PAPER GEOMETRY VS ORANGE GEOMETRY – COMPARATIVE GEOMETRY ON THE PLANE AND THE SPHERE

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In this paper we argue that the simultaneous teaching of geometry of the plane and the surface of the sphere can be carried out in primary and secondary school, as well as at university, provided the material is selected and structured according to the needs of the given age group. We make use of a wide range of tools, including various kinds of spherical fruit, balls, construction tools for spherical geometry, and computer software. To start teaching this material, teachers need to be familiar with only elementary concepts of plane geometry.

Some Basic Ideas

The basic idea behind "paper geometry and orange geometry" is the teaching and learning of geometry of the plane and the spherical surface simultaneously – that is, comparing and contrasting basic concepts such as straight line, circle, distance, angle, polygon, triangle and area, in both geometries.

The simultaneous teaching and learning of plane and spherical geometry is suitable for all age and ability groups, from primary to secondary and tertiary education. Of course, the materials must be selected and structured in accordance with the needs of the given age and ability group.

What Prerequisite Knowledge is Expected From the Teacher?

Teachers need to be familiar with the fundamental concepts of Euclidean plane geometry, and with the concepts of spherical geometry (such as pole points, equator, latitudes and longitudes) that are necessary to understand the geographic coordinate system. Apart from geometry, we only need to make use of the four arithmetical operations at this stage. Comparative geometry can be connected with other competence areas and subjects: first of all geography, later on astronomy, art, chemistry and so on. Later, it can also be extended to advanced topics, from spherical trigonometry to hyperbolic geometry.

What Tools to Use?

For the examples discussed here, it is possible to use spherical shapes of many different kinds, such as fruit or balls, spherical construction materials, or computer software (see, for example, Makara & Lénárt, 2004; Rybak & Lénárt, 2007).

Some Examples

Below are some examples of the types of challenging questions that can be posed when we use spherical surfaces to explore geometric concepts.

Example 1: What is a Straight Line on the Sphere?

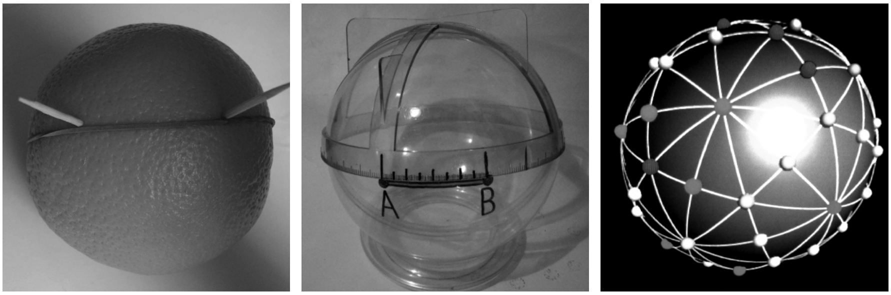

Figure 1. 'Straight lines' on the surface of a sphere

Figure 1 shows some examples of the spherical equivalent of straight lines in the plane. A straight line can be thought of as the line made by a drop of water running down on a ball, a taut string or rubber band on the surface of an orange, a line drawn along the edge of a spherical ruler, or a line on a virtual sphere on a computer screen.

Some Questions Teachers Can Ask

- Can we call this line a spherical straight line? Why or why not?
- What is 'straight'?
- If we can, what properties does this line have?
- Can two of these lines intersect?
- If not, why? If so, how many common points are there?
- Can two of them be perpendicular? What is 'perpendicular'?
- Can two of them be parallel? What is 'parallel'?

Example 2: What is a Circle?


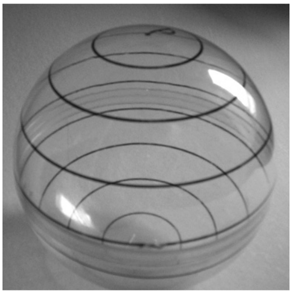

Figure 2. Some examples of lines that might appear to be circles on the surface of a sphere

Look at these lines on the spheres in Figure 2. From the point of view of plane geometry, all of these lines might appear to be circles. But perhaps things work differently on the surface of a sphere?

