argue as being a 'true' representation of social-constructivism) which involves hands-on activities, discussion, cooperative learning, and co-construction of the learning seems highly appropriate pedagogy.

One approach that can be used is based upon C.R.A. (Concrete, Representational, Abstract) sequence of instruction. Although much of the research shows the efficacy of this approach with students who have learning difficulties (e.g. Allsopp, Kyger, & Lovin, 2007; Flores, 2009), the work of Witzel, Mercer and Miller (2003) supported the use of C.R.A. with 'mainstream' students when they showed that algebra students from non-special needs backgrounds benefited more through the use of C.R.A. than they did from a more traditional approach.

Examples of the C.R.A. Approach with Fractions

A Chocolate Dilemma

One activity which has been used to demonstrate fractions as quotients (division) is an activity we refer to as "A Chocolate Dilemma." In this activity there are three chairs at the front of the class and 10 students standing outside of the classroom. On chair one there is one bar of chocolate, chair two has two bars of chocolate and chair three has three bars of chocolate. The bars of chocolate are all the same size. The ten students are invited into the room one at a time and choose to stand behind whichever chair they wish, knowing that when all of the students are standing behind the chair of their choice, they will share the chocolate on that chair. The aim is for the students to maximise the amount of chocolate they can get.

For instance, if two students stand behind chair one they will each receive a half of the bar. If five students stand behind the chair with two bars they will receive two-fifths of a bar each. The remaining three students behind the chair with three bars will end up with one bar each. In this example the people who stood behind the chair with three bars are the happiest! The number of chairs, bars of chocolate and students can be varied to suit and to extend the potential of the problem. Using the students themselves provides a kinaesthetic experience which we interpret as being as 'concrete' as you can get.

Once the students have experienced physically being part of the activity, if required, the activity can then be replicated through modelling with three pieces of paper to represent chairs, six counters of one colour to represent the chocolate bars and 10 counters of another colour to represent the students, or as purely a diagrammatic representation. The diagrammatic form can then be made abstract through the use of symbols, where the denominator is the number of students and the numerator is the number of chocolate bars.

Fraction Estimation

Another activity which employs the C.R.A. approach effectively is a task about estimating fractions. It is adapted from *maths300* (Education Services Australia, 2010). In this activity the concrete element consists of ropes of different lengths and some household pegs. Students are asked to estimate how far along the rope various fractions are situated and they check their estimations by folding the rope into equal partitions. Different lengths of rope are used so that the students recognise that they need to identify what constitutes the whole before they divide it up into equal parts. An important part of this task is the ability of the students to visualise the whole and mentally divide it up into equal parts, then count the number of parts they need. They soon become quite adept at folding the rope into equal parts to check their results.

The recording of this activity is done by representing the rope as a line segment and marking on the line where the fraction is positioned. Different length ropes are represented by shorter or longer line segments to reinforce the idea that the size of the partition is dependent on the size of the whole that is being partitioned. Once students have had lots of experience with the concrete activity and representing them on paper, it is time to move to the excellent software that is available for this activity.

The *Fractions Estimation* software in the maths300 package (Education Services Australia, 2010) has three different representations for fraction estimations, strips (which relate to the rope), towers, and pies. There is an option to include different fractions depending on the level at which students are working. One of the beauties is that there is no pre-partitioning so the students have to visualise how to break the whole up into equal parts. Once students are happy with their choice the computer then partitions the whole into equal parts so that the students can see how close they were to the actual fraction (see Figure 4). There is an option to work backwards which uses all three of the previous representations. For this option students need to enter either the numerator or the denominator of the fraction that is pictured (see Figure 5), which is quite abstract. Four other options that relate fractions to decimal and percentages are included to encourage students to be able to move between them flexibly.

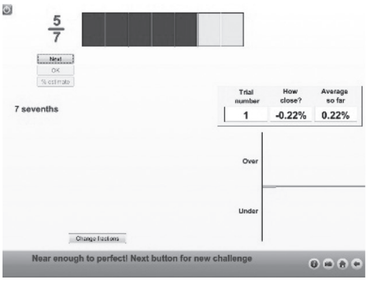

Figure 4. Fraction strip.

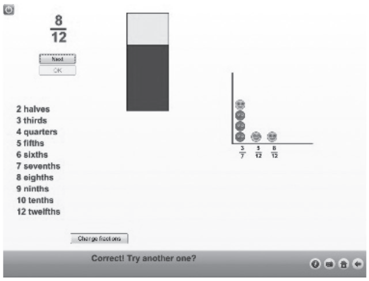

Figure 5. Fraction number.

These two examples demonstrate appropriate pedagogical approaches for use with students in the middle years of schooling to assist them to develop a conceptual understanding of fractions that is so important if they are to succeed in secondary school mathematics.

Some Concluding Remarks

The teaching and learning of fractions is not easy, and the ramifications of the students not developing a sound understanding can be far reaching. Simply assuming that the students who are entering the secondary mathematics classroom have the required understanding to allow them to employ their fraction knowledge in other areas of the mathematics curriculum is perhaps unwise. It could be the case that some time devoted to the conceptual development of fractions will prove to be a wise investment. There could truly be a solid argument that, what could be considered as a step back in the curriculum, will actually mean that later the strides will be greater. We would very much like to suggest that in some instances a re-examination of the pedagogy employed to teach fractions may prove to be fruitful.

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NURTURING NUMBER SENSE **

Derek Hurrell

The University of Notre Dame Australia

Paul Swan

The authors appreciate that the standard 'ten a day' random question lesson introduction is one that is used by many teachers as a manner in which to start the students thinking in a mathematical way. However we would like to suggest a way in which to preserve all of the benefits of the standard ten questions but enhance the range of outcomes.

Introduction

Definitions of number sense vary. For the purpose of this discussion the authors have adopted the following definition:

> A person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful and efficient strategies for managing numerical situations. (McIntosh, Reys, Reys, Bana, & Farrell, 1997, p. 3)

The words *flexible* and *inclination* resonated with the authors who both work with student teachers, many of whom, who in the name of developing 'fluency,' have been shackled by years of boring repetitive basic facts drill and as a result have little or no inclination to pursue mathematics in any form. This is not to belittle the role of fluency which is described by Watson and Sullivan (2008) as readily having facts and knowledge to allow people to carry out procedures accurately, efficiently, flexibly, and appropriately. Indeed the authors are in total agreement with Sullivan (2011) when he describes fluency as one of the six principles on which a sound curriculum should be based. Sullivan noted that fluency may be developed in two distinct ways: "Fluency is important, and it can be developed in two ways: by short everyday practice of mental processes; and by practice, reinforcement and prompting transfer of learnt skills" (p. 29).

Askew (2009) tells us that it is decidedly important for students to be fluent, and that "judicious use of practice can help develop fluency and strategies. It's not a case of practice or problem-solving, understanding or rapid recall, but practice and problem-solving, understanding and (appropriate) rapid recall" (p. 27). Further, Askew (2009) exhorts teachers to keep the activities simple, make sure the activities are done briefly but often, involve everyone, and are focused on the mathematics.

Clearly then, the issue is not whether fluency is important, but rather, which tasks we could offer students to develop fluency, and how we could make the practice meaningful, promote flexibility, and create a disposition in students to use their mathematics. An example of such a task might be ten basic multiplication fact questions. Rather though than randomly set ten questions, the questions might be restructured into logical groupings. The first two questions might involve multiplying by zero or one, questions 9 and 10 might involve the use of the word 'product.' The focus then becomes not on who scored ten out of ten, but rather who experienced difficulty with questions 9 and 10. If a large number of students experienced trouble with these two questions, possibly the word 'product' is the cause of the difficulty and steps may be taken to rectify this difficulty. On the next day the same facts could be given, but 'turned around.' If **3 × 4** was asked on Monday, then **4 × 3** might be asked on Tuesday. The teacher could then use the information that was gleaned to assess the students' knowledge of the commutative property of multiplication.

The purpose of a routine is that the students soon learn the 'rule/s of the routine' and start to focus on the mathematics. The teacher is also freed up to watch how individual students cope with the routine. Eventually students and teachers will begin to adjust or 'tweak' the routines to better suit their needs.

Task 1: Number Hokey Pokey

Four sheets of A3 paper are laid out on the ground as shown in Figure 1 and five children stand around them ready to place a foot on one sheet. There are four active participants and the fifth student is there to monitor and record. The rules are: one person, one foot, one number. For example if one person places a foot on the number two, another on four and still another on eight, the numbers are added and the total is 14. Placing a foot on the number four, another foot on the number two and still another foot on the number one, produces a total of seven.

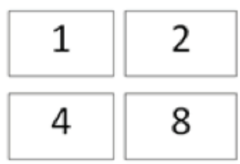

Figure 1. Four sheets of A3 paper.

Set the challenge: "What other totals may be made by adding different combinations of the numbers 'under foot'?" Remember to restrict the players to one foot, one person, one number. Students should soon find that all the numbers from 1 to 15 may be made by placing a foot on one or more of the numbers.

If the teacher wishes to focus on fluency, then he/she would simply call out totals like nine, and watch and listen as students within their groups step on the numbers one and eight. Students would be inclined to participate in such a group task for several reasons. It gives the opportunity for the students to get up of out their seats and work as a group, which helps to breakdown the perception that some students have, that mathematics should always been seen as a solitary, quiet subject, where conversation and co-operation are inappropriate. Further this activity has the twin appeals of gross motor movement and, of course, noise! Other totals could be called out, and the students encouraged to rapidly respond. When the students return to their seats the task can be replicated at an individual level on an A4 version of the board with students placing counters onto the board to represent feet.

However, while this task is interesting it could become so much more. Consider Figure 2: Readers will recognise the four proficiency strands from the Australian Curriculum: Mathematics (ACARA, 2012). If the strands are viewed in a linear or hierarchical fashion then a task will be viewed simply in terms of whether it is a fluency task, *or* a problem solving task, *or* a task that helps develop understanding *or* reasoning.

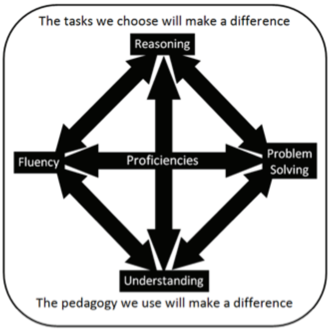

Figure 2. Linking the proficiencies through task selection and appropriate pedagogy (ACARA 2012).

The authors suggest that if the proficiencies were viewed as supporting and building on each other, then the tasks that are chosen and the pedagogy that is applied would help students become more flexible in their thinking. For instance, in this case, Number Hokey Pokey may be primarily a fluency task but the pedagogy employed gives ample opportunity for the students to articulate why they are doing what they are doing, and articulation is an important tool towards understanding and reasoning.

Further, revisiting the Number Hokey Pokey task with a more global view of the proficiencies, it becomes evident that the task is open to variation. Consider what happens if the sheet with '1' written on it is turned over. What totals may be created? What do you notice about the totals? Why are they all even? In these variations there is a clear invitation for the students to engage in problem solving, as simply put, problem solving is where the path to solution is not obvious.

The game may be further extended in various ways, for example:

- Allow more than one foot per number. For example if 4 feet are allowed on any number, what is the largest number that may be formed? What totals may be formed ($4 \times 8 = 32$)? by different combinations of numbers? For example, $2 \times 8 + 1 \times 4 + 1 \times 2 = 22$.
- One number, e.g., 1 may be covered and the original rule one foot, one person, one number applied. Ask the students to determine what sort of totals can and cannot be formed (i.e., odd numbers).
- Play with a single player and allow two feet and one hand to be used.
- Allow different numbers of feet to be used. For example, if 3 feet are used then students would be using the 3 times table, four feet, the 4 times table, 8 feet, the 8 times table.

- Extend to further doubles, 16, 32.
- Recording various totals and investigate them back at the desk.
- Rename the numbers as 2^0 (1), 2^1 (2), 2^2 (4), 2^3 (8), 2^4 (16) etc.

Task 2: Missing digits 45**8

A number is written down with one or more of the digits replaced by an asterisk. The students are then given tasks using this number and all of its possible permutations (due to the possibilities that the asterisk/s could represent). For instance, if 45**8 is the number given, one task could be:

- Replace the asterisks so that this five-digit number can be divided by 9 without leaving a remainder.
- How many answers are there?
- How will you know when you have found them all?

Consider how this task varies from the traditional ' 4591 ÷ 9 =', question. Whereas the traditional question asks the student to make a decision about what the question is asking them to accomplish, and then perform the operation through application of an algorithm or a mental computation strategy, the 45**8 is the number given, one task could be:

- Replace the asterisks so that this five-digit number can be divided by 9 without leaving a remainder.
- How many answers are there?
- How will you know when you have found them all?

Consider how this task varies from the traditional ' 4591 ÷ 9 =', question. Whereas the traditional question asks the student to make a decision about what the question is asking them to accomplish, and then perform the operation through application of an algorithm or a mental computation strategy, the 45**8 question asks for all of this, and also has the added advantages of involving a great deal of problem solving, reasoning, and understanding. Once again the supporting and overlapping nature of the proficiencies is apparent, and, through looking at the task through these different lenses, it shows the richness of the learning that can come from this task. It promotes a sense of both numbers and operations that is perhaps not apparent through merely developing fluency by following a procedure. Experience would also indicate that students find the missing digits task so much more engaging than the standard algorithm task.

Task 3: Estimate

Estimating, defined by Labato (1993) as "guessing with a little bit of problem solving" (p. 350), is an essential component of working mathematically; that is the linking and application of the various proficiencies. More than just problem solving it also requires the application of fluency and reasoning. Consider for example, estimating the result of multiplying 32 by 74. A student might recall a basic fact $3 \times 7 = 21$ as part of performing the estimation, while at the same time reasoning that in reality this calculation involves 3 tens and 7 tens so the result is 21 hundreds or 2100. Thinking the problem through a little further, the student might reason that the answer will be a little larger than 2100 because the numbers in the original calculation were larger than 30×70. Students can then be asked to explore 37×72 and 37×78 and consider what happens to the result when the unit (ones) digits are greater than five, that is, what adjustments have to be made to the estimate to make it more reasonable?

Observing number patterns, which is at the heart of algebraic reasoning and arithmetical reasoning will help alert students to possible calculation errors. Consider what happens whenever the following calculations are performed.

Odd number + odd number,

Even number + even number

Odd number + even number

Consider how observing these patterns would alert a student to an error in a calculation such as 347 + 489. If the student identifies that both numbers are odd, then the answer will have to be even, that is, end in a 0, 2, 4, 6, or 8. If the answer is not even, then the answer is incorrect.

Estimation is not restricted to number. Consider the idea of bracketing measurements. Students experiencing difficulty estimating measures can be taught to refine their estimates via the *Bracketing* game. Students might experience difficulty estimating the length of a typical classroom, so the teacher might provide extreme limits such as "Is it longer than 1 cm and shorter than 1 km?" to which the students must respond "yes" or "no." Further limits are set such as 1 m and 100 m, 2 m and 20 m and so on until a consensus is reached. All the time students are linking their knowledge of measures and reasoning.

Task 4: If I know, then …

The purpose of this routine is to help students make connections between number facts. Establish a single fact. For example, $5 \times 8 = 40$. This fact is written in the middle of the board and the class then participates in a brainstorm of connected facts. Consider how these facts are connected to $5 \times 8 = 40$.

10×8	50×8	$40 \div 5$
10×4	5×80	$40 \div 8$
5×16	50×80	$400 \div 50$
2.5×8	0.5×8	$40 \div 0.5$
		$40 \times \frac{1}{2}$
		$40 \div \frac{1}{2}$

The rich web of related facts not only develops fluency, but improves understanding and provides opportunities for students to reason.

Conclusion

One of the aims of the Australian Curriculum was to make the curriculum 'deep' rather than 'wide' (National Curriculum Board, 2009), and to keep the content descriptions concise and few in number (Sullivan, 2012). This is partly as a recognition that teachers often feel under pressure to 'cover' the curriculum. Even with this in mind, a reasonably informed inspection of the Australian Curriculum: Mathematics (ACARA, 2012) would bring into question whether, when taken individually, all of the content descriptors can be achieved in a single year, in a manner which also satisfies the need to 'cover' the proficiency strands. Clearly, by choosing better tasks, the opportunity exists not only to cover a lot more, but promote flexibility of thought and improve a student's inclination to use and apply the mathematics he or she has been taught. This flexibility of thought and inclination to use the mathematics is key to the development of number sense.

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ENRICHING THE MATHEMATICS CLASSROOM: A VISIT TO THE VIRTUAL ISLAND **

Minh Huynh and James Baglin

RMIT University

Changing the way students perceive mathematics by enriching the classroom experience will assist in redirecting Australia's secondary school students into much needed mathematical and statistical pathways. This paper details a 12 month project that utilised an innovative online virtual world, known as the Island, to engage and educate secondary school students in the statistics components of the Australian mathematics curriculum. The project provided educators with innovative teaching resources that were piloted in Years 8 to 11 mathematics classrooms in the northern suburbs of Melbourne. The resources were designed to improve students' understanding of the role of statistics in scientific research and help develop their statistical skills.

Australian Mathematical Graduates Are in Decline

Australia is facing a deficient mathematical and statistical skill base, with numbers *"falling short of national needs"* (Australian Academy of Science, 2006). As of 2011, only 0.46% of all Australian students graduated with a mathematical or statistical background (Organisation for Economic Co-operation and Development, 2011), well below the OECD average of 1.19%. For Australia to remain a globally competitive economy, something must be done to increase mathematics and statistics graduates.

This decline can be traced back to poor attitudes towards mathematics in secondary education. In a 2012 report, the Australian Chief Scientist stated that high school students are unable to engage with the content taught because it fails to be interesting, realistic, and practical (Chubb, 2012). This is unfortunate because positive attitudes not only drive mathematical learning but are also a key outcome as well (Evans, 2007). Similarly, in statistics education, there exists a consistent positive relationship between statistics attitudes and achievement (Schultz & Koshino, 1998). By improving student attitudes early on (e.g., in middle secondary school), we can enrich their high school experience, which may lead to building tertiary and career pathways in mathematical and statistical disciplines.

Unfortunately, actively engaging students in learning statistics is a challenging task due to the negative connotation that students associate with the subject. Traditional high school statistics classes have been dominated by mathematical thinking (Stuart, 2003), with teachers placing an emphasis on calculations and equations. Existing literature however, advocates that teachers should instead be employing a problem solving approach (PSA) when teaching statistics, which actively engages students in the learning process by contextualising the data collection process (Marriot, Davies, & Gibson, 2009).

Statistics and the Problem Solving Approach

PSA is a four stage process for solving problems using statistics and can be summarised by four activities: specify the problem and plan; collect data from a variety of suitable sources; process and represent the data; and interpret and discuss the results. The activities are cyclic because it may be necessary to refine the initial approach to solving a problem and repeat the process again (Marriot et al., 2009). Garfield and Ben-Zvi (2007) have summarised current research on the PSA and have suggested that teaching statistics through solving problems can improve students' skills as they interact with real data.

Unfortunately, conducting real data investigations within a classroom setting can be a challenge, with concerns about safety and ethics limiting the type of activity that can be carried out. As such, it is difficult for teachers to prepare and design data-based PSA tasks which are interesting and meaningful for students. Hence, there are an abundance of projects and activities revolved around flipping coins and rolling dices, which only vaguely represent real world applications of statistics.

The introduction of technology within the classroom however has provided an enormous spectrum for new approaches to teaching. Research suggests that technology can enhance learning through cognitive, metacognitive, and affective channels (Barkatsas, 2005). For example, simulations have gained popularity in recent years due to their ability to represent "analogies of real world situations" (Prensky, 2007, p. 128). Simulations allow for learning opportunities and experiences that might otherwise be impossible to create within a traditional classroom – learning experiences that are authentic models of the real world. An example of a simulated environment for examining real world situations in statistics is the *Island* (Bulmer & Haladyn, 2011).

The Island

The *Island* is a free, online, virtual human population that can be used for simulating scientific research and data investigations (Bulmer & Haladyn, 2011). Readers can access the *Island* at http://island.maths.uq.edu.au, by emailing james.baglin@rmit.edu.au. The *Island* runs through any device with a web browser and internet connection. The *Island* aims to provide students with an enriching virtual scientific research experience that improves their understanding of the application of statistics in scientific research.

The *Island* platform provides students with the means to visualise data collection by offering them the ability to conduct research in real time with a population of virtual humans living on the *Island*, which is divided into a multitude of towns, suburbs and households (see Figure 1). The Islanders live in geographically dispersed towns, have ancestors, detailed histories and the potential to die from a range of diseases, providing the opportunity for the student to study archival data as well as experimental designs. This program actively engages students in experiencing important statistical concepts, whilst keeping students engaged and interested.

The process of designing an experiment, including recruiting Islanders to participate in their study, randomly assigning Islanders to conditions, manipulating treatment variables to perform an experiment and collecting data to analyse the effect of the treatment are all a part of the Island experience. There are over 200 different independent and dependent variables, referred to as tasks, for students to choose from. Examples of tasks include studying blood pressure and the administration of caffeinated drinks, surveys of anxiety and blood serotonin levels. Figure 2 provides an example of an Islander having their blood pressure measured in real time. The extensive range of potential research designs ensures that there is something to engage every classroom.

The *Island*, implemented extensively at RMIT University, was designed to virtualise, simulate, and even 'gamify' scientific data collection. Positive student feedback from the tertiary setting has indicated enjoyment, interest and improved understanding of the role of statistics and data analysis in science (Baglin, Bedford & Bulmer, 2013). In 2013, a pilot study was conducted utilising the *Island* for a secondary school in Melbourne, Australia (Huynh, Baglin, & Bedford, 2014).

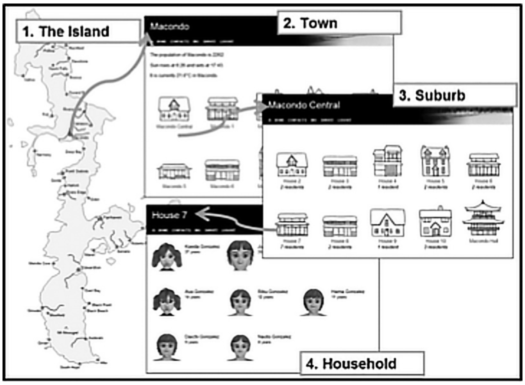


Figure 1. The Island broken up into towns, suburbs and households.

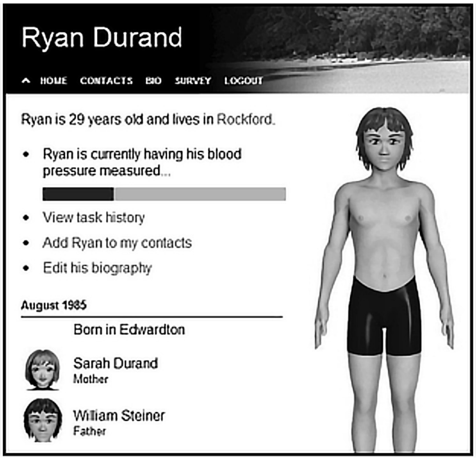


Figure 2. An Islander having his blood pressure measured.

The researchers surveyed students before and after on three self-reported attitude items including: "I enjoy learning about statistics", "Statistics is important in everyday life" and, "I am interested and willing to gain further knowledge about statistics". Students' rated their agreement to each item using a 5-point scale ranging from "Strongly disagree" to "Strongly agree". The pilot study reported that there were significant improvements to students' attitudes after experiencing a one-off *Island* class activity led by the researcher. This activity had students using the *Island* to collect their own individual data and for answering a research question pertaining to the effects of Diazepam on burst sprinting ability.

Feedback from this school was overwhelming, with many students advocating for the use of the *Island* for other classes. One student mentioned that they enjoyed *"collecting the data because it felt like conducting a real experiment."* Usually, collecting data is a tedious activity for high school students because of the age and ethical limitations set upon them. The *Island* program is able to overcome these restrictions by allowing the students to experiment on their own virtual islanders, in a scientific manner. One of the student participants from this study summarised this best when she said: *"It was a great way to test otherwise unethical theories."*

The *Island* in Schools Project

Based on the success of the Island in tertiary education and the promising results of Huynh et al. (2014) in a secondary school, a larger Island in Schools project was initiated. This project aimed to extend use of the *Island* into secondary schools and provide innovative teaching resources that captivate students in applying statistical methods. The project had the following main objectives:

- Use the *Island* to create innovative and enriching teaching resources aligned to the statistics strand of the Australian Mathematics Curriculum
- Enrich and contextualise students' experience of the statistics and mathematics curriculum and thereby improve their attitudes and knowledge of career pathways in the field
- Provide professional development opportunities for secondary school mathematics and science teachers

The project's innovative "Island-based" learning resources are freely available to all mathematics and science secondary school teachers in Australia. Teachers can request electronic copies of the project's resources by sending an email to james.baglin@rmit.edu.au.

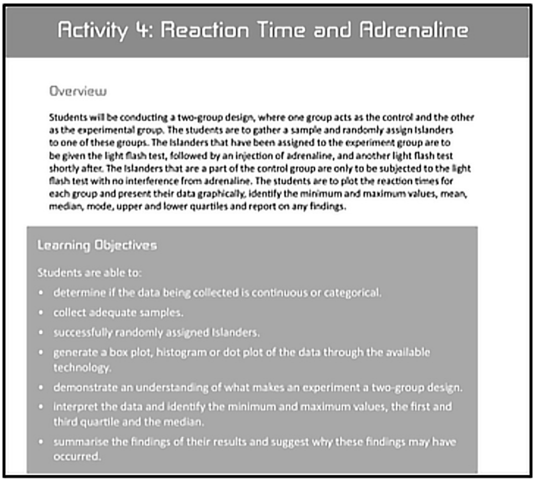
The main outcome of the project was a teacher manual entitled: *The Island in Schools: Innovating the Teaching of Statistics and Data Analysis in Years 9 and 10.* The manual was comprised of 10 *Island*-based activities and a project-based assignment guide, each aligned with the Australian Mathematics Curriculum for Year 9 and 10 students. The ten activities included the following:

- Activity 1: Gathering samples for a population.
- Activity 2: The effect of temperature on exercise performance.
- Activity 3: Balance as a repeated measures design.
- Activity 4: Reaction time and adrenaline.
- Activity 5: Gender and mental health survey.
- Activity 6: Height and liver size.

- Activity 7: Peak flow meter and age.
- Activity 8: Ball bouncing and age.
- Activity 9: Island climate change – longitudinal data.
- Activity 10: Population growth – longitudinal data.
- Extra activity: Open ended research investigation.

Each activity was designed around the PSA approach and the four stages: Plan, Collect, Process, and Discuss. A motivating overview and activity learning objectives are included for each activity (See Figure 3). Teachers will be able to utilise this section to align their planned classroom teaching to the required curriculum (See Figure 4).

Each of the activities also includes a teacher's overview, which outlines the details of the activity and provides example instructions that a teacher can employ when explaining the activity to their students (see Figure 5). The other sections of the activity *(Plan, Process, and Discuss)* also have their own teacher overview component with recommended instructions for students.



Activity 4: Reaction Time and Adrenaline

Overview

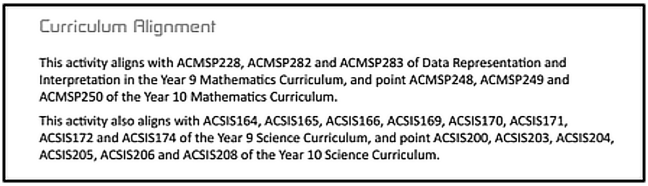
Students will be conducting a two-group design, where one group acts as the control and the other as the experimental group. The students are to gather a sample and randomly assign Islanders to one of these groups. The Islanders that have been assigned to the experiment group are to be given the light flash test, followed by an injection of adrenaline, and another light flash test shortly after. The Islanders that are a part of the control group are only to be subjected to the light flash test with no interference from adrenaline. The students are to plot the reaction times for each group and present their data graphically, identify the minimum and maximum values, mean, median, mode, upper and lower quartiles and report on any findings.

Learning Objectives

Students are able to:

- determine if the data being collected is continuous or categorical.
- collect adequate samples.
- successfully randomly assigned Islanders.
- generate a box plot, histogram or dot plot of the data through the available technology.
- demonstrate an understanding of what makes an experiment a two-group design.
- interpret the data and identify the minimum and maximum values, the first and third quartile and the median.
- summarise the findings of their results and suggest why these findings may have occurred.

Figure 3. Activity 4 overview and learning objectives.



Curriculum Alignment

This activity aligns with ACMSP228, ACMSP282 and ACMSP283 of Data Representation and Interpretation in the Year 9 Mathematics Curriculum, and point ACMSP248, ACMSP249 and ACMSP250 of the Year 10 Mathematics Curriculum.

This activity also aligns with ACSIS164, ACSIS165, ACSIS166, ACSIS169, ACSIS170, ACSIS171, ACSIS172 and ACSIS174 of the Year 9 Science Curriculum, and point ACSIS200, ACSIS203, ACSIS204, ACSIS205, ACSIS206 and ACSIS208 of the Year 10 Science Curriculum.

Figure 4. Activity 4: Alignment with the Australian Mathematics Curriculum [Note the preliminary overlap with the Science Curriculum which will be further explored in future projects].

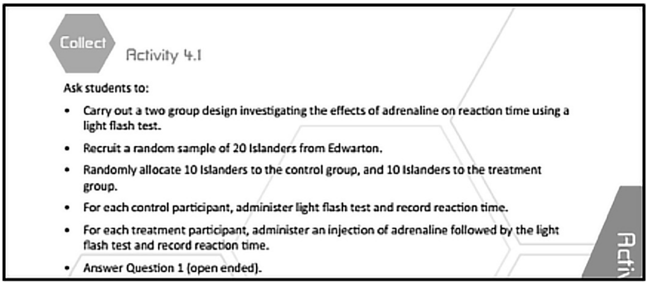


Figure 5. Activity 4: Teacher overview (Data Collection component).

Student worksheets for each task were designed to have students utilise the PSA for answering the research questions. Students are expected to (a) read the research scenario and plan their tasks, (b) collect the data for the research scenario, (c) process and represent the data, and (d) discuss the results of their analysis. As an example, Activity 4 of the *Island* manual requires students to compare reaction times between a group given adrenaline and a group not given adrenaline. The students are required to use the *Island* to acquire their data and answer the questions relating to the research topic (for this activity:). Examples of the worksheet questions include:

- Describing whether a variable is continuous or categorical.
- Calculating the five number summary for the collected data.
- Short answer questions critiquing the research scenario.
- Generating box plots / scatter plots / bar charts with the students collected data.
- Drawing a conclusion about the research question in relation to the students' data / findings.

