Learning Algebra and Developing Functional Thinking - With or Without CAS?

Bärbel Barzel, University of Education, Freiburg
When using CAS, pupils lose by hand skills.
Focusing

1. ... the Topic:
   Algebra & Functional Thinking

2. ... the Technology:
   CAS and MRS (Multi Representational Systems)

3. ... the Tasks & the Teaching:
Algebra and functional thinking: .... problems

... measuring basic algebraic skills in PISA:

Which of the following represent half of a?
Mark the correct answers with a cross.

\[
\begin{array}{ccc}
\frac{a}{2} & \text{Ja} & \text{Nein} \\
\frac{a}{2} - \frac{1}{2} & \text{Ja} & \text{Nein} \\
\frac{1}{2} \cdot a & \text{Ja} & \text{Nein} \\
a - \frac{a}{2} & \text{Ja} & \text{Nein} \\
0.5 \cdot a & \text{Ja} & \text{Nein} \\
a : \frac{1}{2} & \text{Ja} & \text{Nein} \\
\frac{1}{2} a & \text{Ja} & \text{Nein}
\end{array}
\]

in PISA 13 % correct
Algebra and functional thinking: ..... problems

... measuring basic algebraic skills with

For my garden, I bought

- \( r \) red rose bushes and
- \( g \) white gardenia bushes.

The roses cost $4 each.
The gardenias cost $5 each.

Choose the equation that says that the total cost was $70:

- \( 4r+5g=70 \)
- \( 10r+6g=70 \)
- \( r+g=70 \)

Only 34% answered correctly.
Algebra and functional thinking: .... problems

A taxi driver takes 5€ as fix costs and 3€ for every km. Sketch a graph, ...

For which value of x is the y-value 1?
29% wrong answer

Test-Results Semester 1 2008/09

42% sketch a wrong graph

48% sketch a wrong graph

See also: Janvier 1983, PISA 2003, PALMA 2006, Kaput 1994
Algebra and functional thinking: .... problems

Test with about 100 Math-students, Semester 1

The dotted line is drawn from A to B.
$F(x)$ gives the size of the grey area at $x$.

a. Which graph belongs to $F(x)$? Make a cross.
b. Give reasons to your choice.

Results:

a. 66% correct answer
b. 57% correct answer
Algebra and functional thinking: .... problems

Test with about 100 Math-students, Semester 1

Begründen Sie Ihre Wahl:

Die Fläche ist so, wie die Fläche $F(x)$

Graph-as-Picture-misconception

Hoffmann 2007
Algebra and functional thinking: .... problems

\[ f \text{ is a differentiable function with } f(-x) = -f(x). \]

Which of the following is correct?

A) \( f'(-a) = -f'(-a) \)
B) \( f'(-a) = f'(a) \)
C) \( f'(-a) = -f'(a) \)
D) None of the above

A typical answer was: \( f'(-a) = (f(-a))' = (-f(a))' = -f'(a). \)

Eisenberg & Dreyfus (1990)
Algebra and functional thinking: .... What is it about?

<table>
<thead>
<tr>
<th>Learning basic syntactical skills</th>
<th>Getting a sense for symbols and structures</th>
<th>Dealing flexibly with functions and their representations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5 \cdot (2x + 1) = 10$</td>
<td>$5 \cdot (2x + 1) = 2 \cdot 5$</td>
<td></td>
</tr>
<tr>
<td>$10x + 5 = 10$</td>
<td>$f(u) = a \cdot u + a^2$</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Algebra and functional thinking: ..... What is it about?

- Connecting mathematical representations (verbal, graphical, numerical and symbolic representations)
- Developing and connecting different mental images

Presmeg 1997; 2006; Duval 2002; Tall 1997; Fischer 1987; Malle 2000; vom Hofe 1995
Algebra and functional thinking: .... What is it about?

