Suggested Solutions with Marking Scheme to VCAA 2010 Sample Questions

Question 1
Let $f: R \rightarrow R,\ f(x) = x^3 + (k+1)x^2 + kx$.
Solve $f(x) = 0$ for x.

Solution
Method 1

$x^3 + (k+1)x^2 + kx = 0$

$x\left(x^2 + (k+1)x + k\right) = 0$ 1A

$x(x+k)(x+1) = 0$ 1A

$x = 0$ or $x = -k$ or $x = -1$ 1A

Method 2

$x^3 + (k+1)x^2 + kx = 0$

$x^3 + kx^2 + x^2 + kx = 0$

$x^2(x+k) + x(x+k) = 0$ 1A

$x(x+1)(x+k) = 0$ 1A

$x = 0$ or $x = -k$ or $x = -1$ 1A

Question 2
For the simultaneous linear equations:

$ax + 3y = 0$

$2x + (a+1)y = 0$

where a is a real constant, find the value(s) of a for which there are infinitely many solutions.

Solution

Method 1

$\begin{vmatrix} a & 3 \\ 2 & a+1 \end{vmatrix} = 0$ 1M

$a(a+1) - 6 = 0$

$a^2 + a - 6 = 0$

$(a+3)(a-2) = 0$

$a = -3$ **or** $a = 2$ 2A

y-intercept is zero for both lines.
Hence equivalent lines.
For example
When $a = -3$

$-3x + 3y = 0$

$2x - 2y = 0$

When $a = 2$

$2x + 3y = 0$

$2x + 3y = 0$

Method 2

$m = -\dfrac{a}{3} = -\dfrac{2}{a+1}$, y-intercept is zero 1M

$a(a+1) = 6$

$a^2 + a - 6 = 0$

$(a+3)(a-2) = 0$

$a = -3$ **or** $a = 2$ 2A

Question 3

Find the equation of the image of $y = \frac{1}{x}$ under the transformation defined by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ and describe a sequence of transformations that maps the graph of $y = \frac{1}{x}$ onto the graph of its image.

Solution

Method 1

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$x' = 2x \qquad y' = -3y$

$x = \frac{x'}{2} \qquad y = -\frac{y'}{3}$

$y = \frac{1}{x}$

$-\frac{y'}{3} = \frac{1}{\frac{x'}{2}}$ 1M

Image $y = -\frac{6}{x}$ 1A

Method 2

$y = \frac{1}{x}$

$y = -\frac{3}{\left(\frac{x}{2}\right)}$ 1M

Image $y = -\frac{6}{x}$ 1A

- Dilation by a factor of 6 from the x-axis
- Reflection in the x-axis 1A

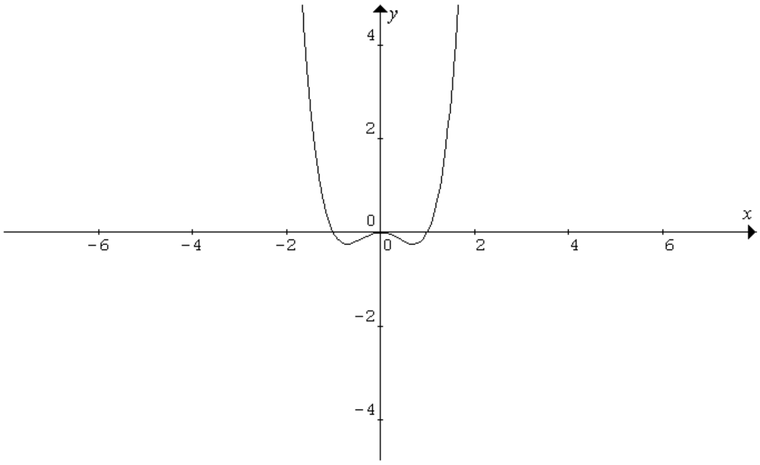
Or an equivalent sequence of transformations such as

- Reflection in the x-axis
- Dilation by a factor of 2 from the y-axis
- Dilation by a factor of 3 from the x-axis 1A

Question 4

State the subset of R for which the graph of the function $f(x) = x^4 - x^2$ is strictly decreasing.

Solution



$f'(x) = 4x^3 - 2x = 0$ 1A

$2x(2x^2 - 1) = 0$

$x = 0$ or $x = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ or $x = -\dfrac{\sqrt{2}}{2}$ 1A

$(-\infty, -\frac{\sqrt{2}}{2}] \cup [0, \frac{\sqrt{2}}{2}]$ or equivalent expression. 1A

Note

A function f is said to be strictly decreasing on a given set if for all a and b in the set

$a < b$ implies $f(a) > f(b)$, hence the use of open and closed endpoints as indicated in the answer even though $g'(\frac{-\sqrt{2}}{2}) = g'(0) = g'(\frac{\sqrt{2}}{2}) = 0$.

