# DIAGNOSIS AND INTERVENTION: THREE DIMENSIONS TO DEVELOPING NUMERACY IN ALL CHILDREN

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*Many students experiences difficulties in learning and applying mathematics and require assistance to overcome the misconceptions or inappropriate ways of thinking they have developed. Underlying causes of any difficulties need to be determined, students need to be led to see inadequacies in their ways of proceeding and thus appreciate a need to change, so that appropriate ways of thinking, generalising and applying mathematical ideas can be developed and implemented through well-focused processes of diagnosis and intervention. In this way, students can be helped to develop the conceptual understanding, fluent processes and self-confidence needed to acquire and use mathematics so as to become numerate.*

## Introduction

Children should have a robust sense of number… this includes an understanding of place value, meaning for the basic operations, computational facility and a knowledge of how to apply this to problem solving. A thorough understanding of fractions includes being able to locate them on a number line, represent and compare fractions, decimals and per cents, estimate their size and carry out operations confidently and efficiently.

Final Report of the National Mathematics Advisory Board, 2008 p. 17 & 18

It is no longer enough to simply study mathematics; mathematical knowledge needs to be able to be used in an ever-widening range of activities. Indeed, those who lack an ability to think mathematically will be disadvantaged, unable to participate in high-level work and at the mercy of other peoples' interpretation and manipulation of numbers and data. As the National Numeracy Review (2008, p. xi) goes on to say, numeracy must be considered an essential goal of education for all. Indeed, both numeracy and literacy are critical components for living full lives in the 3rd millennium, just as Steen predicted at the end of the last millennium when he stated that "an innumerate citizen today is as vulnerable as the illiterate peasant of Gutenberg's time" (Steen 1997). An ability to solve problems, communicate the results and methods used to obtain solutions, to interpret and use the results of mathematical processes and making sense are all essential to being numerate. However, many students fail to achieve even minimal standards of numeracy (DETYA, 2000; National Numeracy Review, 2008; OECD, 2004) and even those who do frequently say that they are 'no good at maths', feel inadequate and are unable to use the elementary mathematics that they have 'acquired'.

The first dimension in building up required levels of numeracy is through well-designed assessment that can reveal how students think and reason about the mathematical ideas they are coming to terms with and thus provide a guide for ongoing teaching. It should also uncover ways students deal with the mathematical tasks they are exposed to and provide feedback on particular learning activities, whether they are appropriate for the students and content in question or whether they need to be adjusted to produce the required learning. Assessment can also inform a student, other teachers, a parent or caregiver about the student's mathematical capabilities and potential. Different outcomes may be noted across different groups of students, different classes and schools, leading to closer examination of the circumstances that might have brought about these results – were the programs dissimilar, were the expectations different, were the activities of the same form, was a similar teaching approach followed, and so on.

## Diagnostic assessment

While it is important to know *what* students know, of even more importance is *how* they know – is their knowledge simply memorised routines, or is there a deep understanding based on well-understood concepts and fluent, meaningful processes applied in appropriate ways? Developing an awareness of how someone knows something is not straightforward. When a student is asked why he or she thinks in a particular way, a usual first response is to provide what the student thinks a teacher wants rather than provide the detail of their own thinking. This is not surprising because being able to reflect on one's thinking while completing a task requires highly developed metacognition to discuss or describe the thoughts and processes being used. A learner struggling to come to terms with a new way of thinking or mathematical topic or is unlikely to be able to talk deeply about what they are doing. Further, when a student responds appropriately to a teacher's question, a teacher may then believe that the response reveals that the student *knows what the teacher wants them to know* whereas the student may actually only *know what the teacher wants*. This shortcoming may not be revealed until later when the understanding needed to apply the mathematics that was assumed to be known or to develop further mathematics is not available.

