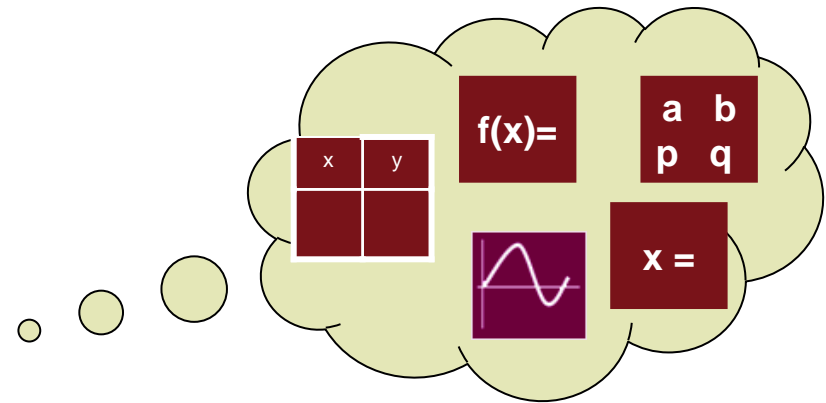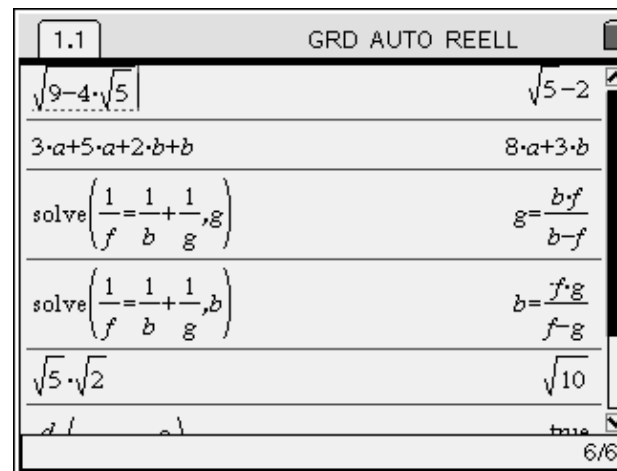Bärbel Barzel,
University of Education, Freiburg

Learning Algebra and Developing Functional Thinking - With or Without CAS?

CAS is too powerful to use it for lower secondary level - only use it in upper secondary level

When using CAS, pupils lose by hand skills

CAS may only be used following the white-box black-box principle
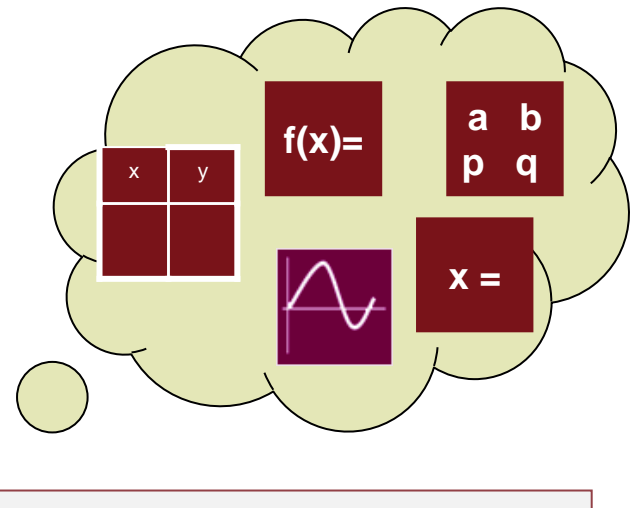
Learning Algebra and Developing Functional Thinking - With or Without CAS?

....

Pupils learn about variables more easily when CAS is integrated

Pupils gain more skills when using CAS

CAS helps to investigate a new topic by using the black-box white-box principle

Topic   Technology   Tasks & Teaching

x	y

f(x)=

a b
p q

x =

Focusing
1. … the Topic:
       Algebra & Functional Thinking

2. … the Technology:
       CAS and MRS (Multi Representational Systems)

3. … the Tasks & the Teaching:

Algebra and functional thinking: …. problems

... measuring basic algebraic skills in PISA:

Which of the following represent half of a?
Mark the correct answers with a cross.

$\dfrac{a}{2}$ Ja ☐   Nein ☐

$a - \dfrac{1}{2}$ Ja ☐   Nein ☐

$\dfrac{1}{2} \cdot a$ Ja ☐   Nein ☐

$a - \dfrac{a}{2}$ Ja ☐   Nein ☐

$0,5 \cdot a$ Ja ☐   Nein ☐

$a : \dfrac{1}{2}$ Ja ☐   Nein ☐

$\dfrac{1}{2}a$ Ja ☐   Nein ☐

in PISA **13 %** correct

Algebra and functional thinking: …. problems

... measuring basic algebraic skills with

For my garden, I bought
- ***r** red rose bushes and*
- ***g** white gardenia bushes.*

The roses cost $4 each.
The gardenias cost $5 each.

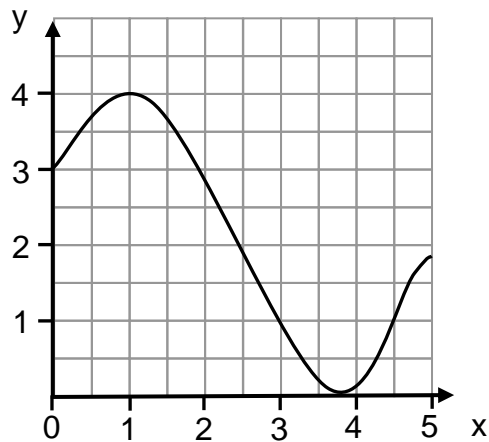*Choose the equation that says that the total cost was $70:*

☐ **4r+5g=70**
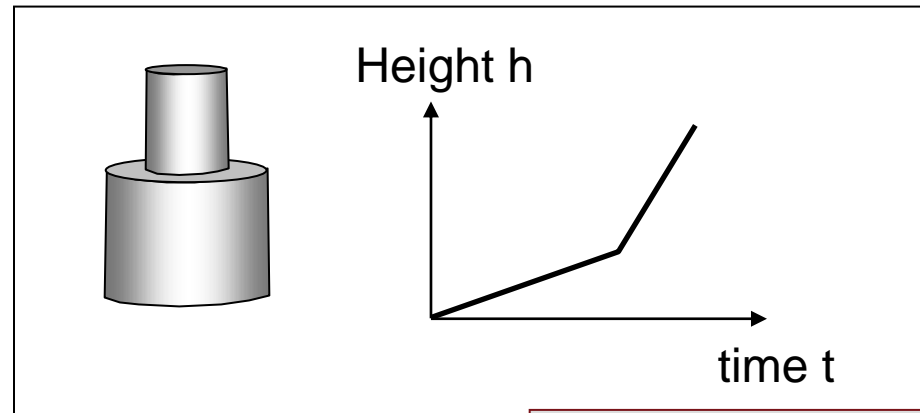
☐**10r+6g=70**   **Only 34% answered correctly.**

☐ **r+g=70**

Algebra and functional thinking: …. problems

For which value of x is the y-value 1?

29% wrong answer

Height h

time t

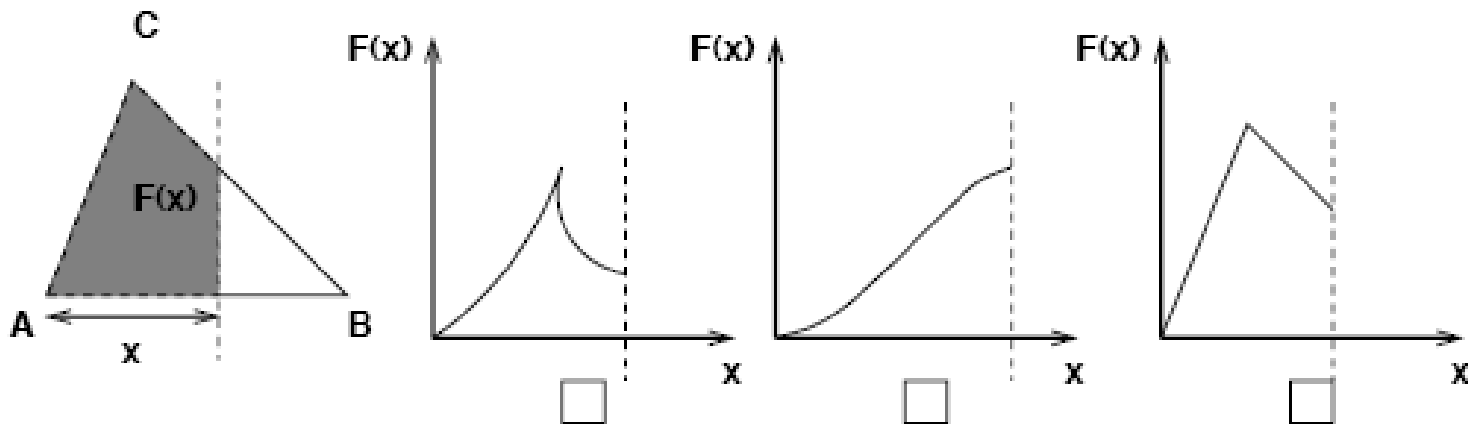**42% sketch a wrong graph**

Test-Results Semester 1 2008/09

A taxi driver takes 5€ as fix costs and 3€ for every km. Sketch a graph, …

48% sketch a wrong graph

See also: Janvier 1983, PISA 2003, PALMA 2006, Kaput 1994

Algebra and functional thinking: …. problems

Test with about 100 Math-students, Semester 1

The doted line is drawn from A to B.