Figure 6 provides a screenshot of the first page of the student worksheet for Activity 4. This figure details the background information for the research scenario that the students should consider *(Plan)* and provides them with a table for their collected data *(Collect)*.

Plan

Specify the Problem and Plan

The race car drivers in Edwarton are wanting to learn more about how they can improve the speed at which they take off after the light turns green. There is evidence to suggest that increasing adrenaline will improve reaction time to the light change; thus, the researchers in Edwarton wish to investigate whether increasing adrenaline improves reaction time. As a researcher in Edwarton, you are required to carry out a two group design investigating the effects of adrenaline on reaction time using a light flash test.

Collect

Collect Data

Activity 4.1 Start by recruiting a random sample of 20 Islanders from Edwarton. Randomly allocate 10 Islanders to the control group, and 10 Islanders to the treatment group. For each control participant, administer the light flash test. For each treatment participant, administer an injection of adrenaline followed by the light flash test. Record your results in the table below.

Control Group		Treatment Group	
Participant name	Reaction time	Participant name	Reaction time
#1		#11	
#2		#12	
#3		#13	
#4		#14	
#5		#15	
#6		#16	
#7		#17	
#8		#18	
#9		#19	
#10		#20	

Figure 6. Student worksheet for Activity 4 (page 1 of 3).

Each *Island* activity includes a marking rubric that teachers can utilise to grade and assess the students work. These rubrics contain a description of the type of task being conducted and an associated comment (for certain items) that teachers can use as sample feedback.

Finally, the Island Manual includes a section pertaining to an open-ended investigation that teachers may like to utilise within their classes. These investigation tasks are Island activities with no specified research question or instructions. Instead, the choice of topic is entirely up to the teacher and / or student. The researchers have noted that these open-ended investigation tasks are extremely effective in contextualising the data and for encouraging students to utilise the PSA for tackling data driven scenarios. Without any explicit instructions or guidelines, the students have to carefully plan their research topic and investigate how to collect the data from the *Island*. By collecting their own data, the activity becomes personalised and unique to the student, making the processing and discussion stages of their work much more meaningful to them.

At the time of writing, the project was piloting the Island-based resources within regular mathematics secondary school classes at four partnered schools in the northern suburbs of

Melbourne. Prior to this implementation, partnered secondary schools were provided with two professional development options to prepare teachers for the implementation: a school-based workshop option prior to the first Island activity, or an expert-led option where teachers sat in a class as an advanced student. The second option was suggested by a teacher at one of the partnered schools as a sustainable professional development model that maximised use of a teacher's time. The benefit of the second model of professional development was that the teachers could observe how expert statistics instructors, from the project team, taught statistical concepts using the *Island* resources. The teachers would then have the knowledge to use the resources in future classes and years. These activities are led by the class teacher with minimal assistance from the researchers.

Future Work and Partnerships

While the current project focused on the Year 9 and 10 statistics component of the mathematics curriculum, some of the partnered schools adapted the resources to Year 8 science classes and Year 11 general mathematics classes. There appeared to be a strong demand for the *Island* resources across the curriculum. Upon completion of the *Island*-based activities, the researchers will be able to examine changes in student attitudes towards statistics. Based upon the results of the pilot study conducted in 2013, the researchers are expecting the *Island*-based resource to also generate an improvement in student attitudes. In addition, the researchers will be seeking feedback from both students and teachers in relation to the *Island* activities (student worksheets, marking rubrics, etc.) to carry out into future phases of our research into the *Island* in schools.

The project has received additional funding to continue into 2015. This will involve further refinement of the resources, additional activities to cover Year 7 and 8 curriculum, mapping learning outcomes to the science curriculum, and a broader program of professional development to existing and newly partnered secondary schools in Victoria and Queensland. Schools interested in joining as project partners are encouraged to contact the project leader, James Baglin (james.baglin@rmit.edu.au).

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MAGIC TRICKS IN THE TEACHING OF THE ARITHMETIC MEAN **

Pearlyn Lim

National Institute of Education, Singapore

Fu Siqiang

Fairfield Methodist School (Primary), Singapore

Pamela Foong Hui Shan

Fairfield Methodist School (Primary), Singapore

Previous studies have found that magic tricks, when chosen thoughtfully, can enhance students' conceptual understanding. However, studies on the impact of magic tricks on particular topics in mathematics are limited. Our study seeks to find out if magic tricks have a positive impact on students' interest in the topic of average, and if they enhance students' understanding of the concept of the arithmetic mean, which we will refer to as 'average' in the paper.

Introduction

There exist several suggestions on how magic involving mathematics can be used in the classroom. However, there are few studies showing how the incorporation of magic tricks has impacted on students in their learning of mathematics. Also, the limited studies we came across were on the topics of algebra and probability, and did not cover the whole range of mathematics topics.

The available literature on the use of magic tricks is encouraging. For instance, Koirala and Goodwin (2000) observed that magic tricks could enhance students' conceptual understanding in algebra. In addition, they found that magic tricks could motivate students to seek additional concepts related to the mathematics explained through the tricks. Lesser and Glickman (2009), in a study on using magic in the teaching of statistics, found that thoughtfully chosen magic tricks could propel students to dig deeper to discuss and understand what was being taught.

Thus, we were motivated to come up with suitable magic tricks to teach the concept of the arithmetic mean, which we will refer to as average in this paper since it is the term used in our textbooks. The topic was chosen as we found that whilst students could easily apply the formula for average ("Total divided by number of items"), they had difficulty articulating what 'average' meant. We hoped that magic tricks would help students understand the meaning of average. We also wanted to find out if the use of magic tricks significantly raised the level of interest in a topic. Our research questions were:

1. Does the incorporation of magic tricks enhance students' understanding of the concept of average?
2. Does the incorporation of magic tricks increase students' level of interest in the topic of average?

Research Design and Methodology

Students

The students involved were 77 Primary 5 students from two mixed ability classes. The first class had 36 students while the second class had 41 students. The two classes were similar in mathematics ability, as could be gauged from their Primary 4 end-of-year and Primary 5 mid-year mathematics results.

Class 1 – The 'Experimental' Group

Two magic tricks were used to introduce the concept of average to the first class. The first magic trick was carried out as follows:

- The teacher showed three ropes of different lengths (Figure 1).
- The magic occurred in the next step where the teacher joined the three ropes together to form a long rope (Figure 2). This was possible as the shorter ropes had magnetic ends.
- Manipulating the parts of the rope, the teacher showed three ropes again, but this time, the ropes were of equal length. He introduced the word 'average' by saying

"This is the average length of the three ropes." (Figure 3).

- The teacher verbally restated what had been done. He told the class that at first he had three ropes of different lengths. He joined the three ropes to form one single long rope. He then separated the long rope into three ropes of equal lengths.
- The teacher then tried to elicit the formula for average by asking students how the length of the long rope could be found. They responded by saying that the three shorter lengths were added. Students continued by responding that when the teacher separated the long rope into three shorter ropes of equal lengths, he was dividing the total length by three.

The teacher wrote down the formula for average on the board:

$$\text{Average} = \frac{\text{Total}}{\text{Number of items}}$$

Figure 1. Three ropes of different length.

Figure 2. Three ropes joined into one long rope.

Figure 3. Three ropes of the same length.

The second magic trick was carried out in the first class as follows:

- The teacher told a story in which six children were reporting their scores on a spelling test to their mother. The respective scores of the children were 1, 2, 5, 2, 6 and 2. The mother repeatedly asked the children what their average score was. Using the formula derived previously, students worked out the average score.
- The teacher brought out six dice and displayed the six children's scores using the dice (Figure 4). He placed the six dice in a box, covered the box and shook the box once.

- When he opened the box, each of the six dice showed the average score of 3, confirming what students had calculated using the formula. (Figure 5)

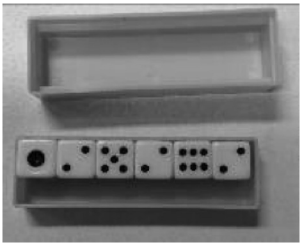

Figure 4. Different scores.

Figure 5. Averaged scores.

After the second magic trick, the teacher brought out the ropes again and reiterated that the ropes were of different lengths to start with. The average length of the three ropes was obtained by finding out the total length of the three ropes, then dividing the total by three. However, it did not mean that the three ropes were of the same length.

Class 2 – The 'Control' Group

In the second class, the teacher taught the concept of average in the traditional way. She gave examples of how average is used in real life and did not make use of magic tricks. The teacher also elicited responses from the students on what they thought average is. From here, the teacher explained that average was a hypothetical evening-out of the total. She then wrote the formula for average on the board.

To ensure that the students understood the concept of average, the teacher arranged the students into groups and each student had to use items that they could find around them to explain the concept to their group mates. The teacher then rounded off the lesson by practising a few questions on average with the class.

At the end of the introductory lesson, students in each class sat for a short quiz that assessed their ability to apply the formula for average. After all the lessons on average had been conducted, the students also completed a feedback form which required responses on how much they enjoyed the lessons and whether the strategies used in the lessons helped them visualise how the formula for average is derived. Students were also asked to write down what they enjoyed most in the lessons on average, and to explain the meaning of average in their own words.

Instruments

As mentioned, the students sat a quiz on average after the first lesson. The quiz comprised the following six questions:

1. What is the formula for average?
2. Find the average of 4, 6, 10, 12 and 18.
3. Find the average of 1.1, 3.2, 6.9 and 0.4.
4. Find the average of 19, 23, 58, 0 and 20.
5. The average of 3 numbers is 10. If the first number is 7 and the second number is 8, what is the third number?
6. The average height of Steve and Bob is 1.75 m. Steve is 0.3 m taller than Bob. How tall is Steve?

The students also completed a feedback form at the end of all the lessons on average. Using a five-point scale, they gave their opinions on the following statements:

1. I enjoyed the lessons on average.
2. I am able to state the definition of average.
3. The lessons helped me visualise how the formula for average is derived.

Students were also asked to write down what they most enjoyed in the lesson and to explain the meaning of 'average' in their own words.

Results and Findings

Quiz Performance

Table 1 shows the percentage of students who gave the correct answers for each question on the quiz in each class. For every question, a higher percentage of students in Class 1 answered correctly. A t-test was conducted to determine if there was a significant difference between the performance of the two classes. A p-value of 0.07 was obtained. The p-value is slightly higher than our chosen significance level of 0.05. (The p-value is the estimated probability of rejecting the null hypothesis of a study question when that hypothesis is true. The null hypothesis in this case is that there is no difference in the performance of Class 1 and Class 2 while the alternative hypothesis is that Class 1 performs better than Class 2.)

Table 1
Quiz Performance of Classes

	Class 1	Class 2
Question 1	72	22
Question 2	97	78
Question 3	86	56
Question 4	86	68
Question 5	100	73
Question 6	25	22

Feedback on Lesson

The feedback form was completed by 34 students from Class 1 and 39 students from Class 2. Table 2 shows the percentage of students giving the respective responses to the statement "I enjoyed the lessons on average".

Table 2
Responses on "I enjoyed the lessons on average."

	Class 1	Class 2
Strongly agree	76	31
Agree	15	36
Neutral	3	26
Disagree	0	2
Strongly disagree	6	5

A higher percentage of students in Class 1 than in Class 2 strongly agreed that they enjoyed the lessons on average. A t-test was conducted and showed that there was a statistically significant difference between the responses of the two classes.

Table 3 shows the percentage of students giving the respective responses to the statement "The lessons help me visualise how the formula for average is derived."

Table 3

Responses on "The lessons help me visualise how the formula for average is derived."

	Class 1	Class 2
Strongly agree	53	33
Agree	29	36
Neutral	6	18
Disagree	6	5
Strongly disagree	6	8

A higher percentage of students in Class 1 than in Class 2 strongly agreed that the lessons on average helped them visualise how the formula for average is derived.

The students also wrote down what they enjoyed most in the lessons. In Class 1, 18 students wrote that they enjoyed the magic tricks the most while 16 students wrote that they enjoyed the story accompanying the second magic trick the most. In Class 2, the responses were varied. 6 students enjoyed the quiz, 6 enjoyed the problem sums they had to do, 3 liked the PowerPoint slides used by the teacher and 2 liked applying the formula to real life situations. The other students mentioned that they liked the way the teacher explained about average or that they enjoyed the topic in general.

Lastly, the students were asked to explain the meaning of 'average' in their own words. In Class 1, 59% of the students gave acceptable definitions. The unacceptable definitions in Class 1 involved ideas of half the total amount or the middle value. In Class 2, 56% of the students gave acceptable definitions. The unacceptable definitions involved ideas of getting equal quantities, the minimum amount someone should get or estimation.

Discussion

Magic Tricks Enhance Students' Understanding

From the students' quiz performance, it could be seen that a higher percentage of students in Class 1 than Class 2 answered every question correctly. The p-value of 0.07 was close to the 0.05 level of significance that we had chosen. As the last question on the quiz required more mathematical manipulation than just applying the formula for average, when removed from the data, the p-value became 0.0166, which now showed that the difference in quiz performance between the two classes was significant. Hence, there is support that the incorporation of magic tricks enhances students' understanding of the concept of average.

In Class 1 and Class 2, 53% and 33% of students respectively strongly agreed that the lessons helped them visualise how the formula for average is derived. This gave further support that magic tricks enhanced their understanding of the concept of average.

We also studied how students defined average in the two classes. A higher percentage of students in Class 1 defined average acceptably than in Class 2. However, the difference in percentage was small, so this finding was inconclusive. Students were only asked to write down the definition of average at the end of the topic, so other activities used by the respective teachers in subsequent lessons could have contributed to their understanding of the topic. How well the teachers explained the problems in subsequent lessons could have also influenced students' ability to define average.

Magic Tricks Increase Students' Level of Interest

From observing students' responses to the statement "I enjoyed the lessons on average", it was evident that Class 1 enjoyed the topic more than Class 2. The p-value of 0.0024 on the t-test supported the hypothesis that the responses of the two classes were significantly different.

In Class 1, the majority of students cited magic tricks as the most enjoyable part of the lesson. In Class 2, the varied responses showed that there was no single factor in the lesson that captivated them. To increase students' level of interest in mathematics, it might be effective to incorporate something memorable or visual such as magic tricks in lessons.

Closing Comments

Our findings indicate that carefully chosen magic tricks can help students understand the concept of average better. To further enhance their understanding of the concept, we suggest that the teacher brings back the ropes at the end and asks the students if the trick really is magic. Discussion can bring out the understanding that joining the three ropes into one and then separating them into three equal parts is simply an illustration of what the average (arithmetic mean) of a set of numbers means.

Finally, the incorporation of magic tricks also increases the level of enjoyment in the mathematics classroom. Therefore, we advocate the use of magic tricks in mathematics lessons and we will continue to explore how magic tricks can be used to teach other topics.

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BIGGEST LOSER PROJECT *

Robert Money and Donald Smith

Ian Lowe

The Mathematical Association of Victoria

This is an update (from Money, 2013) on the mathematics element in the Gambling Issues cross-curricular project that has been piloted in 2014 in Victorian Year 9/10 and VCAL classes. The mathematics unit covers the probability and statistics content of the Year 9/10 curriculum through the contexts of electronic gaming machines ('pokies') and sports gambling. The extensive use of simulations ensures that, in the long term, the 'biggest loser' is the punter.

Context and Goals

Around 5% of adolescents are problem gamblers and the majority of problem gamblers started their gambling in their adolescent years. In the context of increased advertising and on-line accessibility of gambling, schools are increasingly taking on the responsibility to deal with this issue. (Delfabbro, Lambos, King, & Puglies, 2009; Phillips, 2013)

2014 saw some piloting of the Biggest Loser materials in Victorian Year 9/10 and VCAL classes. On the basis of this experience the project will move to more extensive school-based trials early in 2015. The project was initiated by the Mathematical Association of Victoria, in particular in response to weaknesses in other attempts to cover key ideas. The classroom materials include 2–3 week units of work in mathematics and English and one week units in Social Education (Civics and Citizenship) and Health Education. A key concern for the trial will be how best to integrate these units.

In response to criticism of previous gambling education projects (Productivity Commission Report, 2010, p. 9.20), pre- and post-questionnaires are designed to assess any positive impact on student attitudes and misconceptions.

The Mathematics Unit

The Mathematics Unit is designed with options that allow full coverage of the requirements of the AusVELS/Australian curriculum Years 9/10 statistics and probability curriculum. The gambling context also provides students with four key understandings:

- Probabilities, which sum to 1, are distinguished from 'bet to payout' ratios, which sum to more than 1 in unfair games.
- The concept of long term expectation (probability × payout) is used to measure the 'unfairness' in different forms of gambling.
- Gambling simulations are used to focus attention on nine common misconceptions, in particular to persuade students that *Chance has no memory*.
- Study of variability in the outcomes of repeated simulations of unfair games illustrates the maxim *Short term gain – long term pain*. (The longer you play the less chance you have of coming out ahead.)

All the mathematics lessons involve simulation, firstly with dice or cards but eventually with spreadsheets. Worksheets are provided for those teachers who wish to use them. The spreadsheets are focussed towards preparing students for the key discussions that take place as the culmination of the learning experience.

Fair and Unfair Games

A game called *Lucky colours of sunshine* (Smith, 2012) takes on a bit of the excitement and colour of a poker machine. Students 'pay' one dollar to bet on which one of four cards is 'drawn from a hat' and conjecture what would be a fair payout for a win. Spreadsheet simulation is then used to confirm that a fair payout would be \$4 (i.e., a return of the \$1 bet plus winnings of \$3). The probability of a win, is equal to the bet to payout ratio, \$1: \$4.

In a second session, the *Lucky colours of sunshine* game is played again with the payout set at \$2. Extended simulation is used to obtain an average loss per game that is consistent with later theoretical calculations. The students learn:

For the fair game:

$$\text{Expected long term return} = \text{probability multiplied by payout} = \tfrac{1}{4}\times 4 = 100\%$$

For the unfair game:

$$\text{Expected long term return} = \tfrac{1}{4}\times 2 = 50 \text{ cents in the dollar}$$

Discussion should cover the meaning of expected long term return and expected long term loss. The expected long term loss is 50 cents per dollar, but this does NOT mean 50 cents in every dollar – and it does NOT mean that if you keep playing for a long time you will end up losing only 50% of the money you started with.

Students can then be given the opportunity to identify or invent other fair and unfair games. A good example of a fair game is a Melbourne Cup sweep in which all the money collected is eventually returned to the winners.

Sports Betting

Students simulate and then analyse payouts for win/loss/draw outcomes in soccer or win/loss in games such as tennis or netball. Where required an introductory worksheet covers the relationship between payouts and amounts won.

The teacher will have no trouble in providing real data such as those shown in Table 1.

Table 1. *Example of Payouts on a $1 Bet in Football Betting*

	Payouts on a $1 bet		
Match	Home team win	Home team lose	Draw
Arsenal v Reading	1.25	6.00	11.00

In using these data students find that the bet to payout ratios add to more than 1.

In this case $\dfrac{1}{1.25}+\dfrac{1}{6}+\dfrac{1}{11}=1.0575575$

from which the student can calculate the real probabilities and the expected return on a bet as shown in Table 2.

Table 2. *Calculating Real Probabilities*

Calculating Real Probabilities			
Arsenal versus Reading	Home team win	Home team lose	Draw
Real probabilities with a sum of 1 (divide by 1.0575575)	0.75645	0.15759	0.08596

Expected long term return on a bet on a win = probability payout

= $(0.75645×1.25)

= $0.9546

The expected long term loss is 5.44 cents in the dollar.

The teacher can provide real data so that students can find the pattern of expectation built into the offerings of particular sports betting agencies.

Discussion can turn to how the bookies use the amounts wagered by the punters to set the payouts. In the above example, if \$756 was bet on a win, \$158 on a loss and \$86 on a draw, then the bookie would collect \$1000 and the payouts would be:

\$756×1.25 = \$945 for a win: a profit of \$55 for the bookie

\$158×6 = \$948 for a loss: a profit of \$52 for the bookie

\$86×11 = \$946 for a draw: a profit of \$54 for the bookie

Two Bets for Year 9

The AusVELS/ Australian curriculum requirement to cover 'two step chance experiments with repetition' in Year 9 (and 3-step in Year 10) can be addressed in the context of two (or three) people making bets on games of either Unfair Lucky Colours or soccer. Experimental probabilities through simulation can be compared with theoretical probabilities obtained through analysis of tree diagrams, as shown, for example, in Figure 1.

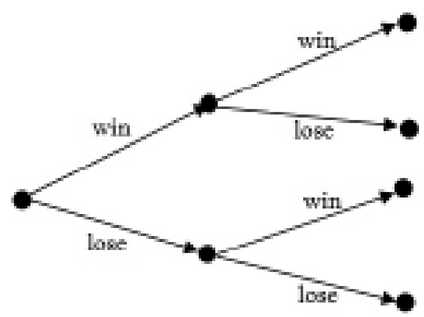

Figure 1. Tree diagram for winning or losing.

If real probabilities and unfair payouts are used for Unfair Lucky Colours then students will be led to the conclusion that there is less chance of 'being ahead' after two games (1/8) than after one game (1/4). So, the longer you play, the more likely are you to end up losing.

The message is *'Short term gain – long term pain'.*

Pokies

A *Pokies* spreadsheet stripped of any colour allows students to see how random numbers are used to generate results. Students can set the expectation and the prize for a win on a \$1 bet and then carry out extended simulations. Using the results they draw up box and whisker plots (Year 9 AusVELS/Australian curriculum) to compare outcomes for 10 trials, 100 trials and 1000 trials.

With the expected long term return set at 85% they will eventually obtain something like the box and whisker plots in Figure 2:

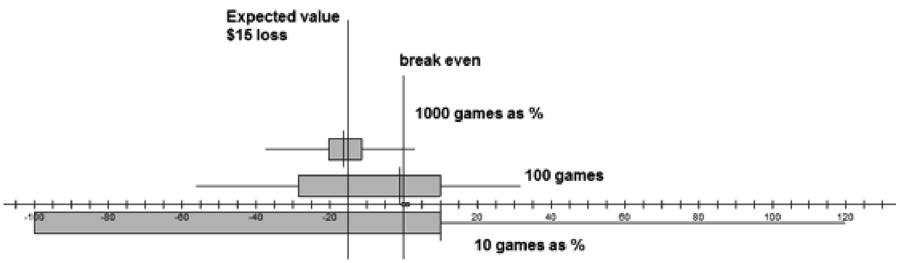

Figure 2. Box and whisker plots for 10, 100 and 1000 games.

In interpreting the box plots students can contrast the possibility of short-term gains (on 10 trials) with the near certainty of long-term losses (on 1000 trials).

[NOTE Year 12 students of binomial probability can compare theses results with theoretical results. The spread is inversely proportional to the square root of the number of trials, so the percentage spread for 1000 trials will be one tenth of the spread for 10 trials.]

Once again, the mantra for the final discussion is *Short term gain – long term pain*.

Going Broke

The *Pokies* spreadsheet, with expectation set at the Victorian standard of 87%, is used to simulate the number of $1 bets required to lose all of an initial $20 stake. Once again, students can use a box and whisker plot to summarize the results – which can include some amazing outliers. The key message for students is that the punter will eventually lose all of the $20, not just 13% of it.

Racing

Students can be encouraged to use what they have learnt so far to undertake a cross-curricular examination of the racing industry. The mathematical component is as follows:

- Stawell Gift. Tossing a die can be used to simulate the final of a six person race in which handicaps are intended to give each runner an equal chance of winning. In such a situation the Year 9/10 AusVELS/Australian curriculum topic of "2 stage chance experiments without replacement" can be addressed through calculating the probability and fair payout for a 'quinella' bet, in which the punter has to nominate which two runners will fill the first two places. (Answer 2/30, not 1/36 or 2/36.)

- Melbourne Cup. A spreadsheet is used for simulation and the conversion of 'bookies odds' or payouts to true probabilities and the long term expected loss to the punter. As an extension advertised quinella payouts can also be analysed.
- TAB. A spreadsheet simulation pretends that each of 25 punters has $100 at the start of a race day. The punters all bet $10 on each of 10 races and after the race the TAB takes its 17% first, with the remainder (the prize pool) shared among the winners. Some punters may end up winning by the end of the day, but the TAB is guaranteed to end up with 17% of $2500, namely $425. The TAB can't lose, so overall, the punters lose.

Misconceptions

Throughout the unit the teacher keeps not only a checklist of AusVELS and numeracy achievements but also of any key gambling misconceptions. In this last lesson, taken some time after the previous classes, students are provided with a spreadsheet showing the wins and losses for ten $1 bets on each of six different poker machines (see Figure 3).

	A	B	D	F	H	J	L
1		All machines are set to the same expectation. In each game you win $4 or lose $1					
2		Machine 1	Machine 2	Machine 3	Machine 4	Machine 5	Machine 6
3	1	-1	-1	-1	-1	-1	-1
4	2	3	-2	3	-2	3	3
5	3	2	-3	2	2	2	7
6	4	1	-4	1	1	1	11
7	5	0	-5	0	0	0	10
8	6	-1	-6	-1	-1	-1	9
9	7	-2	-7	3	-2	3	8
10	8	2	-8	2	-3	2	7
11	9	1	-9	1	-4	1	6
12	10	0	-10	0	0	0	10
13	11						

Figure 3. Wins and losses for ten $1 bets on each of six different poker machines.

Students are then asked to choose which machine they would like to play on for the next ten bets and, for assessment purposes, the teacher insists that they write down their reasons for their choice of machine.

The exercise is repeated a number of times until the teacher has gained enough information to be able to detect any persistence of the following misconceptions (Kahneman, 2011; Waganaar, 1988).

- *Representation bias* – the belief that one of these short term sequences is evidence of the long term probabilities (Tversky & Kahneman,1971).

- *Availability* – the tendency to base judgements on what is most easily remembered, such as previous wins, rather than on losses or objective assessment of the probabilities (Tversky & Kahneman,1973).
- *Personal luck* – the belief that outcomes are influenced by circumstances such as the ‘right’ person, the ‘right’ time or the ‘right’ place (Griffiths, 1994).
- *Illusion of control* – the belief that ‘personal skill’ of an individual can be in control of outcomes, with probabilities influenced by this control (Langer, 1975).
- *Biased attributions* – the belief that external factors, such as ‘bad luck’, are responsible for disappointing results (Gilovich, 1983).
- *Belief in justice* – the conviction that outcomes will eventually be favourable (Lerner & Simmons, 1966).

Lessons from the Pilot Program

Feedback and evaluation of the 2014 pilot program is being used to edit not just the mathematics but all four elements of the project before trialling takes place, after which the final version will be made available to schools. Teachers at the two pilot schools have given valuable feedback, which has contributed to the following conclusions:

- Teachers with flair can use simulation, with or without the aid of spreadsheets, to engage students in the dangers – as well as the fun – associated with gambling.
- More work needs to be done in supporting schools or individual classrooms where cross-curricular opportunities are taken. While most of the piloting has involved just the mathematics, lessons in health education have helped make sure students see something of the more serious side of gambling.
- All classes are mixed ability but some varying more than others. In consequence teachers will adapt and interpret the Biggest Loser resources to meet their particular students’ needs. For some students, experimental simulation will be enough, while for others a further analytical approach is appropriate.
- Different teachers take different approaches in their efforts to promote the key student discussion and explanation that confirms that learning has taken place. While some teachers choose to base these discussions around quite open student activities, other teachers prefer more structured activities, such as completion of worksheets, as the scaffolding for students’ confirmation of their learning. Further development of Biggest Loser resources will allow for such alternatives in teaching and learning styles.

Conclusion

The Mathematical Association of Victoria is fortunate in having the support of the Victorian Responsible Gambling Foundation and of the three key Victorian education sectors. This has allowed for the essential piloting and trialling process that is too often absent from other curriculum initiatives. Hopefully schools will eventually see the Biggest Loser material as not just a context for covering essential AusVELS/Australian curriculum and VCAL goals but also a much needed and timely response to a growing health and safety issue that confronts our students.

Enquiries for 2015 trialling: Contact Ian Lowe at The Mathematical Association of Victoria.

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IMPLEMENTING A SELF-DIRECTED LEARNING APPROACH TO MATHEMATICS IN A PRIMARY SETTING *

Stephanie Nitschke

St Therese Primary School, Torquay

Ensuring that mathematics learning is personalized to all students can be a difficult task. This paper explores the impact that student voice and choice can have on attitudes and outcomes in mathematics. When a self-directed learning approach is implemented into the classroom students are supported to take responsibility for their learning and engagement in mathematics is enhanced. This paper discusses the approach that one school has taken to support students in becoming self-directed learners.

Introduction

A self-directed approach to mathematics allows students to make their own discoveries and connections between mathematical concepts. "By allowing students to interact with and struggle with the mathematics using *their* ideas and *their* strategies, a student-centered approach, the mathematics they learn will be connected to other mathematics and to their world" (Van de Walle, Karp, Lovin, & Bay-Williams, 2006, p. xiv). Through further developing the student-centered approach toward a self-directed approach, students are given choice in their mathematical learning and so a shift in the classroom occurs where students control their learning rather than teachers. Hattie's (2009) research on the influences of student learning discusses that motivation, concentration, persistence, engagement, and reduced anxiety are influences that improve student outcomes. When a student has control over their learning all of these influences are enhanced, resulting in positive learning behaviors and attitudes towards mathematics. Despite the many benefits of a self-directed approach to mathematics, it can be challenging to implement into the classroom. This paper explores the multi-faceted method one school has taken to adopt a self-directed approach to learning and improve student outcomes in mathematics.

School Context

St. Therese Catholic Primary School is set in the coastal township of Torquay in Victoria, Australia. The school comprises 480 students and had an Index of Community Socio-Educational Advantage (ICSEA) value of 1104 in 2013 (Australian Curriculum Assessment and Reporting Authority, 2013a). Until the end of 2013 the school participated in the Contemporary Learning Research Project, an initiative of the Catholic Education Office and Melbourne University. The project supported the school in developing curriculum initiatives, which fostered students to attain the knowledge and skills necessary for responsible citizenship and flexible life-long learning (Catholic Education Office Melbourne, 2009). Although, no longer a part of the project, the school has continued to focus on providing students with a Contemporary Learning environment in order to prepare them for living in the 21st Century. A self-directed approach to mathematics is adopted throughout the whole school community; however, the implementation strategies outlined in this paper were specifically conducted in the 5/6 learning community. The 5/6 learning community consists of 120 students and five teachers. Mathematics sessions are run with 60 students and three teachers for 50 minutes to 1 hour and 15 minutes in length and are focused towards students self-directing their own learning.