In order to uncover a full picture of a student's mathematical knowledge, *Diagnostic assessment* is essential to see what a student knows, to reveal not only what they do not know but also what they need to know. Critically, this will often reveal gaps in a student's mathematical knowledge – essential ideas may not have become central to a student's way of thinking or may not have been included among the sequence used to establish a topic. For example, there are many programs where the aspects of *renaming* are not developed as an extension of place value to provide a basis for number processes such as comparing or rounding, a complete understanding of larger numbers or to underpin computation across whole numbers and fractions.

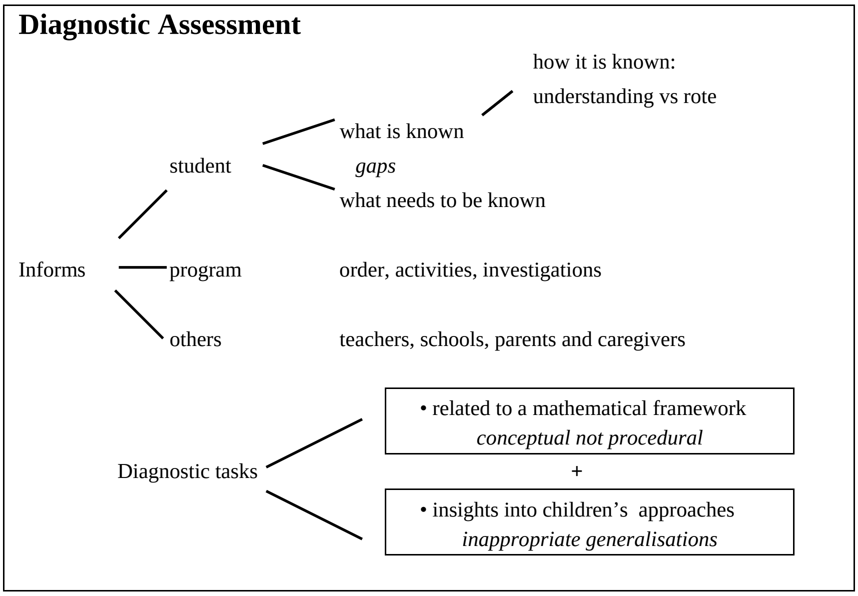



Figure 1. Components of Diagnostic Assessment.

Thus, diagnostic assessment highlights the strengths that a learner brings to a topic as well as the weaknesses in their prior knowledge that may cause misconceptions, errors or difficulties.

Consequently, diagnostic testing is necessarily different from other forms of tests used in class, state or national assessment. Insight is needed into all the steps required to develop facility with a topic rather than just measure how well a final outcome or particular points in the development have been attained. This means that the sequence of questions posed cannot simply follow those used in teaching a topic as this may allow a student to restore their knowledge, concealing the fact that crucial aspects have not become central to their thinking. Frequently, these difficulties arise from initial concepts and thinking introduced at the outset of a topic rather than just an inability in applying the final process. Errors or misconceptions also arise from insufficient understanding of mathematical aspects that were developed outside of a particular topic but are essential to understanding the processes involve. For example, many difficulties with computation are due to aspects of numeration such as zero, place value and renaming that underpin fluent, accurately processes (see Figure 2).

| | | |
|---|---|---|
| 6. 7 | 0 5 0 1. 2 | 7. $^{0}\cancel{0}^{14}\cancel{4}$ |
| + 8. 4 | 7)4 $^{3}$5. 2 $^{7}$9 | - 3. 3 7 |
| 14. 11 | | 4. 0 7 |

Figure 2. Difficulties with process, renaming & zero.

Other errors are based in confusion with and among the rules that have been acquired or else in insufficient understanding of the numbers being worked with. For instance, difficulties with measurement may be due to inadequate understanding of spatial properties or of the significance of zero or decimal fractions in the way measuring instruments are used and the results interpreted. A further source of difficulty is *inappropriate generalisations*, where something that worked in one situation is taken to another setting where the conditions that allowed it no longer apply. For example, *additive thinking* is often used within multiplicative situations for computation, fractions or measurement (see Figure 3).

| | | |
|---|---|---|
| $^{2}$5. 6 | $^{3}$5. 6 | $^{3}$5. 6 |
| x 7. 4 | x 7. 4 | x 7. 4 |
| 1 4. 4 | 3 8. 6 | 2 2. 4 |
| | | 3 9 2. 0 |
| | | 4 1 4. 4 |

Figure 3. Confusion with addition process.