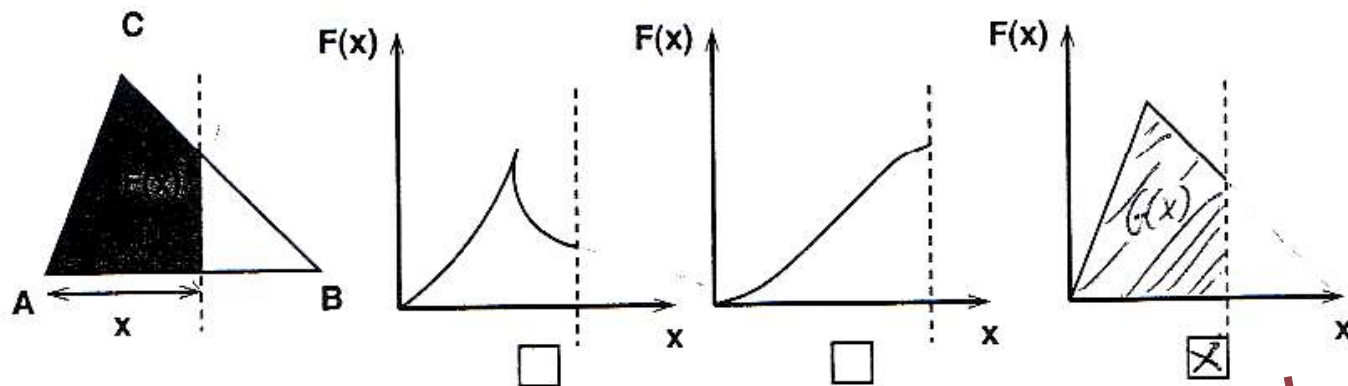*F*(*x*) gives the size of the grey area at x.

a. Which graph belongs to F(x)? Make a cross.
b. Give reasons to your choice.

Results:

a. **66% correct answer**
b. **57% correct answer**

Hoffmann 2007

Algebra and functional thinking: …. problems

Test with about 100 Math-students, Semester 1
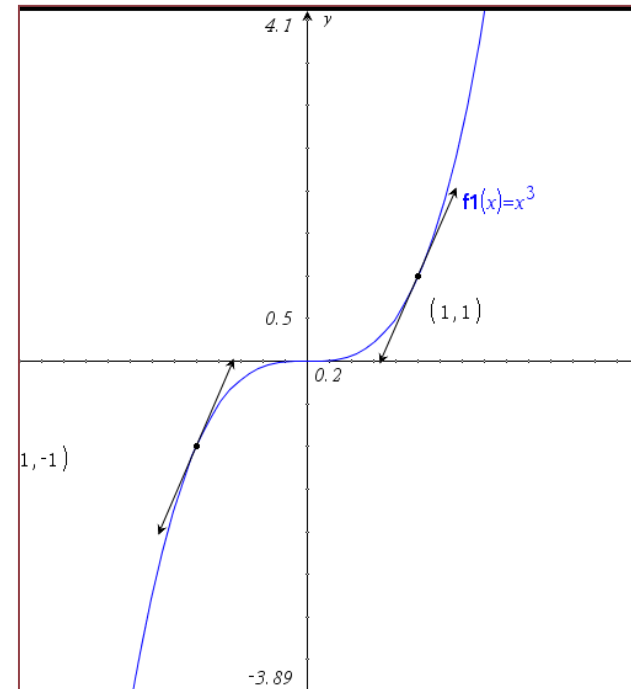

Hoffmann 2007

Algebra and functional thinking: …. problems

f is a differentiable function with *f* (-*x*) = - *f* (*x*).

Which of the following is correct?

A) *f* '(-*a*) = - *f* '(-*a*)

B) *f* '(-*a*) = *f* '(*a*)

C) *f* '(-*a*) = - *f* '(*a*)

D) None of the above

A typical answer was: f '(-a) = (f (-a))'= (- f (a))'= - f '(a).

Eisenberg & Dreyfus (1990)

Topic   Technology   Tasks & Teaching

Algebra and functional thinking: …. What is it about?

| Learning basic syntactical skills | Getting a sense for symbols and structures | Dealing flexibly with functions and their representations |

$5 (2x + 1) = 10$

$ 10x + 5 = 10$

...

$5 (2x + 1) = 10$

$5 (2x + 1) = 2 \bullet 5$

...

$f(u) = a \bullet u + a^2$

Jahnke 1999; Krüger 2000; Schubring 2007 ;
Sierpinska 1992; Arcavi 1992

Topic   Technology   Tasks & Teaching

Algebra and functional thinking: …. What is it about?

| Learning basic syntactical skills | Getting a sense for symbols and structures | Dealing flexibly with functions and their representations |

- Connecting mathematical representations
  (verbal, graphical, numerical and symbolic representations)
- Developing and connecting different mental images

Presmeg 1997; 2006; Duval 2002; Tall 1997;
Fischer 1987; Malle 2000; vom Hofe 1995

Topic   Technology   Tasks & Teaching

Algebra and functional thinking: …. What is it about?

Learning basic syntactical skills	Getting a sense for symbols and structures	Dealing flexibly with functions and their representations

- Connecting mathematical representations (verbal, graphical, numerical and symbolic representation)

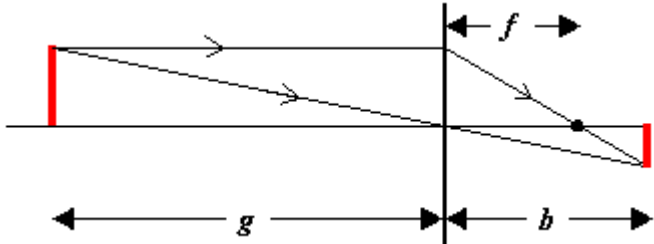- Developing and connecting different aspects/ images of the concept
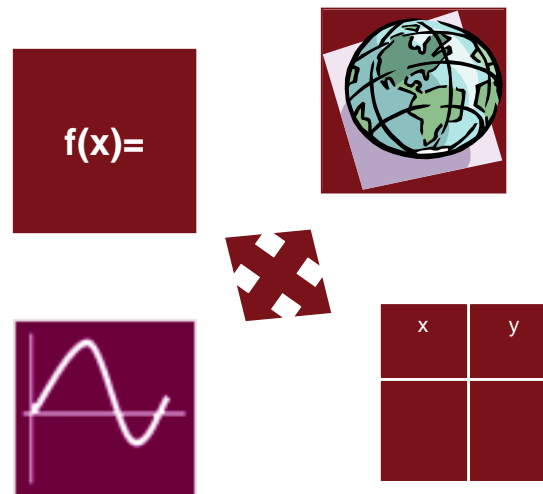
f(x)=

x   y

Topic   Technology   Tasks & Teaching

Algebra and functional thinking: …. What is it about?

| Learning basic syntactical skills | Getting a sense for symbols and structures | Dealing flexibly with functions and their representations |

- Connecting mathematical representations (verbal, graphical, numerical and symbolic representations)
- Developing and connecting different aspects/ images of the concept

´**Variable**´ can be seen

- as a placeholder
- as a generalizer
- as an unknown
- as a changing quantity

´**Function**´ can be seen:

- under the aspect of mapping
- under the aspect of covariation
- as an entity

Malle 1993 & 2000; Fujii 2003

Algebra and functional thinking: …. What is it about?
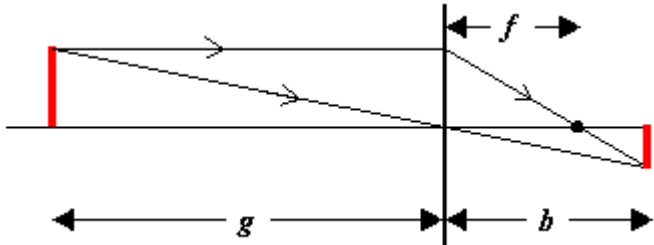
Lens equation

$$\frac{1}{g} = \frac{1}{f} - \frac{1}{b}$$

Drijvers 2005

Focus on structure when typing the equation into the machine, otherwise:

$$\frac{1}{g} = \frac{1}{f - \dfrac{1}{b}}$$

Topic   Technology   Tasks & Teaching

Algebra and functional thinking: …. What is it about?