Benefits of a Self-directed Approach to Mathematics

Student Attitudes

It is a common trait that only those with higher mathematical abilities have positive attitudes towards mathematics. Therefore, it is inevitable that there are many students who have negative attitudes towards mathematics. This is referred to as mathematical anxiety, and can lead to a drop in mathematical performance (Buckley, 2013). Pekrun's (2006) theory on emotions in the classroom discusses that emotions are impacted primarily by how much a student values a task and how much control they feel they have over the task. Turning the ownership of the learning over to students ensures that they have control in their learning and thus fosters positive attitudes towards their mathematical abilities.

Challenging Students at Their Point of Need

One of the obstacles with implementing a self-directed approach to mathematics is that many students do not yet know what it is that they want to learn about. They may have some areas of mathematics that they are interested in pursuing, but can often only stay within their comfort zone – they are unaware of how to challenge themselves to reach a higher level of understanding. Hattie's research on the influences of student learning proposes: "A combination of goal setting plus feedback is most effective. Goals and challenging goals are mutually supportive. The greater the challenge the higher the probability of the student seeking, receiving and assimilating feedback information" (Hattie, 1999, p. 13). The aim of a self-directed approach to mathematics is to have students learning at a level where they are being challenged, but are also experiencing success. For this approach to be successful, students need to be engaged in challenging new learning at their individual point of need. Overcoming this problem – students who are unsure how to adequately challenge themselves – involves developing a flexible curriculum to guide and prompt students in their mathematical investigations.

Constructing a Flexible Curriculum

Students enter a mathematics session with different learning styles, mathematical abilities and prior knowledge. Some students may have mastered certain mathematical concepts, where others have not yet. Having these students repeat what they have previously mastered, as seen in a 'one-size-fits-all' approach, sees them becoming disengaged in their learning and is a missed opportunity for further extension. In order to adopt a self-directed approach to mathematics, this school has developed a flexible curriculum to guide, prompt and challenge students through their learning. The curriculum structure was created using Hattie's (2009) *Visible Learning Theory* and the *Australian Mathematics Curriculum* (ACARA, 2013b). The importance of ensuring the learning is visible, laid the foundations of developing this flexible curriculum, through the use of learning intentions and success criteria. The Australian Curriculum was adapted to create multiple levels of success criteria so that students could choose to enter the learning at their individual point of need. These levels of success criteria were labelled Mild, Spicy and Hot. Originally the Mild, Spicy and Hot success criteria represented levels 5,6 and 7 of the Australian Mathematics Curriculum, respectively (ACARA, 2013b). Through reflection on practice and a focus on a flexible curriculum, the content and structure of the success criteria has varied from one unit of work to another to best accommodate student needs.

Learning Intention: We are learning about mathematical operations and their relationships with each other.

Mild

- Investigate number sequences involving multiples of 3, 4, 6, 7, 8, and 9.
- Recall multiplication facts up to 10×10 and related division facts.
- Develop efficient mental and written strategies of addition and subtraction.
- Develop efficient mental and written strategies of multiplication and division.

Spicy

- Identify and describe factors and multiples of whole numbers and use them to solve problems.
- Use estimation and rounding to check the reasonableness of answers to calculations.
- Solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental, written strategies and appropriate digital technologies.
- Solve problems involving division by a one-digit number, including those that result in a remainder.
- Select and apply efficient mental and written strategies and appropriate digital technologies to solve problems involving all four operations with whole numbers.
- Investigate everyday situations that use integers. Locate and represent these numbers on a number line.

Hot

- Investigate and use square roots of perfect square numbers.
- Apply the associative, commutative and distributive laws to aid mental and written computation.
- Compare, order, add and subtract integers
- Solve problems involving addition and subtraction of fractions, including those with unrelated denominators.
- Multiply and divide fractions and decimals using efficient written strategies and digital technologies.
- Use index notation with numbers to establish the index laws with positive integral indices and the zero index.
- Carry out the four operations with rational numbers and integers, using efficient mental and written strategies and appropriate digital technologies.
- Investigate the concept of irrational numbers, including π.

Figure 1. Mild, spicy and hot success criteria adapted from the Australian Mathematics Curriculum Content Descriptions.

Learning Intention: We are identifying areas we want to improve in mathematics and investigating efficient strategies to achieve this.

Mild

- Show ***understanding*** by creating a detailed show-me of your learning.
- Show ***problem solving*** by applying your learning to some worded problems.

Spicy

- Show ***understanding*** by creating a detailed show-me of your learning.
- Show ***problem solving*** by applying your learning to some worded problems.
- Show ***fluency*** by using appropriate units of measurement and mathematical instruments when solving problems. And by using estimation to check your answers.
- Create your own worded problems on your topic for yourself and others to solve.
- Show ***reasoning*** by explaining the strategies you used to solve problems.
- Show ***understanding*** by planning and conducting a workshop on your topic, use the questions you have written within your workshop.
- Show ***fluency*** by using appropriate units of measurement and mathematical instruments when solving problems. And by using estimation to check your answers.
- Create your own worded problems on your topic for yourself and others to solve.
- Show ***reasoning*** by explaining the strategies you used to solve problems.

Hot

- Show ***understanding*** by creating a detailed show-me of your learning.
- Show ***problem solving*** by applying your learning to some worded problems.
- Show ***fluency*** by using appropriate units of measurement and mathematical instruments when solving problems. And by using estimation to check your answers.
- Create your own worded problems on your topic for yourself and others to solve.
- Show ***reasoning*** by explaining the strategies you used to solve problems.
- Show ***understanding*** by planning and conducting a workshop on your topic, use the questions you have written within your workshop.
- Show ***reasoning*** and ***understanding*** by relating your topic to the real world, what examples and problems can you find. This can be posted on your blog

Figure 2. Mild, spicy and hot success criteria adapted from the Australian Mathematics Curriculum Proficiency Strands.

The success criteria focused on mathematical operations (Figure 1) provides students with a starting point for their investigations. Students can choose to challenge themselves at any level of the success criteria, and may complete all or part of the success criteria depending on their level of prior knowledge. For example; a student may feel they understand and are competent in a number of areas in the Spicy success criteria, and would only like to focus on investigating one or two bullet points they do not yet understand. The style of this curriculum allows students to develop self-assessment skills of their own mathematical abilities and work towards challenging themselves in a chosen area of need. In contrast, the success criteria created using the Australian Mathematics Curriculum Proficiency Strands (Figure 2) was developed to accommodate students working on a range of mathematical concepts (ACARA, 2013b). The students completed a practice National Assessment Program – Literacy And Numeracy (NAPLAN) assessment and were required to self-correct and from this, self-assess and identify areas for improvement (ACARA, 2011). These examples of Mild, Spicy and Hot success criteria demonstrate how a flexible curriculum can be developed and implemented in a mathematics classroom to assist students in self-directing their own learning.

Fostering and Supporting Self-directed Learners

Student-led Workshops

When students teach their peers a mathematical concept, they have the ability to develop a deeper understanding of the concept. This is observed through the way they explain the concept and are able to answer student questions. Hattie (2009) delivers the key message that when students see themselves as teachers, student achievement is enhanced. A Student-led workshop is exactly what its name suggests; a workshop researched, planned and conducted by students to teach their peers a mathematical concept. The students running the workshop are deeply engaged in their learning as it is a mathematical concept of choice and they develop the intrinsic motivation of wanting to perform well and share their learning to their peers. The students who attend the workshops, also do so out of choice. Students choose workshops that focus on areas of mathematics they would like to improve on. These students benefit from the workshops as they experience learning the mathematical concept in a different way, and in 'kid-friendly' language. Thus they have a greater chance of developing an understanding of the concept. These workshops run in conjunction with the Mild, Spicy and Hot criteria to support the self-directed learning environment and provide students with another avenue of mathematical instruction.

Learning Partners

Fielding's (2012) research on the Patterns of *Partnerships* explores the concept of "students as knowledge creators." This involves "students taking a lead role with active staff support, in engaging approaches" (Fielding, 2012, p. 50). As self-directed learners, students begin to take a lead role in their learning, while the teacher's role is to ensure the learner is supported. This can be difficult to achieve when students are working on a range of mathematical learning tasks at a given time. In order to overcome this, and further support student outcomes, learning partners were introduced to mathematics sessions. When students create learning partnerships with one another, their personal and social learning becomes embedded within the mathematics curriculum. The Australian Mathematics Curriculum supports this in the General Capabilities section stating; "The Mathematics curriculum enhances the development of students' personal and social capabilities by providing opportunities for initiative taking, decision making, communicating their processes and findings, and working independently and collaboratively in the Mathematics classroom" (ACARA, 2013b). The learning partner provides students with a first point of call when they have questions or a misunderstanding of a mathematical concept, and additionally supports the development of students teaching students. Learning partners are also utilised to assist students to reflect on and communicate their learning.

Through sharing their learning with their partner, students are encouraged to set future goals for their learning and develop their mathematical reasoning skills. A secondary support was also introduced through the means of an 'Expert wall'. This is a place where students can sign their name up when they feel they have become an 'expert' on a mathematical topic. Other students focusing on the same topic can seek out an 'expert' to assist them in their learning. The 'Expert wall' and learning partners initiatives not only support students in self-directing their learning, but also foster development of self-assessment skills and positive learning behaviors and attitudes.

Student-led Conferences

The importance of students having ownership and responsibility for their learning is evident. For these attitudes to be embedded within the learning community, it is essential that parents are involved in the learning process. The main approach taken to achieve this is through student-led conferences. Bailey and Guskey (2001) assert that student-led conferences enable students to take responsibility for their learning, and aids in increasing parent participation. This method of parent reporting reinforces student attitudes towards learning and in turn enhances student engagement.

Feedback, Teacher Instruction and Assessment

Providing effective feedback is an important tool in supporting self-directed learners. Hattie and Timperley (2007) clarify that effective feedback needs to address three questions for the student; Where am I going?, How am I going?, and Where to next? To satisfy these questions successfully, one-on-one time needs to be spent with students. As mathematics sessions are undertaken using a self-directed approach, this allows time for teachers to check-in with students at least once a week and provides them with essential and effective feedback on their progress. The additional benefit to this one-on-one check-in time is the ability to check and question a student's understanding and methods, and impart in a discussion to support them on correcting any misconceptions. To further support learners, teacher-directed workshops are conducted. The aims of these workshops are to introduce and support new understandings and skills, and are voluntary for students. Making the choice to attend teacher-directed workshops reinforces positive self-directed learning behaviors. Throughout both 'check-ins' and teacher-directed workshops it is necessary to document students' learning, for tracking their progress and reporting purposes. An efficient tool utilised for this purpose is 'Google Docs'. Each student has their own spreadsheet on 'Google Docs' where anecdotal notes are recorded. The benefit of this program is that it is stored online, so all teachers in the 5/6 learning community have access to every student's spreadsheet and can keep informed of their learning progress. The combination of effective feedback and teacher-instruction ensures that self-directed learners are on track with their learning and are displaying clear mathematical understandings and using effective methods to solve problems.

Conclusion

This method of implementing a self-directed approach into a mathematics classroom is by no means a finished product. The nature of the flexible curriculum supports teachers to continually reflect on the curriculum and the support structures that are in place – with staff and students alike – to further develop these, and create a learning environment which enhances learning attitudes and outcomes in mathematics for every student. When students are self-directing their learning, this allows the teacher to have time to have one-on-one meaningful discussions with students to assess their understanding and give quality feedback. In addition, small teacher-led workshops can be conducted that focus on any recognised areas of need. A self-directed approach to learning in a mathematics classroom nurtures students' natural instinct to inquire and investigate and does so through the integration of choice into the curriculum. In order for this approach to experience success, it is important to develop a culture in which students take ownership of their learning within the learning community. The success of this culture relies on applying a number of approaches to support students in their investigations. The result of supporting these students to self-direct their own learning has been an observed increase in positive attitudes towards mathematics, and, in turn, enhanced confidence when solving mathematics problems.

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TRISECTING AN ANGLE *

Karim Noura

Bayside P–12 College

The ancient Greek mathematicians set themselves challenging tasks of completing geometric constructions using only compass and straight-edge. Trisecting an angle using only a compass and straight-edge was one of the unsolvable problems of classic geometry. This paper explores solutions for special cases and Archimedes' 'neusis' solution for an arbitrary angle. Trigonometry and algebra are also used to show how we can calculate the required position of a point for achieving a construction.

The Impossible Problem

The ancient Greek mathematicians set themselves challenging tasks of completing geometric constructions using only compass and straight-edge, that is, straight-edges marked with lengths were not permitted. Euclid showed how a compass and straight-edge could be used to bisect any arbitrary angle (HREF1). By contrast, trisecting an angle, dividing an arbitrary given angle into three equal angles, using only a straight-edge and compass was considered to be one of the hardest problems or an "impossible problem" in the history of mathematics. Many mathematicians attempted to solve this problem, which remained a famous problem of classical Greek geometry (HREF2, HREF3).

In 1837 the French mathematician, Wantzel, proved that the construction was not possible using only a compass and straight-edge (HREF4). Archimedes of Syracuse (circa 287–212 BCE) had presented a geometric construction but it required the insertion of a line segment of given length, so was not a genuine compass and straight-edge construction. Pascal's mechanical angle trisector was based on this method, which can also be achieved using dynamic geometry software such as *GeoGebra*. Trisecting angles using paper folding (origami) also requires the marking off of particular lengths.

Compass and Straight-edge Constructions

Bisecting an Angle

Bisecting an arbitrary angle can be done by construction using compass and ruler. To bisect $\angle ABC$ (see Figure 1), place the compass point at B and draw an arc to cut AB and CB at E and F respectively. Now place the compass point at E and draw an arc. Repeat, with the compass point at F. The two arcs intersect at G. The line BG bisects $\angle ABC$.

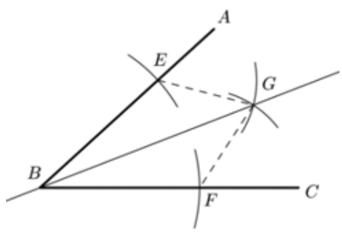

Figure 1. Bisecting an arbitrary angle.

Proof:
In ΔEBG and ΔFBG,
$BE = BF$ (radii of same arc)
$EG = FG$ (radii of same arc)
BG is common
So $\Delta EBG \equiv \Delta FBG$
So $\angle EBG = \angle FBG$
Hence BG bisects $\angle ABC$.

Trisecting an Angle

Similar strategies can be used to trisect a straight angle (180°) or a right angle (90°) utilising compass and ruler. For the straight angle $\angle AOB$ (see Figure 2), we can use compass and ruler to construct two equilateral triangles OMP and ONQ. Therefore angle $\angle AOB$ is divided into three equal angles: $\angle MOP = \angle POQ = \angle QON = 60°$

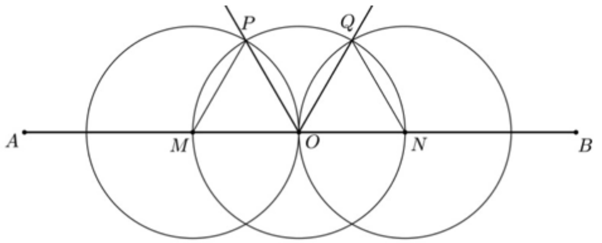

Figure 2. Trisecting a straight angle.

We can trisect a right angle using a similar strategy (see Figure 3). For the right angle, $\angle AOB$, construct an equilateral triangle MON. Bisect $\angle MON$. Therefore the angle $\angle AOM = \angle MOP = \angle POB = 30°$.

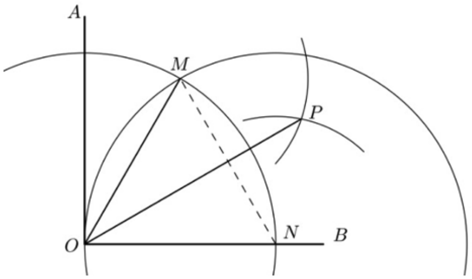

Figure 3. Trisecting a right angle.

However, the strategies used here for 180° and 90° cannot be used to trisect any arbitrary angle.

Origami (paper folding)

An angle drawn on paper can easily be bisected by folding. Japanese mathematicians have used origami to trisect an arbitrary angle (HREF5, HREF6). In Figure 4, segment OB is drawn on the paper to indicate the angle to be trisected, $\angle AOB$. The paper is folded through an arbitrary point Q parallel to the edge of the paper, OA. A second parallel fold is made so that $OQ = PQ$. The paper is then folded so that corner O meets the fold line through Q and point P is on the arm OB of $\angle AOB$ as shown in Figure 4. Fold lines along OO' and OQ' divide $\angle AOB$ into three equal angles.

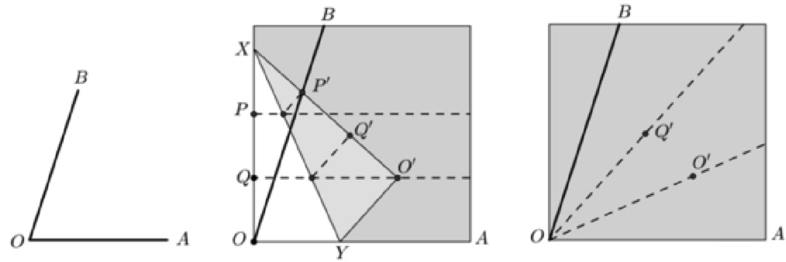

Figure 4. Origami trisection of an angle.

Proof (see Figure 5):

In $\Delta POO'$,

$PQ = QO$

$QO' \perp PO$

So $\Delta POO'$ is isosceles.

So $PO' = OO'$

But $\Delta P'OO'$ is the reflection in fold line XY of $\Delta POO'$.

So $OO' = OP'$

In $\Delta OMO'$, $\Delta OQ'O'$ and $\Delta OQ'P'$,

$O'M = O'Q' = P'Q'$

$OO' = OP'$

$\angle OMO' = \angle OQ'O' = \angle OQ'P' = 90°$

So $\Delta OMO'$, $\Delta OQ'O'$ and $\Delta OQ'P'$ are congruent (RHS)

So $\angle O'OM = \angle O'OQ' = \angle POQ' = \frac{1}{3} AOB$

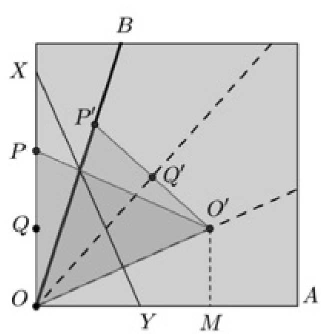

Figure 5. Proof of origami trisection.

Archimedes' 'Neusis' Solution

The ancient Greeks set rules that constructions must be completed using only a compass and straight-edge, that is, without using a marked ruler. Archimedes of Syracuse (circa 287–212 BCE) presented a construction which was not strictly a compass and straight-edge construction because it involved a construction technique known as a neusis construction where a marked ruler was rotated to fit a given length between two given lines.

Given $\angle AOB$ which is to be trisected, draw a circle with centre O through points A and B (see Figure 6). Extend AO. Using a straight-edge marked with the radius, r, of the circle find a position for Q on AO extended so that a line segment can be drawn from B through a point P. on the circle to meet Q in such a way that PQ is equal to the radius of the circle. Hence $QP = OP = OB = r$.

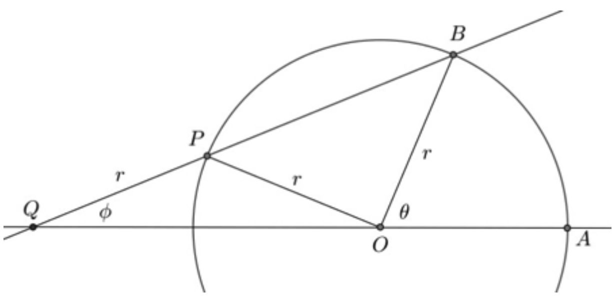

Figure 6. Proof of Archimedes' angle trisection.

Using the properties of isosceles triangles and exterior angles we can prove that

$\angle AQB = \frac{1}{3}\angle AOB$, that is, $\phi = \frac{1}{3}\theta$

Proof:

ΔOBP and ΔOQP are isosceles triangles.

$$\angle OPB = \angle OQP + \angle POQ = 2\phi$$

So $\angle OBQ = 2\phi$

$$\begin{aligned}\angle AOB &= \angle OQP + \angle OBQ \\ &= \phi + 2\phi \\ &= 3\phi\end{aligned}$$

So $3\phi = \theta$, that is, $\phi = \frac{1}{3}\theta$

Finding the Position of *Q*

Using trigonometry and algebra, the position of *Q* can be calculated. In Figure 7, point *O* is at the origin on the Cartesian plane.

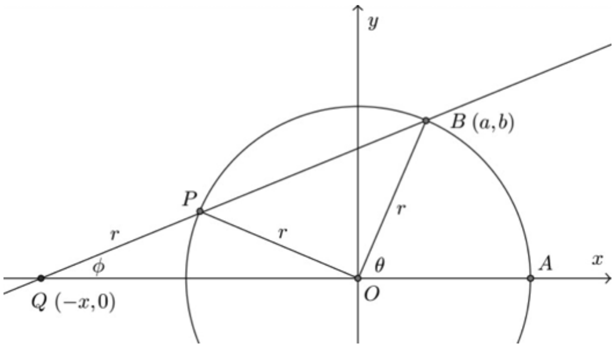

Figure 7. Calculating the position of *Q*.

$\tan\phi = \dfrac{b}{a-x}$ and $\tan\theta = \tan 3\phi = \dfrac{b}{a}$

Using the triple angle formula:

$$\tan 3\phi = \frac{3\tan\phi - \tan^3\phi}{1-3\tan^2\phi}$$

Substituting:

$$\frac{b}{a} = \frac{\dfrac{3b}{a-x} - \dfrac{b^3}{(a-x)^3}}{1-\dfrac{3b^2}{(a-x)^2}}$$

$$\frac{b}{a} = \frac{3b(a-x)^2 - b^3}{(a-x)\left[(a-x)^2 - 3b^2\right]}$$

Simplifying:

$$x^3 - 3x(a^2 + b^2) + 2a(a^2 + b^2) = 0$$

Note that $a^2 + b^2 = r^2$ so that $x^3 - 3r^2x + 2ar^2 = 0$.

If we consider a unit circle, then $r = 1$ and the cubic equation becomes $x^3 - 3x + 2a = 0$. This can be tested using GeoGebra, where a unit circle and the graph of $y = x^3 - 3x + 2a$ are drawn, using the test value $a = 0.9$ (see Figure 8).

The intersection of the cubic graph a $x \approx -1.997$ gives the point Q required for the trisection of $\angle AOB$.

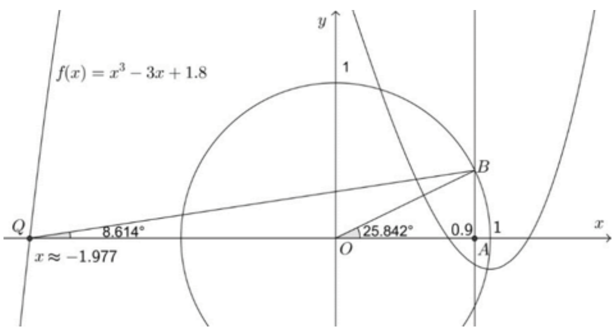

Figure 8. Checking the position of Q for a=0.9.

Pascal designed a mechanical angle trisector constructed from hinged rods (HREF7). This was based on the same geometry as Archimedes' construction. A similar method can be used to construct an angle trisector with dynamic geometry software such as GeoGebra (see Figure 9).

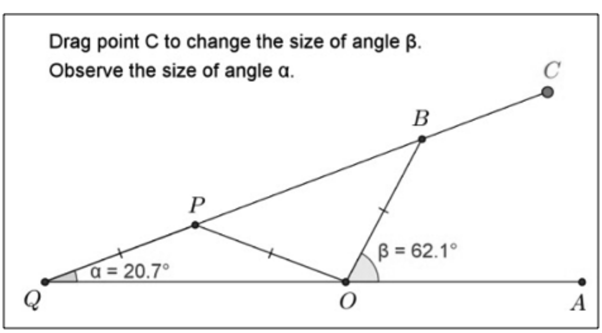

Figure 9. Pascal's angle trisector constructed using GeoGebra.

Conclusion

I hope that this paper provides our students and their teachers with an additional tool in learning and teaching mathematics. Unfortunately, we cannot apply one single and unique strategy for trisecting angles. We may need to deal with each situation separately and make sure that any solution makes sense and provides a logical demonstration.

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EFFECTIVE DIFFERENTIATION: WHERE A GROWTH MINDSET MEETS THE ZPD *

Yvonne Reilly and Jodie Parsons

Sunshine College

The challenge of practically providing each student in a class with the opportunity to work at their own Zone of Proximal Development (ZPD) (Vygotsky, 1978), is often insurmountable to many practitioners. Our model not only alleviates the practical aspects of this challenge, but in addition, creates an environment where students believe that they can improve and an environment where students are expected to identify and select the activity which is 'just right' for their learning requirements.

Introduction

Sunshine College is a multi-campus Government secondary school located within the South Western Victorian Region. It is positioned across four sites and is made up of three junior campuses, including a deaf facility and one senior campus. It is a culturally diverse school with more than fifty language backgrounds. The population, in general, suffers a high degree of disadvantage and a low socio-economic position, with an average Student Family Occupation (SFO) index of 0.8, and a school ICSEA value of 909. Our distribution of students compared with the Australian average is shown in Figure 1.

Distribution of students

	Bottom quarter	Middle quarters		Top quarter
School Distribution	65%	21%	11%	3%
Australian Distribution	25%	25%	25%	25%

Figure 1. Distribution of Students (http://www.myschool.edu.au/SchoolProfile).

In 2008 and after several years of little or no improvement in whole school data (AIM & VCE), and the placement of several numeracy coaches from the now-defunct Western Metropolitan Region, the authors of this paper began to construct a numeracy program which would support the conceptual understanding of all students. Prior to this the majority of mathematics classes at Sunshine College were teacher centred and followed a traditionally recognised structure. The teacher would complete worked examples on the whiteboard, students would copy the examples and then complete a number of almost identical questions from the prescribed textbook. Classes rarely, if ever, used concrete materials; students worked individually; assessment was summative and the opportunity for a tailored and individualised program was the prerogative of the classroom teacher, often resulting in lessons where weaker students were expected to complete fewer questions than the more competent students.

Context

At each junior campus of Sunshine College all students receive five 50-minute lessons of mathematics instruction per week. Each two-week cycle is divided into one of five elements, as illustrated in Table 1. Each element has been developed to specifically address a particular aspect of student learning and have been described in full detail in Siemon, Virgona, and Corneille (2006); Reilly, Parsons, and Bortolot (2009); Reilly, Parsons, and Bortolot (2010); Reilly and Parsons (2011), and in *The Common Denominator* (Jan, 2014). While each element of the Sunshine College Maths Program (SCMP) supports a different aspect of numeracy understanding, they all share a common philosophy, in that all students are encouraged to work within their ZPD in an environment which fosters the growth mindset.

Table 1. *Sunshine College Maths Program (SCMP)*

Week	Lesson 1	Lesson 2	Lesson 3	Lesson 4	Lesson 5
Odd	Differentiated curriculum	Reciprocal Teaching	Differentiated curriculum	Speedy maths	SNMY
Even	Differentiated curriculum	Reciprocal Teaching	Differentiated curriculum	ICT	SNMY

The Results

Prior to the implementation of the SCMP the college was regularly identified as a school where students on average tested two or more years behind the national average for literacy and numeracy. As is demonstrated in Table 2 and Figure 2 the introduction of the SCMP has led to on average a faster rate of growth for students at the college when compared to the average rate of growth demonstrated by the State in the NAPLAN standardised tests for students between grade 7 and grade 9.

Table 2. *The Rate Of Growth Between Years 7 And 9*

NAPLAN School Comparison Report Raw Scores			
Matched School Mean	2011	2012	2013
Ardeer Campus	56	47	45
North Campus	47	51	53
West Campus	45	64	54
School Mean	49.3	54	50.7
State Mean	42	38	39

In February 2014, Sunshine College was noted as a turnaround school by The Grattan Institute. It was heralded as one of four schools in Australia to have students improving at a rate faster than the state average in numeracy (Feb, 2014).

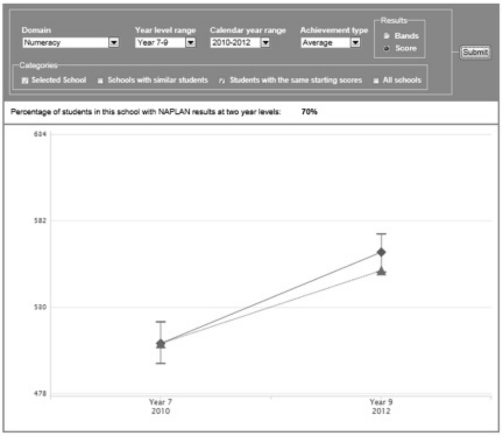

Figure 2. Sunshine College against 'Students at the same starting point', 2010–12 My School Website.

Anecdotal evidence also suggests students are now more confident in their mathematical ability and as a result more students are selecting General Mathematics (Advanced) in Year 11 compared to previous years. When students are questioned as to why they are selecting mathematics they say it is because they like maths or that they are good at it.

It is the authors' belief that a number of factors working together have contributed to the success of the program, most notably a fully differentiated curriculum where students are offered mathematics at their ZPD and the fostering of a growth mind-set within each and every mathematics class.

A Fully Differentiated Curriculum

The mathematics curriculum for schools is governed by The Australian Curriculum Assessment and Reporting Authority. It describes the chronological expectation of student learning in each year of formal schooling. In mathematics this means students in Year 8 are expected to develop an understanding of directed number while students in Year 9 progress to developing an understanding of Pythagoras' Theorem and Trigonometry. This expectation notwithstanding, it is rare to find a classroom where all students are capable of learning a specific concept at the same time and in the same manner. In 2011, the authors described their model for differentiation which avoids the various negative connotations of streaming, grouping and withdrawal of students whilst demonstrating how the model is not just modifying or extending work (Reilly & Parsons, 2011). In addition to these benefits the authors have noted an additional point of difference to their model where the teacher provides for different learning needs by what they refer to as the “Drop-out Model”.