Strengths and weaknesses uncovered in diagnostic assessment then need to be analysed in terms of the mathematical concepts and processes that underpin them so that reasons can be proposed for why they came about. Close observation is essential in allowing insight into the thinking a learner is using to reveal the underlying causes of any difficulties. Usually this will require a task chosen to elicit the ways in which a student is acting, then systematically exploring the possible forms this takes. Nonetheless, any initial attribution of reasons can only be an assumption, usually based on previous experience and will need to be probed further to fully come to terms with an error or misconception.

assumptions

observations probing

↓

causes

Figure 4. Cycle of Diagnostic assessment.

Often several possibilities for an error may need to be considered and the process of probing and observing continued in order to first dismiss one or more before the likely reason or reasons can be determined. In this way, a cycle of observations, assumptions and probing will eventually lead to an understanding of the underlying causes and suggest what is needed to overcome them.

## From Diagnosis to Intervention

A second dimension to building numeracy requires appropriate intervention to build the conceptual understanding and fluent processes needed to acquire and use mathematics. Diagnostic assessment is critical in planning how to teach or re-teach the essential mathematics that underpins numeracy. Once misconceptions, difficulties and gaps in a student’s knowledge have been identified, means to intervene in the learning can be instigated in a manner appropriate to the learner and consistent with the way concepts and processes are best established and consolidated. Describing what is known and needs to be known in terms of the underlying mathematical ways of thinking is critical in providing a basis for this intervention. On the other hand, only providing rules or procedures that may be followed in a less than meaningful way is unlikely to have any long lasting or deep effect on the student's mathematical development.

Constructing new ways of thinking most often begins with the use of materials to show the patterns on which the ideas are developed, linking to a language that provides meaning and only moving to the symbolic expressions that express what is happening succinctly when the learner has adopted the way of thinking as his or her own. Simply showing a student what to do using recorded examples, particularly when only the example that is incorrect is considered, is rarely successful in replacing procedures that have led to errors. At best he or she will try to copy and remember a teacher's approach but any connection to what they do know is often not apparent in the purely recorded form. Rather, engaging and different practice activities, often in the form of games in which learners willingly participate, are an essential part of learning to bring a concept to the forefront of a learner's mind and enable a process to become fluent.

Intervention can then build from the understandings that are essential for the development of further concepts and processes and provide the links needed to extend the ideas to enable applications to new situations, means of solving a range of problems and to develop further mathematics. This process can be summarised:

**The process of intervention**

1. the identification of *understandings and errors* and the description of them in terms of the underlying mathematical concepts and processes

2. uncovering *sources of difficulties* – not only inappropriate thinking but also the degree of understanding of why processes and responses are correct

3. revealing *inadequacies in thinking* to a child in order to build an appreciation of a need for change

4. the implementation of means of *constructing or re-constructing* appropriate ways of thinking

5. *practice* that is focused and motivating to allow a way of thinking to become secure and provide a basis for generalisation to more complex problems and applications or to the development of further mathematics

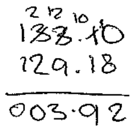
Figure 5. The process of intervention.

## Case study 1

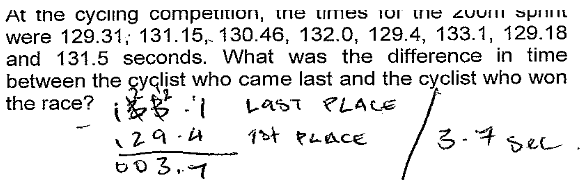
Students were observed to have difficulties solving problems requiring addition with decimal fractions:

| |
|---|
| At the cycling competition, the times for the 200m sprint were 129.31, 131.15, 130.46, 132.0, 129.4, 133.1, 129.18 and 131.5 seconds. What was the difference in time between the cyclist who came last and the cyclist who won the race? |