Some Questions Teachers Can Ask

- Are these lines circles? Why or why not?
- If so, what condition must they fulfill? (They must have at least one point of centre.)
- What is a point of centre?
- How many such points does a spherical circle have? (Two.)

Example 3: What is a Polygon?

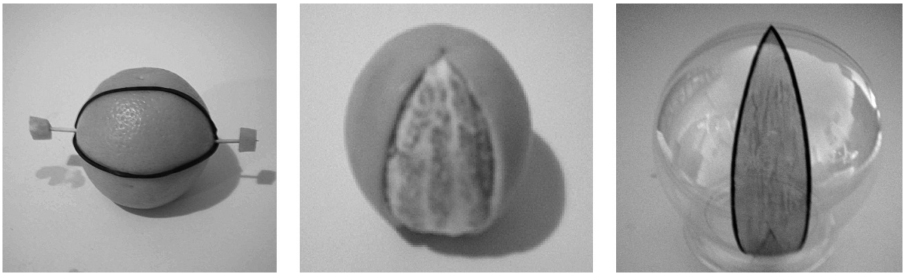

Figure 3. Some examples of possible polygons on the surface of a sphere

Look at examples in Figure 3. We all think we know what a polygon is in the plane. Is there an equivalent on the surface of a sphere? What would it look like?

Some Questions Teachers Can Ask

- Are the shapes in Figure 3 polygons? Why or why not?
- If they are, what properties do they have? For example, are they regular polygons? What is 'a regular polygon'?
- Can you draw such polygons on paper?
- Are the shapes in Figure 4 below spherical polygons? Why or why not?

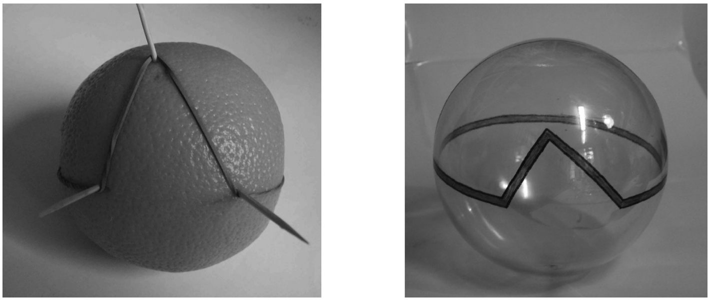

Figure 4. Some more examples of possible polygons on the surface of a sphere

What is the Use of This?

The examples show that we can raise questions in both geometries even about the most fundamental geometric concepts. With the help of models and experiments, students can study these questions and formulate their own answers, on the basis of their own experience and judgment. They are not forced to obey definitions given by the teacher, discovered by authors from ancient times, but instead they can and should create these definitions themselves.

If students only learn about circles on the sphere in Euclidean plane geometry, they do not feel the immediate necessity for giving an abstract definition of a circle. A circle is something that looks like a circle, they think. To please teachers or parents, or to impress their peers, they memorize a verse about 'A circle is a shape consisting of those points ...' but the whole process of definition seems superfluous if the real answer can be found at a glance. However, this strategy does not work in the world of geometry on the surface of the sphere. In order to give a reasonable answer, students must analyse and transfer the definition from the plane to the sphere. Experiment will show that the concept *can* be transferred into spherical geometry, but only with due changes, getting rid of certain biases that seem unquestionable on the plane.

The essence of the process is that students are searching for the truth themselves, through experiments with palpable and virtual models, and through discussion or debate with one another or their teacher. To misquote an often-quoted saying, being a student is not a passive spectator sport. (A personal remark: In Hungarian, a university student is called 'hallgató' which means both 'listener' and 'somebody keeping silence'. I dislike the word, and use clumsy synonyms like 'participant' or 'collaborator' or 'fellow researcher' to address my students.)

Teachers' Objections to the Idea of Comparative Geometry

Teachers often raise objections to the teaching and learning of both plane and spherical geometry. Typical objections include:

- Basically, mathematics teaching is about numbers. Geometry is of secondary importance.
- Geometry is a boring application area of the theory of transformation groups. It does not deserve much attention in teacher training – or in schools.
- I did not learn much about geometry myself, even less about non-Euclidean geometry – how could I teach it?
- You can't be serious about teaching spherical geometry to kids who hardly know the Theorem of Pythagoras on the plane!
- Geometry is not for regular students, but only for students of high ability. It requires a kind of creativity which an average student does not have.

