The square root of two

Quick square
A two digit number can be quickly squared: \(15^2 = (10 + 5)^2 = 100 + 50 + 50 + 25 = 225\) (see diagram). Using place value knowledge \(1.5^2 = \frac{15}{10} \times \frac{15}{10} = \frac{15^2}{100} = 2.25\). Try this with some other numbers such as \(14^2, 27^2\) and \(85^2\). Use a similar method to calculate \(140^2, 150^2, 270^2\) and \(850^2\).

If \(x^2 = 2\), what is \(x\)?
We know that \(x\) must lie between 1.4 and 1.5 since \(14^2 = 196\) and \(15^2 = 225\) so \(1.4^2 = 1.96\) and \(1.5^2 = 2.25\). The table shows part of a systematic evaluation process. Extend this to obtain a value for \(x\) such that \(x^2\) rounds to 2, correct to four decimal places. Explain why it is not possible to find a finite decimal of the form \(1.a_1a_2a_3 \ldots a_n\) where \(a_n\) is non-zero, such that \(x^2 = 2\).

A square with side length exactly root two
The midpoints of a square with side length 2 are joined to form a new shape. Explain why the new shape is also a square, and (without using Pythagoras theorem) why it has side length \(\sqrt{2}\).

Solving \(x^2 - 2x - 1 = 0\) in two different ways
Firstly: \(x^2 - 2x - 1 = 0\) implies \((x - 1)^2 - 2 = 0\) and so \(x = 1 - \sqrt{2}\) or \(x = 1 + \sqrt{2}\). Secondly: \(x^2 - 2x - 1 = 0\) implies \(x^2 = 2x + 1\) and \(x = 2 + \frac{1}{x}\) for \(x \neq 0\). Re-write this as a recurrence relation: \(x_{n+1} = 2 + \frac{1}{x_n}\) and choose an initial value, such as \(x_0 = 2.5\). Values of the sequence \(x_n - 1\) approximate \(\sqrt{2}\). This can be implemented on a calculator using the recursion function, or step by step using the ‘Ans’ memory and repeatedly calculating \(2 + \frac{1}{\text{Ans}}\). After seven iterations this gives \(\sqrt{2} \approx 2.41421 \ldots - 1 = 1.41421\). Find the percentage error.

Decimal expansion of the square root of two
An initial sequence of the infinite non-recurring decimal expansion of the square root of 2 is:

\[1.41421356237309504880168872420969807856967187537694807317667973799073247846210703885038753432764157273501384623091229702492483605585073721264412149709993583141322266592750.\]

Do all the digits \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} in this initial sequence occur with approximately equal frequency?