Lens equation

$$\frac{1}{g} = \frac{1}{f} - \frac{1}{b}$$

as a placeholder:

$$\frac{1}{g} = \frac{1}{f} - \frac{1}{b} \,|\, f = 10 \qquad \frac{1}{g} = \frac{1}{10} - \frac{1}{b}$$

as an unknown:
as a generalizer:

$$\text{solve}\left(\frac{1}{15} = \frac{1}{10} - \frac{1}{b}, b\right) \qquad \text{solve}\left(\frac{1}{g} = \frac{1}{f} - \frac{1}{b}, b\right)$$

$$b = 30 \qquad\qquad b = \frac{-f \cdot g}{f - g}$$

as a changing quantity:

$$b(g) := \frac{10 \, g}{g - 10}$$

"Done"
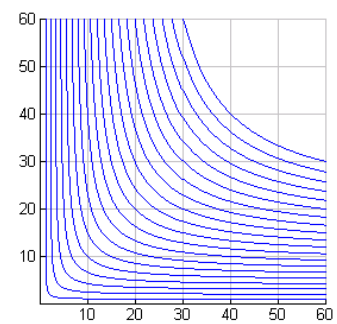
Topic   Technology   Tasks & Teaching

Algebra and functional thinking: …. What is it about?

| Learning basic syntactical skills | Getting a sense for symbols and structures | Dealing flexibly with functions and their representations |

Flexible use of variables, in concrete:

- Calculate with variables in the same way than with numbers, such as: *2a+4a = 6a*

- Knowing that variables are standing for a number not for an object, avoiding a misconception like:
  Six students for one professor: 6s = p

- Accept results with variables as an answer, such as:
  The solution is 2p+1!

Küchemann 1981; Malle 2000;
Mac Gregor/ Stacey 1991;

Which technology supports the learning of algebra?

Technologies are media,
    they have to mediate and to support the
        teaching– and learning process

They are a mediator between mathematics and the pupil

Which technology supports the learning of algebra?

General tools

- DVD
- Internet
- Presentation tools

Mathematical tools

General purpose:

- Computer Algebra (CAS)
- Graphing

- Spread sheet
- Dynamic Geometry

Specific purpose:
- Applets
- Tools for measuring data

Multi-representational system (MRS)
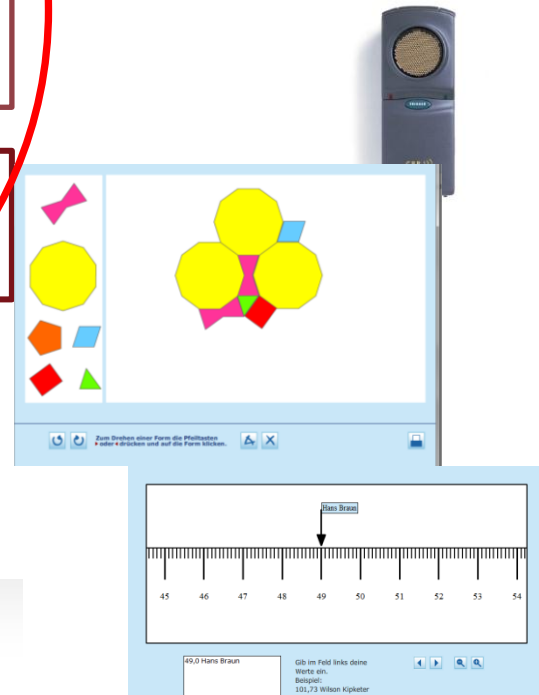
GeoGebra

From CAS to MRS

Multi-representational system

Why does a parabolic mirror work?
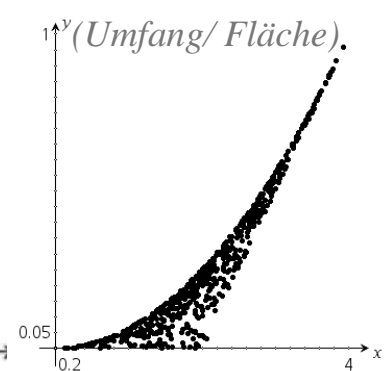
$f(x) = ax^2$

Focus point: $\left(0 \,\Big|\, \dfrac{1}{4a} \right)$
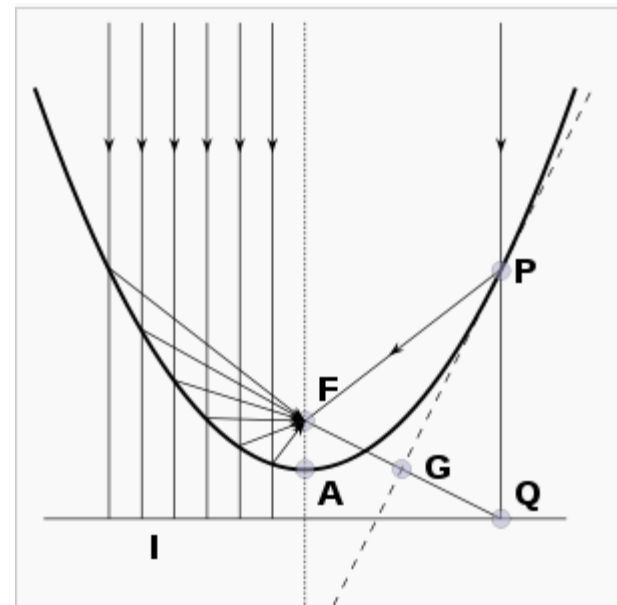
Using variables and functions

	basic	B height	C area	D
♦	=capture('	=capture('	=capture('	
190	10.1499	8.04714	40.8388	
191	10.3448	7.95204	41.131	
192	10.4617	7.89411	41.2929	
193	10.6566	7.79612	41.54	
194	10.8125	7.71643	41.7169	
195	10.8904	7.67615	41.7983	
196	11.3582	7.42838	42.1864	
197	11.4751	7.36481	42.256	
198	11.592	7.30059	42.3143	
199	11.709	7.23572	42.3614	
200	11.7479	7.21395	42.3745	
201	12.2936	6.9016	42.4228	
202	12.4105	6.83282	42.3995	
203	12.5275	6.76339	42.3641	
204	12.6053	6.71682	42.3338	
205	12.6443	6.69338	42.3165	
206	12.6833	6.66988	42.2979	
207	12.8389	6.57528	42.2098	
F198				

21 cm

29.7 cm

h=6.62 cm
A=42.3 cm²
g=12.8 cm

(Basic,Area)

x
35
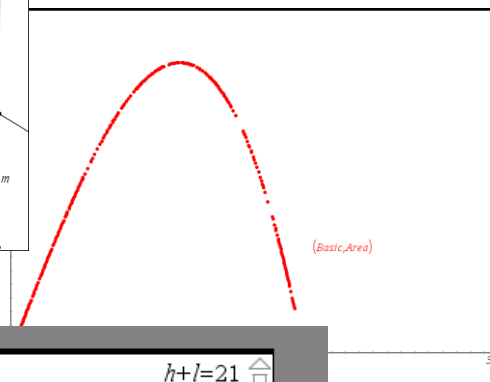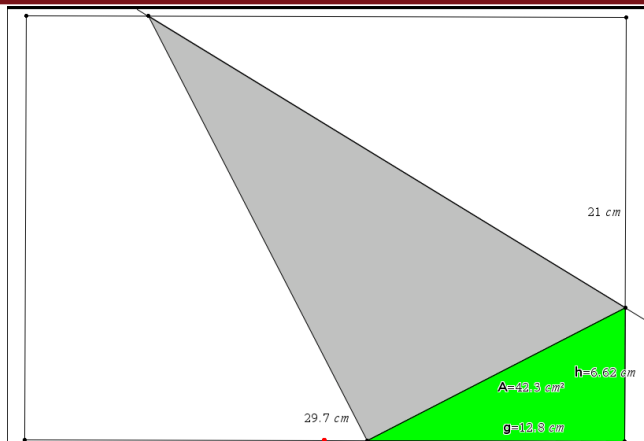
$$h + l = 21 \qquad\qquad h+l=21$$
$$l = 21 - h \qquad\qquad l=21-h$$
$$g^2 = l^2 - h^2 \qquad\qquad g^2=l^2-h^2$$
$$(21-h)^2 - h^2 = g^2 \qquad\qquad 441 - 42 \cdot h = g^2$$
$$\sqrt{441 - 42 \cdot h} \rightarrow g(h) \qquad\qquad Fertig$$
$$\frac{g(h) \cdot h}{2} \rightarrow a(h) \qquad\qquad Fertig$$
$$\frac{d}{dh}(a(h)) \qquad\qquad \frac{\sqrt{-21 \cdot (2 \cdot h - 21)}}{2} - \frac{h \cdot \sqrt{21}}{2 \cdot \sqrt{21 - 2 \cdot h}}$$
$$\frac{d}{dh}(a(h)) \rightarrow a1(h) \qquad\qquad Fertig$$
$$solve(a1(h)=0, h) \qquad\qquad h=7$$