- It is easy – in Hungary! All kids love maths there. But come to *my* classroom and teach *my* kids about your geometry! (Of course some Hungarian teachers say: It is easy - in *other* countries, but come to *my* classroom ...)
- (The Big Objection) It might be a nice topic, but unfortunately *I have no time for it!* The curriculum is so crowded, students must be prepared for entrance examinations!

Is Geometry Unimportant?

No. It is not true that geometry is only an insignificant application of other branches of mathematics. Higher mathematics is very far from suggesting diminution or deletion of geometry in school. Similarly, higher geometry does not support unripe formalization of geometry, the untimely shift from direct geometric experience to coordinate geometry and the algebra of transformations. This shift is one of the reasons for what the National Curriculum Board (2008, p. 5) calls 'alienation of some students from mathematics'.

Let me refer to a personal experience from outside the classroom. I had the good luck to meet painters and sculptors who worked in the field of geometric art. Strangely, many of them were ashamed to confess that school geometry was indifferent or even frustrating for them. Some of them stated that school geometry was totally different from the subject with which they were engaged in their work. These artists were not ignorant in geometry; they had studied Moebius strips, hyperbolic geometry, even the axiomatization of geometry (see also Lockhardt, 2002, who describes very similar complaints – from a mathematician's viewpoint).

Was Geometry Pushed Into the Background?

Yes. For decades, three-dimensional geometry almost disappeared in many curricula. Even Euclidean plane geometry was brutally reduced in many cases. However, in the last few decades many educators recognized that deleting geometry was but the first step to deleting mathematical thinking as a whole; that geometry does have messages that are not only worthwhile, but inevitable to grasp by even those people who will not be directly involved with mathematics in their career. As the National Curriculum Board (2008, p.5) puts it, 'Mathematics is important for all citizens'. The same goes for geometry. Almost all areas of human activity make use of the geometric point of view, in one form or another. I only mention here a special topic: medical geometry. Modern technology is capable to give detailed pictures and projections of the human body. This information can be understood and used only with deep insight into the essence of geometry.

It is a real challenge for a generation of mathematics teachers to teach geometry when they have hardly learned anything about geometry in their elementary and higher studies.

Should We Teach Spherical Geometry to Kids who Hardly Know the Theorem of Pythagoras?

The answer depends on *what* we mean by 'knowing the Theorem of Pythagoras'. Does correct memorizing of the text '*a* squared plus *b* squared …' guarantee correct application of the theorem if one of the legs, not the hypotenuse, is labelled *c*?

An even more important question: Do students *like* it? Are they delighted with the power and elegance of this theorem?

One possible way to move towards this is to ask questions such as: 'Is Pythagoras' Theorem valid on the sphere? What is this theorem all about, anyway? What right-angled triangles exist on the sphere? Can this theorem, not proved, just be transferred from the plane to the sphere?'

These questions can turn the recitation of an ancient theorem into a thought-provoking game between two geometries. Results in different geometries cannot be foreseen without adequate experience and experiments. I have seen many students shining with happiness to hear the sentence: '*Pythagoras' Theorem is false on the sphere!*' The words 'Pythagoras' and 'false' in close proximity to each other enticed and stimulated them. If Pythagoras' Theorem can be false under any condition, then perhaps all theorems of mathematics can be questioned. Perhaps there is still room for the students themselves to work on, even improve on these theorems.

Is Comparative Geometry Only for the Talented Few – the Top Three in a Classroom of Thirty?

No. Comparative geometry is for the majority of the classroom.

The National Curriculum Board (2008, p.4) states:

> An unintended effect of current classroom practice has been to exclude some students from future mathematics study. The goal of equity of opportunity is central to the construction of the mathematics curriculum. This includes consideration of the need to engage more students, the way particular groups have been excluded, and the challenge posed by creating opportunity.

Why and how does comparative geometry contribute to the solution of this vital problem of mathematics teaching?

As we know from various studies, achievement in mathematics largely depends on the socioeconomic background of students. Among other aspects, in a family of highly educated parents, with a good library and ICT resources, a student has incomparably better access to information about classical mathematical-geometric topics than her less fortunate peers.