Question 5

Write down a formula that generates all real solutions of the equation sin(x) + cos(x) = 0.

Solution

$\sin(x) = -\cos(x)$

$\dfrac{\sin(x)}{\cos(x)} = -1$

$\tan(x) = -1$ 1A

$x = n\pi - \frac{\pi}{4}$ where n is an integer. 1A $-\dfrac{\pi}{4}$ 1A + $n\pi$, $n \in Z$

Question 6

For the simultaneous linear equations

$mx + 12y = 12$

$3x + my = m$

find the value(s) of m for which the equations have

a a unique solution

b infinitely many solutions.

Solution

a $\begin{vmatrix} m & 12 \\ 3 & m \end{vmatrix} = 0$ 1A

$m^2 - 36 = 0$

$m \neq 6$ and $m \neq -6$

Alternatively: $m \in R \setminus \{-6, 6\}$ 1A

b When $m = 6$ When $m = -6$

$6x + 12y = 12$ $-6x + 12y = 12$

$3x + 6y = 6$ $3x - 6y = -6$ 1M

Equivalent lines in both cases. This is obvious by looking at $\dfrac{12}{m}$.

Answer $m = 6$ or $m = -6$ 1A

Question 7

Sharelle is the goal shooter for her netball team and during matches has many attempts at scoring a goal. Assume that the success of an attempt to score a goal depends only on the success or otherwise of her previous attempt at scoring a goal.

If an attempt at scoring a goal in a match is successful, then the probability that her next attempt at scoring a goal in the match is successful is 0.84. However, if an attempt at scoring a goal in a match is unsuccessful, then the probability that her next attempt at scoring a goal in the match is successful is 0.64.

In the long term, what percentage of her attempts at scoring a goal are successful?

Solution

Let G be scoring a goal and G' not scoring.

Method 1

$$\begin{array}{c} \\ G_{i+1} \\ G'_{i+1} \end{array} \begin{array}{c} \begin{array}{cc} G_i & G'_i \end{array} \\ \begin{bmatrix} 0.84 & 0.64 \\ 0.16 & 0.36 \end{bmatrix} \end{array} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} x \\ 1-x \end{bmatrix} \quad \text{1A}$$

$084x + 0.64(1-x) = x$ 1M

$-0.8x = -0.64$

$x = 80\%$ 1A

Method 2

$$\begin{array}{c} \\ G_{i+1} \\ G'_{i+1} \end{array} \begin{array}{c} \begin{array}{cc} G_i & G'_i \end{array} \\ \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} \end{array} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} x \\ 1-x \end{bmatrix} \quad \text{1A}$$

$x = \dfrac{b}{a+b}$ 1M

$= \dfrac{0.64}{0.8}$

$x = 80\%$ 1A

Question 8

Show that the graph of $h(x) = \dfrac{x^n}{e^x}$, where n is a positive integer, has a local maximum at $x = n$.

Solution

$h'(x) = \dfrac{ne^x x^{n-1} - x^n e^x}{e^{2x}}$

$= \dfrac{e^x x^{n-1}(n-x)}{e^{2x}}$ 1A

For stationary points, $h'(x) = 0$.

This implies $x = n$ 1M

$n - x$ determines the sign of the solution.

For $0 < a < n$, $h'(a) > 0$

and for $a > n$, $h'(a) < 0$.

Hence there is a local maximum at $x = n$. 1M

Question 9

Let g: $R \to R$, $g(x) = x^2$. Show that $g(u + v) + g(u - v) = 2(g(u) + g(v))$

Solution

$\text{LHS} = g(u+v) + g(u-v)$

$= (u+v)^2 + (u-v)^2$ 1M

$= u^2 + 2uv + v^2 + u^2 - 2uv + v^2$

$= 2u^2 + 2v^2$

$= 2(u^2 + v^2)$

$= 2\big(g(u) + g(v)\big)$ = RHS as required 1M

Question 10

For the functions $f : R \to R,\ f(x) = e^x + e^{-x}$ and $g : R \to R,\ g(x) = e^x - e^{-x}$ show that:

a $[f(x)]^2 = f(2x) + 2$

b $f(x)g(x) = g(2x)$

c $[f(x)]^2 - [g(x)]^2 = 4$

Solution

a LHS = $[f(x)]^2$

$= (e^x + e^{-x})^2$

$= e^{2x} + 2 + e^{-2x}$

$= e^{2x} + e^{-2x} + 2$

$= f(2x) + 2$ = RHS as required 1M

b LHS = $f(x)g(x)$

$= (e^x + e^{-x})(e^x - e^{-x})$

$= e^{2x} - e^{-2x}$

$= g(2x)$ = RHS as required 1M

c LHS = $[f(x)]^2 - [g(x)]^2$

$= (e^x + e^{-x})^2 - (e^x - e^{-x})^2$

$= e^{2x} + 2 + e^{-2x} - e^{2x} + 2 + e^{-2x}$

$= 4$ = RHS as required 1M

Question 11

Find the average value of $y = e^x$ over the interval [0, 2].