Using variables and functions

Felsager 2007

From CAS to MRS
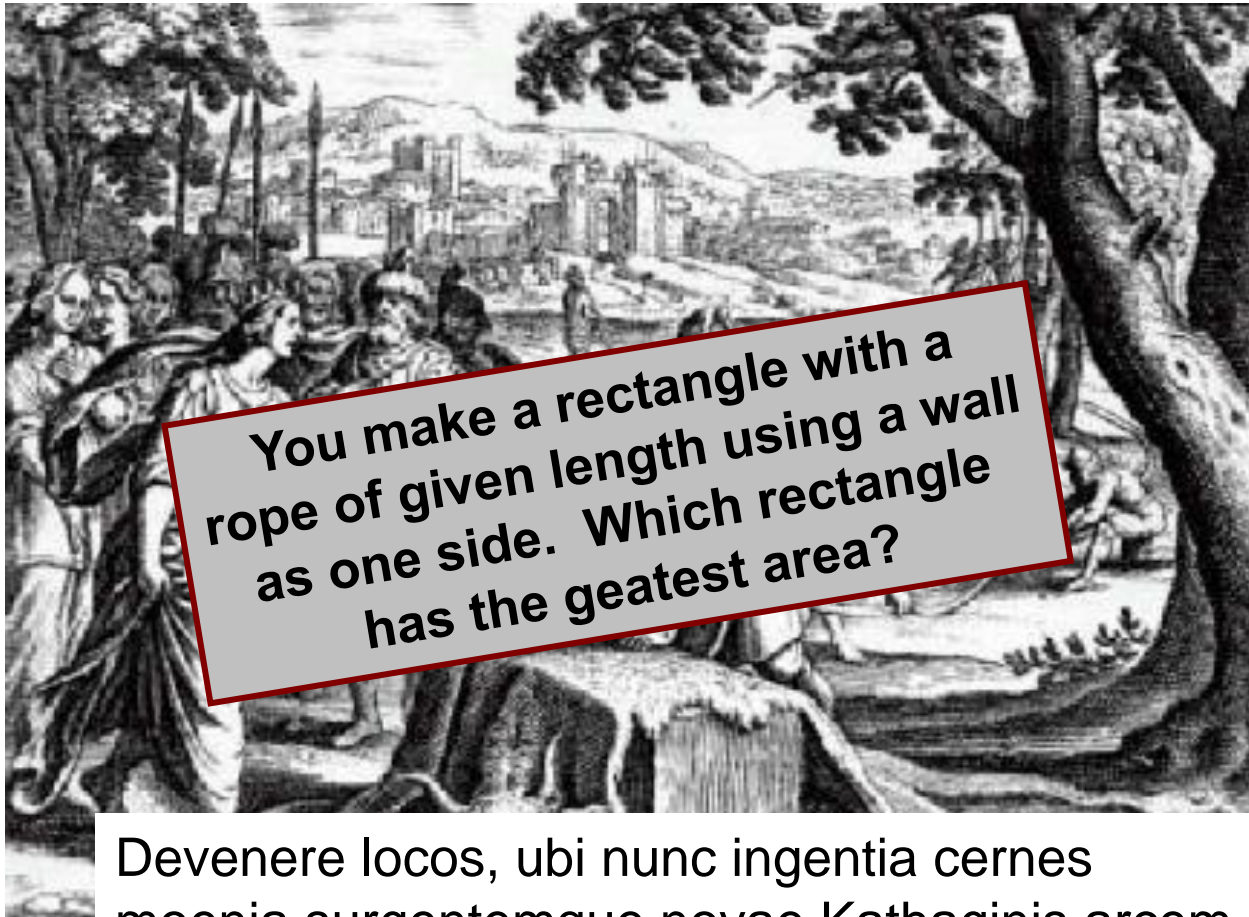
You make a rectangle with a rope of given length using a wall as one side. Which rectangle has the geatest area?

Devenere locos, ubi nunc ingentia cernes
moenia surgentemque novae Kathaginis arcem,
mercatique solum, facti de nomine Byrsam,
taurino quantum possent circumdare tergo (Aneis, Virgil)

Topic   Technology   Tasks & Teaching

Try values

Spread sheet

Numerical approach
First step:
A value!!!

Find an equation

Computer Algebra

Symbolic approach
First step:
An expression!!!

Spread sheet

u make a rectangle with a
of given length usin
one side. Which rec
has the geatest are

Computer Algebra

$f(x):=(13-2\cdot x)\cdot x$ *Fertig*

$\frac{d}{dx}(f(x))$ $13-4\cdot x$

solve(13-

$x=\frac{13}{4}$

solve$(f(x)=0,x)$

solve$(13-4\cdot x=0,x)$

	A	a	B	b	C	ar	D	E	F
◆		=seq(i,i,1,13,.25)		=13−2*a▯		=a▯*b▯			
1		1		11		11			
2		1.25		10.5		13.125			
3		1.5							
4									
					8.5	19.125			
		2.5		8.		20.			
8		2.75		7.5		20.625			
9		3.		7.		21.			
10		3.25		6.5		21.125			
11		3.5		6.		21.			
12		3.75		5.5		20.625			
A1	=1								

Make a sketch

Graphical approach
First step:
A sketch/ a graph!!!

Dynamic Geometry

aa=3.12662 *u*
6.74676 *u*
13 *u*
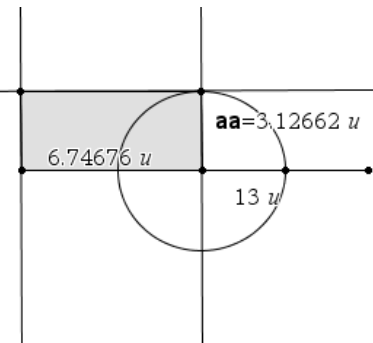
From CAS to MRS

| Spread sheet | Dynamic Geometry | Computer Algebra |

Research study in Norway (Fuglestad 2005)
3 grades (8-10, 14 – 16 years old pupils), 6 classes took part,
The three tools were always available

