FIBONACCI AND PROPORTIONS

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In 1202, Fibonacci wrote "Liber Abaci" to introduce Europeans to the Hindu-Arabic numeral system and the associated arithmetic. He was particularly keen to educate Italian merchants in these matters and discussed problems concerning proportions, such as the following. "A ton of cheese which weighs 22 hundredpounds, that is 2200 pounds, is sold for 24 pounds; it is asked how much 86 pounds are worth?". How did Fibonacci tackle these problems? What are the lessons for teaching our students about proportions?

Introduction

Proportional reasoning plays a critical role in a student's mathematical development (Nabors, 2002). It is considered a watershed concept, which is both the capstone of primary school mathematics, and the cornerstone of higher mathematics (Lamon, 1994; Lesh, Post, & Behr, 1988; Sowder, Phillipp, Armstrong, & Schappelle, 1998). Lesh et al. (1988, p. 93) described proportional reasoning as "a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information". The ability to reason proportionally is considered to be fundamental to the development of algebraic thinking (Cai & Sun, 2002; Chinnappan, 2005), crucial for success in high school mathematics and science (Lamon, 2005), and essential for the development of numeracy in general (Madison & Steen, 2003).

Lamon (2007, p. 629) summarises the matter this way. "Of all the topics in the school curriculum, fractions, ratios and proportions arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites."

How should we introduce such an important concept to students? Lim (2009) notes that proportions have been traditionally taught as "as an equivalence of two ratios" (p. 492). For instance, when the following proportional task is presented: x:16=9:8, students are encouraged to exploit the equivalence of the ratios, and solve for x using algebraic reasoning.

In this paper, we take an historical approach to presenting ideas concerning proportional reasoning.

In 1202, Leonardo Pisano, better known as Fibonacci, published *Liber Abaci* which is now available in English—eight centuries later (Sigler, 2002). Fibonacci introduced the Hindu-Arabic system of numerals to Europeans, who, at the time, tended to use Roman numerals. His work holds a key place in the history of mathematics. In two previous papers (Lenard, Itter, Mills, & Yaneff, 2007; Itter, Lenard, & Mills, 2008) we have discussed the relevance of *Liber Abaci* to the contemporary classroom. In this paper we continue this investigation by describing how Fibonacci deals with proportional reasoning.

Fibonacci's intended audience was "the Italian people above all others" (Sigler, 2002, p. 16). However, it is clear from the problems in the text that he had a special focus on showing merchants how to trade in this new system of arithmetic. Hence it is not surprising that a large part of *Liber Abaci* is devoted to the study of proportions.

He had a very interesting approach to solving problems about proportions based on tabulations. The aim of this paper is to describe his approach.

Fibonacci's approach

The problems that Fibonacci solved were associated with buying and selling or barter of goods, exchange of money and similar mercantile problems. The following problem is typical of those in *Liber Abaci*.

A Pisan hundredweight of corn costs 40 pounds. There are 100 rolls in a Pisan hundredweight. How much would 15 rolls of corn cost? (Note that "roll" is a unit of weight.)

Using the method taught currently in schools, we would use the ratio of cost in pounds to rolls (weight) of corn

$$\frac{pounds}{rolls} = \frac{40}{100} = \frac{p}{15}.$$

Solving the equation for the unknown p gives us the answer that 15 rolls would cost 6 pounds. Fibonacci would solve this problem by using a table with separate columns for the money and weight as in Figure 1.

pounds		rolls
40		100
?		15

Figure 1.

The cost of 15 rolls of corn is found by multiplying 15 by 40, as indicated by the shading, and dividing the result by 100. Thus the cost of 15 rolls is (40×15)/100 = 6 pounds.

Fibonacci extends the use of this tabular method to problems involving three or more categories. The following problem is typical. Two hundred pounds weight of pepper is worth 13 pounds (value) and a hundredweight (100 rolls) of cinnamon is worth 3 pounds (value). How many rolls of cinnamon can be bartered for 342 pounds (weight) of pepper? Fibonacci would set out the problem as in Figure 2 (Sigler, 2002, p. 181).

rolls of cinnamon		pounds (value)		pounds of pepper
?		13		200
100		3		342

Figure 2.

The number of rolls of cinnamon required is found by multiplying 100 by 13 by 342 and dividing the answer by 3 multiplied by 200, as indicated by the shading. Thus the required number of rolls of cinnamon is (100×13×342)/(3×200) = 741. We will refer to this as the "zig-zag method".

