

REAL NUMBERS

David Leigh-Lancaster

Victorian Curriculum and Assessment Authority

The real numbers and related operations form an interesting mathematical structure, however most students and many teachers will have worked mainly with the rational real numbers, and the occasional irrational real number in measurement and function contexts. The real numbers are fundamental to senior secondary study of functions, algebra, calculus and probability, warranting a closer look at their representations and properties.

Describing real numbers

There are significant difficulties and limitations associated with the treatment of real numbers in the senior secondary mathematics curriculum, and these arise in substantial part from the corresponding difficulties with respect to a rigorous treatment of real numbers in mathematics itself. From the pure mathematical perspective, there are four main approaches to dealing with the real numbers: *decimal expansions* - finite, infinite recurring and infinite non-recurring; *equivalence classes* of infinite *Cauchy* sequences of rational numbers (see Estep, 2002, Chapter 11; Lang, 2005, Chapter IX); *ordered pairs of infinite subsets* of rational numbers, called *Dedekind cuts* (see Enderton, 1977, Chapter 5); *nested intervals* with rational endpoints (see Courant and Robbins, 1941, Chapter II pages 68-69).

While various aspects of each of these approaches are treated informally in the senior secondary mathematics curriculum a more comprehensive treatment typically takes place some time early in an undergraduate mathematics study of analysis or algebra and subsequently in the study of aspects of topology, mathematical logic and set theory. The *set* of real numbers, R , has some very important mathematical properties which are not shared with other sets of numbers such as the natural numbers, N , integers, Z , and rational numbers, Q - each of these is a *proper subset* of the set of real numbers:

$N \subset Z \subset Q \subset R$. In school mathematics, R plays the key role of a particular type of universal set in a given context, and necessarily includes elements that are not rational - the set of *irrational* numbers Q' . In senior secondary mathematics, the set of real numbers is required to develop content related to the study of functions and calculus, and the *axioms* and *properties* of the real number system are assumed for this work. Students are typically told that the set of real numbers, R , is the union of the rational numbers, Q and the irrational numbers, Q' , that is $R = Q \cup Q'$. This is often represented in texts, along with the relationship $N \subset Z \subset Q \subset R$ by means of a Venn diagram such as Figure 1, sometimes with illustrative elements included:

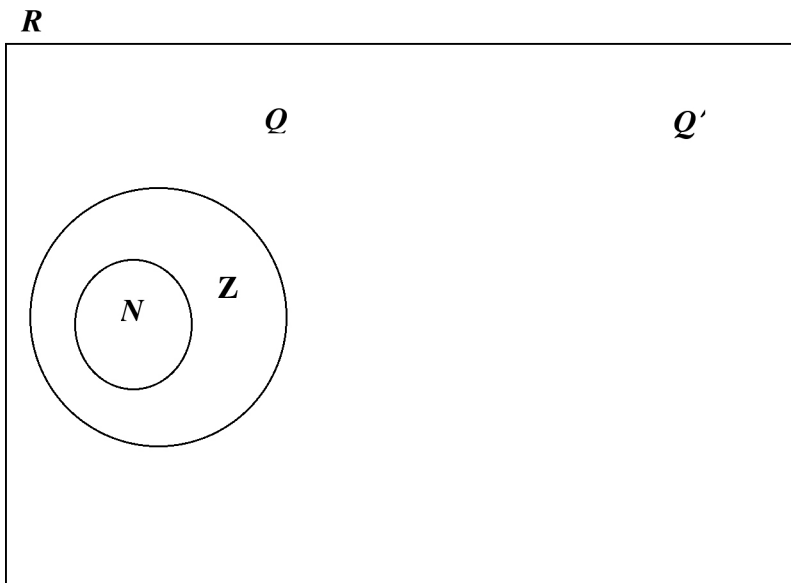


Figure 1: Venn diagram for the relationship between subsets of real numbers

A tree diagram is sometimes used alternatively, however such diagrams say more about the *subsets* of R than R itself, and have the limitation of not indicating the *size* of the sets involved: N , Z and Q are all *countable* or denumerable sets while Q' , and R are *uncountable* sets. That is, Q' and R are sets with larger infinite magnitude than N , Z and Q since there is no one-to-one correspondence between N , Z or Q and Q' or R . Students generally do not have a good appreciation of the greater size of the set of irrational real numbers compared with the set of rational real numbers, indeed some think that there are only a 'few' irrational numbers in existence - and typically quote $\sqrt{2}$ and π as principal examples. That the real

