ALGEBRAIC THINKING: GENERALISING NUMBER AND GEOMETRY TO EXPRESS PATTERNS AND PROPERTIES SUCCINCTLY

George Booker

Griffith University, Brisbane

While algebra has largely been viewed as a formal system met in high school, recent curriculum directions have focussed on the thinking that underpins these ways of operating, recognising that this needs to develop from the earliest days of school. Thus, Algebraic Thinking addresses general mathematical relationships, expressing them in increasingly sophisticated ways as activities move from seeing patterns in number, geometry and measurement to determining solutions to more and more complex problems. Emerging ideas shown with materials, models, tables and patterns of objects lead to verbal descriptions that gradually move from a discussion of what is seen to an ability to describe this in more mathematical terms, using additive, then multiplicative, reasoning.

The significance of algebraic thinking

There is increasing recognition of the importance of algebra as a tool for use in further mathematics, the sciences, business, economics, commerce, computing and the way in which so many relationships are expressed in political and everyday life. These concerns are behind curriculum changes to ensure that everyone has the opportunity to develop algebraic reasoning. For example, algebraic thinking and algebra are proposed as key components of the forthcoming National Mathematics Curriculum (Australian Curriculum, Assessment and Reporting Authority ACARA, 2009). Not only is algebra needed to fully participate in the modern world, it also provides 'an academic passport for passage into virtually every avenue of the job market and every street of schooling' (Schoenfeld, 1995). Moreover, as Barton (in Katz, 2007) reminds us, 'algebra is the key to any success in mathematics at all and abstract algebra is critical to work in advanced mathematics' and the life and work opportunities that come with higher studies across a diverse range of occupations.

Barton and Katz (2007) further propose that initial algebraic thinking might best be developed through problem solving and geometry to enable more students to gain access to algebra. Yet for many students, the development of algebra in high school has often marked the end of enjoyment in mathematics and the onset of a feeling of mathematical inadequacy. Partly this is because a generalisation of number understanding is called for when meaningful conceptions of number and computation might not exist to the extent that they are needed. On the other hand, many students find the formalistic, abstract first approach difficult and unappealing. An entry into the subject akin to the problem-based beginnings of the discipline might prove as attractive to contemporary students as it was to the mathematicians, scientists and merchants of the times when it developed out of practical problems using models and situations that showed the underlying reasoning and patterns.

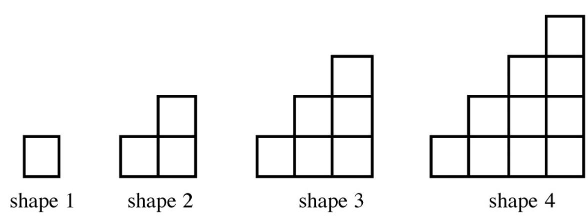
The nature of algebraic thinking

Identifying and using patterns in the solution of problems and in coming to terms with new concepts and process is the essence of mathematical thinking. The study of patterns and relationships fosters children's understanding of large and small numbers and underpins an ability to perform computations fluently, but experiences are also needed to identify, describe, continue and create patterns among numbers, shapes and collections of objects. This need to examine general ideas that underpin a wide body of mathematics has led to calls for the inclusion of algebraic reasoning from the beginning of learning mathematics (Davis, 1985; Mason 1996; Chick, Stacey & Vincent, 2003; Carraher, Schliemann, Brizuela & Earnest 2005; Cai & Moyer, 2008). Not as an introduction to the formal methods usually introduced in high school – algebra early – but as a means of dealing with generalisations and ways of thinking that allow results to be expressed across a range of problem forms rather than simply finding a particular answer to a series of individual problems.

The early algebra being advocated is best referred to as *algebraic thinking*, growing out of a range of number, geometry and measurement activities met in the primary years. Any formal notation should only be introduced when students are ready and there is a need for succinct expressions of the relationships that are revealed. In this way, a basis can be laid for the use of symbols that express generalities concisely and carry meaning independently of the activities with which they were established. This parallels the historical development of algebra, allowing mathematical relationships rather than mathematical objects to come to the fore and provide a tool for dealing with the complexities of today's world.

