THE AFFORDANCES OF TECHNOLOGY FOR STUDENT UNDERSTANDING OF FUNCTIONS

Jill P Brown
The University of Melbourne

The teaching of functions plays a key role in the secondary mathematics curriculum. Electronic technologies afford deepening student understanding of functions by linking multiple representations. Findings from my previous research which investigated the understanding of functions by students in their final two years of secondary school whilst immersed in a graphing calculator learning environment will be presented and discussed. In the RITEMATHS project I am extending this research to consider a range of technologies and how teachers and students can realise the range of affordances offered by these technologies to support the teaching and learning of functions in a multiple representation environment.

Functions

Ferrini-Mundy and Lauten see function as a concept “central to modern mathematics” (1993, p. 155). Functions require multiple linked representations for their full expression. In fact, multiple representations are common and virtually unavoidable in functions (Eisenberg & Dreyfus, 1994). These representation systems include: algebraic, numerical, and graphical (Kaput, 1989, p. 171; Lloyd & Wilson, 1998, p. 252). A single representation cannot adequately represent all aspects of such a complex idea (Ferrini-Mundy & Lauten, 1993; Kaput, 1989). Kaput suggests that multiple linked representations allow learners to combine understanding from different representations in such a way as to build a better understanding of complex ideas and to apply these ideas and concepts more effectively.

The complex nature of functions has resulted in a range of student difficulties. For example, student understanding of the graphical representation of functions has long been of concern. The examiner’s report for the first Pure Mathematics Paper in The University of Melbourne Matriculation Examination for 1948, for example, lamented that “the graphing of functions was rarely done correctly” (Teese, 2000, p. 123). More recent examiners’ comments still target functions but reflect our changing technology: “many students should be better able to determine an appropriate window when drawing graphs” (VCAA, 2001).

The connections and links between the different representations are key aspects of much mathematics. It is very important that a learner has multiple aspects to their concept image (Lloyd & Wilson, 1998). Students need to be able to work both within and across representations. Function sense involves the ability to pass from one representation to another when appropriate, the flexibility to use the most appropriate representation in solving a problem, and the ability to “see” one representation when working in another (Schwarz, cited in Eisenberg & Dreyfus, 1994, p. 46).
A Changed Learning Environment

Whilst the ideas discussed in this paper are relevant to electronic technologies in general, these will be illustrated with graphing calculators. The pedagogical significance of this particular technology should not be underestimated. As Waits and Demana noted, “we have learned that calculators cause changes in the way we teach and in the way students learn” (2000, p. 56). Following Scarantino (2003), the affordances of a teaching and learning environment incorporating graphing calculators will be taken to mean the offerings of such an environment for both facilitating and impeding learning. To take advantage of the opportunities arising, affordances need to be perceived and acted on by teachers and students alike.

The presence of electronic technologies in the classroom can fundamentally change how we think mathematically and what becomes privileged mathematical activity. Classroom tasks must include those that have the chance to be influenced by graphing calculator use, rather than being tasks where the graphing calculator is used to produce or check results but not necessarily contribute to the development of understanding, and concept and skill development. Interpretive and constructive skills must be expected, encouraged, and modelled. Students should be expected to discuss their mathematical understanding and thinking with their teacher and peers as a normal part of their classroom experience.

Student Understanding of Functions in a Graphing Calculator Environment

A study by myself (Brown, 2003a) investigated the cognitive, metacognitive, mathematical, and technological processes senior secondary students used in seeking a complete graph of a difficult cubic function. The term complete graph is used here to describe a by-hand sketch showing a global view of the function and the identification of all key features. Five pairs of students using Texas Instruments (TI) TI-83 or 82 graphing calculators were audio and videotaped solving a problem task. The problem task (adapted from Binder, 1995) was: Work cooperatively, using the graphing calculator, to sketch completely the graph of $y = x^3 - 19x^2 - 1992x - 92$. Show all important features. The reader may find it useful to attempt the task themselves before reading on. The task was selected on the basis that in the standard viewing window of the graphing calculator used no part of the function is visible. The student pairs were expected to work collaboratively to solve the problem. One TI-83 graphing calculator was provided per pair.