numbers are an *uncountable* set, can be shown using proof by contradiction based on what is called a diagonal argument (see Crossley, 2007 pp 54 - 65). A notion of the distinction between rational and algebraic irrational numbers (that is, those which can be expressed as roots of polynomials with rational coefficients, such as $\sqrt{2}$) and transcendental irrational numbers (that is, those which need to be expressed in terms of convergent infinite series, such as e and π) with respect to the entirety of the set of real numbers, can be accessed through the infinite decimal representation of real numbers. An alternative classification of real numbers using a tree diagram is shown in Figure 2:

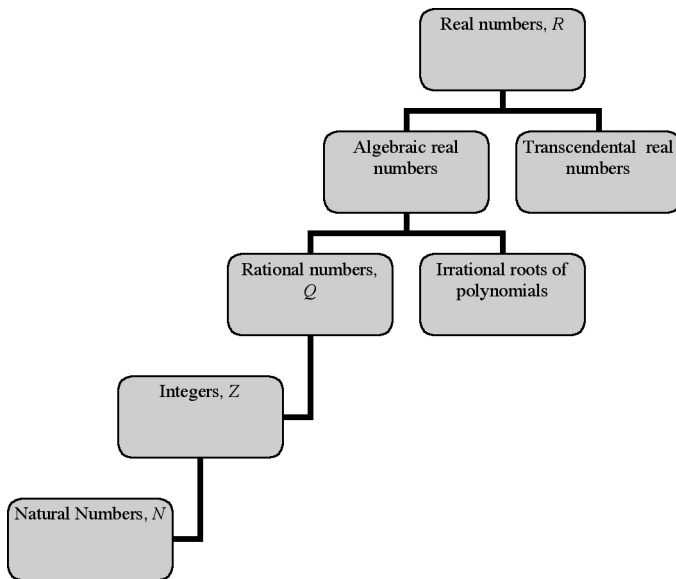


Figure 2: tree diagram for the relationship between subsets of real numbers

The approach to real numbers which underpins the conceptualisation of many Year 7 – 10 students is as a set of numbers within which one can measure continuous data, and carry out related computations in certain measurement contexts involving length, area and volume. For senior secondary mathematics students the approach is likely to be an *informal combination* of:

- decimal expansions and polynomial approximations to the limiting value of certain infinite series;
- the real number-line representation (and graphs of functions of a single real variable on the cartesian axis system $R \times R$ or R^2);

- exact value computation associated with some elements of \mathbb{R} such as certain surds, fractions and multiples of π , logarithms and exponentials that arise in function work; and
- a set of numbers which are substituted into various function and relation expressions (formulas, equations, inequalities) and manipulating accordingly.

Irrational numbers in measurement contexts

Students in Years 7 – 10 are typically introduced to some irrational numbers in the contexts of measurement of certain lengths, areas and volumes associated with triangles, squares, rectangles and circles and three dimensional objects formed from these shapes. These numbers are introduced to provide answers to practical questions such as: given *basic linear measurements* associated with certain two dimensional shapes, such as the length and width of a rectangle, or the diameter of a circle, how does one calculate the corresponding perimeter and area of the shape; or the length of a diagonal or volume of a related three dimensional object? The inverse problem is: given the volume of a cube or sphere, or area of a square or circle, how does one calculate its side length and diameter or radius respectively? In such contexts, number is being used to denote a *measure* associated with geometric objects. Such a measure is made with respect to some unit. It is a reasonable question to ask whether a common measure exists for *any* pair of geometric lengths. This includes special cases such as: Does the length of the diagonal of a square have a common measure with its side? Does the circumference of a circle have a common measure with its diameter?

An affirmative answer to these questions would tell us that one only needs rational numbers for such measurement problems, since, any such ratio produces, by definition, a rational number in positive fraction form. This is not the case – the ratio of the length of the diagonal of any square to its side and the length of the perimeter of any circle to its diameter are not rational – thus, students are typically introduced to certain non-rational, or *irrational* numbers, namely π ; and square roots of numbers that are not themselves perfect squares of a rational number, such as $\sqrt{2}$. In the early and middle years of secondary schooling, their irrational nature is usually described in terms of their decimal expansion - that unlike rational numbers, they have an infinite non-recurring decimal expansion, and this is typically provided as *de facto* knowledge.