Which of the three tools do pupils use when solving tasks,
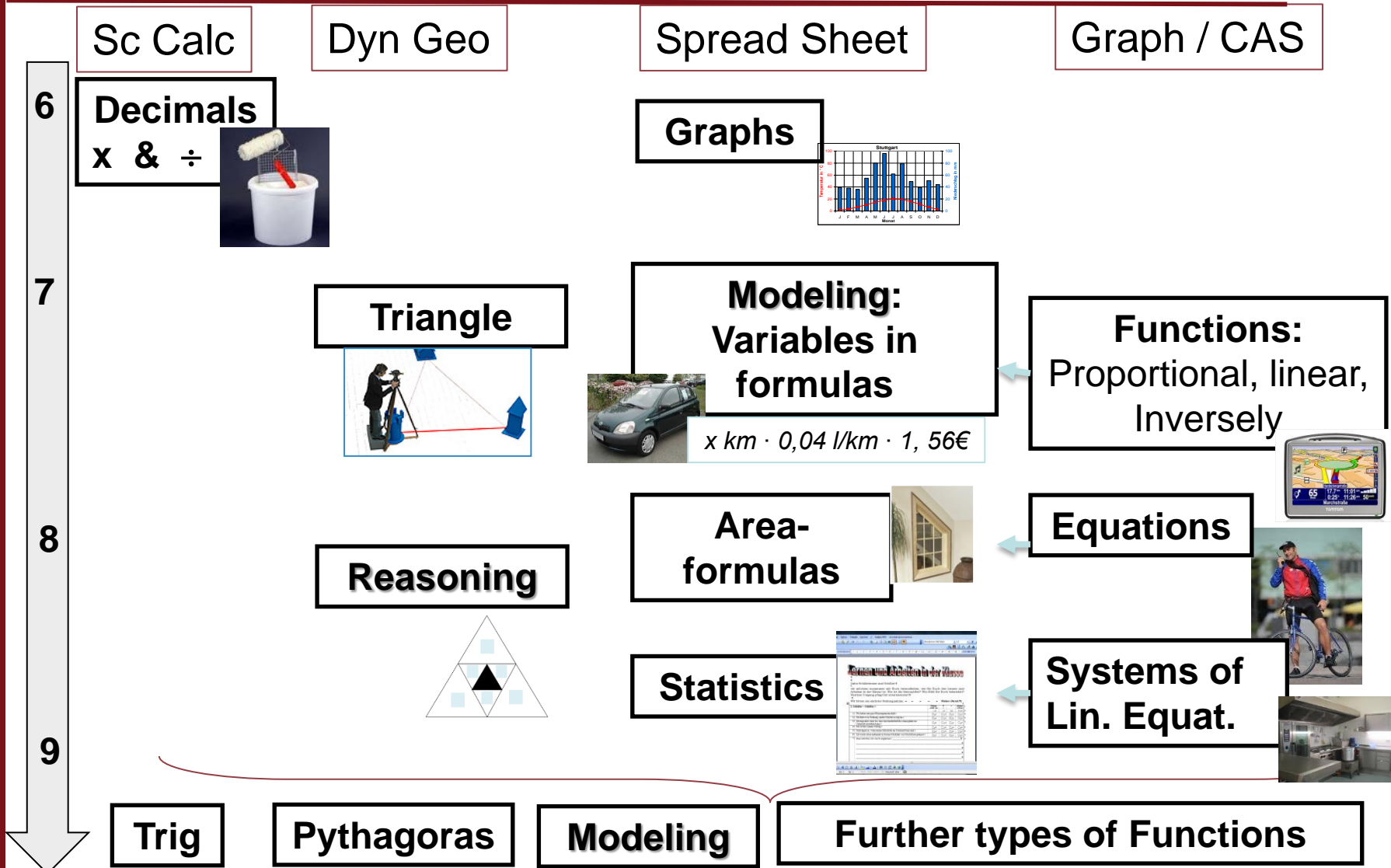**where all tools can support finding a solution?**

All the three tools are used - by different pupils and the
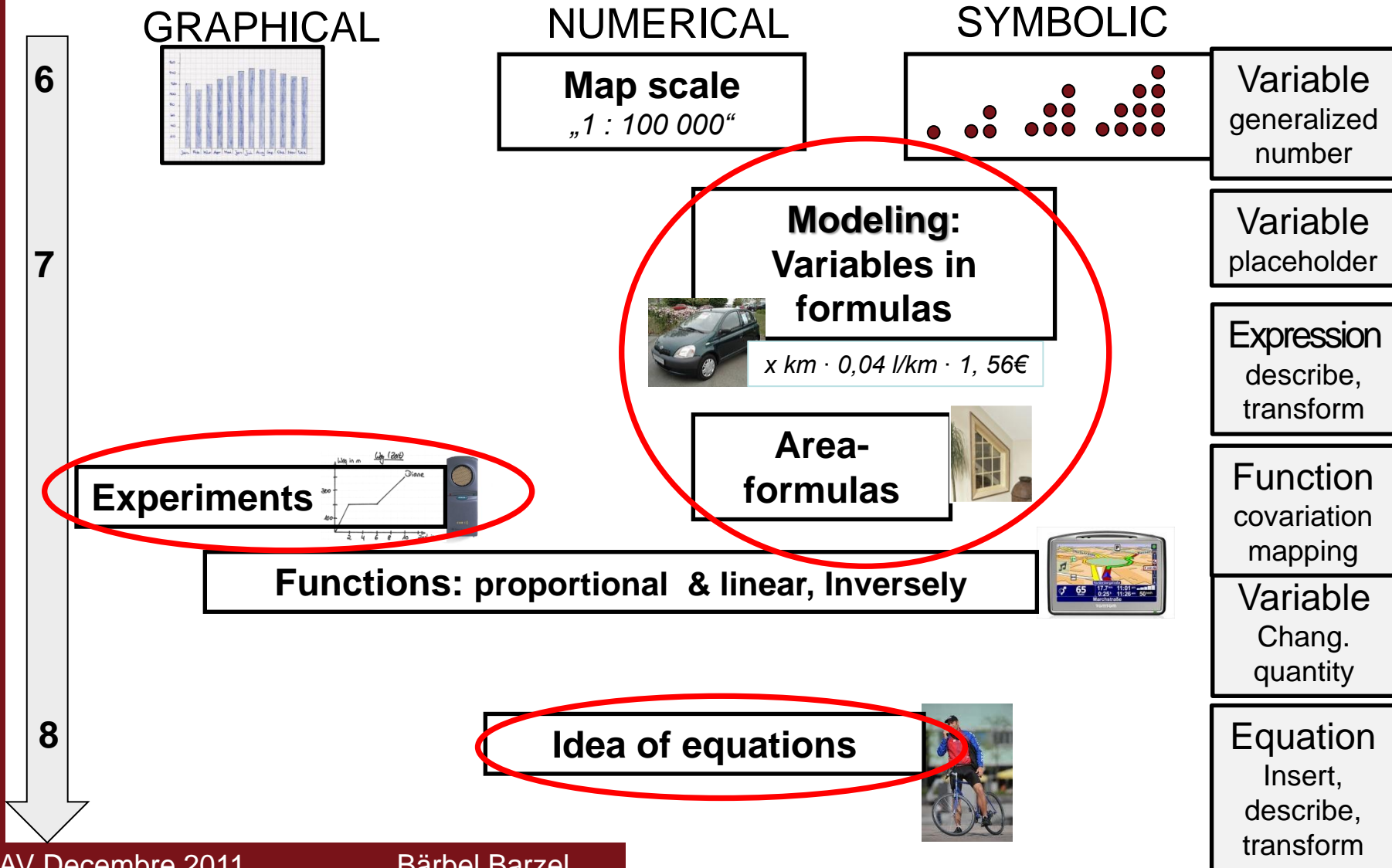**pupils could explain their choice**
according to their solution methods

But : They must be aware of the three tools!!

Topic   Technology   Tasks & Teaching

The relevance of general purpose tools

| Sc Calc | Dyn Geo | Spread Sheet | Graph / CAS |

6 **Decimals** **x & ÷**

Graphs

7 **Triangle** **Modeling: Variables in formulas** **Functions:** Proportional, linear, Inversely

x km · 0,04 l/km · 1, 56€

8 **Reasoning** **Area-formulas** **Equations**

Statistics **Systems of Lin. Equat.**

9

Trig **Pythagoras** **Modeling** **Further types of Functions**

Algebra and functional thinking: How can it be realized?

GRAPHICAL   NUMERICAL   SYMBOLIC

6

Map scale
„1 : 100 000"

Variable
generalized
number

7

**Modeling:
Variables in
formulas**

$x \; km \cdot 0,04 \; l/km \cdot 1, \; 56€$

Variable
placeholder

Expression
describe,
transform

**Area-
formulas**

Experiments

Function
covariation
mapping

Functions: proportional & linear, Inversely

Variable
Chang.
quantity

8   **Idea of equations**

Equation
Insert,
describe,
transform

Introducing "functional thinking" with experiments

Sandra Ganter

Which graph is created by my movement?

Experiments to switch situation & graph

Research question:

How does real experiment supports the conceptualization of a „function"?

Introducing "functional thinking" with experiments

Which graph is created by my movement?