The Drop-Out Model

In this traditional model the teacher often constructs an activity, assessment task or topic test, where the initial questions are easier. However, as the task progresses the questions become increasingly more difficult. While all students work on the same material they drop-out when they have reached their capacity to understand. This model has two potential detrimental effects on student learning. Firstly, it provides the more able student with ample opportunity to demonstrate their knowledge and intelligence as they experience the metaphorical pat on the back as they successfully complete work that was well within their capability, or as with the initial activities, below their ZPD. These same students, who experience success often, when faced with small challenges, have a reduced capacity to cope and can on occasion, crumble. This Drop-Out model can mean that more capable students are not regularly presented with the opportunity to develop perseverance or resilience in their learning — thus supporting the development of a fixed mindset where individuals believe that understanding or intelligence cannot be improved upon (Dweck, 2006).

Secondly, the Drop-Out model indicates very clearly to students who are less able that there are many things which they cannot do. Even when the task is delivered by the most caring teacher saying things like “Don’t worry, just do as many as you can”. The student still hears “You are not as good as everyone else; my expectations of you are much lower than my expectation of the good mathematicians.”

As previously described, (Reilly & Parsons, 2011) the development of an alternative model of differentiation which incorporates ‘just right’ task not only provides learning opportunities for all students to work at their ZPD but also capitalises on the positive learning outcomes associated with a growth mindset.

The Development of a Growth Mindset in Mathematics

There are very few people in this country who would happily admit that they were illiterate, yet many highly educated and well respected individuals will publicly concede that they were “never any good at maths”. There is a widely held perception that being a good mathematician is a particular talent that only a lucky few individuals exhibit and that your natural ability determines or limits your achievement in this area. The psychologist, Carol S. Dweck, PhD, describes this belief system as a fixed mindset, where your level of intelligence is seen as a fixed trait.

Blackwell, Trzeniewski, and Dweck (2007) studied a group of 7th graders in the United States, where after transition from elementary school, the students perceived the work to get harder and as a result the students begin to struggle. In this study, students were identified as having either a fixed mindset or a growth mindset. The authors found students who had been identified as having a growth mindset exhibited a dramatic increase in maths grades when compared to the fixed mindset group (Blackwell et al., 2007).

Dweck (2006) describes how, in order to develop a growth mindset in learners, it is necessary to change the belief that intelligence is a fixed trait, and that if an individual needs to work hard at something it means they are no good at it, as opposed to working hard to become very good at it. Those with a fixed mindset also believe mistakes demonstrate failure as opposed to providing opportunities for learning and understanding. Fixed mindset students have little or no recipe for recovering from failure and instead tend to either give up or blame the teacher. In contrast students who exhibit a growth mindset confront difficulties and seek solutions.

Our Model of Differentiation and a Growth Mindset

In conjunction with providing a differentiated curriculum, Sunshine College guides students towards developing a growth mindset. Students are encouraged to take control of their learning and to develop personal learning goals. In our classrooms the path travelled is as valued as the end result. We encourage students to determine their current level of understanding, help them to set realistic and achievable goals and then guide them to select tasks which best supports their learning, i.e., the task which is 'just right' for them. A 'just right' task refers to a learning activity which allows students to work in small groups on a mathematics problem at their ZPD. All students within the one classroom work on the same learning outcome, e.g., area of composite shapes, but at a level which maximises their opportunity to learn. It is this access to achievable tasks coupled with the perception that less able mathematicians work alongside the better mathematicians on the same learning outcomes which we have observed to have the most substantial effect on student learning.

In order to support the development of a growth mindset while providing a fully differentiated curriculum the authors' believe the following key elements are essential.

- Each task is specifically planned and designed by a team of teachers to address the needs of students across a minimum of three ability levels.
- Students work in concert with the classroom teacher (in the beginning) to select a task which is just right to their learning. 'Just right' tasks focus on conceptual understanding of mathematics as opposed to procedural practice.

- Each task should be designed to teach more than one mathematical concept at a time, reinforcing the complex nature of mathematics.
- When the teacher introduces the tasks to the class, each level is described using the pre-requisite knowledge the students will need to complete the selected task successfully. The teachers avoid labelling the tasks as 'easy', 'medium', or 'hard' because if the task is 'just right' for the student it should feel hard.
- Students work with someone who has chosen the same task as they have and who thinks at the same speed as they do, encouraging equitable mathematic conversation.
- Tasks are created to support conceptual understanding and not just to provide opportunity to practice procedures.
- As students are working at different levels of understanding, teachers are no longer able to stand at the front of the room and explicitly teach to the class as a whole, instead explicit teaching is done at the point of need, with the teacher constructing small groups when necessary to increase efficiency.
- While using this model of differentiation teachers avoid telling the students anything, opting instead to guide them to discovery through effective questioning techniques. As stated by René Descartes in his book *La Géométrie* (1637), "I hope that posterity will judge me kindly, not only as to the things which I have explained, but also to those which I have intentionally omitted so as to leave others the pleasure of discovery." When teachers offer students the short-cut, it is often because the teacher is under pressure to cover the curriculum and believes that the student will understand quicker if given the algorithm or short-cut alongside multiple practice questions, often confusing students who cannot learn without first understanding.
- The use of assessment and feedback to support development of a growth mindset. Student data is shared with the individual student and student improvement data is shared with the class, the emphasis is on improvement above absolute score.

Conclusion

The Sunshine College Maths Program demonstrates that by providing students with various levels of tasks from which they can self-select the most appropriate for their learning within a culture that promotes students responsibility for their own learning, whilst fostering self-confidence and self-belief and where improvement is valued more highly than absolute scores, excellent learning outcomes can be achieved for every student.

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DEVELOPING GEOMETRIC REASONING ABILITIES THROUGH VISUALISATION **

Rebecca Seah

RMIT University, Melbourne, Australia

The development of spatial sense is a key component of understanding the world around us and learning higher mathematics. Making sense of the spatial world involves an ability to visualise shapes, objects, their properties and the relationships among them. Children who are skilled in forming mental images of patterns and relationships can devise solutions to problems more quickly. Using paper folding activities, this session shows how the process of creating different objects allows children to visualise shapes and their properties from different orientations, leading to better understanding of 3-D objects.

Introduction

The ability to interpret visual information is fundamental to human existence (Jones, 2002). It is also important for acquiring advanced knowledge needed for developing expertise in science, technology, engineering, and mathematics (STEM). After tracking 50,000 males and 50,000 females over 50 years, Wai, Lubinski and Benbow (2009) concluded that spatial ability plays a critical role in structuring educational and occupational outcomes in the general population as well as among intellectually talented individuals who go on to achieve advanced educational credentials in STEM. Despite its importance, there has been scant research interest among the mathematics education community to investigate how to promote spatial and geometric reasoning ability. The Australian Curriculum: Mathematics also makes no mention of visual and spatial reasoning (Lowrie, Logan, & Scriven, 2012).

At school, many children's experience with geometry remains static. Typically, children are exposed to a limited range of examples that, in turn, hinders their understanding of geometric properties. One reason is due to material constraints. For example, pattern blocks are often used to teach basic shapes and the concept of symmetry. They can also be used to develop the idea that shapes can be decomposed into other shapes. However, they are less suited for teaching relating to irregular and concave shapes or the relationships between shapes and their angles.

When problem solving is introduced, a lack of spatial visualisation creates further difficulty. Consider the following question taken from Lenchner (2006) (Figure 1):

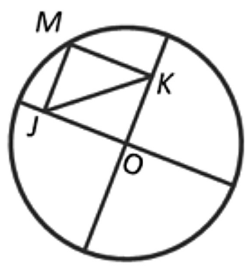

Figure 1. The length of a radius of this circle is 8 cm. Point *0* is the centre. If *OKMJ* is a rectangle, what is the length of *JK*?

Solving this problem requires an ability to interpret this representation as a network of geometrical relationships among elements. In this case, the observer must be able to 'see' that *JK* is the diagonal of the rectangle *OKMJ* and deduce that since the two diagonals of a rectangle have the same length, the length of *JK* must be 8 cm as well (see Figure 2).

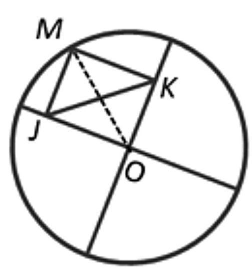

Figure 2. The invisible diagonal *MO* .

These diagrams act as a source of optical data to represent the relationship between the triangle and rectangle. It is depended on an individual's ability to reason geometrically. The observer must have a repertoire of experiences with the attributes of geometric figures in order to infer and interpret the situation.

Paper-folding activities can be a great tool to help children comprehend geometric ideas. When the word origami is mentioned, most people think of it as a Japanese hobby. Originating in China, but popularised in Japan, origami has become an art—the art of paper folding. When mathematics is applied to origami, its usability is immense. The origami stent was created to manoeuvre through narrow veins or arteries and then expand when it reached its intended site to clear a blocked artery. Origami is also used to pack airbags in modern vehicles and to design the Eyeglass Telescope, a 100 m diameter lens to be sent up to space (Lang, 2008). At schools, paper-folding activities have already been used to teach fraction ideas. Evidence suggests that this use has significantly improved children's spatial visualisation ability (Cakmak, Isiksal, & Koc, 2013). The following discussion will first review current understanding of spatial visualisation in order to elucidate how the use of paper folding activities can help children better understand two-dimensional shapes and three dimensional objects, leading to the development of spatial reasoning.

Foundations for Spatial Visualisation

Spatial visualising ability is having the capacity to see, inspect, and reflect on spatial objects, images, relationships and transformations (Battista, 2007). It is the ability to "mentally manipulate, rotate, twist, or invert a pictorially presented stimulus object" (McGee, 1979, p. 893). This ability depends on two factors: (1) the purpose and design of the representations and, (2) the way the viewer sees these representations.

To begin, the learning of mathematics has a long tradition of using objects or diagrams as visual aids, as instructional representations to reason, comprehend, and develop conceptual understanding. When used in geometry, diagrams are not simply representations of actual objects experienced in the world. Rather, they are used in an attempt to take an abstract concept and make it concrete (Phillips, Norris, & Macnab, 2010). A good example would be the use of models to represent platonic solids, five regular, convex polyhedrons expounded in Euclid's Elements. According to Mesquita (1998, p. 183), reiterated by Jones and his colleagues (2012), instructional representations such as diagrams, graphs and figures can be classified as external (embodied materially on paper or other support) or iconical (figurative, centred on visual images). They can also be determined in terms of 'finiteness' (in the sense of finite and diversified forms) and 'ideal objectiveness'.

For example, the triangle in Figure 3 may be considered as a particular shape with sides 2, 3 and 4 in any units—an example of 'finiteness'; the same figure may also depict a geometric figure with three sides, with no reference made to specific concrete material—an example of "ideal objectiveness" (Mesquita, 1998, p. 186).

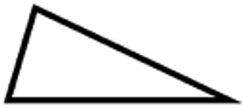

Figure 3. A triangle seen as a finite representation and an ideal object.

Some representations are also more typical than others. For example, many children associate the shape of a square as one that sits with the right angle on a horizontal side but not when it is tilted. A good representation should enable the simultaneous stimulation of multiple relationships, leading to the development of geometrical intuition. However, as Mesquita has cautioned, too much information on the geometrical relationships could appear as 'evident' to students, thus preventing the development of geometrical reasoning.

The ability to visualise does not just depend on the representation; it is also influenced by the way the representation is positioned in relation to the viewer. There are three different ways in which visualisation takes place (Phillips et al., 2010, p. 26). First, physical objects either on paper or the computer screens may be viewed and interpreted for the purpose of understanding something other than the object itself. For example, looking at a net consisting of twelve pentagons and identifying it as a dodecahedron. Second, when 'seeing' an object, the person begins with introspecting some possible images similar to a visualisation object. The accuracy of this mental object depends on how well the individual interprets the information she/he received, which is the third type of visualisation. Collectively, these three visualisations (objects, introspective and interpretive visualisation) place the meaning of any representation within the person's existing network of beliefs, experiences, and understanding. The ability to visualise a figure and reason geometrically is dependent on both the representation as well as the way it is being viewed.

Developing Spatial Visualisation

The human brain is an extraordinarily complex structure designed to store, retrieve and analyse information about objects, places and times. The structures responsible for spatial reasoning are fully functional at a very early age. Early practice on spatial visualisation is an important, even essential part of scaffolding for later development (Gersmehl & Gersmehl, 2007). Activities for teaching visualisation take two forms: visualisation for understanding, and visualisation for analysis (Phillips et al., 2010).

Visualisation for understanding helps children to encode important information in multiple ways. Language and a common vocabulary play a vital role in the development of geometric ideas (Booker, Bond, Sparrow, & Swan, 2014). Teaching for understanding requires activities be done conjointly with language-rich instruction (Phillips et al., 2010). Meanwhile, visualisation for analysis is intended to develop problem solving ability. The quality of the design is vital to help children see and interpret important features of the problem.

Paper folding is an excellent way of exposing children to the language of geometry and shaping their geometric thinking. Basic folding activities allow young children to learn two-dimensional shapes by creating regular and irregular polygons and identifying them. Thinking and making the folds sensitise children to the idea of concave and convex shapes. They learn to name shapes according to the number of corners found in the fold. Using a square paper, children can investigate how many different types of shapes making one fold can create. One fold can make a shape with three 'corners', in four different orientations, and multiple numbers of four - five and nine-sided shapes but not seven-or-eight-sided shapes. Comparing the result of making only one fold with a triangular or a rectangular paper extends children's understanding of the properties of these shapes.

A simple task of making a square paper from an A4 size paper (Figure 4) by folding one side to the adjacent side requires understanding that a square has two right-angled triangles joined together on its longest side (hypotenuse). Folding a five-pointed star from a pentagon or a six-pointed star from a hexagon enables children to see the geometric properties of these shapes. These activities provide opportunities for children to study regular and irregular shapes from different orientation. Misconceptions caused by exposure to prototypical shapes such as triangle, square, and oblong can then be resolved.

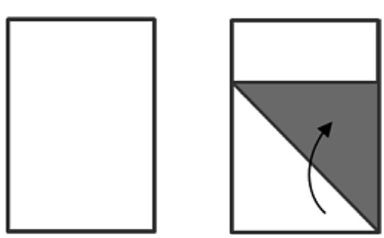

Figure 4. An example of folding a square using A4 paper.

Like the Möbius strip, the hexaflexagon or flexagon (Figure 5) is a fun way of getting children interested in geometry. Discovered by Arthur H. Stone (HREF1) in the 1930s, a flexagon is a flat model in the shape of a hexagon constructed by folding a strip of paper that can be flexed or folded in a certain way to show faces besides the two that were originally on the front and back. It develops an understanding that a hexagon can be made by six equilateral triangles or three parallelograms and why certain folds will not allow the model to flex. Painting the faces with different patterns allow children to investigate symmetry and be fascinated by the transformation.

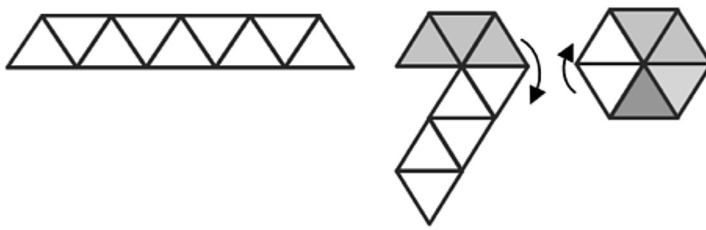

Figure 5. Sequence for folding a flexagon.

The idea of measuring the amount of turn from one direction to another is something many children find it hard to grasp. Typically, they struggle to comprehend the orientation of the drawn angle and the length of the lines enclosing the angle. When the relationship between polygons and the sum of interior angles is explored, it is often done through either measuring the angles of regular and irregular shapes printed on paper or dissecting polygons into triangles and make inference. Paper folding activities give children concrete representations of regular and irregular shapes to manipulate and measure and thus assist children to come to terms with these ideas (see Figure 6).

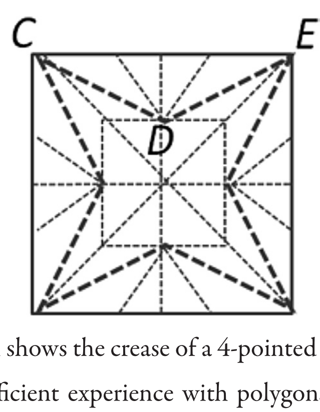

Figure 6. The diagram shows the crease of a 4-pointed box. What is $\angle CDE$?

Once children have sufficient experience with polygons, they will be ready to learn three-dimensional objects. Seeing a three-dimensional representation from a two-dimensional format is a critical skill for understanding the world around us as well as operating in situations where mathematics is needed including engineering, designs, and constructions (Booker et al., 2014). Nets, sticks, Geofix and other commercially produced materials have been used to construct three-dimensional objects such as the platonic solids. Constructing module origami is another engaging activity for children. These modules are constructed by joining multiple numbers of basic building blocks called a Sonobe unit (Figure 7). Folding the models can be simple whereas assembling them together will require some introspective visualisation of what the intended object will look like.

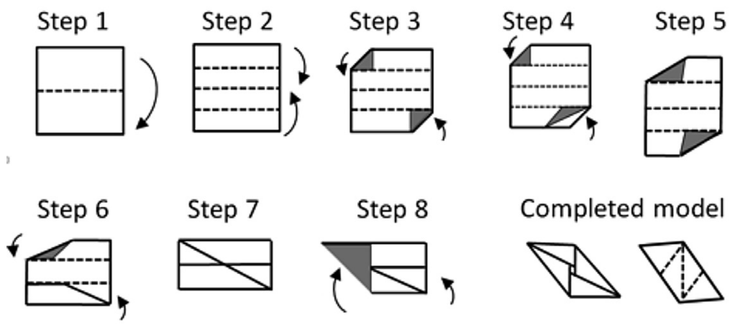

Figure 7. A Sonobe unit created by Mitsunobu Sonobe
[Simon, Arnstein, & Gurkewitz, 1999].

Encouraging children to generate their own visual objects, through drawing or construction of 3-D modules, is a crucial step in developing visualisation. It helps to determine how well the person interprets the information she/he received. Sharing these images with other children can help to develop communication skills and learn that there are many ways of interpreting an image. It is personally meaningful and relevant to students' understandings and ascertainment of their own prior knowledge.

Developing Spatial Visualisation

From an educational perspective, spatial ability provides information that other means of instruction do not, allowing children to develop deeper and richer concepts than they otherwise might (Phillips et al., 2010). The ability to visualise is dependent on both the design of the representation as well as individuals existing network of beliefs, experiences and understanding. Paper folding activities help develop visualisation and support understanding of essential geometric principles. They have the advantage of engaging children of all ages to learn geometry in a stimulating and enjoyable way. This in turn may encourage more children to engage with learning other aspects of mathematics rather than the number work that often dominates school mathematics.

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ALIGNING VALUES, ENERGISING MATHEMATICS LEARNING **

Wee Tiong Seah

Monash University

The convictions which teachers and students value in the teaching and learning respectively of mathematics in the classroom may be similar, but may also be different. The interactions within any mathematics lesson necessarily involve many instances each of which represents the coming together of these values, and there is thus the potential for contestation and conflicts. Expert teachers seem to be adept at negotiating about such value differences/conflicts to achieve values alignment, thus energising their students' learning experiences. Three mathematics classroom-based episodes exemplifying the strategies of redefining, reprioritising and complementing are presented in this article to demonstrate what values alignment might look like.

The Classroom, Where Values Come Together

Many teachers of mathematics would probably agree that the teaching and learning environment in many school mathematics classrooms has much room for improvement, in that students can be even more enthusiastic about learning mathematics, even more engaged in the subject, and demonstrate even better performance in applying the mathematical knowledge they have acquired.

Indeed, one of the most commonly-asked questions posed to mathematics teachers is probably 'when are we ever gonna use this maths?' (or its equivalent). This is probably an example of many a situation in the mathematics discourse of lessons which reflects differences in what teachers and their students value. The students may be valuing *application*, in that they find it important that what they have been learning can be applied to other things in life, so that they can decide how much effort or interest to dedicate to the topic. However, their teacher may be teaching, say, proof because she may be valuing *understanding* (when knowledge of the proof enhances students' understanding of the related mathematics content) or *rationalism* (see Bishop, 1988) (in that learning proof can develop learners' logical thinking skills).

Thus, the mathematics classroom represents a pedagogical space where the different values of students and teachers come together, and this happens many times over the duration of a lesson. When a teacher breaks the class up for a group discussion, the teacher's decision may reflect her valuing of, say, *collaboration* or *communication*. However, not all the students may necessarily share the same valuing. During the group discussion, the teacher may make a conscious effort to walk amongst the desks to familiarise herself with the students' knowledge and fluency, and to provide individual instruction if needed. Whatever that she values in this pedagogical approach, it is likely that not all students share this valuing too, and so this is another example of a value difference situation in the same lesson. Likewise, value differences can take place when the teacher and her students possibly value different convictions when she responds to particular students' answers to her questions, when she attends to a particular student in class who almost always never raises his hand, when she decides how much homework to allocate, and so on.

Value Differences Beyond the Classroom

On the other hand, it remains socially acceptable – preferable even – amongst teenagers and young adults to openly declare one's own dislike for and/or weakness in school mathematics. There is also a shortage of mathematics graduates in the workforce across many economies, with implications for key professions such as accountancy and engineering to lose their leadership and competitive edge in an internationally tough environment.

These take place despite that, over the years, educational research has provided teachers and educators with a huge range of resources and constructs which help us to understand better how children learn (mathematics), how mathematical ideas are cognitively processed and applied, and how affective factors regulate teachers' and learners' attitudes, confidence and beliefs.

It does seem reasonable to argue that we need to instill in learners the notion that mathematical knowledge and skills are important to us (whatever the reasons might be), that there is a need to learn it well, before the many cognitive and affective approaches to mathematics pedagogy will start to make real differences to learners' approaches to mathematics learning. Learners need to want to learn, and they need to acquire the values which give them the stubbornness to want to succeed in this learning when the going gets tough. That is, there is a need for us as teachers and educators to inculcate in our learners the will to learn (mathematics), and to guide the applications of some of their values to bring about purposeful and productive learning experiences.

The significance of the role of learners' will to learn – that is, their valuing of mathematics learning – had been discussed elsewhere (see, e.g., Seah & Wong, 2012). The purpose of this article, however, is to focus on the aspect of directing / guiding learners' applications of what they value in mathematics learning in mathematics lessons, as means of facilitating effective understanding of, and/or high performance in, mathematics. Firstly, there is a need to define what 'values' means in the context of school mathematics education. The article will then introduce the notion of 'values alignment' as it has been applied in other disciplines, before we look at examples of mathematics classroom discourses to see how different values alignment strategies might be employed by teachers to negotiate about value difference / conflict situations so as to optimize their students' mathematics learning experiences.

Values and Valuing in Mathematics Education

Research into the roles of values and valuing in mathematics learning and teaching can be said to have begun with Alan Bishop's seminal book in 1988, entitled 'Mathematical enculturation: A cultural perspective on mathematics education'. Bishop (1996) defined values in mathematics education as the

> deep affective qualities which education aims to foster through the school subject of mathematics. They appear to survive longer in peoples' memories than does conceptual and procedural knowledge, which unless it is regularly used tends to fade. (p. 19)

Ongoing research in the last 25 years or so has led to an alternative conception of values as being a volitional variable, a perspective which will be subscribed to in this article:

> Values are the convictions which an individual has internalised as being the things of importance and worth. What an individual values defines for her/him a window through which s/he views the world around her/him. Valuing provides the individual with the will and determination to maintain any course of action chosen in the learning and teaching of mathematics. They regulate the ways in which a learner's/teacher's

> cognitive skills and emotional dispositions are aligned to learning/teaching. (Seah & Andersson, in press)

As volitional variables, values and valuing represent convictions and will which in turn stimulate courses of action. As written by the philosopher Ayn Rand, "a being of volitional consciousness has no automatic course of behaviour. He [sic] needs a code of values to guide his actions" (1961, p. 97). This is not to say, however, that values are always expressed in the form of observable actions. Rather, the potential for action is the basis for valuing.

It is reasonable to imagine that an individual's commitment, conviction and will – that is, values – have both cognitive and affective components. Values have cognitive components because we can often explain and defend what we find important. And since we often find ourselves activating these mental processes with passion and devotion, the affective aspect is evident as well.

As a volitional construct, values and valuing support the development of cognitive processes and affective states. We can imagine, for example, a learner's attempts at understanding and doing well in vector mechanics because of her valuing of *duty*. For this learner, academic success may be regarded as a personal obligation to her parents for the sacrifices they had made to afford for her, say, an independent school education. Here, the learner's valuing of *duty* supports the development of mental reasoning skills in a particular mathematics topic that she has found difficult. Of course, *duty* is but one example of a value here; different learners value different convictions when learning the very same topic. On the other hand, values and valuing also support affective growth, such as when a learner's valuing of any one or more of Bishop's (1988) six mathematical values – *rationalism, objectism, control, progress, mystery, and openness* – potentially support the development of positive attitude to school mathematics.

Interactions in the Mathematics Classroom

The (mathematics) classroom represents a pedagogical space where teachers and their students interact and engage with one another. What are conveyed and exchanged (verbally or otherwise) would reflect what individuals involved in the discourse are valuing. This discourse would also represent negotiation attempts by all involved; learners too are aware of their capacities to adopt, resist or reject discursive positions. For example, when a learner asks the familiar question, "when will we ever need to use this mathematics?", it reflects her valuing of, say, *relevance*. In the same way, the teacher's response will reflect particular values too, which may or may not be in harmony with what the learner values.

However, these instances in the classroom discourse when different values come together suggest the potential for contestation and conflicts. Value differences and value conflicts are understandably not conducive to mathematics learning and teaching activities. In order for a lesson to 'move forward' productively, teachers cannot choose to ignore the value differences/conflicts. One way or another, consciously or not, teachers devise different strategies and practices to negotiate about the competing values such that there is achieved a certain level of alignment amongst the values concerned.

Values Alignment in the Mathematics Classroom

Values alignment is a term used by Peter Senge (2006) to describe how learning organisations (including schools) create shared visions within their structures, so that they survive and thrive. The underlying belief is that

all relationships – between one person and another, between the present and the future, between customer and product, a team and its goals, a leader and a vision – are claimed to be strengthened by aligned values. (Branson, 2008, p. 381)

Thus, for a teacher, being able to facilitate values alignment between what she and her students value differently promises to strengthen the relationships, and is one of the keys to nourishing teaching and learning practices. Indeed, if some teachers are effective (however the reader may understand this term) whichever classroom they are allocated, whereas others perform well in particular classrooms with particular learner characteristics only, this could be because the former have a way with attaining values alignment in whatever classroom situation they find themselves in.

However, values alignment is not about ensuring that students' values are the same as their teachers'. It is thus different from values inculcation. Rather,

> building … values alignment is about providing a cooperative and collaborative process whereby the members of the organisation can develop strategies, systems and capabilities that not only support those values that have previously been clarified as being essential for the ultimate success of the group as a whole but also are supported by the majority of the people within the group as acceptable guidelines for directing their behaviour. (Branson, 2008, p. 383)

That is, values alignment facilitates the co-existence of different values that are held by different people interacting together. In so doing, learners perceive that their knowledge, skills and dispositions are valued, thereby potentially deepening their sense of ownership in relation to their learning of mathematics.

Teacher attempts at aligning the values in the class can result in one of three possible scenarios. Firstly, a values alignment attempt may be successful, in that the teacher and her students are satisfied and comfortable with their respective values being in harmony with one another, so that productive mathematics learning and teaching can take place. Secondly, although the various competing values may be negotiated such that they are aligned with one another, the resultant scenario does not facilitate effective mathematics learning and/or teaching. A common example here would be when a teacher might give in to students' valuing due to her own valuing of, say, *peace*, such that she allows students to run the curriculum. The third scenario refers to the situation when the values alignment attempt does not succeed, and the different (conflicting) values remain unresolved.

It is reasonable to argue that the second and third scenarios represent unsatisfactory values alignment attempts, in that the quality of mathematics learning would be negatively affected in the absence of valuing that scaffolds meaningful cognitive and affective developments. How then do teachers (and students) go about negotiating the differences in what they value generally and in mathematics learning/teaching more specifically, so as to achieve values alignments that facilitate mathematics learning which can be represented in the first scenario above? The following classroom examples, based on mathematics lessons observed recently, demonstrate several such strategies.

Values Alignment Through Redefining

Case 1. Michael (a pseudonym) was a mathematics teacher in a secondary school. His attempts to scaffold students' learning with the use of manipulatives such as geoboards and pattern blocks were often met with student resistance. "These are for young kiddies, sir!" his students would say. Michael has since found a way round this issue, and his students are now exploring and understanding geometrical concepts using software programs such as dynamic geometry software, as well as online resources such as those hosted by the National Library of Virtual Manipulatives.

In this case, Michael's use of concrete manipulatives reflects his valuing of *visualisation*. However, this teaching approach was resisted by his students whose values were not aligned with the image of teenagers 'playing with blocks'. There was a potential here of a value conflict between Michael and his students, which could possibly result in the learners 'switching off' and becoming disengaged. In resolving the potential value conflict, Michael had redefined what he and his students valued, having understood that his valuing of *visualisation* actually reflected the valuing of something more internal and deeper, that is, *exploration*. This was crucial, since the students' values were aligned with *exploration* as well; it was just that they did not want to feel like small kids playing with blocks. By redefining his valuing of *visualisation* with the use of digital learning technologies, Michael was able to plan and execute his lessons such that the dynamic geometry software and the online apps provided the students with opportunities to explore – and thus visualise – the relevant geometrical ideas and concepts in a form that was aligned with what the students valued. Michael's valuing of *visualisation* had given him the will to resolve the value difference situation in ways which still allowed for student visualising to take place, only that the means of actualising this valuing were now acceptable to the teenage students, who were understandably wanting to behave more like adults and doing adult tasks. For his students, their positive response to the ICT use was an endorsement of their common valuing of *exploration*.

In this instance, values alignment was achieved through Michael's redefining what he valued such that its expression became aligned with what his students valued.