Comparative geometry is new not only for the student, not only for the parent, but even for the teacher. So the above-mentioned advantage of better background is not as big in this case.

On the other hand, the way of working through the material is based on direct experience and experiments. Creativity, common sense, initiative and mental courage prevail over advantages due to background rather than personal excellence. Basically, everybody starts from the same starting line.

Results are unexpected, often startling. Some eleven- and twelve-year-olds outperform some high school students, both in originality and logical thinking. Children or teenagers who were labelled as 'mathematically untalented' are often more productive, more free and more open to communicating with others than their 'talented' peers.

In this manner, comparative geometry can provide the joy of independent discovery, delight in mathematics, and the feeling of success to students who have never, or very rarely, had such experiences in their school career.

'No Time for it!'

It is commonsense that we have time for what we want to have time for. So the question is whether the whole enterprise takes more time than it really deserves. Again, the answer depends on the teacher's goals. If we want to make our students recite a definition, then all this activity is superfluous or even harmful. However, if our aim is to make them really understand the meaning of 'straight', 'circle', 'angle' or 'square', then, astonishingly, *comparative geometry is not a loss, but a gain of time!* The advantages are seen not only when we introduce a new concept, but later on, when we need to build on a concept already learnt. We will find students' knowledge deeper, yet more flexible, than knowledge gathered by mere cramming. Also, when students are experimenting with the models, the teacher has more time to deal with students individually or in small groups, according to their needs.

Knowledge and the Role of the Teacher

As mentioned above, introducing this material hardly requires any more background knowledge from the teacher than that which is expected for understanding the basics of the geographic coordinate system. Through much of the work, we only use the equivalent of synthetic geometry of the Euclidean plane, with many geometric experiments and relatively few computations, mostly addition and subtraction.

Still, the task is not easy.

One reason is that this is new material anyway. From her valuable time, the teacher must spend hours and days becoming a bit familiar and more comfortable with the basics of spherical geometry. Similarly, she must give time for the students to work, experiment, discuss and debate; also, to find space in a crammed room for models and tools.

The most difficult task is the change of roles.

Let me give a personal example. I have been doing research for forty years in spherical geometry; and for thirty years I have been working on the educational consequences of comparative geometry. If I succeed to raise the interest of a teenager for the topic, and she does a search on the internet, then the words 'spherical geometry' will give about one and a half million results in half a second. Only some hundreds of thousands are relevant among these; but even this amount is far beyond the capacity of my brain and life to explore. So, a week later, in front of the whole class, this same teenager can ask me a lot of questions about which I have no or hardly any idea.

This means that the change of times and flow of information has deprived me, the teacher, from my Know-All role. I have but two tools left. One is my personality, my conviction that geometry is an interesting and human area of experimenting, thinking and communicating. The other, most important thing is to leave the main role in the educational play, on the educational stage, to my students. Each of them must feel that the whole process is for her and by her; that her discoveries and failures are the central points of the whole activity. Any author of the past or present in this field is her partner in finding the truth, including the teacher.

Euclid said: 'There is no royal road to geometry.' True – but there is no slavish route either. There is but one way to geometry and all sciences: the role of equal ranks. Not because we think of ourselves as great as those authors (who knows?), but because those authors expected us, their readers, to be their equal partners, not rote learners of their texts.

Conclusion: Beyond Mathematics

Teaching comparative geometry means that we offer different approaches to the same topic to our students. The message of geometry teaching changes from the monologue of one system into the dialogue of two or more different systems. Students can learn the method and spirit of creative debate, looking at a partner with a different opinion not as an enemy but as a partner in searching and finding the truth. In this manner, geometry may help educate people who can communicate, work and live together on this globe.

I would like to draw the attention of Australian colleagues to comparative geometry in the hope that this project can be an effective and enjoyable tool to achieve the goals described in National Curriculum Board (2008, pp. 1–2):

> A fundamental goal of the mathematics curriculum is to educate students to be active, thinking citizens, interpreting the world mathematically, and using mathematics to help form their predictions and decisions about personal and financial priorities. In a democratic society, many substantial community, social and scientific issues are raised or influenced by public opinion, so it is important that citizens can critically examine those issues from mathematical perspectives.

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