1. **Step:**
Spontaneous movements and interpreting the graph

2. **Step:**
Match a given graph
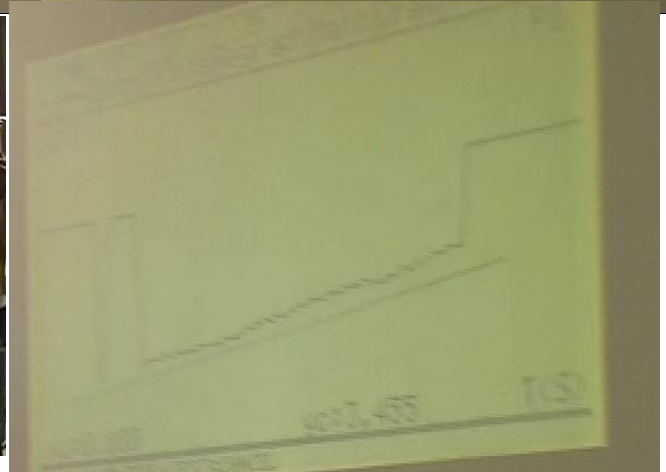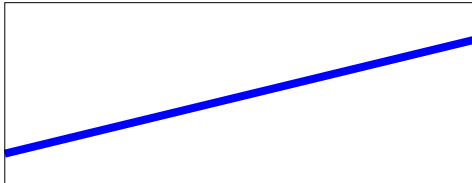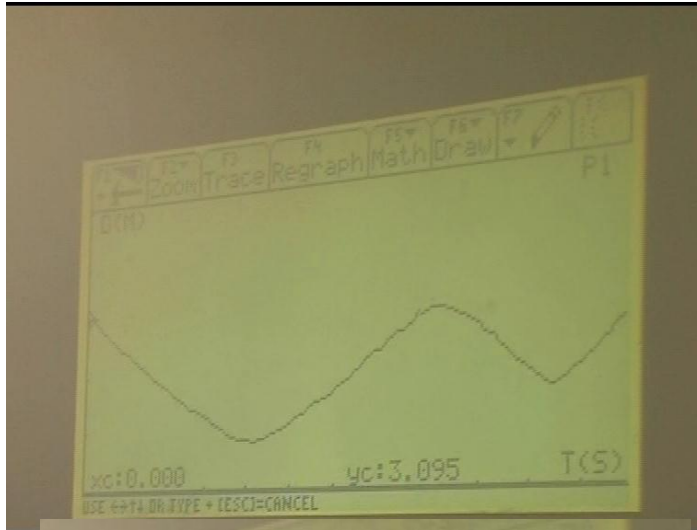
Introducing "functional thinking" with experiments

Which graph is created by my movement?

At first:

a very strong picture in mind as a pattern

graph-as-a-picture misconception

They overcome faults and argue in different ways, e.g.:

Statically: in the sense of mapping points

Dynamically: in the sense of covariation
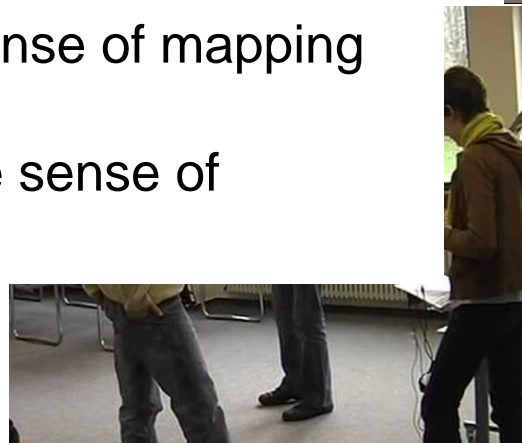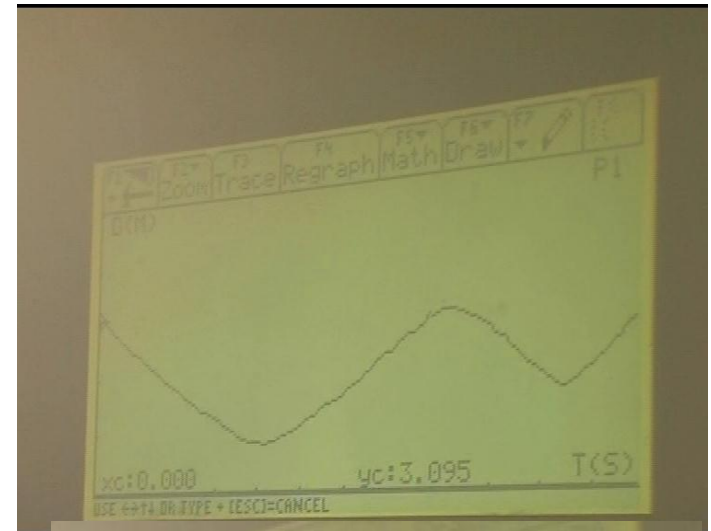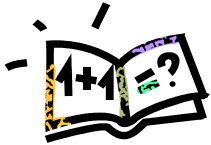
Introducing "functional thinking" with experiments

Three groups:

Pre- & Posttest-Design

Experimental-group **(n=77)**	Independent student experiments (in groups)
Control group **(n=86)**	Experiments shown on film
Zero group **(n = 41)**	Traditional teaching with textbook

Introducing "functional thinking" with experiments

Learning gains in the three groups

Topic   Technology   **Tasks & Teaching**
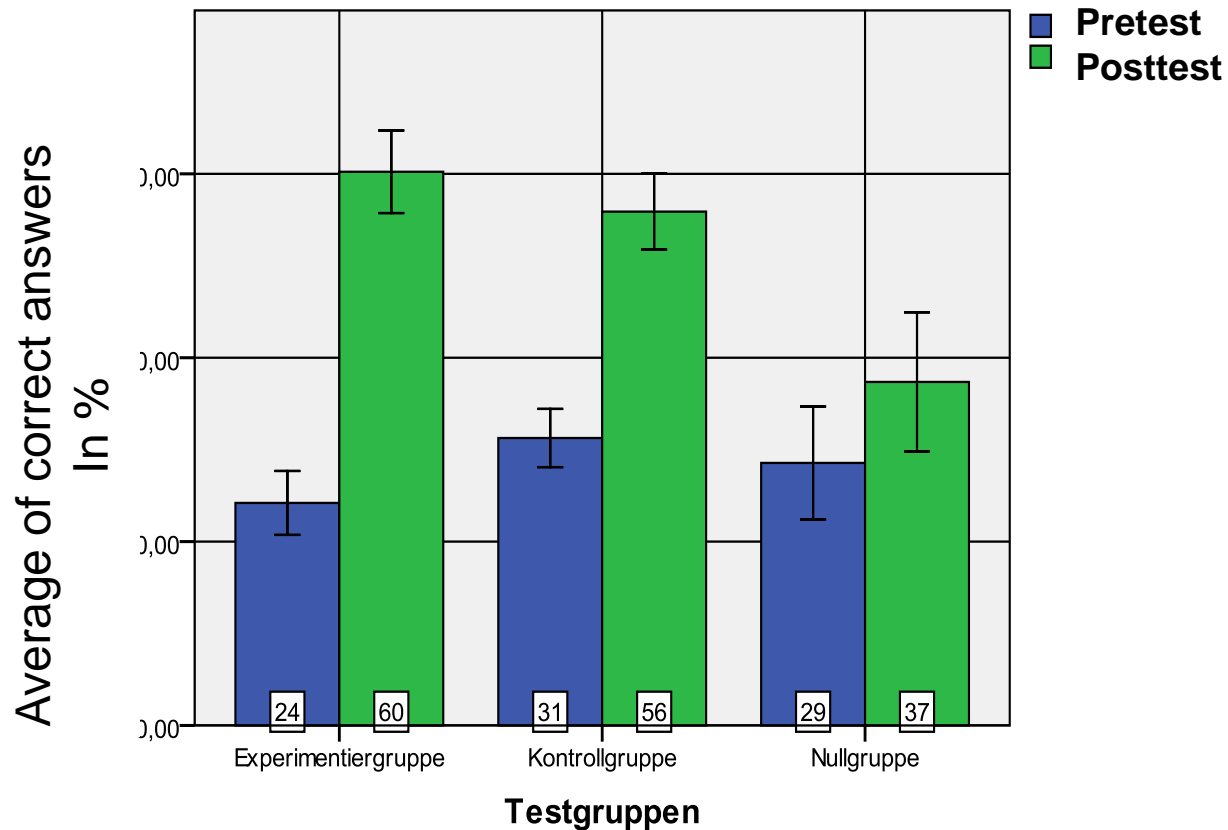
Introducing "variables" ……

| **Modeling:**
Variables in formulas |

What does it cost to go by car, by bike and by train? Compare.