The students in this study were experienced users of the graphing calculators unlike many studies where students are learning to use the technology or the many studies where students’ experiences and expertise is not reported. The students in this study were selected on the basis that they were expected to be able to solve the task. The reason for this purposive selection (Merriam, 2002) being that it would allow behaviour and actions undertaken, pertinent to informing future teaching, to be observed. The methodology used allowed a view of pairs of students as they worked through a problem task to be recorded, in contrast to many studies that merely consider the aggregated results of students on tasks undertaken individually under test conditions (e.g., Shoaf-Grubbs, 1994). In order to obtain an improved record of the results of students’ actions, videotaping of graphing calculator screen outputs supplemented audio recordings. The graphing calculator screens were thus used as windows into the minds of the students. Together with observational notes, and students’ scripts, these allowed new insight into student understandings to be gained.

Students’ responses to the task were examined and analysed in order to gain a greater insight into the processes students undertake when using a graphing calculator to solve such a problem. The focus was on understanding the kind of knowledge gained from learning about functions in the classroom in a graphing calculator environment and the extent to which this knowledge (both mathematical and technological) was exhibited in solving the particular task. Transcripts of verbal exchanges augmented by observational notes, were matched with graphing calculator screen shots and the students’ written scripts. These were subjected to intensive qualitative macroanalysis and microanalysis using tools developed by me from Schoenfeld’s work (1985). Initially, the transcripts were coded locally according to the activity being undertaken and then more
globally by episode. The activity categories used were reading, organising or planning, selecting a viewing window, searching for or identifying a local feature, considering the global view, adjusting scale marks, evaluating, and recording. In addition, for each activity the representations that were available or were being used or considered by the students were coded (i.e., algebraic, graphical, numerical). Episodes were periods of time when the students were engaged in one large task or a series of closely related tasks in the pursuit of some goal. Episodes were classified according to their specific purpose in the task solution such as observing the function in the standard window or searching for a complete graph.

The analysis based on this coding, in conjunction with the use of diagrammatic tools, led to a number of defining moments becoming apparent in the solution processes of the student pairs. A defining moment refers to an important or momentous event that may have had the potential to facilitate or impede the solution process. It was interpreted as being a circumstance where some action, cognitive or physical, or a decision (i.e., a metacognitive action), or a series of these may have had the potential to facilitate or impede the solution process. Defining moments, therefore, occurred at critical points in the solution process. Similar circumstances may have occurred in the solution processes of other pairs but it was the responses of the pairs to these circumstances that determined whether or not they became defining moments for a particular pair’s solution. Each circumstance can be described in terms of the situation, the condition, and the response action, as illustrated in Figure 1, for the defining moment: use of scale marks.

With regard to the defining moment: use of scale marks, only one response action was identified. This action occurred in response to four different conditions (Figure 1). In turn these four conditions were identified in three separate situations. In some circumstances students have to “learn to perceive” (Scarantino, 2003, p. 954) an affordance, that is, they must be taught or somehow learn themselves to become attuned to what specifies an affordance. Unfortunately, some students “truly ascribed affordance[s]” (p. 957) related to the use of scale marks, whilst others “falsely ascribed” an affordance (p. 957). The former case involved two pairs of students, with one pair using the scale marks to improve their view of the function by setting them at zero and effectively turning them off, whilst the other pair carefully edited the scale marks to both facilitate their by-hand sketch and to allow greater precision in their estimation of key features coordinates. In the latter case, another pair upon being confronted with an unexpected view of the function on several consecutive occasions resorted to editing the scale marks in the apparent belief that this would potentially change the view of the function.

The defining moments identified during the macroscopic analysis were related to: (a) how students responded to particular views of the function, including apparent no view, apparent vertical lines, and other unusual or unexpected views; (b) use of the numerical representation; (c) how students made use of the scale marks, including their misuse and interactions between scale marks and the view; (d) use of opportunistic planning; (e) engagement in discussion; and (f) identification of key features of the function. Further details can be found in Brown (2003b) and Brown and Stillman (2003).

All students in this study demonstrated they had an archetypal image of a cubic function for which they were searching, a necessary requirement for the confident use of the calculator to graph functions according to Anderson, Bloom, Mueller, and Pedler (1999, pp. 490-491). Their previous mathematical experiences
allowed the students to include the only possible shapes of a cubic function in their toolbox and hold onto this view even when their ideas were challenged by unexpected views of the function on the calculator screen. From their actions, dialogue, and solution it can be inferred that all students in the study correctly understood that both a global view and the identification of all key features are required for a complete graph of a function to be determined.