Generalisations and relationships that come to light from these activities can be shown with materials, patterns of objects, models, diagrams, and tables before verbal descriptions are made of them, gradually moving from a discussion of what is observed to an ability to describe this in more mathematical terms. Additive descriptions are likely to appear at first, but need to be extended to take up multiplicative reasoning before any move to representations with abbreviations or symbols as a shortened way to portray patterns or communicate properties concisely.

These shapes have been made using toothpicks



What is the perimeter of each shape?
What would be the perimeter of shape 5
What about shape 8, shape 10 and shape 12?
Describe what is happening as the shapes grow.

Figure 1: Growing shapes

A first level of description would be that each new shape has an additional column of toothpicks with as many toothpicks as the column it builds on and 3 more toothpicks – the perimeter increases by 4 toothpicks as the shape grows. This insight provides an additive way of thinking; the first shape has a perimeter of 4 toothpicks, the second shape has a perimeter of 8 toothpicks, the third shape has perimeter of 12 toothpicks, and so on. The pattern is to add 4 toothpicks to the shape before it: 4, 8, 12, 16, 20, … .

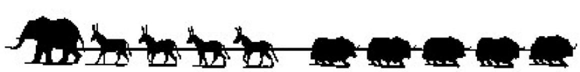
Reflecting on this pattern shows that the fifth shape has 20 or 5 x 4 toothpicks for its perimeter, the perimeter of the tenth shape is 10 x 4 or 40 toothpicks, the perimeter of the twelfth shape is 12 x 4 or 48 toothpicks. This multiplicative way of describing the pattern gives the result for each shape much more directly and can lead to a general statement – the perimeter of a shape is given by multiplying the shape number by 4. Later, when the general shape can be named as the nth shape, the pattern can be expressed concisely as 4 x n (or 4n) where n is the shape number.

Problems based on situations where different relationships among the objects in the problem are used also assist in building this thinking:

Tug of War

Five oxen are as strong as eight donkeys

An elephant is as strong as one ox and four donkeys



Who will win this tug of war?
Use materials or a drawing to explain how you made your choice

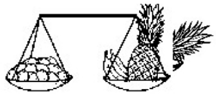
Figure 2 Tug of War (Adapted from the Dutch MiC algebra program 1998)

Balancing Fruit

13 kiwifruit weigh as much as two bananas and one small pineapple

Four kiwifruit and one banana have the same weight as one small pineapple

How many kiwifruit are needed to balance a small pineapple?



Show how you worked this out

Figure 3 Using a balance to invoke algebraic thinking (Booker, Bond, Sparrow & Swan, 2009)

These problems can be readily modelled using different coloured counters to represent and organise the different stages in the solutions. The reasoning is algebraic in character as the changes are made while acting on the unknowns rather than having numerical answers to work with.

Another form of problem solving that assists the development of algebraic thinking, is to use a model to solve problems involving fraction concepts (Lencher, 2005; Ferruci, Kaur, Carter & Yeap, 2008).

Buying Bread

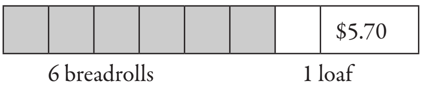
Larry and Jerry went to the Organic Bakers to buy some bread. Larry bought 6 whole meal bread rolls and one multi-grain loaf of bread for $5.70. Jerry bought 4 whole meal bread rolls and two multi-grain loaves of bread for $8.60.

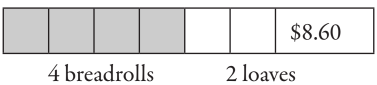
What was the price of a multi-grain loaf of bread?

What would be the cost of a dozen whole meal bread rolls?

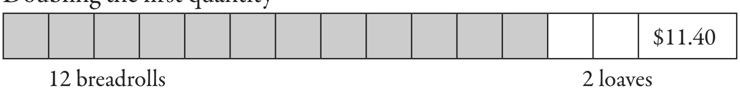
Figure 4 A problem with several overlapping relationships (Booker et al, 2009)

While a solution could be found using a process of 'try and adjust', using rectangles to model the data and reveal connections among them introduces a form of algebraic thinking which underpins the use of equations at a much later stage:





Doubling the first quantity



Comparing the second quantity and the new third quantity shows that 8 bread rolls cost ($11.40 - $8.60) or $2.80. 12 bread rolls will cost ($2.80 + $1.40) or $4.20. So 6 breadrolls cost $2.10 and 1 loaf would cost ($5.70 - $2.10) or $3.60.