DB Die Bahn
BahnCard 100: 3800€
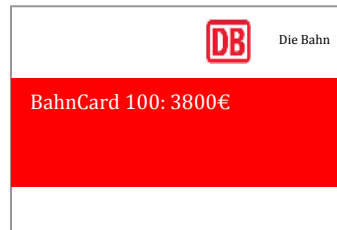
$$w + 4 \cdot r + k \cdot 0{,}04 \cdot b$$

fx = B14+C14*D14*0,04+4*F14

X .20 +500 900
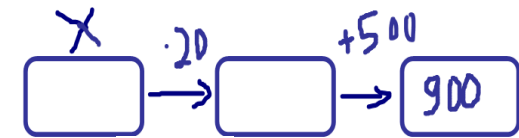
	Tabellenkalkulationsblatt zur Aufgabe Erkunden 4: Was kostet das Auto?					
	Wertverlust in 4 Jahren (in Euro)	Benzinpreis (in Euro pro Liter)	Kilometerzahl in 4 Jahren (im Kilometer)	Benzinkosten in 4 Jahren (in Euro)	Jährliche Reparatur / Steuer / Versicherung (in Euro)	Gesamtkosten in 4 Jahren (in Euro)
14	3000	2,05	40000	3280	900	9880

Introducing "variables" ……

Area-formulas
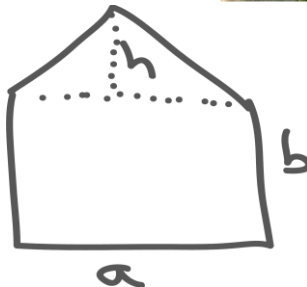
Invoice of a glazier

	A	B	C	D	E	F	G
	E15		⊗ ✓ ⊝ *fx*	5,6			
1							
2					Müller Fensterbau		
3					GmbH & Co KG		
4	**Abrechnung für rechteckige Fenster**						
5	**Fenstermaße**	Höhe			0,8	m	
6		Grundseite			2	m	
7							
8						Preis	
9	**Fensterfläche**	Preis pro qm			3	€ / m²	
10		berechnete Fläche			1,6	m²	
11		Materialpreis	gesamt				4,80 €
12		Arbeitszeit Zuschnittt pauschal			20	€	20,00 €
13							
14	**Rahmen**	Preis pro laufendem Meter			10	€ / m	
15		berechneter Umfang			5,6	m	
16		Materialpreis Rahmen					56,00 €
17		Fenstergriff und -aufhängung			50	€ / Stück	50,00 €

Expressions to measure the area of this window:

Till	$a \cdot b + \frac{1}{2} \cdot a \cdot h$
Merve:	$a \cdot (b + h)$
Paul:	$b \cdot a + h$
Sverre:	$a \cdot (b + h) - b \cdot h$

Introducing "equations" ……

| Idea of equations |

What is equality?....

…when looking at a table

…when looking at graphs

$f2(x)=0.19 \cdot x+10$
$f1(x)=0.15 \cdot x+12$
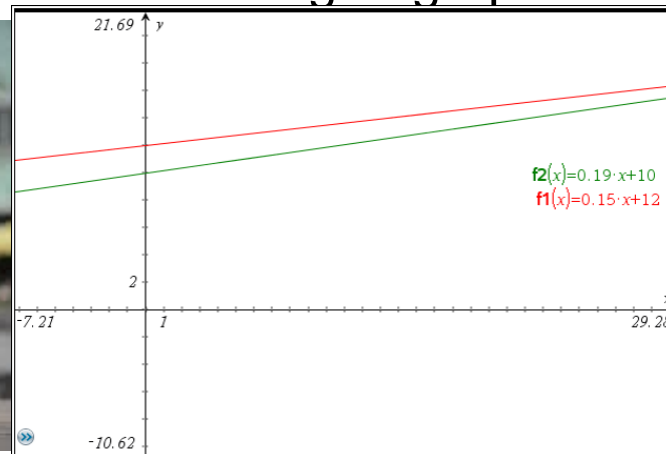
…and sometimes you need an exact value:

$$3 \cdot x+0.25=2 \cdot x+3$$

$$[3 \cdot x+0.25=2 \cdot x+3]-0.25$$

$$[3 \cdot x=2 \cdot x+2.75]-2 \cdot x$$

$$\text{solve}(3 \cdot x+0.25=2 \cdot x+3,x)$$

$$3 \cdot x+0.25=2 \cdot x+3$$

$$[3 \cdot x=2 \cdot x+2.75]$$

$$[x=2.75]$$

$$x=2.75$$

Using variables and functions

CAS when learning algebra
- Yes or No?

$$c(a)=ye+n$$

Mattias Zeller

| TI-nspire CAS | versus | TI-nspire |

Topic   Technology   **Tasks & Teaching**

Using variables and functions

$$c(a) = ye + n$$

TI-nspire CAS	versus	TI-nspire

Using variables und functions

$$c(a)=ye+n$$

TI-nspire CAS	versus	TI-nspire

TI-nspire CAS

1.1	BOG AUTO REELL
$1 \cdot a + 2 \cdot a + 3 \cdot b + 4 \cdot b$	$3 \cdot a + 7 \cdot b$
$\text{factor}(12 \cdot x + 36)$	$12 \cdot (x+3)$
$\text{expand}(12 \cdot (y+3))$	$12 \cdot y + 36$
$12 \cdot z + 36 \mid z = 2$	60
$(12 \cdot t + 36 = 96) - 36$	$12 \cdot t = 60$
$\dfrac{12 \cdot t = 60}{12}$	$t = 5$

1/6

TI-nspire

1.1	BOG AUTO REELL
$a := 7$	7
$3 \cdot a$	21
$\text{nSolve}(12 \cdot y + 36 = 96, y)$	$5.$

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Using variables und functions

Applied problems

$$c(a) = ye + n$$

Party organizers

After graduation, you want to organize a party. Several offers of local providers are available:

	YUKI EVENT	PARTYMAD	FLASH
meal / person	24 €	15 €	20 €
rent for the room	400 €	2300 €	900 €
stereo	350 €	400 €	included
decoration	included	200 €	300 €

Compare the offers.

These points might help you:
- How many guests do you expect? What will the costs be at which offer?
- When do two offers have an equal price?
- Present an overview of your results.

Additional task:
Another provider has a big party room. His offer is the cheapest, so long as more than 500 guests come. How much money can he charge, while still being the cheapest?

Using variables und functions

Applied problems

$$c(a) = ye + n$$

10 → 3

6 → ?

x → 7

7098423 → ?

$$y = m x + b$$

… if I add 1 to the first number, than I have to … to the other number …

Using variables und functions

Observations

$$c(a) = ye + n$$

1. The step from arithmetic to algebra:
- Non-CAS-pupils perceive a difference in the underlying rules of arithmetic and algebra.
- CAS-pupils accept results with variables as an answer
  – Non-CAS-Pupils quite often do not.

2. Using different representations
- Non-CAS-pupils avoid algebraic work,
  because it can not be done by the machine.
- CAS-pupils are more motivated and keen to use variables,
  Non-CAS-pupils find the use of variables to be difficult.

Using variables und functions

Observations

Topic   Technology   Tasks & Teaching

Using variables und functions

Analysing pupils' solutions

Which graph belongs to the
function f with
f(x) = x (x+2) ^{2} ?
Give reasons!

CAS-group :   26
Comparison group
without technology :   80

Using variables und functions

Analysing pupils' solutions

From the CAS-group:

$$f(x) = x\,(x+2)^2$$

Using variables und functions

Analysing pupils' solutions

From the CAS-group:

$f(x) = x\,(x+2)^2$

Using variables und functions

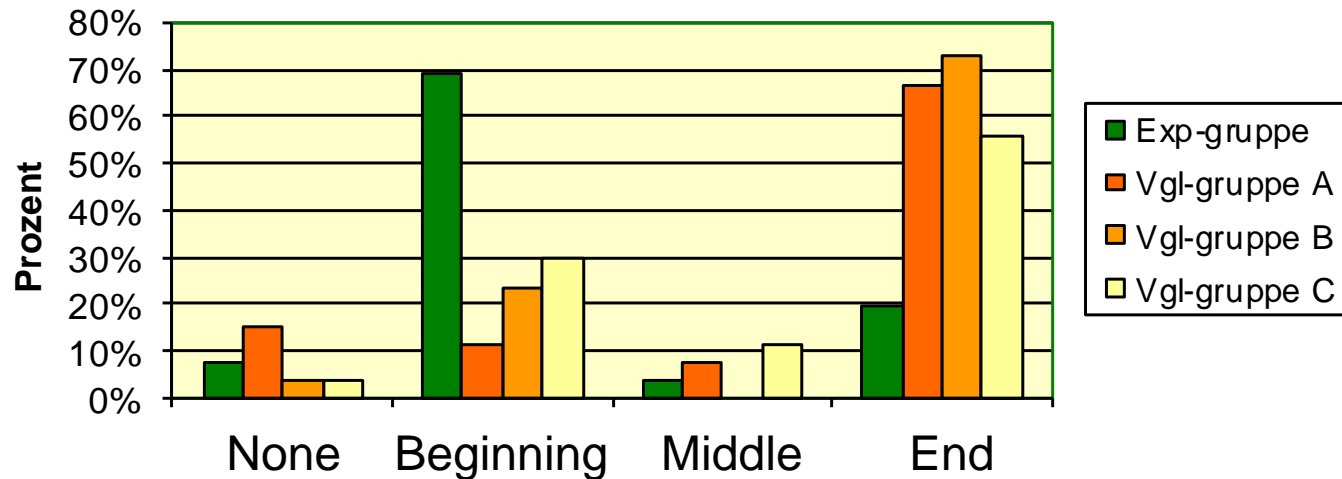Analysing pupils' solutions

From the comparison group:

$$f(x) = x \, (x+2)^2$$

Using variables und functions

$$f(x) = x \, (x+2)^2$$

Where is the statement of the answer?