- Connecting mathematical representations (verbal, graphical, numerical and symbolic representation)
- Developing and connecting different aspects/images of the concept

Learning basic syntactical skills
Getting a sense for symbols and structures
Dealing flexibly with functions and their representations
Algebra and functional thinking: .... What is it about?

- Connecting mathematical representations (verbal, graphical, numerical and symbolic representations)
- Developing and connecting different aspects/ images of the concept

´Variable´ can be seen:
- as a placeholder
- as a generalizer
- as an unknown
- as a changing quantity

´Function´ can be seen:
- under the aspect of mapping
- under the aspect of covariation
- as an entity

Malle 1993 & 2000; Fujii 2003
Algebra and functional thinking: .... What is it about?

Focus on structure when typing the equation into the machine, otherwise:

\[
\frac{1}{g} = \frac{1}{f} - \frac{1}{b}
\]

Drijvers 2005
Algebra and functional thinking: .... What is it about?

Lens equation

\[ \frac{1}{g} = \frac{1}{f} - \frac{1}{b} \]

as a placeholder:

\[ \frac{1}{g} = \frac{1}{f} - \frac{1}{b} \quad | \quad f = 10 \]

as an unknown:

as a generalizer:

as a changing quantity:

solve \( \frac{1}{15} = \frac{1}{10} - \frac{1}{b} \)

\[ b = 30 \]

solve \( \frac{1}{g} = \frac{1}{f} - \frac{1}{b} \)

\[ b = \frac{-fg}{f - g} \]

\[ b(g) := \frac{10 \cdot g}{g - 10} \]

"Done"
Algebra and functional thinking: .... What is it about?

Flexible use of variables, in concrete:

- Calculate with variables in the same way than with numbers, such as: $2a + 4a = 6a$
- Knowing that variables are standing for a number not for an object, avoiding a misconception like: 
  \textit{Six students for one professor:} $6s = p$
- Accept results with variables as an answer, such as: 
  \textit{The solution is} $2p + 1$!
Which technology supports the learning of algebra?

Technologies are media, they have to mediate and to support the teaching– and learning process. They are a mediator between mathematics and the pupil.
Which technology supports the learning of algebra?

General tools
- DVD
- Internet
- Presentation tools

Mathematical tools
General purpose:
- Computer Algebra (CAS)
- Graphing
- Spread sheet
- Dynamic Geometry

Specific purpose:
- Applets
- Tools for measuring data

Multi-representational system (MRS)
From CAS to MRS

**Multi-representational system**

Why does a parabolic mirror work?

\[ f(x) = ax^2 \]

Focus point: \( \left( 0 \mid \frac{1}{4a} \right) \)
Using variables and functions

\[
\begin{align*}
\text{Using variables and functions} & \\
\text{Using variables and functions} &
\end{align*}
\]
Using variables and functions

Felsager 2007
From CAS to MRS

You make a rectangle with a rope of given length using a wall as one side. Which rectangle has the greatest area?

Devenere locos, ubi nunc ingentia cernes moenia surgentemque novae Kathaginis arcem, mercatique solum, facti de nomine Byrsam, taurino quantum possent circumdare tergo (Aneis, Virgil)
Learning Algebra & Developing Functional Thinking - With or Without CAS?

Topic Technology Tasks & Teaching

Bärbel Barzel

1. Einstieg

Topic

Technology

Tasks & Teaching

Numerical approach

First step:
A value!!!

Spread sheet

Precise approach

First step:
A value!!!

Spread sheet

Symbolic approach

First step:
An expression!!!

Computer Algebra

Dynamic Geometry

Graphical approach

First step:
A sketch/ a graph!!!

Dynamic Geometry
Research study in Norway (Fuglestad 2005)
3 grades (8-10, 14 – 16 years old pupils), 6 classes took part,
The three tools were always available

Which of the three tools do pupils use when solving tasks,
where all tools can support finding a solution?