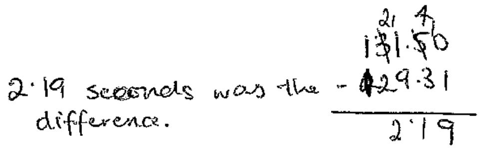
As well as subtraction to find the difference in time, this problem requires students to first identify the time taken by the winning cyclist and the time taken by the cyclist who came last. The fastest cyclist takes 129.18 seconds and the slowest cyclist takes 133.1 seconds.



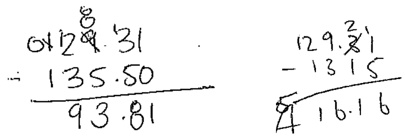

Students may be able to complete correctly the subtraction they set out to solve, but often are unable to determine which numbers to use. A common error is to see 129.4 seconds as the fastest time since 4 is less than 19.


At the cycling competition, the times for the 200m sprint were 129.31, 131.15, 130.46, 132.0, 129.4, 133.1, 129.18 and 131.5 seconds. What was the difference in time between the cyclist who came last and the cyclist who won the race?

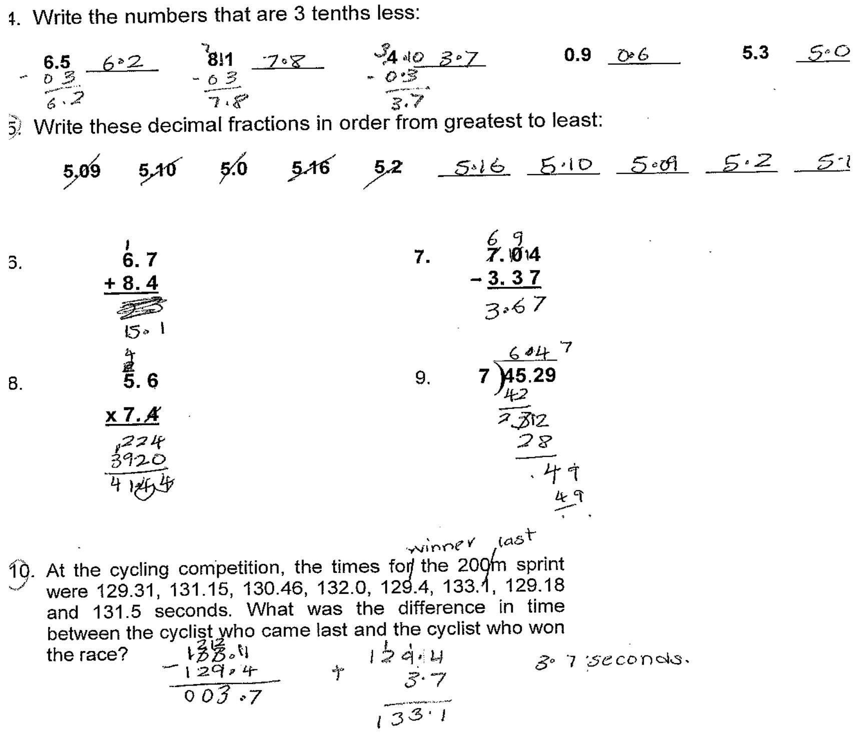

Others are unable to interpret the problem and choose the first and last times listed in the problem:



Some reverse the order of subtraction yet still obtain an “answer”:




The fact that most calculation examples were completed correctly does not actually indicate an understanding of decimal fraction computation, nor does it reveal where misunderstandings are sourced. The cause of their problem solving difficulties in fact lies in an inability to order decimal fractions, in turn revealing that it is a lack of decimal place value that needs to be addressed.