Solution

Average value $= \frac{1}{2-0}\int_0^2 \left(e^x\right)dx$ 1A

$= \frac{1}{2}\left[e^x\right]_0^2$

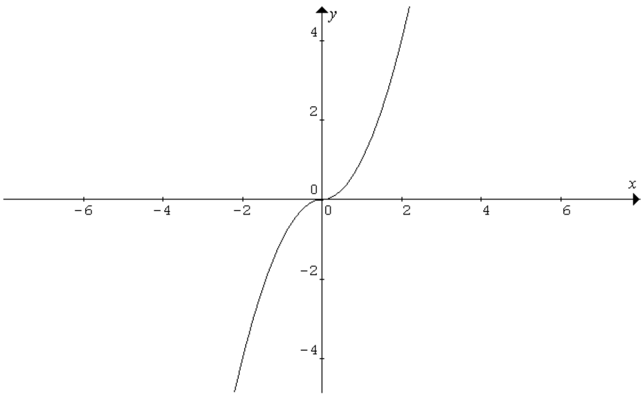
$= \frac{1}{2}(e^2 - 1)$ 1A

Question 12

Part of the graph of the hybrid function

$$f(x) = \begin{cases} -x^2, x \le 0 \\ x^2, otherwise \end{cases}$$

where x is a real number is shown below:

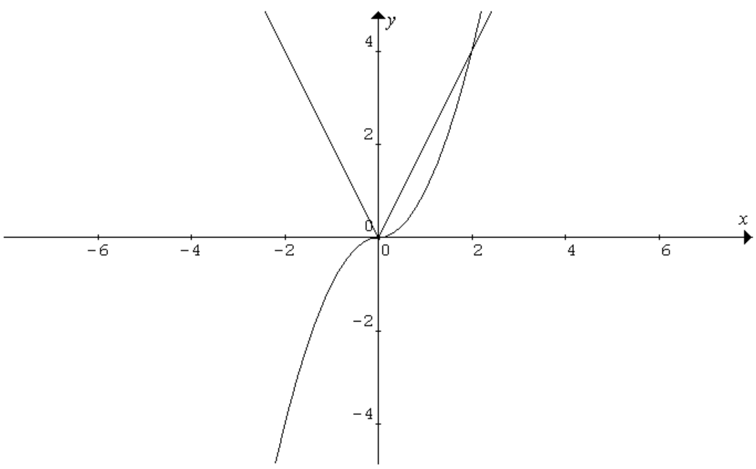


a Draw the graph of the derivative function f' on the same set of axes

b Write down a rule for the derivative function.

Solution

a



Right-hand branch 1A
Left-hand branch 1A

b $f'(x)=\begin{cases} -2x, x\le 0 \\ 2x, otherwise \end{cases}$

Correct rule and domain 2 x 1A

alternatively : $f':R\to R, f'(x)=2x\,\text{sgn}(x)$.

Correct domain 1A

Correct rule 1A

Note

The sgn function is defined by:

$$\text{sgn}(x)=\begin{cases} 1 \text{ where } x>0 \\ 0 \text{ where } x=0 \\ -1 \text{ where } x<0 \end{cases}$$

The hybrid function *f* is differentiable at $x = 0$ with $f'(0)=0$, so f' can either be specified as a hybrid function, or alternatively using the sgn function, as indicated in the answer, in which case $f'(x) = 2x\ \text{sgn}(x)$.

Question 13

The speed *v*, in metres per second, of an object moving in a straight line is given as a function of time *t*, in seconds, by $v(t)=\dfrac{24}{t+1}$ where $t\ge 0$.

- **a** State the initial speed of the object.
- **b** Find the values of *t* for which the speed is *less than* 2 metres per second.
- **c** Find the distance travelled by the object in the first 10 seconds.

Solutions

a $v(0)=24$ m/s 1A

b $\dfrac{24}{t+1}<2$

$24<2t+2$

$t > 11$s 1A

c $\displaystyle\int_0^{10}\left(\frac{24}{t+1}\right)dt$ 1M

$= 24\left[\log_e(t+1)\right]_0^{0}, t > 0$

$= 24\log_e(11)$ m 1A