All the three tools are used - by different pupils and the
pupils could explain their choice
according to their solution methods

But : They must be aware of the three tools!!
The relevance of general purpose tools

<table>
<thead>
<tr>
<th>Sc Calc</th>
<th>Dyn Geo</th>
<th>Spread Sheet</th>
<th>Graph / CAS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decimals</td>
<td>Variables in formulas</td>
<td>Equations</td>
<td>Functions: Proportional, linear, Inversely</td>
</tr>
<tr>
<td>Triangle</td>
<td>Area-formulas</td>
<td>Systems of Lin. Equat.</td>
<td>Further types of Functions</td>
</tr>
<tr>
<td>Reasoning</td>
<td>Statistics</td>
<td></td>
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</tr>
</tbody>
</table>

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Algebra and functional thinking: How can it be realized?

**GRAPHICAL**

Experiments

**NUMERICAL**

Map scale „1 : 100 000“

**SYMBOLIC**

Variable generalized number

Variable placeholder

Expression describe, transform

Function covariation mapping

Variable Chang. quantity

Equation Insert, describe, transform

Modeling: Variables in formulas

Area-formulas

Functions: proportional & linear, Inversely

Idea of equations
Introducing "functional thinking" with experiments

Which graph is created by my movement?

Experiments to switch situation & graph

Research question:
How does real experiment supports the conceptualization of a „function“?
Introducing "functional thinking" with experiments

Which graph is created by my movement?

1. Step:
   Spontaneous movements and interpreting the graph

2. Step:
   Match a given graph
Introducing "functional thinking" with experiments

At first:
a very strong picture in mind as a pattern
graph-as-a-picture misconception

They overcome faults and argue in different ways, e.g.:
Statically: in the sense of mapping points
Dynamically: in the sense of covariation

Which graph is created by my movement?
Introducing "functional thinking" with experiments

Three groups:

- **Experimental group** (n=77)
  - Independent student experiments (in groups)

- **Control group** (n=86)
  - Experiments shown on film

- **Zero group** (n=41)
  - Traditional teaching with textbook

### Topics
- **Technology**
- **Tasks & Teaching**

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Introducing "functional thinking" with experiments

Learning gains in the three groups

<table>
<thead>
<tr>
<th>Testgruppen</th>
<th>Pretest</th>
<th>Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experimentiergruppe</td>
<td>24</td>
<td>60</td>
</tr>
<tr>
<td>Kontrollgruppe</td>
<td>31</td>
<td>56</td>
</tr>
<tr>
<td>Nullgruppe</td>
<td>29</td>
<td>37</td>
</tr>
</tbody>
</table>
Introducing "variables" ......

**Modeling:** Variables in formulas

What does it cost to go by car, by bike and by train? Compare.

\[ w + 4 \cdot r + k \cdot 0.04 \cdot b \]

**Example:**
- **Car:**
  - Warteverlust: 3000€
  - Benzinpreis: 2.05€ pro Liter
  - Kilometerzahl: 40000 km
  - Jahreskosten: 3280€
  - Gesamtkosten: 9880€

- **Bahn:**
  - BahnCard 100: 3800€

- **Bike:**
  - 500€

**Excel Table:**

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>3000</td>
<td>2.05</td>
<td>40000</td>
<td>3280</td>
<td>900</td>
<td>9880</td>
</tr>
</tbody>
</table>

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Introducing "variables" ......

Area-formulas

Invoice of a glazier

Expressions to measure the area of this window:

Till: \(a \cdot b + \frac{1}{2} \cdot a \cdot h\)

Merve: \(a \cdot (b + h)\)

Paul: \(b \cdot a + h\)

Sverre: \(a \cdot (b + h) - b \cdot h\)
Introducing "equations" …..

**Idea of equations**

What is equality?....