1. Write the numbers that are 3 tenths less:

6.5 6·2 8.1 7·8 4.0 3·7 0.9 0·6 5.3 5·0

5. Write these decimal fractions in order from greatest to least:

5.09 5.10 5.0 5.16 5.2 5·16 5·10 5·09 5·2 5·1

6. 6. 7 + 8. 4 = 15·1

7. 7. 0 4 − 3. 3 7 = 3·67

8. 5. 6 x 7. 4

9. 7 ) 45.29

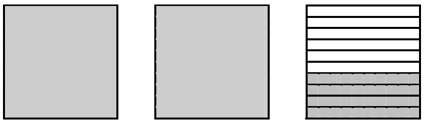
10. At the cycling competition, the times for the 200m sprint were 129.31, 131.15, 130.46, 132.0, 129.4, 133.1, 129.18 and 131.5 seconds. What was the difference in time between the cyclist who came last and the cyclist who won the race?

winner last

133·1 − 129·4 = 003·7 129·4 + 3·7 = 133·1 3·7 seconds.

In order to provide *intervention* on the underlying difficulties the diagnosis has revealed, place value for decimal fractions needs to be built up. This requires teaching to

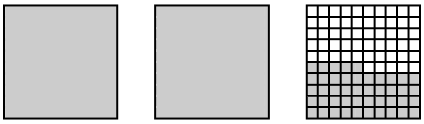
- name fractions with ones and tenths



2 ones and 4 tenths – 2 and 4 tenths

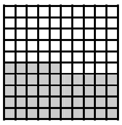
| ones | tenths | |
|---|---|---|
| 2. | 4 | 2.4 |

- name fractions with ones and hundredths



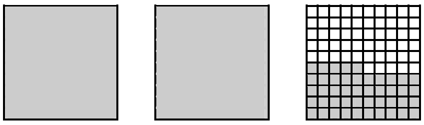
2 ones and 45 hundredths – 2 and 45 hundredths

- rename hundredths as tenths and hundredths



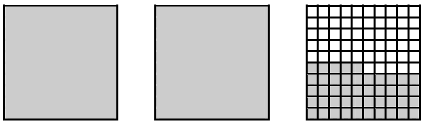
1 tenth is 10 hundredths
45 hundredths is 4 tenths 5 hundredths

- name fractions with ones tenths and hundredths

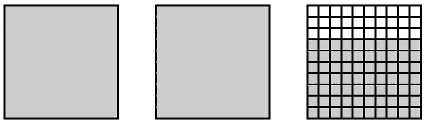


| ones | tenths | hundredths | |
|---|---|---|---|
| 2. | 4 | 5 | 2.45 |

- compare fractions with ones tenths and hundredths



| ones | tenths | hundredths | |
|---|---|---|---|
| 2. | 4 | 5 | 2.45 |



| ones | tenths | hundredths | |
|---|---|---|---|
| 2. | 6 | 0 | 2.6 |

Elicit that 2.6 is greater than 2.45 because it has more tenths

- compare fractions using symbols only, drawing on place value

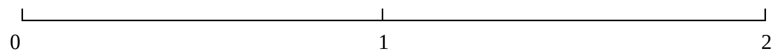
- Apply this place value understanding to another problem of the same form – firstly to determine the least and greatest times taken, then to the renaming needed to complete the subtraction meaningfully

Note that the examples chosen to (re-)establish the way of thinking are different to those that were completed incorrectly so as to focus on the underlying ideas rather than simply be seen to correct an example that was answered erroneously. When several examples like these have been examined and the student has developed an understanding of what is needed, then the problem can be looked at again to see that the student can not only determine which numbers to use but is able to see what was done inappropriately when he first attempted the problem.

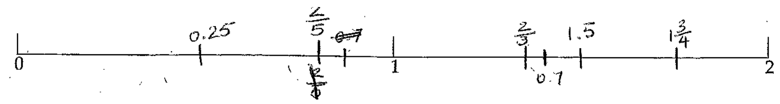
## Case study 2

Another area where students are often observed to have difficulties is in placing fractions on a number line. In other words, they have an inability to see fractions as numbers among the whole numbers and often view them as two numbers arranged according to 'decimal' or 'fraction' rules – a way of using whole numbers rather than extending their understanding of numbers.