The following problem shows how one could extend Fibonacci's method to four categories. One hen is worth 15 eggs, and one cow can be exchanged for 17 hens. If 2 cows cost 3 pounds in gold, how many eggs could be purchased for ½ pound of gold? Fibonacci would set out the data for this problem as in Figure 3.

1		15		3		2
17		?		½		1

Figure 3.

Using the zig-zag method, we find that the number of eggs that could be purchased for ½ pound of gold is:

$$(17\times15\times(½)\times2)/(1\times3\times1) = 85.$$

The key detail in Fibonacci's approach is the *order* of the columns in the table.

Format of Fibonacci's tables

Fibonacci dealt with conversion problems slightly more complex than the standard problems found in today's school textbooks. Most of his examples involve the different currencies to be found in markets all over the region. Thus, he sought to teach a reliable, quick, and easily remembered method for converting the values of goods and money.

Fibonacci's genius was in tabulating the conversion data in a particular format for which the zig-zag method is ideally suited. At heart Fibonacci's method is similar to the arrow chasing approach found in some modern texts for converting one unit of measurement to another. We could recast the problem of hens, cows, gold, and eggs as in Figure 4.

$$\begin{tikzcd} \text{hens} \arrow[d, "\times 15"'] & \text{cows} \arrow[l, "\times 17"'] \arrow[d, "\times 3/2"] \\ \text{eggs} & \text{gold} \arrow[l, "?"] \end{tikzcd}$$

Figure 4.

This diagram is self-explanatory: to convert from hens to eggs in the direction of the arrow we multiply by 15, and to convert from eggs to hens against the arrow we must divide by 15 (that is, multiply by 15^{-1}). By chasing the arrows in the anti-clockwise direction in Figure 4, we can convert gold to eggs. Thus, ½ pound of gold would be equivalent to

$$½\times(3/2)^{-1}\times17\times15 = 85 \text{ eggs.}$$

It is important to note that the order in which Fibonacci tabulates the items is not entirely arbitrary, and indeed the reader of *Liber Abaci* has to be careful to get the correct order of the columns in any particular problem. Fibonacci does not explicitly explain a general rule for ordering the item headings, so we are left to infer his procedure. Figure 4 suggests a clue to formatting Fibonacci's tables.

Contiguous headings must occur in pairs which are directly related in the data—with the understanding that the first and last headings are contiguous for this purpose (wrap-around). Consider Figure 3. To begin, the leftmost heading is arbitrary – hens in this case. After that we work by pairs: hens and eggs adjacent in the first row; also in the first row come gold and cows in that order because we know the relationship between cows and hens, so they must be adjacent. In the second row the data for hens and cows are inserted. Only two entries remain unfilled, namely eggs and cows in the second row. Since we want the number of eggs for ½ pound of gold, ½ is entered for gold, and we now use the zig-zag method. Thus the order hens-eggs-gold-cows is consistent with the data and leads to a table where the Fibonacci's method gives the correct answer.

Note the order hens-cows-gold-eggs is also consistent with the data, and the zig-zag method leads to the correct answer. See Figure 5.

hens		cows		gold		eggs
1		2		3		15
17		1		½		?

Figure 5.

However, the order hens-gold-eggs-cows is not consistent with the data, and the zig-zag method does not lead to the correct answer. For example, in Figure 6, the zig-zag method yields the answer 5/68. This error occurs because hens and eggs are related in the data but are not placed in continguous columns.

hens		gold		eggs		cows
1		3		15		2
17		½		?		1

Figure 6.

There may be more than one correct order, but not all orders are necessarily correct.

Conclusion

In *Liber Abaci*, Fibonacci deals with proportion problems which are, in essence, the same as proportion problems that we find in contemporary textbooks. The problems that he considers vary in difficulty from the sorts of problems that we will present to Year 8 students to more complex problems. His problems are further complicated by the range of currencies that confronted merchants in Italy in the 13th century.

He shows his genius by developing a mechanical approach to setting out these problems. Fibonacci's approach, which is based on tables, is quite different from the contemporary approach based on algebra. By reading *Liber Abaci,* we develop a deeper appreciation of the history and continuity of mathematics.

Readers are invited to compare Fibonacci's method and their own method on the following problem with a 21st century flavour.

Some entrepreneurial children decided that 1 chocolate frog was worth 10 jelly beans, 2 packets of wizz fizz were worth 7 chocolate frogs, and 5 packets of bubble gum were worth 3 packets of wizz fizz. How many jelly beans would 2 packets of bubble gum be worth?

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