...when looking at a table

<table>
<thead>
<tr>
<th>42</th>
<th>43</th>
<th>44</th>
<th>45</th>
<th>46</th>
<th>47</th>
<th>48</th>
<th>49</th>
<th>50</th>
<th>51</th>
<th>52</th>
<th>53</th>
</tr>
</thead>
<tbody>
<tr>
<td>18.3</td>
<td>18.45</td>
<td>18.6</td>
<td>18.75</td>
<td>18.9</td>
<td>19.05</td>
<td>19.2</td>
<td>19.35</td>
<td>19.5</td>
<td>19.65</td>
<td>19.8</td>
<td>19.95</td>
</tr>
</tbody>
</table>

...when looking at graphs

...and sometimes you need an exact value:

```
3·x+0.25=2·x+3
[[3·x+0.25=2·x+3]-0.25
[[3·x=2·x+2.75]-2·x
solve(3·x+0.25=2·x+3,x)
```

x=2.75
Using variables and functions

CAS when learning algebra - Yes or No?

\[ c(a) = ye + n \]

Mattias Zeller

TI-nspire CAS  versus  TI-nspire
Using variables and functions

\[ c(a) = ye + n \]
Using variables und functions

\[ c(a) = ye + n \]

**TI-nspire CAS versus TI-nspire**

1. \( 1 \cdot a + 2 \cdot a + 3 \cdot b + 4 \cdot b \)
2. factor(\(12 \cdot x + 36\))
3. expand(\(12 \cdot (y + 3)\))
4. \( 12 \cdot x + 36 \mid x = 2 \)
5. \( (12 \cdot y + 36 = 96) - 36 \)
6. \( 12 \cdot y = 60 \)
7. \( 12 \cdot y = 60 \)
8. \( 12 \cdot y = 60 \)
9. \( 12 \cdot y = 60 \)
10. \( 12 \cdot y = 60 \)
11. \( 12 \cdot y = 60 \)
12. \( a = 7 \)
13. \( 3 \cdot a \)
14. nSolve(\(12 \cdot y + 36 = 96, y\))

**MAV Decembre 2011**

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Using variables and functions

**Party organizers**

After graduation, you want to organize a party. Several offers of local providers are available:

<table>
<thead>
<tr>
<th></th>
<th>YUKI EVENT</th>
<th>PARTYMAD</th>
<th>FLASH</th>
</tr>
</thead>
<tbody>
<tr>
<td>meal / person</td>
<td>24 €</td>
<td>15 €</td>
<td>20 €</td>
</tr>
<tr>
<td>rent for the room</td>
<td>400 €</td>
<td>2300 €</td>
<td>900 €</td>
</tr>
<tr>
<td>stereo</td>
<td>350 €</td>
<td>400 €</td>
<td>included</td>
</tr>
<tr>
<td>decoration</td>
<td>included</td>
<td>200 €</td>
<td>300 €</td>
</tr>
</tbody>
</table>

Compare the offers.

**These points might help you:**
- How many guests do you expect? What will the costs be at which offer?
- When do two offers have an equal price?
- Present an overview of your results.

**Additional task:**
Another provider has a big party room. His offer is the cheapest, so long as more than 500 guests come. How much money can he charge, while still being the cheapest?
Using variables und functions

Applied problems

... if I add 1 to the first number, than I have to ... to the other number ...

\[ \text{c}(a) = y e + n \]

\[ y = m \times x + b \]
Using variables und functions

Observations

1. **The step from arithmetic to algebra:**
   - Non-CAS-pupils perceive a difference in the underlying rules of arithmetic and algebra.
   - CAS-pupils accept results with variables as an answer – Non-CAS-Pupils quite often do not.

2. **Using different representations**
   - Non-CAS-pupils avoid algebraic work, because it can not be done by the machine.
   - CAS-pupils are more motivated and keen to use variables, Non-CAS-pupils find the use of variables to be difficult.
Using variables und functions

Observations

\[ y_{UK1} = 24 \cdot x + 750 \]
\[ \downarrow \text{weiß} \downarrow \text{mehl} \]

\[ \text{FLASH} = 20 \cdot x + 900 \]
\[ \downarrow \text{weiß} \downarrow \text{mehl} \]

\[ \text{Party und} = 15 \cdot x + 2900 \]
\[ \underline{\text{das Neue Apartment weiβ mehr}} \]

Wenn mehr deutlich wird, dass
Einen billiger und die Miete mehr.