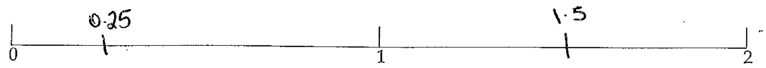
Place these fractions on the number line below: 1.5, $\frac{2}{3}$, 0.7, 1 $\frac{3}{4}$, $\frac{2}{5}$, 0.25



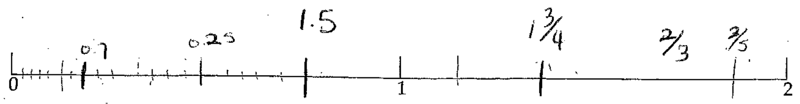

Some would interpret the number line simply as a line and place some fractions where they believed they should be on the whole length and then place the others among them



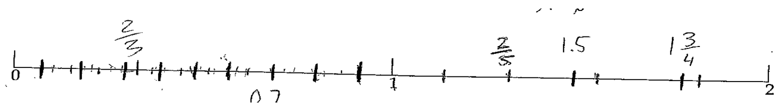

Others could only place decimal fractions

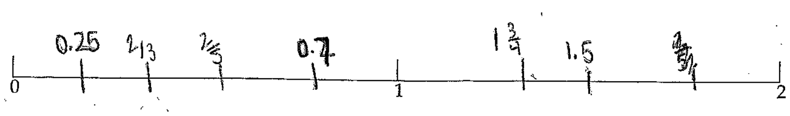


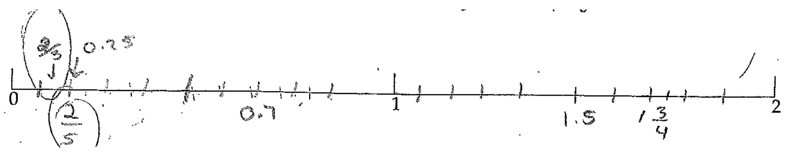

Fractions were often clustered around the whole numbers 0, 1 and 2

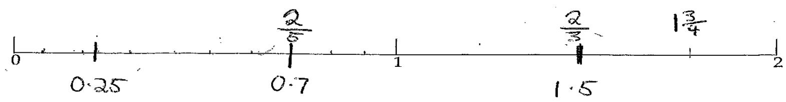



When the line was divided to show parts, some fractions were placed correctly then others were simply put among them



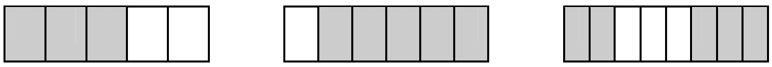




Difficulties with number lines often stem from their use in the early years to count on and back. On the one hand this use focuses on the answers rather than counting on usually achieved by using the points on the number line to count ‘1, 2, 3 – the answer is nine’ rather than counting on ‘6, 7, 8, 9’. This emphasis on the points on the line rather than the distances between the numbers leads to difficulties with using a ruler to measure length and the idea of placing fractions along the number line. Later, difficulties will arise when the number line is extended to provide meaning for negative as well as positive numbers.

Thus one of the first approaches to overcoming the underlying difficulties the diagnosis has revealed would in fact be to overcome premature uses of a number line in the first years of school and instead leave them for the development of fraction ideas in the first place and then to show how real numbers are extended from the whole numbers. This would also provide a basis for understanding scale and other notions of proportional reasoning.

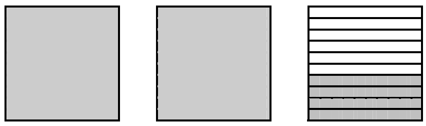
In order to provide *intervention* on the underlying difficulties the diagnosis has revealed, the fraction concept needs to build from the use of region models to show proper fractions and mixed numbers (whether decimal or common fractions) to the way in which they can also be represented on a number line.