The existence of *simple* arithmetic approaches to evaluating a *rational approximation* for $\sqrt{2}$ indicates an important distinction between $\sqrt{2}$ and π as irrational numbers - they

are each a different kind of irrational number, $\sqrt{2}$ is *algebraic* and π is *transcendental*. What does this distinction mean? A real number is said to be *algebraic* if it is the root of some polynomial equation with integer coefficients, that is, it is the solution to some equation of the form: $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0$, where the a_i are integers and n is a natural number. In particular, $\sqrt{2}$ is a solution to the equation $x^2 - 2 = 0$, and any rational number $\frac{m}{n}$ is an algebraic real number since it is the unique solution to a *linear* polynomial equation $nx - m = 0$. The area of a circle also provides a situation where both surds and π are naturally involved, for example, if we wish to form a circle with an area of 200 square metres, then the radius is found by solving the equation $200 = \pi \times r^2$, which gives $r = \sqrt{\frac{200}{\pi}} \approx 7.98$ metres choosing the positive square root since the radius is a length.

Decimal expansions and the number line

As the decimal expansion of a rational number either terminates, for example, $\frac{5}{8} = 0.625$ or has an infinite recurring pattern, for example, $\frac{5}{7} = 0.714285714285\dots = 0.\overline{714285}$, the decimal expansion of an irrational real number must be infinite and *non-recurring*. One can readily write down the decimal expansion of some irrational real numbers using a ‘rule’ or ‘process’ by which there will be no infinitely recurring sequence of digits, for example, the real number formed when the digits of each natural number are written down from left to right, side by side: $r = 0.12345678910111213\dots$. This number was first described in 1933 by David Champernowne and is now known as Champernowne’s number. It was shown to be transcendental in 1937. This number can be used straight away to generate an infinite *countable* sequence of other transcendental irrational real numbers of the form $n + r$, where n runs through the natural numbers: $1 + r = 1.12345678910111213\dots$, $2 + r = 2.12345678910111213\dots$ and so on. Indeed, several other well known irrational numbers are: $0.122333444455555\dots$; $0.101001000100001\dots$; and $0.23571113171923\dots$. Every real number can be said to be expressed by its *infinite decimal expansion* if we consider the ‘terminating’ decimal expansion for a fraction such as $\frac{5}{8} = 0.625$ to be $0.625000\dots$ (or $0.624999\dots$). Thus, all real numbers can be classified as having either an infinite recurring decimal expansion (the rational real numbers) or an infinite non-recurring decimal expansion (the irrational real numbers).

Students will also be familiar with the *metaphor* of the *number line*, that is, a geometric line with a specific point, O , identified with zero, another specific point identified with 1, and other points identified by with the integers, fractions and surds. The term *metaphor*

is used here (see Crossley, 2007) since numbers and geometric points are not the same thing. However, a number line can be used to provide a coarse grain visual indication of the relative size of the sets of rational and non-rational real numbers, for example if the set of points corresponding to a sequence such as $\{\frac{1}{n}\}$ is deleted from the interval $[0, 1]$ the corresponding image still looks 'full'. The relationship in this context between *geometric constructs* such as lines, segments and points and *number constructs* such as real numbers and the set of real numbers is not as obvious as it might at first appear. A more thorough consideration of what it means to 'locate' certain numbers (or more precisely points which are associated with certain numbers) in some sort of constructed sequence from a given point of reference on a line shows that the process involves some subtlety.