2. B. Neun:
\[ 14 \cdot x + 3000 \]
\[ \underline{\text{Rent}} \]

Preis:
\[ p(e, g, x) = e \cdot g + x \]
\[ \underline{\text{Ermen Gute}} \]

\[ \text{solve} \left( p(15, 500, 2900) = p(a \cdot 500 + g), g \right) \]
\[ g = 10 \cdot 400 - 500 \cdot a \]
\[ \underline{\text{Rent neuer Mietpreis}} \]

Das neue Apartment kostet i. B. 18, 200
und 2900 für den Rent.
Using variables und functions

Which graph belongs to the function $f$ with $f(x) = x(x+2)^2$? Give reasons!

CAS-group: 26
Comparison group without technology: 80
Using variables und functions

From the CAS-group:

\[ f(x) = x(x+2)^2 \]

Analyzing pupils' solutions
Using variables und functions

From the CAS-group:

\[ f(x) = x \cdot (x + 2)^2 \]

Analysing pupils‘ solutions
Using variables and functions

Analysing pupils' solutions

From the comparison group:

\[ f(x) = x(x+2)^2 \]

\[ f(x) = x(x^2 + 4x + 4) \]
\[ f'(x) = 3x^2 + 8x + 4 \]
\[ f''(x) = 6x + 8 \]
\[ f'''(x) = 6 \]

\[ 3x^2 + 8x + 4 = 0 \]
\[ x^2 + 2\frac{2}{3}x + 1\frac{1}{3} = 0 \]
\[ x = -1 \frac{2}{3} \pm \sqrt{\frac{8}{9} - 1 \frac{1}{3}} \]
\[ x_1 = -\frac{2}{3}, \quad x_2 = -2 \]

\[ f''(-\frac{2}{3}) = 4 > 0 \quad \text{Min} \quad \text{Max} (-\frac{2}{3}, -2, 0) \]
\[ f''(-2) = -4 < 0 \quad \text{Max} \quad \text{Max} (-2, 0) \]

\[ f(x) = \frac{(\frac{-2}{3})^2 + 4(\frac{-2}{3})^2 + 4 \cdot (\frac{-2}{3})}{2} \]
\[ = \frac{\frac{8}{27}}{2} + \frac{8}{9} - 2 \frac{2}{3} \]
\[ = -2.67 \]

\[ f(-2) = 0 \]

The graph c is a graph of the function.

Da das Maximum mit dem Graphen übereinstimmt.
Using variables und functions

\[ f(x) = x(x+2)^2 \]

Where is the statement of the answer?

- None
- Beginning
- Middle
- End
Using variables and functions

Possible explanations for students writing the statement of the answer first

\[ f(x) = x (x+2)^2 \]

Reasons

- Focus on expression
- Not clear
- Change graph & expr

in %

Exp-gruppe
Vgl-gruppe A
Vgl-gruppe B
Vgl-gruppe C
Using variables and functions

Possible explanations for students writing the statement of the answer first

Did they follow the "recipe"?

<table>
<thead>
<tr>
<th></th>
<th>Exp-gruppe</th>
<th>Vgl-gruppe A</th>
<th>Vgl-gruppe B</th>
<th>Vgl-gruppe C</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Yes</strong></td>
<td>120</td>
<td>60</td>
<td>40</td>
<td>30</td>
</tr>
<tr>
<td><strong>Not clear</strong></td>
<td>80</td>
<td>30</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td><strong>No</strong></td>
<td>40</td>
<td>10</td>
<td>10</td>
<td>40</td>
</tr>
</tbody>
</table>

\[ f(x) = x(x+2)^2 \]
Learning Algebra and Developing Functional Thinking - With or Without CAS?

Focusing
1. ... the topic
2. ... the technology
3. ... the tasks & teaching

Promote deep understanding.
Use CAS, but not just the symbolic features
Allow variety of approaches