Values Alignment Through Reprioritising

Case 2. Diane was an immigrant secondary school mathematics teacher from Canada, teaching in a small country town in Australia. When one of her students suggested that he would 'just chuck in *c*, just chuck in the *c*' when working with an algebraic equation, Diane responded that he was being too casual with his use of mathematical language. Her own mathematics learning experience in Canada had instilled in her a valuing of *formality* (in mathematics), a tradition that she felt needed to be upheld. Thus she would have preferred her students to talk about "adding the constant, *c*".

Yet, at the same time, Diane was deeply aware and concerned that she was teaching a 'weak' class, and that meant that it would not be wise to get "too caught up in those formal, scary things". She was mindful that for these students, a valuing of *fun* would be a key motivator for them. As such, she made a conscious effort to "sacrifice 'plus *c*' for 'chuck in *c*' ".

Here, Diane had realised that 'pushing' her students to share her valuing of *formality* through the use of formal terminology in classroom discourse would be counter-productive. This group of students needed first and foremost to be able to be interested enough in the subject, and to develop some confidence to acquire the skills and concepts required of them. The students' valuing of *fun* would support the cognitive and affective growth that they needed, in that it would at least draw them to mathematics instead of them not wanting to have anything to do with it. This understanding led Diane to reprioritise her valuing of *formality* and *fun*, thereby achieving values alignment between herself and her students. This reprioritisation of Diane's values was evident when she talked about the relative importance of notations/formality and understanding/enjoyment, and how it would be her willing sacrifice to interchange the order of priority for the sake of facilitating her students' learning.

In this second case, values alignment was achieved when one group of people (in this case, Diane) in interaction reprioritises what they value to achieve a common valuing in the whole group. Diane had not given up her valuing of *formality*. However, she also shared students' valuing of *fun*, and a reprioritisation between these two values within herself has resulted in an alignment of what she and her students valued.

Values Alignment Through Complementing

Case 3. Amy taught in a primary school in a south-eastern suburb in Melbourne, where there was a high concentration of Asian migrant professionals in its demographics. There was a valuing of *competition* and *grades* in the community within which Amy's school was sited, and the school had embraced these values as well. However, Amy had grown up valuing *co-operation* and (conceptual) *understanding* instead.

Over time, Amy started to develop mathematics lessons which reflect the valuing of both *grades* and *understanding*, as well as the valuing of both *competition* and *co-operation*. Thus, her students strive to understand concepts while also engaged in daily practice in answering mathematics questions. They are also able to work together and help one other, while enjoying pitting their mathematical skills against one another.

For Amy, the need to confront the value differences between herself and the school/community was quite urgent, for she knew that she would not be able to teach mathematics effectively and professionally satisfyingly if she did not negotiate these differences soon enough. She talked to colleagues and some parents, and she referred to relevant literature. While she was not ready to give up what she had grown to value, she was also getting to understand how what the students valued were culturally powerful agents of engagement and motivation. At the same time, she felt that her students needed to learn to value *co-operation* too as a means of humanising competition, and that their developing meaningful understanding of (mathematical) concepts would further enhance their capacity to achieve even better grades in assessments. Over time, Amy developed pedagogical strategies that not only allowed for these potentially conflicting values to not just co-exist, but also to further support each other's development within her class.

In this case, values alignment for Amy and her students was achieved through an acknowledgement of the different values, and a purposeful consideration of how they could co-exist and indeed, complement each other. This complementarity reflects one of Hofstede's (2001) cultural value continua, that is, *masculinity / femininity*. Here, the students' valuing of *masculinity* in the form of *grades* and *competition* has struck a balance through alignment with Amy's valuing of *femininity*, in the form of *understanding* and *co-operation.*

Concluding Ideas

In this paper, the volitional construct of values and valuing was introduced. Teacher-student and student-student interactions in the mathematics classroom are framed within the context of instances when different values are expressed by the teacher and the students, with the potential for contestation and conflicts to take place. The extent to which teachers of mathematics are able to negotiate about such value differences / conflicts to achieve alignment of related values would affect the effectiveness of the mathematics lessons. Three mathematics classroom-based episodes exemplifying different value alignment strategies – that is, redefining, reprioritising and complementing – were presented above. Ongoing research work with expert teachers of mathematics is expected to add breadth and depth to these value alignment strategies, paving the way for future professional development activities with teachers to further optimise learners' mathematics learning experiences.

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FUN WITH DYNAMIC GEOMETRY IN PRE-SCHOOLS AND PRIMARY SCHOOLS **

Pumadevi Sivasubramaniam

Teacher Education Institute, Raja Melewar Campus, Malaysia

This paper describes how a graphing calculator project using dynamic geometry was planned and executed for pre-school and primary school students. The facilitators for these projects are undergraduate teacher trainees. The paper also describes the higher order thinking skills aimed to be developed through the project and the outcome of the first semester interactions. The paper includes a brief outline of the pedestrian training provided for the facilitators to equip them with the necessary skills to serve as facilitators for the project.

Introduction

Graphing calculators are hand-held educational tools used basically in secondary schools and higher education institutes. Hand-held graphing technology was first introduced into the secondary school curriculum in 1986 with the Casio fx-7000G (Waits & Demana, 1994). The technology through the years has proved to support the creation of new visions for mathematics education, many of which demand for broader access to deeper mathematics for all students (American Association for the Advancement of Science, 1993; National Research Council, 1989).

There have been several studies of young children using dynamic geometry software in computer environments (Healy, Hoyles, & Noss, 1997; Olive, 2000; Sinclair, Moss, & Jones, 2010). Olive describes children drawing free-hand stick figures using the circle and segment tools in Geometer's Sketchpad. They were challenged to construct a row of 'paper dolls' similar to the row obtained when cutting out a stick figure from folded paper. The children enjoyed moving parts of their stick figures to make them 'dance' together. Dynamic geometry software is now readily available as computer software and in graphing calculators. However, there has been little reported use of the dynamic geometry functionality of graphing calculators in primary schools. At the 2014 Teachers Teaching with Technology International Conference, Marsha Burkholder presented a workshop titled *"TI-Nspire Technology is a Virtual Manipulative for the Elementary Student"*. The teacher used graphing calculators for pupils in grades 5 and above. In Malaysia two projects called the *TI-Nspire project for Primary Schools* and the *TI-Nspire project for Preschool* were initiated this year. The piloting project was carried out in October, 2013. Preparation for these two projects started in 2012, including planning for a year of training for teacher trainees to serve as facilitators for the projects. This training for the teacher trainees was conducted in 2013 (January to September).

The project did not employ the graphing calculator as a tool for performing mathematical calculations nor for deeper mathematics. The dynamic geometry application software in the graphing calculator was used as a medium for individuals to develop and demonstrate their higher order thinking skills.

The provision of this dynamic geometry application makes this hand-held technology suitable for primary school (aged 7 to 10) and pre-school (aged 5 and 6) pupils. The projects aimed to develop higher order thinking skills among primary school and pre-school pupils using the geometry application as the tool to support the development and demonstration of these higher order thinking skills in a fun filled context outside the demands of the national curriculum.

The Project and HOTS

Higher-order thinking is a concept of education based on learning taxonomies such as Bloom's Taxonomy. In Bloom's Taxonomy, skills involving recall and comprehension are classified as lower order thinking skills (LOTS). On the other hand, skills involving analysis, evaluation, and synthesis (creation of new knowledge) are thought to be of a higher order, requiring different learning and teaching methods compared with the learning of lower order thinking skills. Higher-order thinking skills (HOTS) are activated when individuals encounter unfamiliar problems, uncertainties, questions, or dilemmas. Successful applications of HOTS result in logical explanations, accurate decisions, unique performances, and creation of new products that are valid within the context of available knowledge and experience and that promote continued growth in intellectual skills. Higher order thinking skills are grounded in lower order thinking skills and are linked to prior knowledge of subject matter content of the individual. Appropriate teaching strategies and learning environments facilitate the growth of HOTS as do student persistence, self-monitoring, and positive attitudes (King, Goodson, & Rohani, 1998).

It is claimed that using HOTS requires more cognitive processing than LOTS and also has more generalised benefits. Higher order thinking involves the learning of complex judgmental skills such as critical thinking and problem solving. Higher order thinking skills are more challenging to learn or teach and are also more valuable because such skills are required to enable an individual to solve problems in novel situations (situations unfamiliar to those in which the skill was learned).

With the goal to foster higher order thinking skills, the TI-Nspire project challenges individuals to be critical and creative thinkers by providing the basic technological knowledge to use the graphing calculator and an open platform to employ the learnt knowledge to be creative. The project classified the levels of activities into a hierarchy of three categories based on the difference in demand of the cognitive processes involved in performing the activity. Psychomotor skills are also involved but are similar for all levels. The three levels of activities are as follows:

Level 1: Static Drawing

This level requires an individual to first draw by hand on a slate what they intend to create. Then make a representation of their drawing using the geometry tools in the TI-Nspire CX graphing calculator. (See Figure 1.)

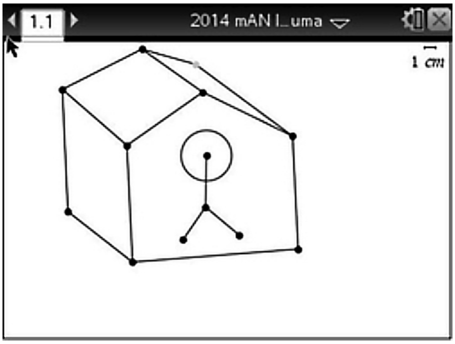

Figure 1. Stick drawing of a man in a house (static drawing).

Cognitive process involved: Decision making in selecting the appropriate geometric shapes to create the desired product with the limited facilities provided by the software.

This process demands logical and critical thinking to manipulate the facilities offered in the graphing calculator to produce the desired outcome. Such thinking represents application and analysis levels of Bloom's Taxonomy. Application is involved because the individual has to apply the acquired knowledge in a new situation. Analysis is involved because the individual has to determine which shape, size, order, and colour is to be selected so that all the shapes will appear on the screen to produce the desired product. Drawing and colouring also requires the selection of appropriate tools. Objects drawn using the wrong tools cannot be coloured.

Psychomotor process involved: Drawing using the graphing calculator is not the same as drawing using a pencil and paper. The difficulty when using pencil and paper is the demand on the artistic skills of the individual. Using the graphing calculator requires a different skill – the ability to draw a virtual image of the object you desire. Dexterity plays an important role when manipulating the cursor pad in the graphing calculator to produce an accurate image that one desires. For example, the correct amount of pressure has to be applied by the individual using his/her finger on the graphing calculator pad to 'grab' a point or segment using the cursor and then gently slide it to the desired position.

Level 2: Animating an Object

For this level an individual has to be able to do all that is required in Level 1 and then animate the created object or objects. This level has several sub-levels of challenges. Animating a geometric shape is the first sub-level.

It is based on the fundamental step of animating a point. To animate a shape has different levels of cognitive demands. The simplest is to animate a circle by animating the centre of the circle on a straight line or on the circumference of another circle. Animating a drawn object is even more difficult to achieve and makes greater cognitive demands.

Cognitive process involved: All cognitive processes in Level 1 are required. However to animate the object drawn in Level 1 requires a higher level of organisation which has to be engineered by intelligent cognitive processes in the drawing process so that the object drawn animates as desired. To draw so that the desired animation takes place requires organisational analysis of the manner in which the animation works, followed by synthesis of a plan to try out. From the outcome of the trial, the individual would have to make judgements of the alterations that are necessary to make the plan feasible. For example, in Level 1 the stick man was drawn using a circle and a few segments, and could be drawn in any order. To animate the drawn object – that is to make it move within the frame of the graphing calculator – requires two straight parallel segments and not two parallel lines (Figure 2). Then the circle must be drawn with the centre point on one line to represent the head. On the other segment the centre of another circle is drawn, and the legs must be drawn from the centre of the circle to the circumference. Animating each centre then will enable the whole stick man to move along a straight line. Testing it may result in distortion of the object because the segments may not be parallel and the centres may not be perpendicular to each other. Hence, further reasoning is required to make appropriate alterations.

Other animations, such as shooting an arrow to hit a specific target, are cognitively less demanding but require careful selection of the correct shape to make the path of movement of the arrow appear as it takes place in the real world. It also requires using an arc and not a straight line for the path of movement of the arrow and setting the angle of the bow and the target to be hit at the appropriate positions (Figure 3).

Psychomotor process involved: Similar to those involved in Level 1.

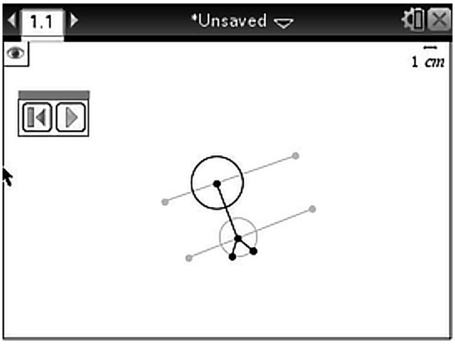

Figure 2. Animating the Stick Man.

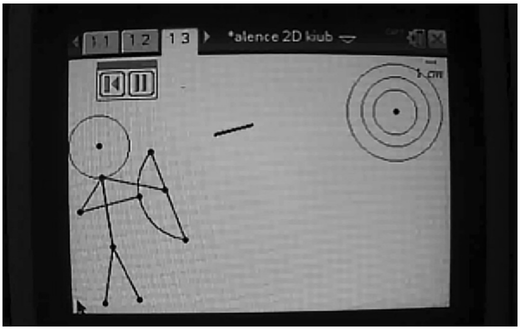

Figure 3. Arrow animated to hit the target from the bow.

Level 3: Using the Trace Tool to Draw a Real World Object

The Trace facility was chosen as the tool with which students could display their higher order thinking skills. Creating a real world object such as the coconut tree in *Figure 4* makes the greatest cognitive demand. It makes all the cognitive demands of Level 2 but further requires careful organisational analysis combined with creative ability to relate the repeated trace of a shape involved in an animation process to produce a real world object. Many individuals are unable to create a real world object using the Trace facility.

Psychomotor process involved: Similar to those involved in Level 1.

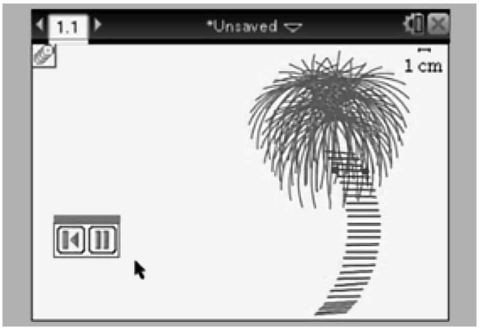

Figure 4. A coconut tree drawn using the Trace facility.

Executing the Project

With the three levels of fun filled tasks set for the project, the first stage of the project was to train 22 trainee teachers (mathematics majors) who then served as facilitators for the primary school and pre-school projects. The trainees were basically given introduction to on, off and use of the basic tools in the geometry application in the graphing calculator by their lecturer. They were challenged to search the internet to obtain further information on how to use the geometry application in the TI-Nspire CX graphing calculator. The trainees were informed that that they would have to produce Level 2 and Level 3 products to qualify to enter the TI-Nspire competition. The competition served as a purpose and motivation for the trainees while the criteria set for each activity served as a guide of the skills required to be acquired to prepare them as qualified facilitators for the projects. They were allowed to keep the graphing calculators for a period of six months. At the end of the six months they were to submit a video and typed text in softcopy of the steps involved in creating their two products. The rule stated that if only one product was submitted then the candidate would be disqualified. All their products were then used to produce an e-book and a hardcopy printed book. The products were aimed to be concrete evidence of the skills acquired by the trainees to qualify them as facilitators for the primary and pre-school projects. The products also displayed each individual's knowledge of using the geometry application. Each individual was then to explain to the others and share their knowledge, hence creating a learning community among the trainees to enhance their skills in using the graphing calculator. Prior to the products being produced they were unaware what they could share with each other.

The next stage was the piloting of the project. Piloting for the pre-school and primary school projects was carried out on pre-school pupils from the Raja Melewar campus pre-school. Each trainee worked individually with a pupil so that the teaching and learning processes were at the pace of the individual pupil. There were 25 pairs of trainees and pupils involved in the piloting. Each trainee and pupil pair was supplied with a graphing calculator and a set of notes of the technical skills to be taught. Pupils in the pre-school were required to do Level 1 and Level 2 activities. The piloting done on pre-school pupils was also an indication of the feasibility of the project for primary school pupils.

The pre-school and primary school projects commenced in 2014. The pre-school project involved 20 pupils during their school hours and the primary school project involved 20 pupils after school hours. The primary school project was conducted by 20 trainees for all five sessions. Hence, there was a trainee to pupil ratio of one to one for all the primary school sessions.

For the pre-school project, the first session was conducted by the lecturer, while the second session was conducted by 17 trainees. Hence, the first pre-school session was a one teacher to twenty teacher pupil ratio. The second session had three groups with a one trainee to two pupils ratio, while the remaining groups had a one trainee to one pupil ratio. Five interactions per semester were set for the primary school project with Levels 1 to 3 as the target, while two interactions were set for pre-school project per semester with Levels 1 and 2 as the target.

The second semester's interactions are basically for a competition and will not involve further teaching and learning. It will solely involve independent pupils working with the graphing calculator to produce products that fulfil the criteria set for the competition. The primary school interactions were to be 60 minute sessions, while the pre-school sessions were planned to be about 40 minutes per session. However, these time limits were never adhered to for every session – sessions always extended even as much as 60 minutes when not regulated.

Findings

Pedagogical Aspect

The first interaction for both projects (the primary school and pre-school projects) was very motivating. The pre-school pupils claimed they had never used a calculator, let alone a graphing calculator. All the participating pupils were highly motivated and keen to learn. The opportunity for immediate response to their inquiries of how to operate the various tools led to an active learning environment. This immediate attention is vitally important for teaching the technical aspects for manipulating the geometry application in the graphing calculator. When only one teacher was dealing with 20 pupils during the first pre-school interaction, the teacher was seated on the floor so that the pupils could sit around and touch the teacher to draw her attention to respond to their inquiries. The number of tools that could be explained to the pupils was limited because the psychomotor level requires the teacher to hold the child's hand and guide the child to exert the required pressure to command the calculator. This attention was required for all pre-school pupils and lower primary pupils; that is, pupils aged 5 to 9. The pupils above 9 years old were capable of discovering for themselves by trial and error the skills of manipulating the cursor using the pad on the graphing calculator. However a range of 1 to 5 pupils per teacher is suitable for the teaching of the technical skills to use the geometry tools in the graphing calculator.

Affective Domain

In both projects pupils were enjoying the interactive nature of the graphing calculator. A pupil in the pre-school stated, "This (the graphing calculator) is playing with me. I like this toy". During every session for both projects, the pupils never requested to stop and this always led to the extension of the length of time for each session.

Cognitive Domain: Level 1 Activity

As pupils were introduced to the tools in the geometry application, they were eager to draw a house or a playground and they used the segment tool. When they wanted to colour their drawings they were taught to draw geometric shapes such as circles, triangles, rectangles and other polygons (Figure 5). They often discovered on their own the order to draw and colour the objects. If they wanted a red circle to be drawn inside a yellow rectangle, then they had to draw the yellow rectangle first and then the red circle on top. If they drew the yellow rectangle after drawing the red circle, only the outline of the circle would be seen. The colour would be covered by the yellow colour of the rectangle. All pupils in the pre-school and primary school had no difficulty doing Level 1 activities. Of the 20 pre-school pupils all

20 could produce products for Level 1 after about an hour. On the other hand, the primary school pupils were able to do Level 1 activities in a shorter span of time – less than one hour.

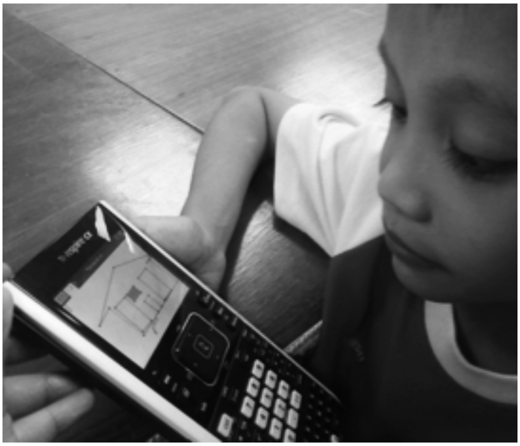

Figure 5. A pre-school pupil's screen drawing of a house.

Cognitive Domain: Level 2 Activity

The animation tool was the most fascinating tool for all the pupils in both projects. Once they learnt to animate a point then a circle, they wanted to animate everything they drew. The challenge to animate any drawn object requires organisational skills of varying levels depending on the object drawn. The fact is that the drawing as done in Level 1 could not be animated as desired. Hence the process in this level begins with being clear with what you want to animate. Then organising the drawing steps with the realisation that all animations are based on animation of a point on a path on which the point exists. This realisation comes from exploring and reasoning and remembering the results of exploration. Initially, the pupils had a tendency to turn to the trainee teacher to provide immediate help. During the subsequent sessions trainees were pre-warned not to supply the solution. It was interesting and astounding how the pupils were able to solve their problems through deep thinking and reasoning, and achieve the desired outcome when no help was given.

Cognitive Domain: Level 3 Activity

This level is the most challenging level. Of the pupils who participated, none was able to produce a real object using the trace facility. Even trainees found this extremely challenging. Only three out of the 22 trainees were able to produce a real world object for this level while others merely produced patterns.

Conclusion

These projects to develop HOTS are fun filled sessions and have proved to be successful. Pupils did develop HOTS but the time frame and number of limited sessions for pupils to use the graphing calculators provided limited opportunity for greater achievements such as dealing with a Level 3 activity. A longer period of time with more frequent sessions may prove to be more fruitful.

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DOUBLE DIVISION **

Pumadevi Sivasubramaniam and Mohd Ariff Jasmi

Teacher Education Institute, Raja Melewar Campus, Malaysia

This paper explains how the conventional long division method posed difficulty for students to do division of four to six digit numbers by divisors which are two digit numbers. To help these primary school students to overcome their difficulty a method called double division was used. The extent to which this method helped the students in doing and understanding division of large numbers and the extent to which the experience of doing division using this method enabled students to deal with higher order thinking skills questions are described in this paper.

Introduction

The long division algorithm is the most popular method used in primary schools in Malaysia to divide large numbers. In order to use this algorithm, pupils must have the knowledge and understanding of the following basic concepts and skills (Wilson, 2005):

- Ability to estimate the quotient,
- Knowledge of multiplication and division fact, and
- Ability to subtract.

The algorithm for long division is one of the most complicated and meaningless procedures that pupils in the primary level have to master and hence many are unable to do long division (Wilson, 2005). When a trainee taught a Year 4 class the long division algorithm to do division of large numbers many were incapable of doing it. Among the errors made were the following: (a) not using zero as a place holder (Figure 1), (b) writing the answer in the wrong place value position (Figure 1), (c) unable to estimate the correct quotient. (Figure 2), and (d) mere careless mistakes (Figure 3).

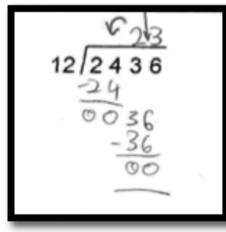

Figure 1. Sample of a student's work illustrating zero not used as a place holder and writing the answer in the wrong place value position.

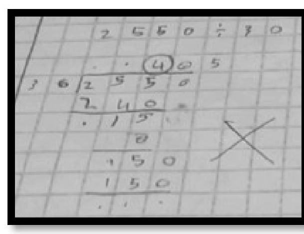

Figure 2. Sample of a student's work illustrating estimation of an incorrect quotient.

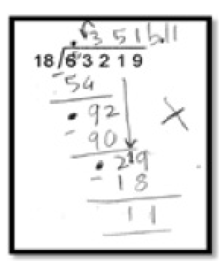

Figure 3. Sample of a student's work illustrating a careless mistake.

In the determination to find a method to help his pupils do division of large numbers, the trainee reviewed literature and came across a method called the double division method suggested by Wilson (2005). This method does not require estimation skills as a prerequisite. The prerequisite for using the double division method are addition and subtraction with and without regrouping. Figure 4 shows the division of 21 639 by 77 using the double division method.

To obtain 2×77, just add $77 + 77 = 154$. Like wise to obtain 4×77, add $154 + 154 = 308$, to obtain 5×77 add $308 = (4 \times 77)$ and $77 = (1 \times 77)$ and so on. Hence one does not have to memorise the basic multiplication facts to use the double division method to do division of large numbers.

$$
\begin{array}{llcrr}
1\times 77 = 77 & & & 26139 & \\
2\times 77 = 154 & (77+77) & & -\underline{15400} & 200 \\
4\times 77 = 308 & (154+154) & & 6239 & \\
8\times 77 = 616 & (308+308) & & -\underline{6160} & 80 \\
 & & & 79 & \\
 & & & -\underline{77} & \\
 & & & 2 & \underline{+\ 1} \\
 & & & & 281
\end{array}
$$

Answer: $26139 \div 77 = 281$ remainder 2

Figure 4. Using the double division method to divide 21639 by 77.

This study describes the effects of the double division method on the performance of pupils and their understanding of the algorithm when doing division of large numbers, that is, 4-digit, 5-digit and 6-digit dividends by 2-digit divisors. The study also examines the effect of the double division method on pupils' ability to deal with higher order thinking skills questions related to division.

Methodology

This study involved six Year 4 pupils from a rural primary school in Malaysia. They were selected based on a pre-test administrated to a class of 33 pupils. The test consisted of two parts – Part A and Part B. Part A consisted of 15 long division questions, all with two digit divisors but five with 4-digit dividends, five with 5-digit dividends and five with 6-digit dividends.

Part B consisted of 10 questions aimed at determining pupils' ability to deal with higher order thinking skills questions. The 10 questions were classified using Bloom's revised taxonomy levels (Anderson et al., 2000) as shown in Table 1. Questions in levels classified as applying, analysing, evaluating, and creating are higher order thinking skills questions.

Table 1. C*lassification of Part B Questions Based on Bloom's Revised Taxonomy*

Question Number	**Question**	**Level in Bloom's revised taxonomy**
1	_ _ _ _ _ ÷ 11 = 4035	Understanding
2	Find the value of 40350 × 11 based on your working for question 1. Explain your answer	Applying
3	500600 ÷ ____ = 2503	Analysing
4	4438500 ÷ ____ = 4035	Analysing
	(Instruction: For questions 5 to 8, select (circle) without doing any calculations 'True' if the answer is possible and 'False' if the answer is not a possible answer.	
5	1000 ÷ 25 = 4000 True / False	Evaluating
6	1100 ÷ 25 = 44 True / False	Evaluating
7	4500 ÷ 15 = 300 True / False	Evaluating
8	0750 ÷ 15 = 500 True / False	Evaluating
9	_ _ 2 _ ÷ 21 = _ _ 01 remainder 2	Creating
10	8423 ÷ _ _ = _ _ 0 _ remainder 2	Creating

Pupils were selected for the study based on their scores in Part A of the test using the following criteria: Two pupils whose percentage score is above 55%, two with scores between 20% to 55% and two with scores below 20%. Hence two pupils with a score of 10, two with a score of 8 and two with a score of zero were selected as participants for this study. The six selected pupils were then interviewed to obtain insight about their line of thinking when doing the questions using the long division algorithm. The next stage was the teaching and learning of the double division method. During the first teaching and learning session division of four up to six digit dividends by single digit divisors were taught. This was followed by Exercise 1 which consisted of 6 questions of four up to six digit dividends and single digit divisors. Two days later the second learning session commenced. Division of five and six digit dividends by 2-digit divisors were taught. This was followed by Exercise 2 which consisted of 8 questions of five digit dividends and 2-digit divisors and Exercise 3 which consisted of 4 questions of six digit dividends and 2-digit divisors. Two days after the second teaching and learning session, the post-test was administered followed by an interview of the pupils to obtain insight about their line of thinking when doing the questions using the double division method. A reflective journal was kept of every interaction with the pupils, in addition to field notes taken during interactions of any observable behaviour of the pupils.

Findings

The Pre-test and Post-test (Part A)

Table 2 shows the overall percentage score of the six participants of this study in Part A of the pre-test and post-test. The data shows all pupils improving in their test scores. Pupils 1, 2, 4, 5 and 6 improved their score by more than 20%.

Table 2. *Part A Pre-test and Post-test Percentage Scores*

Pupil	Pre-test score	Post-test score	Post-test – Pre-test
1	67	93	+26
2	67	100	+33
3	53	60	+7
4	53	87	+34
5	0	87	+87
6	0	100	+100

Pupil 1 made multiplication errors for questions 4, 6 and 15, careless mistake -subtraction error for question 5 and a careless mistake - writing the wrong number (0 instead of 9) for question 12 in the pre-test. In the post-test he made only one subtraction error for question 15. All multiplication errors were overcome because only addition was used to obtain the products in the post-test (Figure 5).

Pupil 2 did not use zero as a place holder for questions 7 and 12 and made careless mistakes for several questions in the pre-test. Using zero as a place holder does not arise when using the double division method and hence questions 7 and 12 posed no problem (Figure 6) in the post-test for Pupil 2.

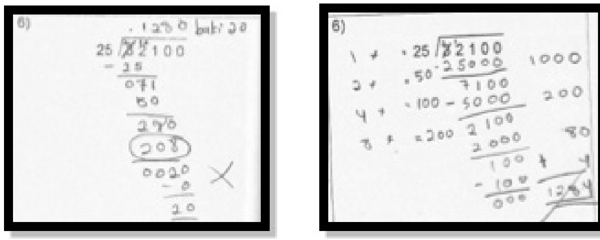

Figure 5. Pupil 1's working for question 6 (pre-test: left; post-test: right).

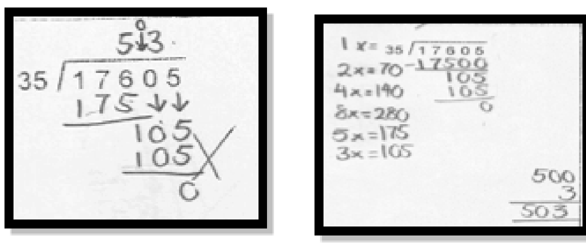

Figure 6. Pupil 2's working for question 7 (pre-test: left; post-test: right).