Algebraic thinking, through its focus on generalising from patterns, ways of representing relationships and a corresponding analysis of changes that are discerned, is building ways of thinking that forge connections among the various topics and prepare for the more formal content of algebra that can be developed when appropriate understandings are available:

"see"
patterns – materials, models,
tables, graphs, diagrams ...

algebraic thinking

language
describe patterns and
reasons for them

symbols
concise representation
of what is "known"

Figure 5 A model for developing algebraic thinking

Historical origins of algebra

Algebra is the science that teaches how to determine unknown quantities by means of those that are known.

Euler 1767

In examining the origins of algebra, historians of mathematics have looked beyond the particular problems of early mathematics and the results provided, seemingly without reasons, to see the thinking about the general that can be characterised as algebraic thinking

A problem from the Rhind Papyrus (C.1650 BC)
A 'heap' whose seventh part is added to it becomes 19
In order to solve this problem, the answer was first to assume that the answer is 7, suggesting the heap would become 8, which is too small. The adjustment needed to change 8 to 19 was then applied to the initial attempt of 7 to find the required number.

This method then became known as the 'rule of false position', used universally to solve problems of this form.

Figure 6 Reasoning with unknowns in Ancient Egypt (Chase, 1979)

Thus, algebra did not begin with the symbolic reasoning most associated with the name given to us by al-Kwarizmi in the title of his book *Hisab al-jabr w'al-muqabala* concerning the solution of equations, but has been separated into three distinct phases (Bashamakova & Smirnova 2000). At the beginning algebra was largely rhetorical involving the use of words and sentences, lasting from the earliest beginnings of mathematics until around 250 AD where both the problem and its solution were expressed solely in words. While the problems shown in Babylonian and Egyptian works were often characterised as arithmetic, when the numbers did not represent specific objects and operations were required on unknown quantities, this has come to be considered algebraic in nature, showing general patterns not specific results.

Arabic mathematicians focussed on finding methods for solving more complex problems than could be easily dealt with using the arithmetical methods of the day. The power of their thinking lay in showing that a particular method of solution, and hence the solution of certain classes of problems, was needed only once when words or symbols related to general numbers and operations, rather than specific instances. The general quantities used in these early forms of algebraic reasoning were indicated in a variety of ways – 'heap' in Egyptian writing, 'length' or 'area' by Babylonian and Greek writers, 'thing' or 'root' in Arabic – but all revealed the thinking about generalisations that characterises algebra.

In time, this evolved to a syncopated form where the words and actions were expressed in abbreviated forms that stood for the words and sentences used previously, beginning around 250 AD in the work of Diophantus. From this point on, algebraic thinking focussed on equation solving with an emphasis on finding numbers that satisfied specified relationships. The algorithms of Diophantus, the Hindu number-based solutions for quadratics and the *al jabr* (completion) and *muqabala* (Balancing) of Al-Khwarizmi, that arose in this pursuit of general answers were translated to Europe as methods and viewed as a 'universal arithmetic'.

Weighing fish (Calandri 1491)

The head of a fish weighs $\frac{1}{3}$ of the whole fish,

the tail weighs $\frac{1}{4}$, and the body weighs 300 g.

How much does the whole fish weigh?

Figure 7 A problem posed by Calandri in 1491 to promote algebraic reasoning

There are many ways in which this problem can be solved but one that uses a model and understanding of fractions is algebraic in nature:

	head	body	tail
Information about the fish	$\frac{1}{3}$	300g	$\frac{1}{4}$
Express as like fractions	$\frac{4}{12}$		$\frac{3}{12}$

Since the head and tail make up 7 twelfths of the fish, the body must be 5 twelfths of the fish – 1 twelfth must be 60g and the fish weighs 720g.

Calandri's solution to this problem used the 'rule of false position', assuming an answer of 120g and then adjusting to find the correct result. Since this implied that the head weighed 40g, the tail 30g then the body would weigh only 50g. This result needed to be multiplied by 6 to get 300g so the fish weighed 6 x 120g or 720g.