- Use rectangular models to establish the fraction concept as the number of parts out of the total number of equal parts



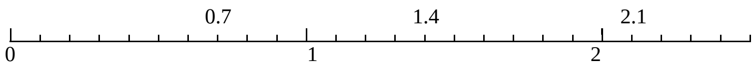
3 fifths 5 sixths 5 eights

- Introduce meaning for decimal fractions as in the case study above



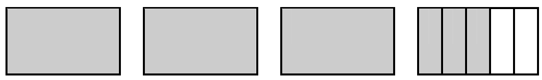
2 and 4 tenths – 2.4

- Place decimal fractions onto a number line with 10 divisions between whole numbers

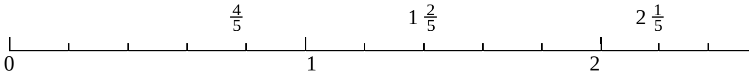


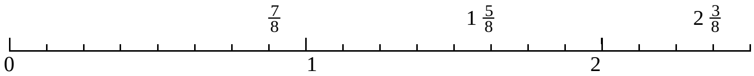

- Use rectangles to introduce improper fractions and rename them as mixed numbers

$3\frac{3}{5}$ or $\frac{18}{5}$

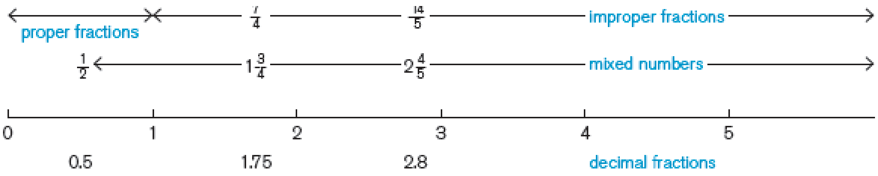


- Use a number line to place different common fractions on the number line by drawing divisions to show the number of parts in each one




- Show how all fraction forms can be represented on a number line




- Have children determine the relative positions of the various fraction forms on a number line

## Conclusion

Building numeracy in all students is a critical aspect of contemporary schooling. Understanding how concepts and processes are constructed and connected provides a basis for overcoming misconceptions and inappropriate ways of thinking that may have developed (Hiebert & Grouws, 2007). Appropriate intervention programs can then be planned and implemented to build students' competence and confidence with fundamental mathematical ideas. Confidence in knowing what to do, when to use their mathematics and why it is appropriate to use it provides the third dimension to building numeracy. When all 3 dimensions – diagnosis of difficulties, intervention to build meaningful ways of thinking, and developing confidence in their mathematical ability – are present, students will be prepared to engage with further mathematical ideas and be inclined to use their knowledge of mathematics in the many everyday and work contexts where reasoning and sense making will be required.

## References

Booker, G. (2011). *Building Numeracy: Moving from Diagnosis to Intervention* Melbourne: Oxford University Press

Booker, G; Bond, D., Sparrow, L., & Swan, P. (2010). *Teaching Primary Mathematics [Fourth Edition]* Sydney: Pearson Education Australia

Booker, G. (2009). *Booker Profiles in Mathematics–Revised: Numeration & Computation*, Milton, Qld: Abacus Press.
DETYA (2000). *Numeracy, A Priority for All: Challenges for Australian Schools*, Canberra: Commonwealth of Australia.
Hiebert, J. & Grouws, D. (2007). '*The effects of classroom mathematics teaching on students'* learning', in F. Lester (ed.), Second Handbook of Research on Mathematics Teaching and Learning, Charlotte, NC: Information Age Publishing, pp. 382–392.
National Numeracy Review Report DEWR (2008). accessed at <www.coag.gov.au/reports/docs/national_numeracy_ review.pdf>
OECD (2004). '*Learning for tomorrow's world: First reports from PISA 2003'*. Paris, France: OECD.
Steen, L. (ed.) (1997). *Why Numbers Count: Quantitative Literacy for Tomorrow's America*, New York: The College Board.
Steen, L. (ed.) (2001). '*Embracing Numeracy'*, *Mathematics and Democracy*, Washington DC: National Council on Education and the Disciplines.