Pupil 3 made careless mistakes in addition in the pre-test and post-test. Hence his performance did not differ much from the pre-test to the post-test (Table 2). Pupil 4 made several multiplication and subtraction errors in the pre-test. Pupil 4 also made a poor estimation of the quotient for question 14 in the pre-test but this was not repeated in the post-test (Figure 7).

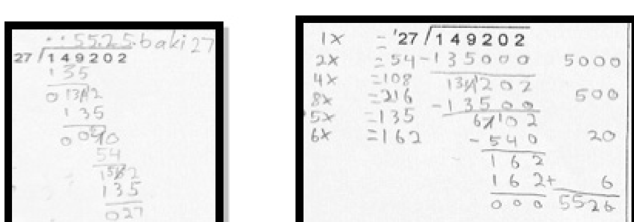

Figure 7. Pupil 4's working for question 14 (pre-test: left; post-test: right).

Pupils 5 and 6 did meaningless working for all questions in the pre-test. In the post-test Pupil 5 only made two errors which were careless mistakes while Pupil 6 obtained full score. The working shown in the post-test by Pupils 1, 3, and 6 for some of the questions illustrated that they realised that if for example their first estimation of a quotient in the hundreds place value was too small, another suitable estimate of a quotient in the hundreds place value could be added on when using the double division method (Figure 8). However, using the long division algorithm would require redoing and not adding on (Figure 8). Hence, the long division algorithm is not as flexible as the double division method.

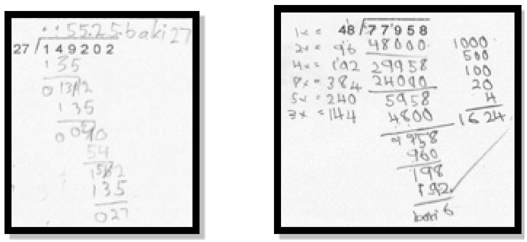

Figure 8. Pupil 4's question 14 (left) and Pupil 6's question 10 (right) working showing flexibility difference.

The double division did help Pupils 1, 2, 4, 5, and 6 to overcome most if not all of their difficulties in dividing large numbers. This method however did not help Pupil 3 to overcome his careless mistakes in addition. The addition operation plays a major role in the double division method hence Pupil 3's performance did not improve much with the use of the double division method.

The Interview After the Pre-test and Post-test (Part A)

During the interview after the pre-test, Pupil 1 stated the incorrect dividend for each of the corresponding digits in the quotient. For example, for question 1 ($1599 \div 13 = 123$), he stated that the dividend 13 and 26 produced the digits 1 and 2 respectively in the quotient. This is because the digit 1 in the quotient is assumed to be the quotient of the dividend 13 divided by the divisor 13, and not 1300 divided by 13 to obtain a quotient equal to 100, although digit 1 in the answer is written in the hundreds place value position when doing long division. The teacher when teaching had explained that 13 divided by 13 equals one, and so this is the pupils' understanding of the long division algorithm. Hence, the only correct response was for the digit in the ones place value position; that is, 3 in the quotient was from the dividend 39. Pupils 2, 3, and 4 gave similar responses. As for pupils 5 and 6, the interview after the pre-test revealed that both pupils were thoroughly confused with the conventional long division algorithm.

In the interview after the post-test all six pupils were able to state the correct dividend that produced each of the digits in the quotient (with a few exceptions which will be discussed later in this section). This was because the double division algorithm deals with the exact value of the dividend and quotient. For example, for question 1, the first step involves the dividend 1300 and the quotient 100. Then subtracting 1300 from 1599 leaves a remainder of 299, and the next quotient has to be estimated. The quotient is 20 and the dividend is 260. Hence, the pupils were able to state the correct dividend that produced each of the digits in the quotients. However, Pupil 1 was unable to state the dividend that produced the digit 0 for question 2, but later in the interview for questions 7 and 12 she managed to state 0. This difficulty arose because the double division does not show how a zero appears in the quotient except as a result of adding all the quotients of the parts which results in the sum which is the final quotient (the answer) having a zero.

Pupils 1, 3, and 6 for all questions with working similar to that in Figure 9 were unable to match the digits in the quotient with the correct dividend during the interview after the post-test.

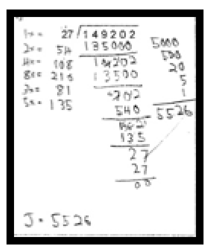

Figure 9. Pupil 6's working for question 14 in the post-test.

Pupil 6 stated 27 as the dividend that produced the digit 6 in the quotient and not 162. Similar responses were given by Pupils 1, 3, and 6 for all questions done in this manner. The interview revealed that although the double division method provided flexibility to obtain the final quotient, it confused pupils about the relationship between the digits in the quotient with the correct dividend. Pupils 2, 4 and 5 who did not show this type of working were able to correctly connect the dividend and quotient.

All six pupils during the interview after the post-test stated that they preferred using the double division method to the conventional long division.

Part B – The 10 Questions

The results in the pre-test and post-test for Part B are in Table 3. The results show that all pupils have improved marginally from their pre-test to post-test scores for Part B. In the pre-test all six pupils answered questions 5, 6, 7 and 8. During the interview after the pre-test all six pupils admitted that they guessed the answers. Pupils 1, 2, 3, 5 and 6 guessed incorrect answers for all the other questions. Pupil 4 had correct answers for questions 1 with the correct working of $4035 \times 11 = 44385$. However for question 2, Pupils 3 and

4 multiplied 40350 by 11 and obtained the correct answer. Pupil 3 did not connect the answer in question 1 with that in question 2.

Table 3. *Part B, Pre-Test and Post-Test Percentage Scores*

Pupil	**Pre-test score**	**Post-test score**	**Post-test – Pre-test**
1	30	50	+20
2	40	60	+20
3	40	60	+20
4	50	60	+10
5	40	50	+10
6	20	40	+20

In the post-test, all pupils obtained the correct response for questions 5, 6, 7 and 8 but when interviewed, all used calculation of multiplying the quotient with the divisor. None of them could reason out without calculation. For example for question 5, none of the pupils were able to state that a 4-digit number divided by a 2-digit number can only result in a 3-digit or 2-digit quotient.

In the post-test, question 1 was answered by all 6 pupils using the calculation 4035 × 11 and question 2 using 40350 × 11 Although all six pupils stated these calculations for questions 1 and 2, none of the pupils connected the question 2 answer with the question 1 answer. It was hoped that they would state that since 4035 × 11 = 44385 in question 1, 40350 × 11 = 443850 in question 2.

Questions 3, 4, 9 and 10 were all incorrect in the pre-test for all six pupils. When interviewed after the pre-test, they claimed that they did not have a clue of how to do these questions. In the post-test all six pupils did not get questions 3 and 4 correct. However during the interview Pupil 4 stated that for question 3 and 4, the divisor can be obtained by dividing the dividends 50060 (question 3) and 4438500 (question 4) by their respective quotients 2503 and 4035. This division obviously was too tedious for the pupil, and hence no answer was written. None of the pupils used reasoning such as the dividend of 5-digits has produced a quotient of 4-digits hence the divisor must be a 2-digit number. Then $50 \div 2 = 25$, but $0 \div 5 \neq 25$ and hence the divisor must be less than 25 and so on and so forth.

Question 9 was done correctly by Pupil 4 in the post-test. However, during the interview Pupil 4 claimed that he had guessed. His dividend was 4223 and quotient as 0201 R2. Pupils 1, 2 and 6 also had guessed the same dividend and quotient for the first step, but failed to add the remainder of 2 and hence got the wrong answer 4221 (Figure 10). The fact that they wrote a zero for the first space for the quotient illustrated that they had insight that when a 4-digit number is divided by a 2-digit number, the quotient cannot be a 4-digit number. However, none of them realised that the first two digits of the dividend could also be 63 or 84, and that the first digit of the quotient could be 3 or 4 respectively.

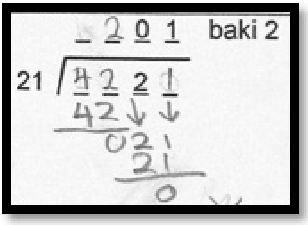

Figure 10. Pupil 2's working for question 9, Part B.

All the six pupils failed to answer question 10 correctly even in the post-test. They did not realise that question 9 and 10 were related. The first sensible step would be to subtract the remainder from the given dividend which would give 8421 as the resulting dividend. Then, using 4 as the first digit in the second space of the quotient, and using 21 as the divisor, would have resulted in the correct final answer. All six claimed that they did not know how to do question 10.

Conclusion

The double division method did improve pupils' performance in division of large numbers involving 2-digit divisors. The double division method provided a meaningful algorithm, and helped the pupils to avoid encountering the difficulties of determining the appropriate place value position of the digits of the quotient, using zero as place holder and recalling their times tables. Hence, the difficulties posed when using the long division algorithm were not evident when using the double division method. The double division method also improves pupils' understanding of division of large numbers. Hence, the double division method was a suitable method for the pupils in this study to overcome their difficulties, and to improve their attitude and performance in doing division of 4, 5 and 6 digit dividends by 2-digit divisors. The double division method to a lesser extent did improve pupils' ability to deal with higher order thinking skills questions, but this is inconclusive as the marginal improvement may be due to the use of the same questions in the pre-test and post-test – the effect of repeated exposure.

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USING ITEMS FROM THE PISA INTERNATIONAL MATHEMATICS ASSESSMENT IN TEACHING *

Kaye Stacey

University of Melbourne

This paper will look at some of the items used in PISA 2012, the OECD's 2012 assessment of mathematical literacy for 15 year olds. The media attention is always on the rankings and comparisons with other countries, but PISA also provides a range of resources that can be adapted for teaching and need to be better known. There are many publically released items, paper-based and computer-based, as well as frameworks to help teachers analyse the cognitive demand of mathematics items. PISA items test whether students will be able to use mathematics in their personal lives, as citizens and for work.

PISA: Its Purpose and Outcomes

Since 2000, the OECD has conducted regular surveys of students to ascertain the extent to which schooling systems around the world prepare their students for life after school. This Programme for International Student Assessment (PISA) examines whether students are prepared for life-long learning and all spheres of life: personal life (e.g., personal budgeting and purchasing, moving around), being a citizen (e.g., voting, understanding tax systems, community decision making), working in occupations requiring high or low level mathematical skills, and for understanding the broad scientific and technical world. Because the OECD is an organization of governments, its main focus is on the education systems' role in developing citizens who will have productive and satisfying lives and who will be able to make an economic contribution to their countries. The PISA surveys measure how well educational systems meet this important goal, and reveal some of the characteristics of the systems which perform well.

PISA is a huge educational study. Surveys are conducted every three years, with a random sample of 15-year-old students. In 2012, nearly 519,000 students in 65 countries participated in the main survey covering mathematics, reading, science, general problem solving and the core background questionnaires, with many undertaking the optional components including computer-based assessment of reading and mathematical literacy, financial literacy. The survey was delivered in 85 different national versions in 43 different languages, using rigorous processes to ensure that the items are free from cultural and linguistic bias, so that the data are as comparable as possible (Stacey & Turner, 2015). Because of the scale of the surveys and the complexity of data integrity and analysis, the first results of the latest survey, PISA 2012, were not released until December 2013 (OECD, 2013) and findings will continue to be published for many years (e.g., Ainley & Gebhardt, 2014).

In accordance with OECD policy, a great deal of information is freely available about PISA, past and present, especially through the OECD website. There is now a very large bank of publically released items which can be used and adapted by teachers (OECD, 2006, 2009, 2013a). A great deal of information about Australia is available from the ACER's 'ozpisa' website. The most detailed analysis of Australia's mathematics results are ACER's reports of the two surveys which have examined mathematics most intensely: PISA 2012 (Thomson, De Bortoli, & Buckley, 2013) and PISA 2003 (Thomson, Cresswell, & De Bortoli, 2004). All of the publications in the reference list are available online. Results from PISA are used in many different ways: to compare the performance of different countries, to examine the differential performance of students belonging to different subgroups within a country (e.g., Australian states, school sectors, migration groups), to track performance over time and to link features of the learning environment to student performance. The surveys are designed so that scores (not ranks) from different survey administrations are directly comparable, so it is now possible to examine trends in achievement for mathematics from 2003 to 2012. Australia's performance has always been above average, consistent with our position as a rich, stable country with compulsory education and a well-trained teacher workforce. However, our PISA scores have steadily decreased since 2003 and a statistically significant gender gap has recently opened up (Thomson, Hillman, & De Bortoli, 2013).

For the first time, PISA 2012 included an optional computer-based assessment of mathematical literacy (CBAM) (OECD, 2013b) and Australia participated. CBAM had two purposes. The first was to enhance the assessment of 'traditional' mathematical literacy e.g., by using a dynamic stimulus for an item about movement or by providing a rotatable three dimensional image to mimic the way in which a real object can be handled. The second purpose was to assess the new types of mathematical literacy required in a computationally rich world. Sample CBAM items are available from ACER (2012).

Mathematical Literacy and Categories of Problems

Because of its focus on mathematics for life, PISA assesses a particular type of mathematical knowledge and skills, which it calls *mathematical literacy,* with both paper-based and computer-based items. The 2012 definition (OECD, 2013b) is as follows:

> Mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens. (p. 25)

It follows from the definition that mathematical literacy is inherently involved in modelling the real world and using mathematics in context: although pure mathematics is an important part of school mathematics curricula, testing it is not within PISA's focus, except in service of mathematics in context. Another key point is that mathematical literacy is not intended to be a low level skill, but that individuals will both possess and require mathematical literacy of different degrees of complexity, from basic life skills to very high levels of technical work.

The emphasis on describing, explaining and predicting real world phenomena (including from the social world) means that PISA is inherently linked to mathematical modelling. PISA problems are chosen to assess aspects of the mathematical modelling cycle, shown in Figure 1. Teaching students to apply their mathematical knowledge in the real world also needs to attend to each of these processes – how to make assumptions and identify relationships in order to formulate a real world problem as a mathematical problem, how to employ intra-mathematical knowledge and skills to obtain a mathematical solution to the problem (the traditional focus of so much mathematics teaching), how to interpret that result sensibly to answer the real world problem, and how to evaluate whether the formulation of the problem has been adequate to give a useful answer. In 2012, Australian students performed better on items focusing on interpreting results than on formulating problems mathematically or solving the intra-mathematical problem. They also performed better on items testing Change and Relationships and Uncertainty and Data, than on items testing Space and Shape or Quantity (Thomson, De Bortoli, & Buckley, 2013). This guides where to put new emphases in teaching.

Another aspect of the PISA framework (OECD 2013b) is that it describes a set of fundamental mathematical capabilities (sometimes called competencies) which students need to develop for success in all mathematical work. PISA mathematics problems are analysed in terms of these capabilities, and they are used to describe what the increasing levels of mathematical proficiency mean. The mathematical capabilities are: *communication, representation, devising strategies, mathematisation, reasoning and argument, using symbolic, formal and technical language and operations, using mathematical tools* (including digital tools).

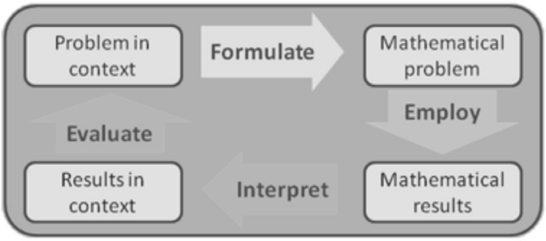
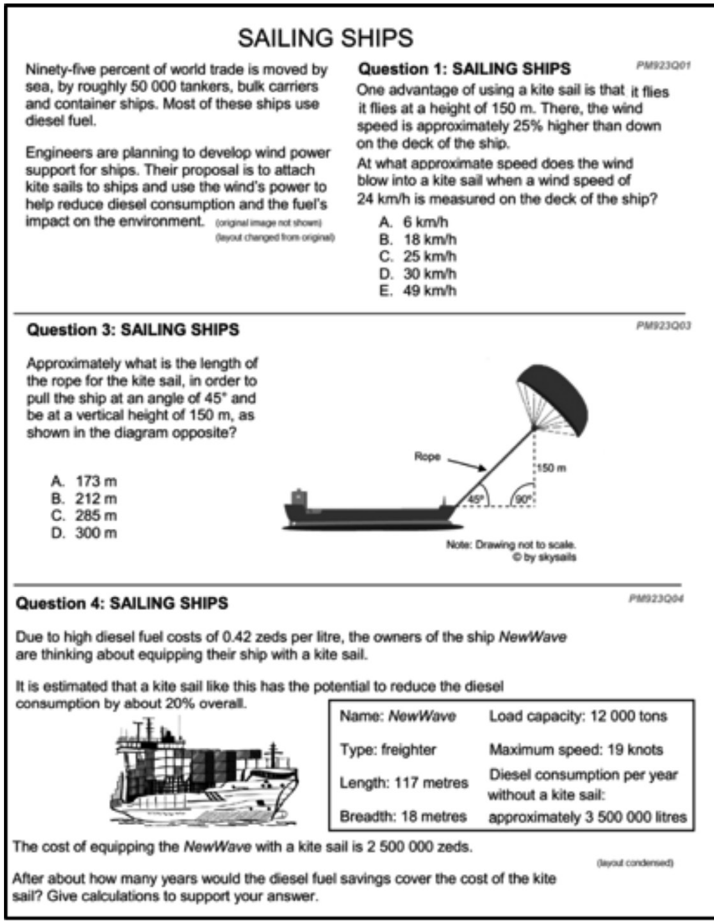


Figure 1. PISA 2012 model of mathematical modelling. (OECD 2013b).

Sample PISA Items

Sailing Ships

Items from PM923 Sailing ships are shown in Figure 2. The Skysails Company (http://www.skysails.info/english/power/) makes sails to supply green power from the wind to drive ships and for power generation at sea. This authentic situation provides the stimulus for items involving percentage change (Question 1), real world interpretation of algebraic formulas (Question 2 not released), Pythagoras's theorem (Question 3), and a multi-step calculation involving rates (Question 4). Question 3 involves creating a mathematical model of the real situation and then applying intra-mathematical thinking. Questions 1 and 2 have greatest cognitive demand in the *Employ* process of Figure 1, but Question 3 was judged to have greatest demand on the *Formulate* process. Question 1 was easy, Question 2 was below the average difficulty of PISA items and Question 3 was difficult.



SAILING SHIPS

Ninety-five percent of world trade is moved by sea, by roughly 50 000 tankers, bulk carriers and container ships. Most of these ships use diesel fuel.

Engineers are planning to develop wind power support for ships. Their proposal is to attach kite sails to ships and use the wind's power to help reduce diesel consumption and the fuel's impact on the environment. (original image not shown) (layout changed from original)

Question 1: SAILING SHIPS PM923Q01

One advantage of using a kite sail is that it flies it flies at a height of 150 m. There, the wind speed is approximately 25% higher than down on the deck of the ship.

At what approximate speed does the wind blow into a kite sail when a wind speed of 24 km/h is measured on the deck of the ship?

A. 6 km/h
B. 18 km/h
C. 25 km/h
D. 30 km/h
E. 49 km/h

Question 3: SAILING SHIPS PM923Q03

Approximately what is the length of the rope for the kite sail, in order to pull the ship at an angle of 45° and be at a vertical height of 150 m, as shown in the diagram opposite?

A. 173 m
B. 212 m
C. 285 m
D. 300 m

Question 4: SAILING SHIPS PM923Q04

Due to high diesel fuel costs of 0.42 zeds per litre, the owners of the ship *NewWave* are thinking about equipping their ship with a kite sail.

It is estimated that a kite sail like this has the potential to reduce the diesel consumption by about 20% overall.

Name: *NewWave*	Load capacity: 12 000 tons
Type: freighter	Maximum speed: 19 knots
Length: 117 metres	Diesel consumption per year without a kite sail:
Breadth: 18 metres	approximately 3 500 000 litres

The cost of equipping the *NewWave* with a kite sail is 2 500 000 zeds.

(layout condensed)

After about how many years would the diesel fuel savings cover the cost of the kite sail? Give calculations to support your answer.

Figure 2. PM923 Sailing ships from PISA 2012 (paper-based) (OECD, 2013a).

Body Mass Index

The computer-based item CM038 Body mass index Question 1 (see Figure 3) involves the Uncertainty and data content category, and the *Interpret* process (make inferences from a set of graphs). Students work with a website where they can click on the buttons to show or hide any of the six graphs. All six graphs are displayed by default, but not all of them are required to answer the questions. The first statement in particular requires students to adopt an efficient sorting strategy, guided by their statistical knowledge. It was observed in the initial trials that students appeared motivated to work harder on computer-based items than on paper-based items. However the interactive features can go well beyond motivation, to provoke and display mathematical thinking in a way that would be much harder to assess with a paper-based item. The released computer-based items can be used by students from the ACER website (ACER 2012).

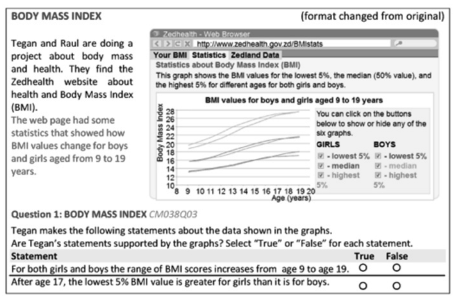


Figure 3. CBAM item Body mass index Question 1 (ACER 2012).

Cable Television

The scoring scheme for PISA items has to be very robust, because it must be carried out relatively quickly by teams in many countries operating in different languages. Training and assistance are offered to make this as reliable as possible, but it is also important that the scoring should be as simple as possible. The released items show the scoring key, which can be a guide for teachers who want to use the items with their own students.

The coding for the second question of PM978 Cable television in Figure 4 provides an example of the schemes given for released items. Two types of responses received full credit. Code 11 indicated responses saying that Kevin needed to take into account the actual number of households with TVs for the two countries. One of several examples is: "Because the population of France is about 10 times more than Norway and there is only about 3 times as many households that subscribe to cable TV in Norway compared to France." Code 12 indicated responses based on calculation of the actual number of subscribers in the two countries. One given example is "Because France has 24.5 × 0.154 = approximately 3.8 million households that subscribe to cable TV, while Norway has 2.0 × 0.427 which is approximately 0.8 million households. France has more cable television subscribers." All other responses received no credit.

CABLE TELEVISION PM978

The table below shows data about household ownership of televisions (TVs) for five countries.

It also shows the percentage of those households that own TVs and also subscribe to cable TV.

Country	Number of households that **own TVs**	Percentage of households that **own TVs** compared to **all households**	Percentage of households that **subscribe to cable television** compared to households that **own TVs**
Japan	48.0 million	99.8%	51.4%
France	24.5 million	97.0%	15.4%
Belgium	4.4 million	99.0%	91.7%
Switzerland	2.8 million	85.8%	98.0%
Norway	2.0 million	97.2%	42.7%

Source: ITU, World Telecommunication Indicators 2004/2005
ITU, World Telecommunication/ICT Development Report 2006

Question 2: CABLE TELEVISION PM978Q02 – 00 11 12 99

Kevin looks at the information in the table for France and Norway.

Kevin says: "Because the percentage of all households that own TVs is almost the same for both countries, Norway has more households that subscribe to cable TV."

Explain why this statement is incorrect. Give a reason for your answer.

(answer space omitted)

QUESTION INTENT:

Description: Understand proportionality based on data provided in a table
Mathematical content area: Uncertainty and data
Context: Societal
Process: Interpret

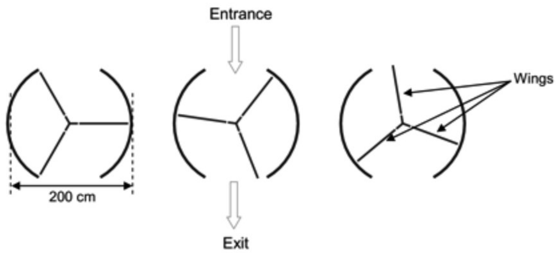
Figure 4. PM978Q02 Cable television with categorisation (OECD 2013a).

Revolving Door

PM995Q01 Revolving Door Question 1 (see Figure 5) requires a simple calculation, but understanding the real world diagrams is challenging, so the fundamental mathematical capability of *representation* is being called upon. Interestingly, the field trial did not indicate that living in a country where such doors are common gave an advantage. PM995Q03 Revolving Door Question 3

REVOLVING DOOR

A revolving door includes three wings which rotate within a circular-shaped space. The inside diameter of this space is 2 metres (200 centimetres). The three door wings divide the space into three equal sectors. The plan below shows the door wings in three different positions viewed from the top.



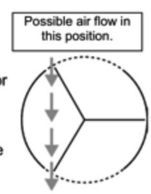
Question 1: REVOLVING DOOR *PM995Q01 – 0 1 9*

What is the size in degrees of the angle formed by two door wings?

Size of the angle: ....°

Question 2: REVOLVING DOOR *PM995Q02 – 0 1 9*

The two door **openings** (the dotted arcs in the diagram) are the same size. If these openings are too wide the revolving wings cannot provide a sealed space and air could then flow freely between the entrance and the exit, causing unwanted heat loss or gain. This is shown in the diagram opposite.



What is the maximum arc length in centimetres (cm) that each door opening can have, so that air never flows freely between the entrance and the exit?

Maximum arc length: .... cm

Question 3: REVOLVING DOOR *PM995Q03*

The door makes 4 complete rotations in a minute. There is room for a maximum of two people in each of the three door sectors.

What is the maximum number of people that can enter the building through the door in 30 minutes?

A. 60
B. 180
C. 240
D. 720

Figure 5. PM995 Revolving door (OECD 2013a).

involves multiple instances of proportional reasoning. Students have to construct a model of the situation (probably implicitly) to make a path such as from total time (30 mins) to total revolutions (120) to total entry options (360) to total people (720). This draws heavily on the fundamental mathematical capability of *devising strategies.* Assembling all of these relationships systematically to solve the problem is complex and so the item is classified as *Formulate.*

The item PM995Q02 Revolving door Question 2 was one of the most difficult items, with only 4% of students successful. This item makes heavy demands at the formulation stage. It addresses the main purpose of revolving doors, which is to provide an airlock between inside and outside the building, and it requires substantial geometric reasoning followed by accurate calculation. The real situation has to be carefully analysed, and this analysis of air flow needs to be translated into geometric terms and back again to the contextual situation of the door multiple times during the solution process. The question draws very heavily on the *reasoning and argument* fundamental capability. It is unclear in this problem when *Formulate* ends and the *Employ* process begins, because of the depth of geometric reasoning required. A careful analysis of the solution of an individual in terms of the modelling cycle would probably find it often moving from the *Formulate* arrow (what does it mean in mathematical terms to block the air flow?) to the *Employ* arrow and back again. The modelling cycle depicted in Figure 1 is only a model of mathematical thinking, and reality is more complex.

Teaching With PISA Items

The main purpose of this article has been to provide teachers with some background information about PISA, to point out some of the key concepts, to provide a sample of the items, and to give references to some of the resources that have come out of the PISA project. Whereas the media usually report just country ranks, mathematics teachers can profitably look at the overall purpose of the test, get ideas for teaching and assessment from the released items, and use some of the framework concepts, such as the modelling cycle and the fundamental mathematical capabilities, to direct their teaching emphases. In some instances, PISA items will be very appropriate to use exactly as they are. Very often, though, the items will provide ideas for teachers who wish to embed a PISA mathematical literacy philosophy in their teaching. Real world contexts, such as the sails for saving fuel for container ships, could be used in many ways, for short items just like PISA or as a stimulus for substantial project work. Additionally, some teaching will focus, like the PISA items, on one of the processes of mathematical modelling but other teaching will give experience of the whole modelling cycle. Almost certainly, teachers will often make more sophisticated grading schemes, so that students get good feedback to improve their work, and criteria such as accuracy of mathematical expression are valued. School assessment has different priorities to that of international assessment and this is reflected in the exact questions asked, what to write in a solution (e.g., just an answer or working) and in the grading. However, both PISA items and PISA's mathematical literacy philosophy have something to offer mathematics teaching, beyond the famous country ranks.

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MORE DOWN TO EARTH MATHS *

Leigh Thompson

Bairnsdale Secondary College

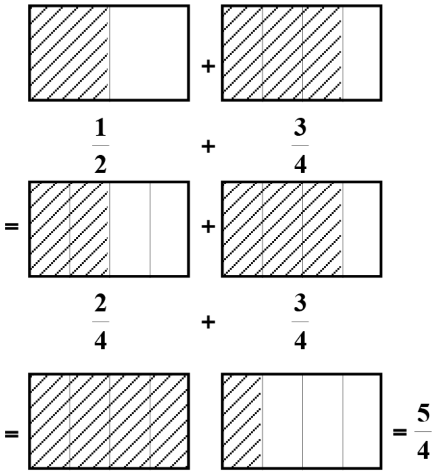
Jacqui Kerr

Curtin University

Some of the difficulties students experience with fraction operations, multiplication tables, arithmetic and indices may be addressed by understanding fraction operations using diagrams and relevant field laws; using patterns and strategies to reduce the number of multiplication facts that need to be memorised; and arranging indices in decreasing order to enable generalisation.

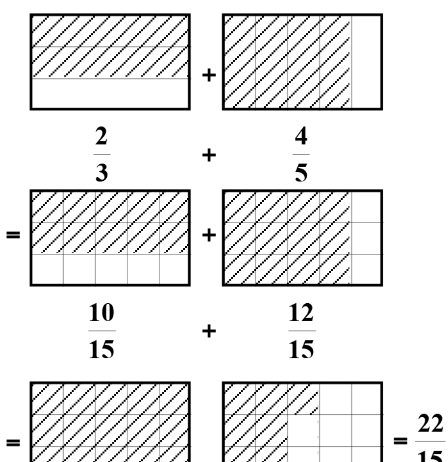
Use of Diagrams to Aid Fraction Understanding

A major problem for many mathematics students at senior level and some at junior level is doing simple processes such as fraction operations. Showing addition of fractions with the same denominators is easily demonstrated using diagrams and arithmetic steps. Diagrams are particularly useful when showing addition of fractions with different denominators (Figures 1 and 2). The step between the second and third lines in both computations can bring about the following questions, "Why does $\frac{1}{2}\times\frac{2}{2}=\frac{1\times 2}{2\times 2}$, $\frac{2}{3}\times\frac{5}{5}=\frac{2\times 5}{3\times 5}$, $\frac{4}{5}\times\frac{3}{3}=\frac{4\times 3}{5\times 3}$ and generally $\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}$?" A detailed and comprehensive proof of the last of these (HREF1) involves the Field Laws with six instances of associativity, two of commutativity, two of multiplicative inverse and one of multiplicative identity. This proof is unlikely to be understood by many mathematics students at senior level who have forgotten fraction operations. Application of the Field Laws can assist their comprehension, while being able to name the relevant laws is unlikely to do so.



$$
\begin{aligned}
\frac{1}{2}+\frac{3}{4} &= \frac{1}{2}\times 1+\frac{3}{4}\\
&= \frac{1}{2}\times\frac{2}{2}+\frac{3}{4}\\
&= \frac{1\times 2}{2\times 2}+\frac{3}{4}\\
&= \frac{2}{4}+\frac{3}{4}\\
&= \frac{5}{4}\\
&= 1\frac{1}{4}
\end{aligned}
$$

Figure 1. Addition of fractions with different denominators.