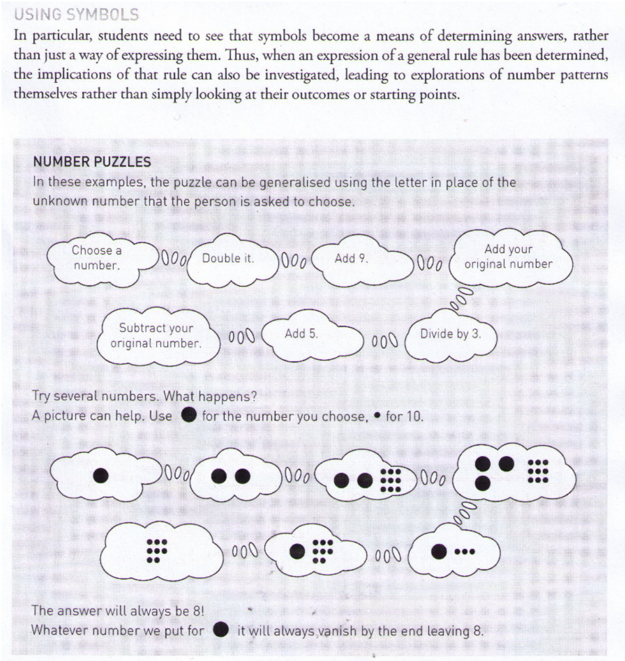
Only when these new and general ways of solving problems were internalised could the modern conceptions of algebra in terms of symbols, functions and structures arise. While it has been tempting to move as soon as possible to this formalistic approach as the basis for school algebra, this move has not given much appreciation or meaning of algebra to many learners. For instance, as long ago as 1969, Skemp noted that such an approach provided only the final product of the mathematical discovery and did not generate in the learner the processes by which mathematical discoveries are made, as such teaching 'the mathematical thought not the mathematical thinking'. Similar comments were made by Arcavi (1995) when he noted the many difficulties that arose when students are faced with the results and the solutions arrived at historically 'without having been given time to struggle with the motivation and the issues behind the problems'. Consequently, both Katz (2007) and Sfard (1995) have pointed to the need for a more careful building through the stages and conceptions revealed in the historical development if students are to acquire ownership and power over the symbolic algebra they will meet and need in later study and work.

Nonetheless, it is worth keeping in mind the comment by Hobbes (1588-1679), a philosopher and mathematician, on the difficulties symbols create – surely a view echoed by many school algebra learners today!

> *Symbols, though they shorten the writing do not make the reader understand it sooner than if it were written in words … there is a double labour of the mind, one to reduce the symbols to words, another to attend to the ideas they signify*

Building from algebraic thinking to algebra

Notions of algebraic thinking have been discussed for many years as a forerunner of formal algebra as well as a guide to building fundamental algebraic processes. The essence of algebraic thinking appears with the move from particular numbers and measures towards relations among numbers and measures (Carraher et al, 2005). In this way, mathematical relationships rather than mathematical objects become the objects of study with generalising, inverting and reversing operations, treating computational processes in general ways and reasoning about patterns the means to build this way of thinking (Lee, 2001; Sfard, 1995). Algebraic thinking, then, focuses on general mathematical relationships, expressing them in increasingly sophisticated ways – seeing patterns, describing them with words or diagrams, before leading to the use of symbols that can express generalities concisely and carry meaning independently of the activity with which they were established.



USING SYMBOLS

In particular, students need to see that symbols become a means of determining answers, rather than just a way of expressing them. Thus, when an expression of a general rule has been determined, the implications of that rule can also be investigated, leading to explorations of number patterns themselves rather than simply looking at their outcomes or starting points.

NUMBER PUZZLES

In these examples, the puzzle can be generalised using the letter in place of the unknown number that the person is asked to choose.

Try several numbers. What happens?
A picture can help. Use ● for the number you choose, • for 10.

The answer will always be 8!
Whatever number we put for ● it will always vanish by the end leaving 8.

Figure 8 Using a number puzzle to foster algebraic thinking (Booker et al, 2009)

Squaring and Multiplying

Enter a two-digit number into your calculator, eg 63, square it and put the result in memory.