Using variables und functions

$$f(x) = x\,(x+2)^2$$

Possible explanations for students writing
the statement of the answer first
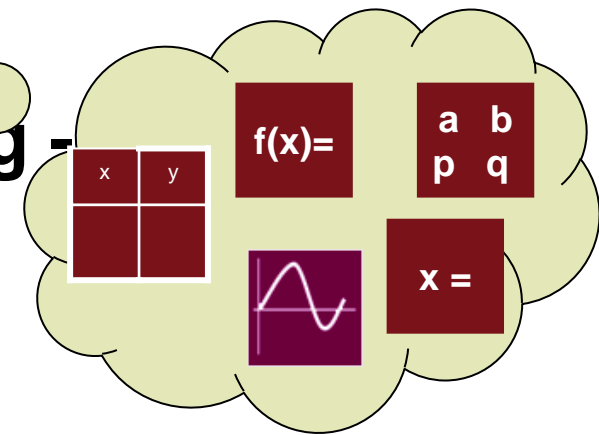
Using variables und functions

$$f(x) = x\,(x+2)^2$$

Possible explanations for students writing
the statement of the answer first

Did they follow the "recipe" ?

in %

- ■ Exp-gruppe
- ■ Vgl-gruppe A
- ■ Vgl-gruppe B
- □ Vgl-gruppe C

Yes   Not clear   No
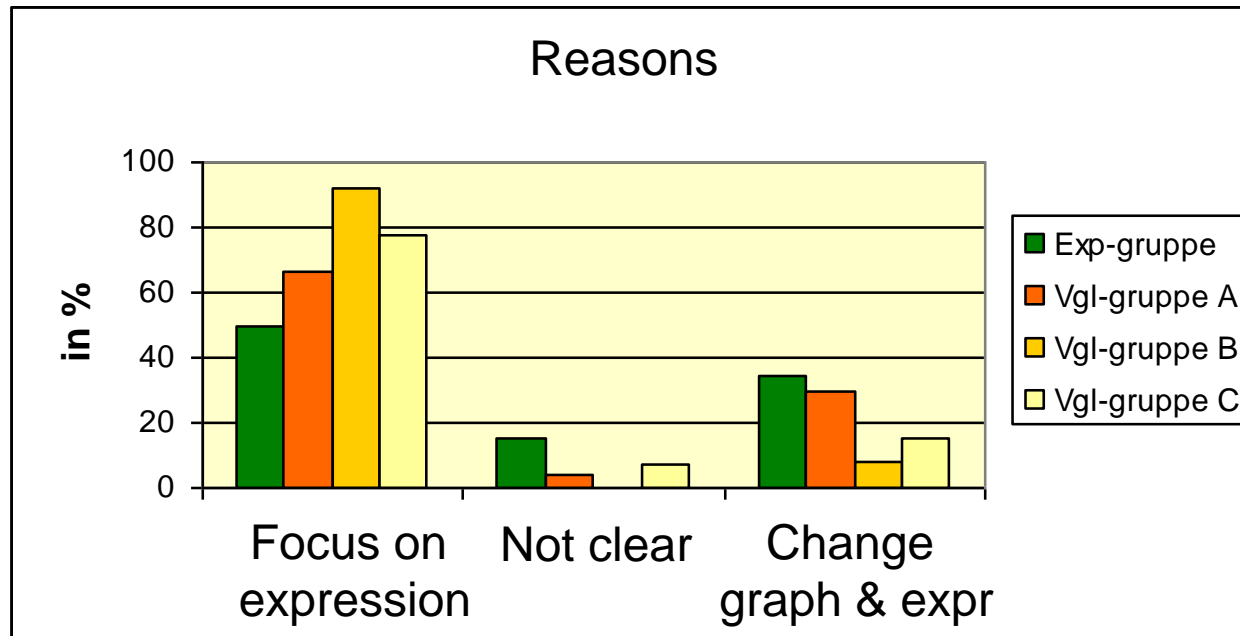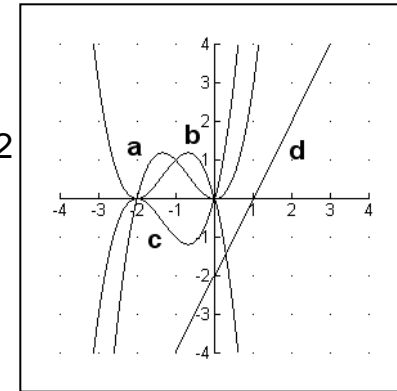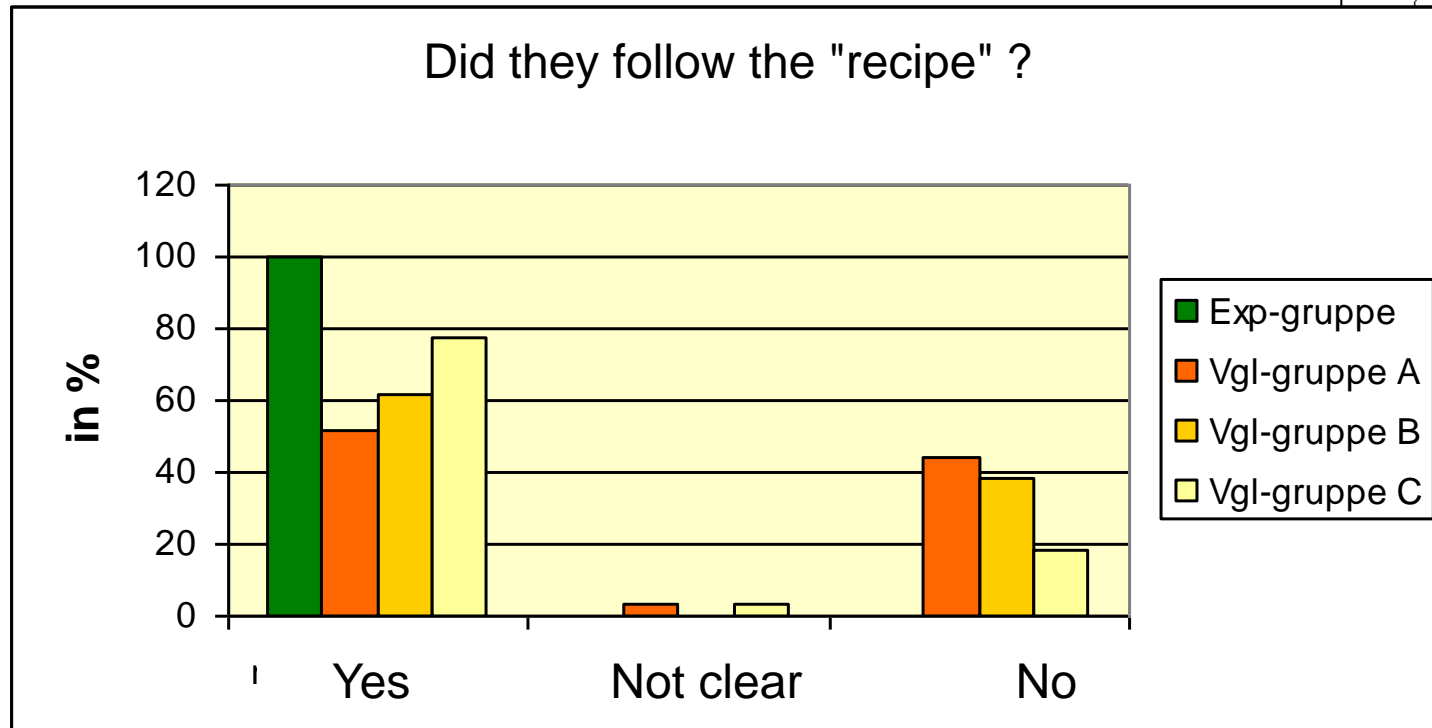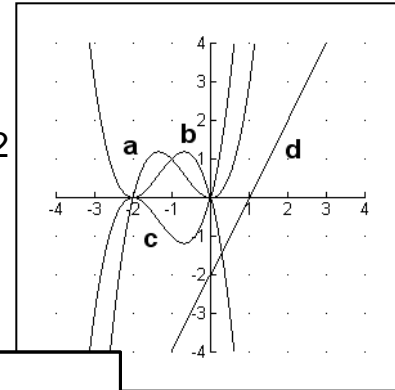
Learning Algebra and Developing Functional Thinking - With or Without CAS?

Focusing

1. … the topic

2. … the technology

3. … the tasks & teaching

Promote deep understanding.

Use CAS, but not just the symbolic features

Allow variety of approaches