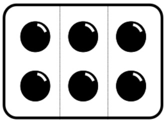


$$
\begin{aligned}
\frac{2}{3}+\frac{4}{5} &= \frac{2}{3}\times 1+\frac{4}{5}\times 1\\
&= \frac{2}{3}\times\frac{5}{5}+\frac{4}{5}\times\frac{3}{3}\\
&= \frac{2\times 5}{3\times 5}+\frac{4\times 3}{5\times 3}\\
&= \frac{10}{15}+\frac{12}{15}\\
&= \frac{22}{15}\\
&= 1\frac{7}{15}
\end{aligned}
$$

Figure 2. Addition of fractions with different denominators.

Why 'of' Means Multiply

Diagrams help students understand why we use 'multiply' for 'of' (see Figure 3).



If balls come packaged 6 to a box then
two **of** these boxes totals $2 \times 6 = 12$ balls
a third **of** these 6 balls is $\frac{1}{3} \times 6 = 2$ balls

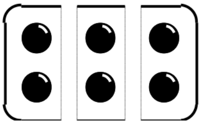


Figure 3. Why 'of' means 'multiply' in mathematics.

Multiplication of Fractions

The diagrams in Figures 4, 5 and 6 should help students to see that $\dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{a \times c}{b \times d}$.

$$\frac{1}{2} \times \frac{3}{4} = \frac{1}{2} \text{ of } \frac{3}{4}$$

$$\frac{3}{4} \qquad \times \qquad \frac{1}{2} \qquad = \qquad \frac{3}{8}$$

$$\begin{aligned} \frac{1}{2} \times \frac{3}{4} &= \frac{1 \times 3}{2 \times 4} \\ &= \frac{3}{8} \end{aligned}$$

Figure 4. Multiplication of fractions.

$$\frac{2}{3} \times \frac{4}{5} = \frac{2}{3} \text{ of } \frac{4}{5}$$

$$\frac{4}{5} \qquad \times \qquad \frac{2}{3} \qquad = \qquad \frac{8}{15}$$

$$\begin{aligned} \frac{2}{3} \times \frac{4}{5} &= \frac{2 \times 4}{3 \times 5} \\ &= \frac{8}{15} \end{aligned}$$

Figure 5. Multiplication of fractions.

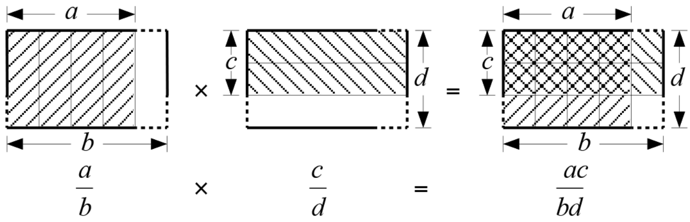


Figure 6. Generalised multiplication of fractions.

Commutative Law

Early years students, once able to count, can perform simple additions by using counters (or their fingers). They can practise writing these across or down. Using a calculator to confirm the answers generally has no more novelty value than the counters, fingers or written calculations for these very young students. By adding the larger number to the smaller number, then adding the smaller number to the larger number, they begin to realise that the latter is much easier and quicker (see Figure 7). The Commutative Law for addition can be generalised as $a + b = b + a$.

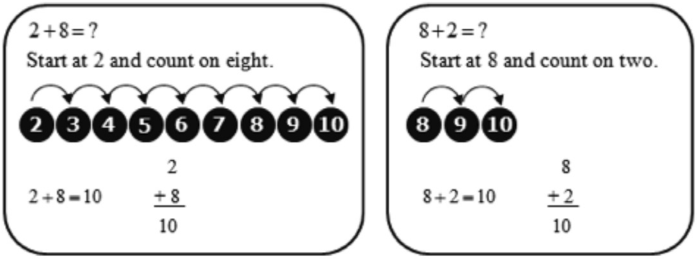


Figure 7. Simple addition showing commutativity.

The Commutative Law for multiplication can be generalised as $a \times b = b \times a$. When young children multiply by counting in groups, they may count by the smaller number the larger number of times, or by the larger number the smaller number of times. They will find the latter, whilst not necessarily easier, is much quicker (see Figure 8).

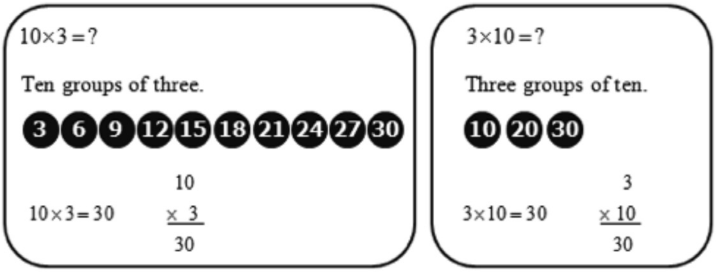


Figure 8. Simple multiplication showing commutativity.

Times Tables and the Distributive Law

Multiplication by 0, 1 and 10 need to be known and can be quickly learnt. Multiplication by 11, 12 and beyond can be done by long multiplication, with the first two providing easy initial practice of an application of the Distributive Law. Commutativity (turnarounds) reduces the number of times tables products to be learned from 64 to 36 (see Figure 9).

x	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

x	2	3	4	5	6	7	8	9
2	4							
3	6	9						
4	8	12	16					
5	10	15	20	25				
6	12	18	24	30	36			
7	14	21	28	35	42	49		
8	16	24	32	40	48	56	64	
9	18	27	36	45	54	63	72	81

Figure 9. Commutativity reduces the number of times tables products from 64 to 36.

Strategies for the 9 Times Table

Multiplication by 9 can be done by numbering the fingers as shown in Figure 10, then folding down the finger labelled with the same number as the number of nines required. Holding down finger 8 leaves 7 fingers to the left of the bent finger and 2 fingers to the right of the bent finger, so $8 \times 9 = 72$. This method can be used for $1 \times 9 = 09$ through to $10 \times 9 = 90$.

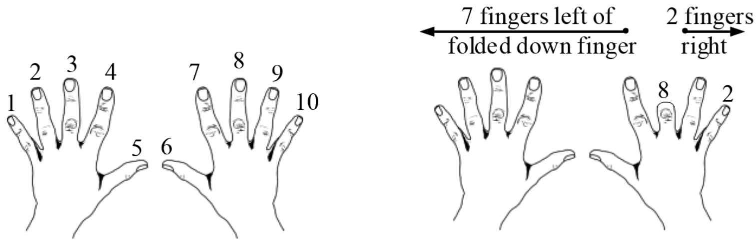


Figure 10. Using hands to multiply by 9. e.g. 8 × 9 = 72

Simple algebra can be used to explain why this works. Consider the product $n \times 9$ where n is an integer and $1 \le n \le 9$.

$$\begin{aligned} n \times 9 &= n(10-1) \\ &= 10n - n \\ &= 10n - 10 + 10 - n \\ &= 10(n-1) + (10-n) \end{aligned}$$

Hence the *tens digit* of the product $n \times 9$ is $n-1$ which also is the number of fingers to the *left* of the bent finger. The *ones digit* is $(10-n)$ which is the number of fingers to the *right* of the bent finger. The *digital sum* of a whole number is the sum of its digits repeated until a single digit is obtained. For all whole number multiples of nine the digital sum is always nine.

Digital sum of 531 is $5+3+1=9$ and $531 = 59 \times 9$
Digital sum of 828 is $8+2+8=18$, digital sum of 18 is $1+8=9$ and $828 = 92 \times 9$
Digital sum of 1242 is $1+2+4+2=9$ and $1242 = 138 \times 9$

531, 828 and 1242 are Gippsland AM radio station frequencies that were changed in 1978 from 530, 830 and 1240 respectively to comply with the new standard 9 kHz spacing (from 10 kHz).

Patterns in the Seven Times Table

If the seven times table is arranged in a 3 × 3 table (Figure 11), patterns can be seen in the rows and columns. The tens digits in the first row are 0, 1, 2. The second and third rows start with the digit used in the previous row's last column, so the tens digits of second and third rows are 2, 3, 4 and 4, 5, 6 respectively. The ones digits also form a pattern: 1, 2, 3 down the third column, with 4, 5, 6 and 7, 8, 9 down the second and first columns respectively.

Tens digits pattern

1 × 7 = **0**7	2 × 7 = **1**4	3 × 7 = **2**1
4 × 7 = **2**8	5 × 7 = **3**5	6 × 7 = **4**2
7 × 7 = **4**9	8 × 7 = **5**6	9 × 7 = **6**3

Ones digits pattern

1 × 7 = 0**7**	2 × 7 = 1**4**	3 × 7 = 2**1**
4 × 7 = 2**8**	5 × 7 = 3**5**	6 × 7 = 4**2**
7 × 7 = 4**9**	8 × 7 = 5**6**	9 × 7 = 6**3**

Figure 11. The seven times table.

After students have learnt as far as the five times table, and worked on the strategies for 7 times and 9 times tables and their turnarounds, there remain only 6 × 6 = 36, 8 × 6 = 48 and 8 × 8 = 64 to be learnt.

Strategies for Some Particular Products

Students can find rote learning of tables tedious and uninspiring but being able to multiply $4\frac{1}{2} \times 4\frac{1}{2}$, or 5.5 × 5.5 , or 85 × 85 , or 78 × 82 in their heads will motivate some students. To square single digits and a half, find them on a number line, multiply the numbers either side and add a quarter (see Figure 12). To square these as decimals students need to know that $\frac{1}{2} = 0.5$ and $\frac{1}{4} = 0.25$. For 85^2, students can first find 8.5^2, that is, 72.25, so $85^2 = 7\,225$.

$4\frac{1}{2} \times 4\frac{1}{2} = 4 \times 5 + \frac{1}{4}$
$= 20\frac{1}{4}$

$4\frac{1}{2}$

$5.5 \times 5.5 = 5\frac{1}{2} \times 5\frac{1}{2}$
$= 30\frac{1}{4}$
$= 30.25$

$$a^2 - b^2 = (a-b)(a+b)$$
$$\text{so} \quad a^2 = (a-b)(a+b) + b^2$$
$$\left(4\tfrac{1}{2}\right)^2 = \left(4\tfrac{1}{2} - \tfrac{1}{2}\right)\left(4\tfrac{1}{2} + \tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2$$
$$= 4 \times 5 + \tfrac{1}{4}$$
$$\left(4\tfrac{1}{2}\right)^2 = 20\tfrac{1}{4}$$

Figure 12. Squaring numbers ending in $\frac{1}{2}$ or 0.5.

Differences of two squares can also be applied in calculating certain products, for example, 78 × 82.

$$\begin{aligned} 78 \times 82 &= (80-2)(80+2) \\ &= 80^2 - 2^2 \\ &= 6\,400 - 4 \\ &= 6\,396 \end{aligned}$$

Some multiplications and additions can be more easily done by a combination of reordering and use of the commutative and associative laws (Figure 13).

$$\begin{aligned} 4\times17\times25 &= 17\times4\times25 \\ &= 17\times(4\times25) \\ &= 17\times100 \\ &= 1\,700 \end{aligned}$$

$$\begin{aligned} 9-15+7 &= 9+-15+7^{*} \\ &= 9+7+-15 \\ &= 16-15 \\ &= 1 \end{aligned}$$

* subtracting 15 is equivalent to adding negative 15

Figure 13. Simplifying multiplication and addition.

Applying Arithmetic

Using Diagrams Can Help Avoid Errors When Applying Arithmetic

Students need more than competency in arithmetic to effectively apply their skills. The following two questions are frequently answered incorrectly by students who rely only on arithmetic but rarely by students who draw diagrams.

(1) A 21 metre veranda is to be supported by a post every 3 metres. How many posts are required? 21 ÷ 3 = 7 but 8 posts are needed.

(2) How many people are there between the fifth and eighth person in a queue? 8 – 5 = 3 but there are only two people, that is, the sixth and seventh people in the queue.

Using the Multiplicative Identity When Converting Between Units

The multiplicative identity (1) is useful when converting between units. The examples in Figure 14 show how expressing the multiplicative identity as a fraction with equal numerator and denominator can help students convert between units correctly.

Convert 3 metres to millimetres.

$$\begin{aligned} 1\text{ m} &= 1000\text{ mm} \\ 3\text{ m} &= 3\text{ m}\times1 \\ &= \overset{3}{\cancel{3\text{ m}}}\times\frac{1000\text{ mm}}{\underset{1}{\cancel{1\text{ m}}}} \\ &= 3\times1000\text{ mm} \\ &= 3000\text{ mm} \end{aligned}$$

Convert to radians.

$$\begin{aligned} 180° &= \pi\text{ radians} \\ 30° &= 30°\times1 \\ &= \overset{1}{\cancel{30°}}\times\frac{\pi\text{ radians}}{\underset{6}{\cancel{180°}}} \\ &= \frac{\pi}{6}\text{ radians} \end{aligned}$$

Figure 14. Using the multiplicative identity when converting units.

Indices: Looking for Patterns

As $2^2 = 4$ and $2^1 = 2$, what do 2^0, 2^{-1} and 2^3 equal? Common student (incorrect) answers are: $2^0 = 0$, $2^{-1} = -2$ and $2^3 = 6$. Explaining that $2^1 = 2$, $2^2 = 2 \times 2$ enables students to see $2^3 = 2 \times 2 \times 2 = 8$. The pattern shown in Figure 15 enables students to evaluate zero and negative indices. Each time the index decreases by one, we halve the previous value. Similarly, for powers of 3, each time the index decreases by 1, we divide the previous value by 3. For powers of 10, as the index decreases by 1, we divide the previous value by 10.

2^4	$= 2\times2\times2\times2$	$=16$	3^4	$= 3\times3\times3\times3$	$=81$	10^4	$=10000$
2^3	$= 2\times2\times2$	$=8$	3^3	$= 3\times3\times3$	$=27$	10^3	$=1000$
2^2	$= 2\times2$	$=4$	3^2	$= 3\times3$	$=9$	10^2	$=100$
2^1	$=2$	$=2$	3^1	$=3$	$=3$	10^1	$=10$
2^0	$=?$	$=1$	3^0	$=?$	$=1$	10^0	$=1$
2^{-1}	$=?$	$=\frac{1}{2}$	3^{-1}	$=?$	$=\frac{1}{3}$	10^{-1}	$=0.1$
2^{-2}	$=?$	$=\frac{1}{4}$	3^{-2}	$=?$	$=\frac{1}{9}$	10^{-2}	$=0.01$

Figure 15. Evaluating zero and negative indices by following a pattern.

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Maths Rocks 2014

Summary Papers

OUR MATHS ROCKS! DEVELOPING FLUENCY IN EARLY NUMBER CONCEPTS THROUGH FUN AND ENGAGING ACTIVITIES

Johny Alagappan

Early Years Mathematics Intervention Teacher

Young children arrive at school already having experienced a variety of situations involving mathematical thinking and problem solving. The classroom and surrounds of the school provide a rich learning environment. The activities for developing mathematical understanding, when connected to real life experiences, appear to generate great interest and provide a natural extension to the knowledge that children already bring from home.

Introduction

At the beginning of this year, I took on the role of a specialist mathematics intervention teacher in addition to working as a classroom teacher. My intervention work involves mostly working with year one children. Planning differentiated mathematics lessons for the year two children while supporting and scaffolding the learning of those at an earlier stage of development has provided me with unique insights into their mathematical learning. I have gained valuable knowledge about not only their stages of development but also which activities interest them, aid their growth and vastly improve their perceptions about mathematics.

Growth Points

Young children arrive at school already having experienced a variety of situations involving mathematical thinking and problem solving. Children from different cultural backgrounds are exposed to mathematical concepts such as shapes, quantities, number recognition as well as counting objects. The development of *Growth Points through the Early Years Numeracy Project* (ENRP) (Clarke, 1999) was a milestone that enabled teachers to gain insights into children's mathematical development and plan effective as well as targeted learning experiences for them.

The Growth Points are based on extensive international research and succinctly describe children's stages of mathematical understanding and development through their first three years of school. It is also a valuable resource when formulating school curriculum plans and tracking the development of children's knowledge; often helping identify children who need further intervention.

Children develop early mathematical understandings (Queensland Studies Authority, 2006) in number, patterns and algebra, measurement, chance and data, and space by investigating and communicating about:

- quantities and their representations, and attributes of objects and collections.
- position, movement and direction.
- order, sequence and pattern.

The ability to identify the number of items in a collection requires an ability to count items as well as being able to recognise that the last number counted is the actual quantity of the collection. I have observed that some children employ strategies such as arranging items in rows and columns when counting, while others do not appear to be able to correlate the object to the actual number. Very few have the ability to visualise and count objects arranged in small quantities such as by twos and fives. Priority must be given for developing these visualisation techniques through activities such as using tens frames and Subitising cards on a daily basis.

The Classroom and Surrounds as a Rich Learning Environment

I place large emphasis on utilising the immediate context of the classroom and surrounds of the school that provide a rich learning environment. The activities for developing mathematical understanding, when connected to real life experiences, appear to generate great interest and provide a natural extension to the knowledge that children already bring from home. Our day begins with counting how many bags have arrived at school. During roll call, children are counting how many have arrived for the day. Questions such as "What time did you get out of bed?", "How many bicycles did you see on the way to school", and "Were there more red cars than blue cars on the way to school?" set the tone for mathematical thinking for the rest of the day. Subsequent classroom tasks always involve the children to think mathematically and solve problems that a classroom teacher encounters on a daily basis. Tasks such as calculating the total number of children present, after accounting for any absences, generates mathematical thinking even before the children have set foot inside the classroom.

All subsequent routines and learning procedures require the thinking and problem solving capabilities of the students themselves. The money needed for placing lunch orders is worked out (counting quantities of money), the day of the month is mentioned, with any important dates noted (calendars), the week of the term is noted, the number of shoes in the classroom (doubling) and also the banker for the day assigned to give our prize money and keep an account of the transactions.

A typical mathematics lesson could be combining Physical Education with counting. Skipping ropes, hoops, and balls have become associated more with mathematical learning than any particular sport in my classroom. The distinctions between the disciplines are blurred to the point that children now believe mathematics is actually doing a whole lot of fun stuff like playing games and solving actual problems that arise in their daily lives! Children develop their counting skills by counting how many times they can bounce a tennis ball without letting it run (authentic task). They collaborate in bouncing a basketball within a circle and attempt to get the highest number possible within a time frame. Individuals develop their motor coordination skills as well as counting backwards by utilising a skipping rope. They might start at a number such as 50, and attempt to skip while counting backwards – a task challenging even for some adults!

Children are encouraged to count flowers in the school gardens, leaves on trees by estimating the number of leaves in groups, estimating the quantities of bricks used in one section of the classroom walls, counting how many steps taken to reach the toilets and back by doubling, and finding arrays of objects within the classroom as well as the larger school. Data gathering activities could typically involve studying the cars parked in the staff parking lot, or researching the type of fruit brought for morning recess. Authenticity is not a token but rather the norm in all mathematics learning. The children do not see a distinction between mathematical problem solving in real life and learning mathematics inside the classroom; they both merge to create an authentic learning environment. There are no textbooks, and algorithms are only a means to an end.

A beginning activity when learning place value concepts has been the research of early counting in other cultures to determine the causal factors for developing a counting system. By taking on the role of shepherds and merchants through short plays involving dress-up boxes, the children get a glimpse into the distant past that has shaped the evolution of the present day number system. They become emotionally invested in the process; they own it and believe they are the guardians of this system.

Conclusion

When children are provided with rich, authentic problem solving tasks on a daily basis, they are being placed in a position of continuing their prior learning that has taken place before entering the school. It is a natural continuation of a process for which the brain is wired. Mathematical learning can be and should be natural, exhilarating, sensory, authentic, relevant, and always related to a meaningful context.

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INFORMATICS

Mike Clapper

Australian Mathematics Trust

The Australian Curriculum has now embraced Digital Technologies as a strand in the Technologies Learning Area. Students in Years 5 and 6 are expected to be able to design, modify and follow simple algorithms represented diagrammatically and in English involving sequences of steps, branching, and iteration (repetition), whilst by Year 8, they should Implement and modify programs with user interfaces involving branching, iteration and functions in a general-purpose programming language. This potentially will have a large impact not only on Technology in schools but also on the way mathematics needs to be taught. The Australian Informatics Competition is probably the most significant resource material available in this area which does not require programming skills and this session will introduce this resource to teachers in a practical way. Incorporating Informatics problems into the maths curriculum has the potential to identify problem-solvers and to stimulate logical thinking.

What is Informatics?

Informatics is the science underlying computing, communication and the Internet. Alternatively, informatics is the study of the structure and behaviour of natural and artificial systems that generate, process, store, and communicate information. Informatics also includes the study of the cognitive, social, legal, and economic impact of information systems.

Students studying informatics learn the basic algorithms, data structures and computational techniques that underlie information and communication, and demonstrate their learning through computer programming tasks.

Informatics and the Australian Curriculum

Digital Technologies is now firmly embedded in the Australian Curriculum and a key element of this is the requirement to develop algorithmic thinking, starting in Foundation Year. In particular, in Years 3 and 4, students are expected to:

- 4.4 Define simple problems, and describe and follow a sequence of steps and **decisions (algorithms)** needed to solve them
- 4.5 Implement digital solutions as simple visual programs with **algorithms involving branching (decisions)**, and user input

(Numbering refers to sections in the Digital Technologies Scope and Sequence Document, HREF1. The first defines the stage level, the second the order of the outcome within the stage. Emphasis is mine.)

In Years 5 and 6 students should be able to:

- 6.4 Define problems in terms of data and functional requirements, and identify features similar to previously solved problems
- 6.6 Design, modify and follow simple algorithms represented diagrammatically and in English involving sequences of steps, **branching, and iteration (repetition)**
- 6.7 Implement digital solutions as simple visual programs involving **branching, iteration (repetition)**, and user input

And in Years 7 and 8:

- 8.5 Define and decompose real-world problems taking into account functional requirements and economic, environmental, social, technical and usability constraints
- 8.7 Design algorithms represented diagrammatically and in English; and trace algorithms to predict output for a given input and to identify error
- 8.8 Implement and modify programs with user interfaces involving branching, iteration and functions in a general purpose programming language

Finally in Years 9 and 10:

- 10.7 Design algorithms represented diagrammatically and in structured English and **validate algorithms** and programs through tracing and **test cases**
- 10.8 Implement **modular programs**, applying selected algorithms and data structures including using an **object oriented programming language**

Whose Responsibility?

I think this is well-intentioned and an important initiative if Australia is to develop its own programmers and a general wider understanding of digital technology. However, I have grave concerns about the current capacity to deliver this curriculum in our schools. In Primary settings, will every teacher be expected to deliver this, or will a specialist have to be employed (and where are these specialists to be found)? In secondary, very few schools have IT personnel who can program competently (due to the shift from Computer Science courses to applications-based IT courses) and in any event, schools often have no curriculum time slot in the junior years for any formal information technology teaching. In my view, those best placed to deliver this curriculum will be maths teachers, who may well protest at doing so because their own curriculum is already crowded enough.

AMT Informatics Program

At the least, what teachers will need are some resources. For a number of years, the Australian Mathematics Trust has provided an Informatics program which covers all of these curriculum requirements and which might be viewed by teachers as a useful starting point for teaching algorithmic thinking. The AMT Informatics program is designed to encourage algorithmic and logical thinking in all students and to identify and encourage potential programmers. It consists of the following components:

- AIC (in March) – a non-programming competition in algorithmic thinking, on-line, Years 5 – 12
- Free programming training module through AMT website - anytime
- AIO (September) – open programming competition
- AIIO (February) – invitational programming competition
- Selection schools and further invitational competitions
- IOI (International Olympiad in Informatics)

For the purposes of this presentation, as a support to teachers, the most important of these components is the first.

Informatics Competition

- Late March – 1 hour, 15 questions
- Currently has 3 divisions: Junior (7/8), Intermediate (9/10) and Senior (11/12)
- New Upper Primary Division starting in 2015
- Emphasises algorithmic thinking.
- Will appeal to some students who do not shine in conventional mathematics.
- Will identify potential programmers.
- Going online in 2015.

Structure of Paper

- First six questions – 3 marks each – traditional multiple-choice (5 options)
- Next nine questions - three-stage tasks – 2marks per stage
  - A three-stage task consists of a small problem to solve where there are three sets of data
  - The first data set is small or simple enough to be susceptible to ad-hoc techniques, but hopefully provides a basis for students to get a feeling for the problem and to develop an algorithm to be used in the remaining data sets.
  - The answers are numbers in the range 0-999.

Question Types

- Applying Rules: Students are required to apply well-defined rules to a set of data. These questions are always multiple choice, and are typically the first or second question in a paper. They are less frequently used in the Senior paper
- Logic: Usually multiple choice and early in a paper, these questions use non-algorithmic puzzles to encourage rigorous reasoning and case analysis
- Analysis questions: many analysis questions require students to determine the number of operations a particular algorithm requires on a set of data. This gives students an introduction to complexity analysis of the algorithm. Another popular analysis question asks students to count the number of different routes through a network. Most analysis questions are multiple choice, although some are three-stage
- Algorithms: Algorithm questions are the heart of the AIC. About half of the multiple choice questions are algorithmic and they dominate the three-stage questions. All are optimization questions of one sort or another. Breadth first searches are popular, and when used in three-stage questions students can be guided to discover the technique for themselves. Dynamic Programming and Two-Person Game questions are also common, and the problem setting committee takes delight in a good ad-hoc problem that requires its own special purpose algorithm.

Sample Questions

Sample questions (and solutions) are available on the AMT website (http://www.amt.edu.au/AICsampleset2.pdf) A book of past papers is available from the AMT bookshop (also through the website) at $45. For 2015, we are introducing a GetSet training program for the AIC which will be available for $2 per student with competition registration. The on-line programming course and training site is available at http://www.amt.edu.au/eventsaioc.html

References

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KEEPING PROBLEM SOLVING AT THE CENTRE

Mike Clapper

Australian Mathematics Trust

There are many ways in which teachers can ensure that all students engage in classroom problem solving and enrichment activities. Whilst there are ample resources available for problem solving and enrichment, these are often poorly sequenced and structured. This paper illustrates the use of Australian Mathematics Trust, and other, materials which are well-sequenced and aim to develop problem-solving skills and to encourage students to think mathematically.

What Do We Mean by Enrichment?

And who is it for? This is an important point of discussion because many teachers believe that enrichment is only for students who show particular aptitude for a subject, a view that I would regard as out of step with current curriculum design.

Australian Curriculum

- In the new Australian Curriculum, there is a greater emphasis on problem solving and on working mathematically.
- In my view, this means that all students should be actively engaged both in problem solving and problem formulation. This should precede, rather than follow, the development of technical skills

I have identified in italics below what I regard as the components of the proficiency strand which require students to actively engage in problem-solving, reflection and problem formulation, none of which can be achieve through technical skill development alone. Whilst you might not agree with everything I have picked out, it would be hard to deny that there is a strong emphasis on the deeper understanding that can only come from this type of active engagement.

Australian Curriculum Proficiency Strands

Understanding

Students build a robust knowledge of *adaptable and transferable* mathematical concepts. They make *connections* between related concepts and progressively apply the familiar to develop new ideas. They develop *an understanding of the relationship between the 'why' and the 'how'* of mathematics. Students build understanding when they *connect related ideas,* when they represent concepts in different ways, when they *identify commonalities and differences* between aspects of content, when they describe their thinking mathematically and when they *interpret mathematical information*.

Fluency

Students develop skills in choosing appropriate procedures, carrying out procedures flexibly, accurately, efficiently and appropriately, and recalling factual knowledge and concepts readily. Students are fluent when they calculate answers efficiently, when they recognise robust ways of answering questions, when they *choose appropriate methods* and approximations, when they recall definitions and regularly use facts, and when they can manipulate expressions and equations to find solutions.

Problem Solving

Students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively. Students formulate and solve problems when they use mathematics to represent unfamiliar or meaningful situations, when they design investigations and plan their approaches, when they apply their existing strategies to seek solutions, and when they verify that their answers are reasonable.

Reasoning

Students develop *an increasingly sophisticated capacity for logical thought* and actions, such as analysing, proving, evaluating, explaining, *inferring, justifying and generalising*. Students are reasoning mathematically when they explain their thinking, when they *deduce and justify strategies used* and conclusions reached, when they *adapt the known to the unknown, when they transfer learning from one context to another*, when they prove that something is true or false and when they compare and contrast related ideas and *explain their choices.*

Answers to the Opening Questions

So my answers to our opening question are:

- What do we mean by enrichment?
- Activities which provide appropriate challenge for all students at all levels, engaging them in both problem formulation and solution.
- Who is enrichment for?
- Everyone (however, I acknowledge that the materials referred to here tend to suit more able students).

Primacy of Problem Formulation

By focusing on technical skills, we run the risk of neglecting problem formulation. Ideally, we start with a problem, and the technical skills arise out of a need to deal with the type of problem faced. For instance, if we are coming across a lot of problems involving Area and Perimeter of rectangles, we will find that we generate a lot of quadratic equations. This gives a context in which developing techniques for solving these equations makes sense and the answers will also make sense.

The Value of Context

Students often panic when they get to worded problems, because they have been socialised to think they are harder. Teachers can shy away from this problem but in doing so they are disenfranchising their students from the possibility of actually using the mathematics they learn. I would argue that context is more often than not, helpful in giving students a picture of what we are trying to achieve and hence, their intuition can work in harness with their technical skill.