- multiply the numbers that are 1 more and 1 less than your number.
- compare it with the square of your original number.
- try some other two-digit numbers

What do you notice?

Try some other two-digit numbers but multiply the numbers that are 2 more and 2 less

- what pattern can you see now?

Try some other two-digit numbers but multiply the numbers that are 3 more and 3 less

- what is the pattern now?

What do you think will happen if you try 4 more and 4 less?

What about 5 more and 5 less?

Figure 9 Number patterns that promote algebraic thinking

Conclusion

Working on, representing and solving structurally related problems in a variety of ways prepares students to think algebraically as they articulate and generalise their solutions. Initial verbal descriptions can give way to more mathematically based explanations, preparing for the more concise, symbolic arguments that will eventually develop into algebra as it is used in further mathematics. In particular, students can be helped to construct algebraic notation in a meaningful way through their representations using materials, diagrams, models, tables and graphs in their search for patterns and generalisations. An understanding of why and how the concepts of patterning and algebra have emerged in mathematics can then provide a richer background to algebraic thinking to teacher and students alike.

Algebraic thinking should not be seen as a new topic or strand added to those already in the curriculum but as a means of 'tightly interweaving existing topics of early mathematics' to provide opportunities for later learning (Carraher, Schlieman & Schwartz, 2008) and provide an entry into the very symbolic contemporary and future world.

References

Australian Curriculum, Assessment and Reporting Authority (ACARA), (2009) *Shape of the Australian Curriculum: Mathematics* Canberra: Commonwealth of Australia

Arcavi, A., (1995) 'Teaching and learning algebra: Past, present and future' *Journal of Mathematical Behavior* 14, 145-162

Bashamakova, I. and Smirnova, G., (2000) *The Beginnings & Evolution of Algebra* Washington: Mathematics Association of America

Booker. G., Bond, D., Sparrow, L. and Swan, P., (2009) *Teaching Primary Mathematics* [Fourth Edition] Sydney: Pearson Education Australia

Cai, J., and Moyer, J., (2008) 'Developing algebraic thinking in earlier grades: some insights from international comparative studies' in National Council of Teachers of Mathematics (2008) *Algebra and algebraic thinking in school mathematics* (70th yearbook) Reston Virginia: NCTM, 169-193

Carraher, D., Schliemann, A., and Schwartz, J., (2007) 'Early algebra is not the same as algebra early' in Kaput, J., Carraher, D. & Blanton, M. (Eds) *Algebra in the Early Grades* Mahwah, New Jersey: Erlbaum

Carraher, D., Schliemann, A., Brizuela, B., and Earnest, D., (2005) *Arithmetic and algebra in early mathematics education*

Chase, A., The Rhind Papyrus, NCTM 1979

Chick, H., Stacey, K., & Vincent, J. 2003 (Eds), Proceedings of the twelfth ICMI Study Conference: *The future of the teaching and learning of algebra*, University of Melbourne Press, Melbourne

Davis, R., (1985) 'Algebraic thinking in the Early Grades' in *Journal of Children's Mathematical Behaviour*, 4, 198-208

Ferruci, B., Kaur, B., Carter, J. and Yeap, B., (2008) 'Using a model approach to enhance algebraic thinking in the elementary school mathematics classroom' in National Council of Teachers of Mathematics (2008) *Algebra and algebraic thinking in school mathematics* (70th yearbook) Reston Virginia: NCTM, 195-209

Katz, V., (2007) 'Stages in the history of algebra with implications for teaching' Educational Studies in Mathematics 66, 185-201

Lee, L., (2001) 'Early algebra – but which algebra?' in Chick, H., Stacey, K., & Vincent, J. 2003 (Eds), Proceedings of the twelfth ICMI Study Conference: The future of the teaching and learning of algebra, University of Melbourne Press, Melbourne, Australia, 392-399.

Schoenfeld A (1995) Report of working group 1 in Lacampagne C, Blair W,& Kaput J (Eds) The algebra initiative colloquium (Vol2 11 – 18) Washington DC.

Sfard A (1995) 'The development of algebra: Confronting Historical and Psychological Perspectives' Journal of Mathematical Behavior 14, 15-39

Skemp R (1969) The psychology of learning mathematics Harmondsworth: Penguin