Axioms and real numbers

The axiomatic definition of real numbers defines the real numbers as a set which has certain basic properties, specified by axioms. The idea behind this approach is that one writes down a minimal set of properties which are necessary and sufficient to capture the mathematical structure of the desired number system. Some of these structural properties of the real numbers are also common to the rational numbers and the complex numbers, but others are not, and these give the real numbers their distinctive, and unique, characterisation. The rational numbers, the real numbers and the complex numbers satisfy the *field* axioms. That is, for all numbers x , y and z : $x + y$ and $x \times y$ are always defined elements of the set (closure); $x + (y + z) = (x + y) + z$ and $x \times (y \times z) = (x \times y) \times z$ (associative); there exists an element 0 such that $x + 0 = x = 0 + x$ and there exists an element 1 such that $x \times 1 = x = 1 \times x$ (identity); there exist numbers $-x$ and x^{-1} such that $x + -x = 0 = -x + x$ and $x \times x^{-1} = 1 = x^{-1} \times x$ (inverse); $x + y = y + x$ and $x \times y = y \times x$ (commutative); and $x \times (y + z) = (x \times y) + (x \times z)$ (distributive). Both rational and real numbers, but not complex numbers, also satisfy the *order* axioms for a total (linear) order relation ' $<$ ': either $x < y$ or $x = y$ or $y < x$ (trichotomy); $x < y$ and $y < z$ implies $x < z$ (transitive); $x < y$ implies $x + z < y + z$ (translation); and if $0 < z$ then $x < y$ implies $x \times z < y \times z$ (scaling). Q and R are both densely ordered sets, that is, for any two elements of Q there is another element of Q that is between the two given elements with respect to the order relation, and similarly for any two elements of R there is another element of R that is between the two given elements with respect to the order relation. The set of rational numbers Q is also dense in R , that is for each real number there are arbitrarily close rational numbers. The fact that the rational numbers

are dense in the real numbers means that one can always approximate the location of any computable real number on a number line. A set of numbers is said to have an upper bound if there is a number k for which all elements of the set are less than or equal to k . The axiom that distinguishes R from Q is the *completeness* axiom: any non-empty subset of R that is bounded above in R has a *least* upper bound in R .

The *least upper bound* property characterises the real numbers as being *complete*, and does not apply to the rational numbers, for example, any increasing rational sequence of numbers that converges to $\sqrt{2}$ has $\sqrt{2}$ as an upper bound, but this number is not an element of Q . It is, however, the irrational real number that is the *least upper bound* for all such sequences. An analogous statement follows for the notion of greatest lower bound for every non-empty subset of R that is bounded below. The completeness axiom ensures that the set of real numbers is *continuous* - there are *no gaps* between the numbers in R . It can also be shown that the real numbers form the *unique* complete (Archimedean) ordered field (see, for example, Cohn, 2002, pp 275 – 279; HREF1).

Computation and some stuff we don't know

How are certain transcendental irrational numbers 'computed'? Essentially by using polynomial approximations to infinite series that rapidly converge to the required real number in a suitably small number of computations. Related algorithms are built into the scientific functionality of calculators and other digital technology. The approach commonly used by teachers is based upon an *informal* treatment of the idea of an infinite power series expansion for a function f of a single real variable that converges over a given subset of R . Thus, if it is the case that for such a function f :

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots + a_nx^n + \dots = \sum_{i=0}^{\infty} a_i x^i$ over some non-empty subset of R , then a polynomial *approximation* to $f(x)$ is given by:

$$f(x) \approx a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots + a_nx^n = \sum_{i=0}^n a_i x^i.$$

Polynomial functions are readily evaluated by repeated application of combinations of arithmetic operations. For practical computation purposes the function f is expressed as the sum of a finite approximation function and a remainder function, with the latter being used to provide a bound on the error associated with the approximation. For a fuller treatment of convergence and error see Chapter 8 of Courant and Robbins (1941); Chapter 6 of Hirst

(2006) and Chapters 36 and 37 of Estep (2002). For example, the well known infinite series expansions for the exponential function $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$ enables one to compute e through evaluation of $f(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots$. While a range of real have been established as transcendental, the status of others is currently unknown in this regards. It is known that if x and y are transcendental real numbers, then at least one of $x + y$ or $x \times y$ is also transcendental. However it is not known which of $\pi + e$ or $\pi \times e$ and $\frac{\pi}{e}$, π^{π} , e^e , π^e are transcendental or not. It is known that the transcendental functions, such as the exponential function, output transcendental numbers when the input values are algebraic and in the natural domain of the function, with some obvious exception such as $e^0 = 1$ and the like. Proofs of transcendence are typically more difficult than proofs of irrationality, and both of these, with the small number of exceptions with respect to irrationality, are beyond the scope of senior secondary mathematics courses. However, it is important for students to be aware that mathematics is not a 'closed domain' and there are indeed many questions about real numbers open to further investigation such as the problem of 'identifying' computable real numbers from their first few digits (HREF2).

References

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