What Enrichment Is Not

- **You've finished the LHS, try the RHS.** We've all done this in the immediacy of coping with classroom realities, but we must never pretend that 'busy work' is any kind of enrichment.
- **Move onto the next exercise, the next chapter, the next year level.** This may be extension, or acceleration, but it is not enrichment and it tends to compound the problem of potential boredom further down the track.
- **Go and play on the computer.** Apart from the fact that this tends to 'reward' the more able students and compound the boredom factor for others, most 'enrichment' websites are poorly organized and sequenced, so that the enrichment experienced is not a systematic one which will lead to the development of problem-solving skills. There are some notable exceptions, such as the nrich.org site.

The Reality of the Classroom

Most teachers are having to deal with:

- a wide range of abilities with the classroom
- time pressure to 'get through' the curriculum
- behavioural issues
- the need (particularly for young or starting teachers) to focus on classroom management

Any focus on providing enrichment opportunities needs to take these realities into account. Working with good existing resources may be more realistic than trying to always invent or develop your own.

Australian Mathematics Trust Materials

1. Australian Mathematics Competition
2. Informatics Competition
3. Mathematics Challenge for Young Australians
4. Mathematics Challenge Enrichment Series

Australian Mathematics Competition

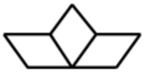
In my view, the Australian Mathematics Competition:

- is largely wasted if used only as a snapshot
- is a massive resource of problems at every level and organised by topic,
- provides clear solutions (with alternatives)
- is deliberately designed to have use as classroom investigation starters
- provides many pre and post competition possibilities

Some sample problems are now used to illustrate the richness of the problems.

Rhombus Tiles (2012 – Middle Primary Question 30, Upper Primary Question 20, Junior Secondary Question 28)

A rhombus-shaped tile is formed by joining two equilateral triangles together. Three of these tiles are combined edge to edge to from a variety of shapes as in the example given.

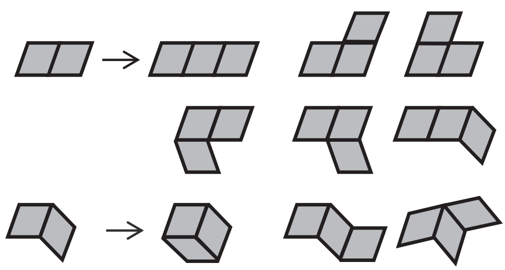


How many different shapes can be formed? (Shapes which are reflections and rotations of other shapes are not considered different)

This is a fairly difficult question in the context of a 60 minute or 75 minute competition, but a very rich investigation for a class involving problems of rotation and reflection, as well as the need for systematic counting – try it!

Solution

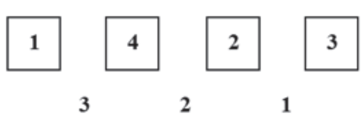
Find the two arrangements of two rhombuses, then position the third rhombus systematically around the edge, ensuring no repeats by rotation or reflection.



Now try 4 tiles!

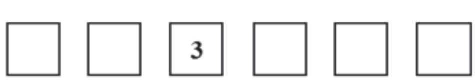
Differences of Numbers (2011 – Junior Secondary Question 27, Intermediate Secondary Question 25)

An arrangement of numbers is said to have different differences when the differences between neighbours are all different. For example, the numbers below have differences of 3, 2 and 1 (all different)



Differences:

If the numbers from 1 to 6 are arranged with different differences, and with 3 in the third position, what are the last three digits?



This question is fairly language dependent, which is challenging for students in the test environment, but once students have grasped what is being asked, they can attack the problem in various ways. Many students reach an answer by trial and error but this question provides an opportunity for students to try to explain why their solution is unique (though it is the three digit answer required which is unique, not the whole arrangement). The answer is left as an exercise for the reader.

This also presents a 'problem formulation opportunity'. Would it change the problem if the 3 was in a different box, or if another number was in the box? This gives rise to some interesting discussions about symmetry (e.g. 3 in the 4th box, 4 in the 3rd box). What if there were more boxes? Students need to be encouraged to use problems to suggest other problems.

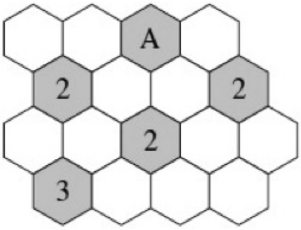
Further examples of such problems were given in the workshop.

Informatics Competition

- Deals with the Digital Technology strand of Australian Curriculum
- Emphasises algorithmic thinking
- Will appeal to some students who do not shine in conventional mathematics
- Will identify potential programmers
- Going online in 2015
- New Upper Primary Division

Garden Plots 2008 – Junior Secondary Question 1

A vegetable garden is laid out in hexagonal plots. In the diagram below, the shaded plots are growing tomatoes. Marigolds will be planted in some of the adjacent plots to deter pests. The number in each shaded plot is the number of adjacent plots that will be planted with marigolds.



Which number should be in plot A?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Cities 1 2007 – Junior Secondary Question 3

The land of Straightopia has four cities, all built along a single straight highway. The distances between the cities are as follows:

Distance	City P	City Q	City R	City S
City P		3km	3km	1km
City Q	3km		6km	4km
City R	3km	6km		2km
City S	1km	4km	2km	

You are travelling along the highway from one end to the other. In which order might you travel past the four cities?

(A) Q, P, S, R (B) Q, R, P, S (C) R, P, S, Q (D) S, R, P, Q (E) S, R, Q, P

These questions and many more (and solutions) can be found on the AMT website (http://www.amt.edu.au/aicsample.html). Collections of past papers are also available. The on-line training site allows students to prepare for the AIO, an open programming competition. This can be accessed at: http://www.amt.edu.au/eventsaioc.html.

Mathematics Challenge for Young Australians

The Challenge is a series of problems taken over a three-week period using mathematics from different topics within the curriculum but designed to develop problem-solving techniques. There are four levels, Middle Primary (Years 3/4), Upper Primary (5/6), Junior (7/8) and Intermediate (9/10). Primary papers contain 4 questions, whilst secondary papers contain 6 questions. Students are able to work co-operatively. Comprehensive notes, extension problems and solutions are provided for teachers. The problem-solving techniques might include:

- Understanding the problem
- Build a model
- Draw a diagram
- Trial and improvement
- Exploring all cases
- Make a systematic list
- Solve a simpler problem
- Work backwards
- Use logic
- Use algebra (or a spreadsheet)

Examples of Challenge problems can be found on the AMT website at:
http://www.amt.edu.au/wumcya.html

MCYA Enrichment Series

The MCYA Enrichment series is a systematic course in problem-solving with an optional competition element. It introduces students to some new mathematical content which typically forms the basis for Olympiad style problem-solving competitions. Each student receives a mini-text for the series they are doing and a set of student problems based on the text. Teachers also receive a Teacher Guide with fully worked solutions and a marking guide.

The Enrichment Series can be used as a whole-class activity or for students to work through alone (with occasional assistance from a mentor). The series has six levels:

5. Newton – typically Years 5/6
6. Dirichlet – typically Years 6/7
7. Euler – typically Years 7/8
8. Gauss – typically Year 8/9
9. Noether – typically Years 9/10
10. Polya – very good Year 10 students

Chapters can be divided into five topic areas, Number Theory, Combinatorics, Geometry, Algebra and Problem-Solving techniques. As an example, chapter 7 in the Euler book introduces modulo arithmetic, whilst chapter 11 introduces Dirichlet's Pigeon-hole principle. There is a large number of well-sequenced and challenging problems in each chapter and the competition problems are closely connected with the chapter topics. Detailed chapter headings and problem examples are available on the AMT website, along with instructions for managing and directing students taking the Enrichment stage. Examples can be found at: http://www.amt.edu.au/mathematics/mcya/

CREATING FORMULAS IS MORE IMPORTANT THAN USING THEM: TECHNIQUES TO TEACH "REASONING"

Michaela Epstein

Hume Central Secondary College

Andrew Worsnop

Velvet Learning

Every maths classroom should be teaching mathematical 'Reasoning'. As well as being a required proficiency strand in the Australian Curriculum, Reasoning facilitates the development of a deeper conceptual understanding and holistic view of mathematics. Key to this is supporting students to create and explore formulas rather than solely memorising these mathematical shortcuts.

In this session Andrew and Michaela present concrete tactics and strategies so that Reasoning isn't another item to add in a crowded curriculum. Instead it's a powerful way to increase students' long-term recall and ability to handle non-routine problems.

What is Mathematical Reasoning?

The Australian Curriculum relates "Reasoning" to a set of verbs: "analysing, proving, evaluating, explaining, inferring, justifying and generalising". It is the application of mathematical knowledge and procedures and the discussion, communication and negotiation around mathematics. Understanding ideas at a deep conceptual level is important, but Reasoning is where students put them to use.

Why is Mathematical Reasoning Important?

The ability to reason mathematically is important when students encounter problems at school and beyond. Mathematical reasoning hinges on the use of logical thought and justification of ideas. For this reason it is important in contexts where novel problems are faced and when decisions about strategies and methodology need to be made.

Indeed, Sullivan (2011, p. 19) highlights that studies have found formal school mathematics to be hardly used in the workplace. Instead more intuitive methods are relied upon, for which reasoning, as well as problem solving and estimation is essential.

What Are Issues with Implementing Reasoning Effectively?

Although reasoning is one of the four proficiency strands that runs through the Australian Curriculum, approximately only 7% of the content elaborations refer to reasoning, with a disproportionate amount referring to fluency and understanding (Atweh, Miller, & Thornton, 2012). Therefore, for teachers using the new curriculum to guide classroom practice, reasoning is implicitly presented as being of lesser importance.

This skewed emphasis does not benefit mathematics classrooms, particularly given the 'distraction' of fluency that already exists in external assessments, textbooks and general teaching practice. Stacey (2010, quoted in Sullivan, 2011, pp. 5-6) has reported that the consensus view from leading educators, curriculum specialists and teachers "is that Australian mathematics teaching is generally repetitious, lacking complexity and rarely involves reasoning." Disproportionate level of attention is given to "answer-getting" rather than the processes used and the logic behind problem solving. As Swan (2011) states, mostly in the classroom "reasoning remains invisible."

All this indicates that there exist external, as well as internal, factors that systemically influence the choices teachers make and the learning tasks they use, such that reasoning is under-prioritised.

Techniques and Strategies for Implementing Mathematical Reasoning in the Classroom

Principles

- Reasoning should be assessed, or it won't be valued: some of the easy activities listed below can quickly go in a topic test. Richer assessment might include presentations or interviews.

- Reasoning should be part of the natural classroom activities almost every lesson – not as standalone events once per topic.
- Tasks should be structured to emphasize mathematical communication.

Easy Reasoning Activities

Given that planning time is a struggle, here are some easy reasoning activities. You can use at least one of these every class every lesson, by getting students to:

- explain ideas: ask questions that force students to not just perform a skill or recall a fact but explain an idea. Get them to explain why the formulas they use work, explain their choice to solve an equation a certain way, explain why this concept is different or similar to another concept.
- evaluate a mathematical statement: Test whether something is true, false or sometimes true, sometimes false.
- categorise, compare and contrast: given a group of math ideas, sort them into two or more groups and explain the grouping. This can be done with shapes, algebraic terms, types of maths problems, vocabulary terms.
- solve problems more than one way: this can be done in plain way (e.g. solve a quadratic equation using three different techniques, and comment on the differences in using each technique) or by placing additional constraints for the next solution.
- analyse an incorrect response and give feedback like a teacher: The emphasis should be on how to help the hypothetical student who got this problem "wrong" get another similar problem right and not just solve the immediate mistake.

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ROCK MATHEMATICS WITH SOME BUBBLE AND SQUEAK

Rama Ramakrishnan

Elsie-Rajam Private School, WA

The Australian Curriculum: Mathematics aims to ensure that students:

- *are confident, creative users and communicators of mathematics, able to investigate, represent and interpret situations in their personal and work lives.*
- *develop an increasingly sophisticated understanding of mathematical concepts and fluency with processes*
- *recognise connections between the areas of mathematics and other disciplines and appreciate mathematics as an accessible and enjoyable discipline to study.*

To achieve this, an eclectic approach in the teaching of mathematics is essential. This paper provides some examples for a task which is not as easy as it may sound!

Elements of this Discussion

Bubble and squeak is a traditional English dish made with the shallow-fried leftover vegetables from a roast dinner. The main ingredients of this discussion are solving problems – a variety of them – defining the concept of 'lumeracy' in mathematical pedagogy, and the use of technology in mathematics classes. The settings for all of these elements are varied, from diverse sources not following any particular teaching philosophy.

Problem Solving

The Australian Curriculum clearly talks about Mathematics having its own value and beauty and emphasises a focus on developing sophisticated and refined mathematical understanding, fluency, logical reasoning, analytical thought and problem-solving skills in students. These capabilities of students must respond to familiar and unfamiliar situations by employing mathematical strategies to make informed decisions and solve problems efficiently (ACARA, 2012)

In this context the following classical problems will be explored. First, we consider the following quoted in Stacey (1995):

> Antony, aged nine (9) sat staring at a word problem for several minutes, apparently doing nothing. To break the silence, I asked what was wrong. Pointing at the first word, he blurted out, 'But I don't know what Georgina is!' (p. 208)

and a comment from a Victorian secondary teacher, also quoted in Stacey (1995).

> At our school, we think it is important to learn how to teach problem solving. That is where mathematics is heading. But the skill to teach them about problem solving is going to take teachers two or three years to acquire. The need to become competent in clearly defining the problem so the kids aren't sitting there saying they can't do this and then the teacher ends up doing it for them. And it is going to take a long time to get kids to be able to present their work so that someone else can understand it. (p. 208)

Princesses and Pearls

A rajah on his death left to his daughters a certain number of pearls with instructions that they be divided up in the following way: his eldest daughter was to have one pearl and a seventh of those that were left. His second daughter was to have two pearls and a seventh of those that were left. His third daughter was to have three pearls and a seventh of those that were left. The youngest daughter went before the judge and told that this complicated system was unfair as she may not get any! The judge whom was well versed in problem solving said that the proposed division was just so all of them will get the same amount (Tahan, 1993, pp. 175–177). (See also Gossett, 2014).

Bhaskara and Lilavathi: The Swarming Bees

A fifth part of bees rested on the flower Kadamba, a third on Silinda. Three times the difference between these two numbers flew over the flower Krutaja. One bee alone remained in the air attracted by perfume of jasmine and a bloom! Tell me beautiful girl, how many bees were in the swarm! (Tahan, 1993, p. 137).

Diophantus's Riddle

"Here lies Diophantus," the wonder behold. Through art algebraic, the stone tells how old: 'God gave him his boyhood one-sixth of his life, One twelfth more as youth while whiskers grew rife; And then yet one-seventh ere marriage begun; In five years there came a bouncing new son. Alas, the dear child of master and sage after attaining half the measure of his father's life chill fate took him. After consoling his fate by the science of numbers for four years, he ended his life.' (Diophantus, born sometime between 201 and 215 CE; died aged 84 sometime between 285 and 299 CE).

Stated in prose, the poem says that Diophantus's youth lasts 1/6 of his life. He grew a beard after 1/12 more of his life. After 1/7 more of his life, Diophantus married. Five years later, he had a son. The son lived exactly half as long as his father, and Diophantus died just four years after his son's death. All of this totals the years Diophantus lived (Mathworld, 2014).

In all these problems the model that is useful in unpacking the problem is the Clarify, Choose, Use, Interpret and Communicate framework. (Perso, 2001; Polya, 1945).

Lumeracy

The word *lumeracy* was coined in 2011 by the author (teacher & educationalist) who defined it as a word to represent being educated with knowledge to read, write and use numeracy, manage information, express ideas and opinions, communicate in an ethical manner, make decisions and solve problems (Ramakrishnan, 2011).

The issue of language is best addressed by the concept of *lumeracy*. There is also still a great deal to learn about effective ways of using technology in the teaching of mathematics. While technology can enhance and /or show multiple representations, improve the mathematical pedagogy, language becomes critical in clarifying and interpreting problems and the results obtained using mathematical methodology. This obstacle could be overcome by making students lumerate, as opposed to literate and numerate in the traditional sense.

What is considered as a lumeracy resource is one that is rich in thinking in mathematical ways and crosses all learning areas of knowledge. It connects natural world with its mathematical beauty, the history of mathematics along with the mathematicians and the culture of their time with current world connections. Lumeracy resources make the learning of Mathematics interesting, absorbing, enjoyable and above all fun.

Some examples of the resources, most of which can be obtained online are:

- Mathematics – a human endeavour (Jacobs, 1992).
- Mathematics (Time-Life books – Bergamini,1970).
- The man who counted (Tahan, 1993).

- Math through the ages: a gentle history for teachers and others (Berlinghoff & Gouvéa, 2002).
- Anno's mysterious multiplying jar (Anno & Anno, 1983).
- The number devil: A mathematical adventure (Enzensberger & Berner, 1998).
- Fractals, googols, and other mathematical tales (Pappas, 2010).
- What's your angle, Pythagoras? A math adventure (Ellis & Peacock, 2004).
- The adventures of Penrose the mathematical cat (Pappas, 1997b).
- Mathematics appreciation (Pappas, 1993).
- The magic of mathematics: Discovering the spell of mathematics (Pappas, 1994).
- The music of reason: Experience the beauty of mathematics through quotations (Pappas, 1995).
- Mathematical scandals (Pappas, 1997a).
- Math-a-day: A book of days for your mathematical year (Pappas, 1999).

Technology

Technology in the form of Computer Algebra Systems (CAS) and/or CAS along with Texas Instrument's *Navigator* wireless communication system goes hand in hand with mathematical pedagogy. It provides multiple representations of functions, dynamic geometry and sophisticated statistical analysis in addition to routine mathematical computations. For those who are unfamiliar with the 'navigator' system, the Texas Instrument (TI)'s Computer Algebra System (CAS) hand-held calculator which is called the *TI-Nspire*, communicates wirelessly with the teacher and students through a wireless desktop unit that displays all CAS screens and information on to the whiteboard through software installed on the computer via a data projector.

Setting up of a class takes a few minutes. A class list with names is typed into a new list and students assign their own passwords. When the class is started by clicking on the "start the class" command, students receive a login screen and they can communicate by typing in information. The teacher can send files to students and can receive answers in different available formats. The teacher may choose to run the class with all students' screens on the whiteboard. In addition, the teacher can make one student the presenter (showing only that student's screen in real time).

Technology can free us from an algorithm-driven curriculum, and it is an important challenge for mathematics educators and their ability to create curriculum experiences which help students construct deep conceptions of how numbers and operations work and how they interact with each other. Students' success in this area will determine the extent to which they are able to tap into the power of algebra as a symbolic system, and through it a range of new mathematical tools that it can provide (Stacey, 1995).

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MATHEMATICAL FIBRE ART ROCKS

Katherine A. Seaton

Department of Mathematics and Statistics, La Trobe University

The current resurgence in interest in knitting and crochet has gripped parts of the mathematical world and led to some beautiful and informative pieces of mathematical fibre art, often based on a few simple stitches and rules for combining them. Fibre art can be used to explore biology and mathematics, in particular geometry.

Curvature

Imagine in your mind a bowl of salad, with cherry tomatoes, slices of cucumber and some frilled lettuce. First, think about a tomato, a little sphere. We recognise it as curved, and we know that the surface area is given by $4\pi R^2$, where R is the radius of the sphere. This type of curvature is called *positive curvature*. For a sphere, the curvature is said to be constant: $1/R^2$. This accords with our experience: the surface of a beach ball is less curved than that of a tomato. Now consider the slice of cucumber. The surface is planar, and we say it has *zero curvature.* We could imagine bigger and bigger flat surfaces, and they would never close up, unlike the skin of a tomato.

The lettuce is curvy, but not in the same way as the tomato. If we flattened a leaf, there would be folds and overlaps, too much surface. This is *negative curvature;* it is the opposite of the curvature a sphere has. A surface that has constant negative curvature is called a hyperbolic plane. It has non-Euclidean geometry.

Fibre Art

In 1997, the mathematician Daina Taimiņa realised that models of the hyperbolic plane can be constructed using crochet (Taimiņa, 2009). Her robust models, as well as being beautiful (and exhibited in galleries), have been used to prove mathematical results (HREF4). Her method is simple (see Figure 1 for the result):

- Crochet a short starting chain (the work will grow exponentially).

- Choose a number N, say 5.
- Begin to double crochet the second row, doing one double crochet in each of N stitches. In stitch $N+1$, do **two** double crochet stitches.
- Continue this pattern, keeping the counting going as you go from one row to the next.

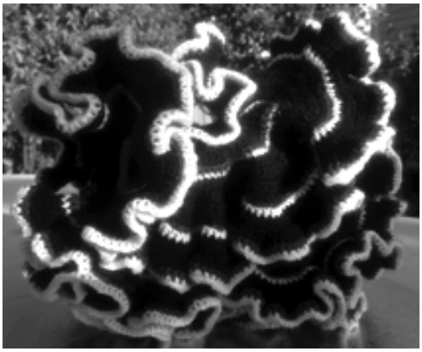

Figure 1. Australian Summer, a piece of hyperbolic crochet [Katherine Seaton, 2014].

Negative curvature abounds in nature, inspiring the Hyperbolic Crochet Coral Reef project. The curators of this project (the ex-patriate Australian sisters, Margaret and Christine Wertheim) describe it as "a woolly celebration of the intersection of higher geometry and feminine handicraft, and a testimony to the disappearing wonders of the marine world" (HREF3). Other mathematicians and fibre artists are exploring the interaction between the two fields of endeavour; starting points for inspiration, photos and further patterns (including knitting), can be found at the websites (HREF1, HREF2).

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A DAY AT THE MUSEUM (OF MATHEMATICS)

Andrew Wrigley

Somerset College, Queensland

The Museum of Mathematics in New York is the only Mathematics Museum in North America. It is a cornucopia of Mathematics designed to inspire the minds of students (and adults). Enjoy a journey through some of the exhibits, ideas and innovations which can be readily transferred to the classroom.

Introduction

The first thoughts that probably come to mind about a museum of mathematics are of a dimly lit room containing a collection of old manuscripts like Euclid's Elements, Newton's Principia or various volumes of the Vedic sutras. Maybe you envisage mathematical marvels like an early Babylonian abacus, a set of Napier's Bones or a slide rule that has been to the moon and back. The Museum of Mathematics in New York has none of these. It occupies a two storey site on 23rd Street, in the Flat Iron district of Manhattan and is bright, a delight and sure to excite.

In the foyer you are confronted by square wheeled tricycles. Yes, square wheels are quite functional if the surface is an inverted catenary. The wheels then give a smooth ride over a bumpy surface as shown in Figure 1. (A catenary is the shape of a hanging chain – a famous one is the double inverted catenary of the McDonalds' logo).

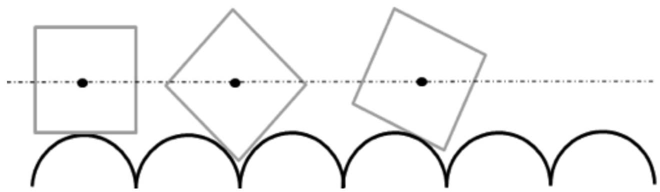

Figure 1. Square wheels on a catenary.

You can also get a smooth ride riding along on top of a number of 'Reuleaux' tetrahedra called the 'Coaster Roller'. These are shapes of constant diameter but are not spheres! A 2D version of these Reuleaux shapes is found in coins like the British 50 pence and 20 pence pieces (heptagonal) or a Bermudian dollar (triangular). They have a constant width so they don't get stuck in coin-operated machines. A Reuleaux triangle bit can even be used to drill square holes (with rounded corners)! Who said that 'squaring the circle' was impossible? (see Figure 2).

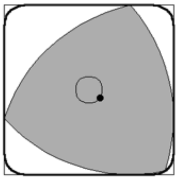

Figure 2. Drilling a square hole.

In the stairwell, between the museum floors zero and negative one, you can admire an amazing way to multiply using the 'String Product'. All you need is the parabola $y = x^2$ and a maze of strings connecting points on the graph. When you multiply any two x values (one negative, one positive) a straight line connects their (x,y) coordinates on the parabola. Where this line crosses the y-axis is the answer! This is an impressive demonstration of a *nomogram*, with the y-axis stretching over 6 metres high.

In Figure 3, the string product is demonstrating that $2\times3=6$.

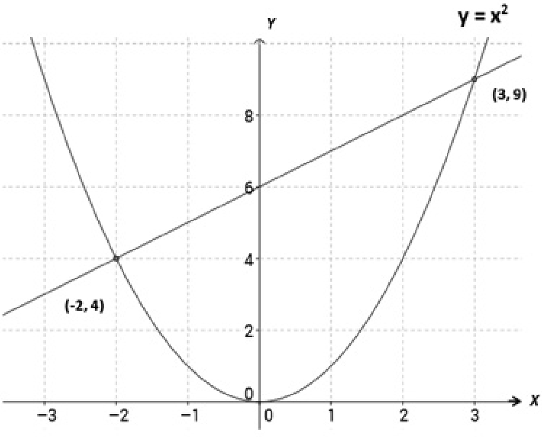

Figure 3. String product.

The 'Hyper Hyperboloid' is made up of two sets of straight cables which are anchored at the top. The base is swivelled around by the operator who sits in a chair in the middle of the chamber and watches straight lines turning into spirals. This concept of building curves with straight beams is used in tall structures like industrial cooling towers and, of course, the Sydney Tower. In Spain, Gaudi's *Sagrada Familia* has hyperbolic vaults and windows. It's often remarked that the only straight lines in this church are the queues of tourists waiting to go in. Started in 1882, this awe inspiring basilica has its completion date set for 2026. Gaudi famously commented "My client is not in a hurry".

'The Human Tree' uses software that takes a mathematical 'selfie', projects it onto a screen and then replaces each of your arms with a copy of your body. On each of those copies, the arms are replaced by smaller copies, and so on, and so on, to create a very personal fractal tree. The pattern is governed by the same rules that influence the shapes of real trees, frost crystals and even our own lungs.

Mathematics and music can be investigated through the 'Harmony of the Spheres'. The spheres light up when touched and emit notes to create a (musical) chord. As you move around, the notes change to create either harmony or dissonance. Major chords, minor chords and harmonies can be explored as the pattern of coloured lights move through space.

The museum has unique exhibitions which each emphasise a mathematical concept that can be understood at different levels. There are over 30 exhibits which provide an ideal mix between the electronic wonders and good old fashioned manipulatives that you can actually hold and operate with more than just two thumbs. And if you need a break from all these exhibits then the Enigma Café allows you to rest your legs (but not your brain) and work on a variety of innovative puzzles. It is inspiring to see both parents and children becoming totally absorbed in a problem together.

The Museum of Mathematics also teaches classes, holds public lectures, runs a shop and organises special events. Last year on 5th December they attracted more than 2000 mathematics fans to surround the nearby Flat Iron building with glow sticks to check that it obeyed Pythagoras' theorem. *It did!* The date 5/12/13 (or 12/5/13 in the US) was significant as that represents a Pythagorean Triad $5^2 + 12^2 = 13^2$. The next opportunity for this to occur is on 3rd April 2105 (3/4/05) or maybe 17 August 2015 if you are not worried about the numbers being in order.

Now, with all the 'hands on' happening at the Museum of Mathematics, the exhibits do need some 'feet up' at times, in which case they have displayed on them a simply coded notice; *1, 2, 3, 5, 4*. (decryption: *out of order*).

MATHEMAGICAL MARVELS TO LIVEN UP LESSONS

Andrew Wrigley

Somerset College, Queensland

An interactive and entertaining stroll through a variety of mathematical ideas to spark interest and discussion. Basic number operations, algebra, geometry and probability are covered and a calculator might be useful. Participants will be invited to share their own 'tricks of the trade'.

Introduction

Can you see symmetry in 2014? What if it is written 98×99–(609+6969+111)+1? What are the prime factors of 123456789? What about 987654321? Playing around with numbers can be fun with or without a calculator. But why do we need a calculator? Other cultures have developed their own methods of multiplying numbers together, which can make our own system seem complicated.

Mathematical Tricks

Maths tricks are a great learning experience, especially when they are simple enough for students to demonstrate themselves. They can also illustrate interesting mathematical ideas which allow them to be readily understood. Choose any three digit number, for example, 317. Enter it in to the calculator twice to give 317317. Now divide it by 7, divide the result by 11 and divide the result of that by 13. The answer is always the original number (317). The reason for this is found by reversing the operations because 7×11×13 = 1001, which then provides an opportunity to discuss place value. There are variations of this trick using, for example, the fact that 137×73 = 10001. And 73 is the favourite number of *The Big Bang Theory's* Sheldon Cooper for an assortment of reasons. The classic 1089 trick is harder to explain but relies on multiples of 99. Finding the cube root of a perfect cube can be straightforward, if you can spot a simple pattern.

Magic Squares

Magic Squares have been a source of mathematical interest for over 2000 years and constructing them can be easy with a little knowhow. But why is a square with a magic constant of 176 so magical? And how many magic hexagons exist? It was Euler, of course, who produced the first magic square of squares (see Figure 1).

68^2	29^2	41^2	37^2
17^2	31^2	79^2	32^2
59^2	28^2	23^2	61^2
11^2	77^2	8^2	49^2

Figure 1. Euler's magic square of squares.

Mathematical Cartoons

The domain of cartoons can help animate mathematics. Discover how Homer Simpson found counter examples to Fermat's Last theorem, how Spiderman relies on Pythagoras' theorem and why Sponge Bob Square Pants really needs to get a maths lesson. The world of proofs can be witnessed quite easily, especially when they fail. *Two* can be proven to equal *one* using Year 9 Algebra and π can even be shown to equal *two*!

Probability

Probability is also a great source of trickery. Try to answer this multiple choice question.

Question: If you choose an answer to this question at random, what is the chance you will be correct?

A. 25%
B. 50%
C. 60%
D. 25%

Probability and Gambling lead to that $64 000 question; is there a winning system in Roulette? And this runs neatly on to that thorny subject of error and the words of English Mathematician Charles Babbage "Errors using inadequate data are much less than those using no data at all".

The Joy of Six

Use only mathematical operators (no numbers) to complete these equations (see Figure 2). (The third row is already completed).

0		0		0	=	6
1		1		1	=	6
2	+	2	+	2	=	6
3		3		3	=	6
4		4		4	=	6
5		5		5	=	6
6		6		6	=	6
7		7		7	=	6
8		8		8	=	6
9		9		9	=	6

Figure 2. The joy of six.

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THE MATHEMATICAL
ASSOCIATION OF VICTORIA