The findings were: (1) all students demonstrated understanding of the local and global nature of functions and the synthesis of these in determining a complete graph; (2) a range of mathematical and graphing calculator knowledge was applied in seeking a global view of the function with their combined application being more efficient and effective; (3) an understanding of automatic range scaling features facilitated efficient finding of a global view; (4) all pairs demonstrated having a clear mental image of the function sought and the possible positions of the calculator output relative to this; (5) students were able to resolve situations involving unexpected views of the graph to determine a global view; (6) students displayed understanding of local linearity of a function; (7) when working in the graphical representation, students used the algebraic but not the numerical representation to facilitate and support their solution; (8) scale marks were used to produce more elegant solutions and facilitate identification of key function features to produce a sketch but some students misunderstood the effect of altering these; (9) pairs differed in the proportion of cognitive and metacognitive behaviours demonstrated with question asking during evaluation supporting decision making; (10) correct selection of an extensive range of graphing calculator features and use of dedicated features facilitated efficient and accurate identification of coordinates of key function features.

Teacher Understanding of Functions in a Graphing Calculator Environment

The graphing calculator screen was used in the previous study as a window into the minds of students, however we rarely have the same luxury when it comes to considering teachers using technology. There are forums where teachers share ideas and approaches and the MAV conference is one such place. At the 2002 MAV annual conference many of the participants at an option presented by myself were prepared to share with me, their initial approach in solving the same problem task as used in my study with the five pairs of students. The responses of these teachers to the problem task provide some insight into the thinking of teachers using technology when confronted with tasks such as this. The analysis of these responses (Brown, 2004) provides an insight into the variety of methods that teachers have potential access to in order to facilitate student learning. As teachers have long known, whatever method they present to their class, some students will use a variety of other methods so being mindful of a variety of methods will be of benefit.

Initial View of the Function

The greatest effect on task progress is often the result of one’s initial actions. Undertaking particular initial actions that led to differing initial views were shown in my previous work with students to have differing consequences for subsequent actions and ease of solution (Brown, 2002). Hence, the initial actions of the teachers and the subsequent resulting view were analysed. Seven initial action categories and seven initial view categories were observed in the data collected. A one to one correspondence between the action and view categories did not exist with some actions leading to more than one possible view for the function and vice versa.

The varied initial actions of respondents demonstrate the myriad approaches used by teachers in finding a global view of a given function in a graphing calculator environment. The initial overt action of respondents was classified into categories listed in descending order of frequency of use: editing the WINDOW settings; using Zoom Standard; exploring explicit mathematical information by hand, algebraically and/or graphically; direct selection of Zoom Fit (an automatic range scale feature for a given domain); using TABLE; using GRAPH (uses previous WINDOW settings); or editing WINDOW settings in conjunction with Zoom Fit.
The initial windows seen by the users were classified into categories dependent on the visible aspects of the function or the position of which could easily be inferred. The categories were: a global view (GV) which showed all global aspects of the function including all key features; an almost global view (AGV) where the majority of a global view was seen and the position of unseen features was immediately apparent; significant view (SV) where all or the majority of key features were unsighted, however, the function was clearly visible across the majority of the viewing domain; no apparent view (NAV) where no view of the function was obvious; apparent vertical lines (AVL); function coincident with y axis and no other part of the function visible (FCY); and unknown (neither window settings nor view recorded). Examples of the six known initial viewing window categories are shown in Figure 2. From a teaching perspective the first three categories should allow students to unproblematically progress toward a solution whilst the remainder are potentially problematic for students (Brown, 2003a).

Implications for Teaching

Along with the myriad approaches undertaken by the teachers, it was clear that no one action, initial or otherwise, could be singled out as being best from a teaching and learning perspective, and therefore teachers need to provide a variety of specific experiences for students to maximise their opportunities to develop expertise in such a range. The diverse approaches undertaken by the teachers demonstrate the difficulties any one teacher may have considering the varying complexity for different learners undertaking the same task in the classroom. It is only with focussed experiences that students will be able to develop sufficient expertise with tools, and techniques and flexibility in using these to meet the challenges they could face in their mathematical studies and to minimise the number of possible views of a function observed in the path to producing a complete graph.

The approaches used by the teachers reinforce that it is a combination of graphing calculator and mathematical knowledge that allow a user to seamlessly determine a complete view of the graphical representation of a function. In particular, with regard to the technology, both teacher and students alike need an understanding of a range of features and flexibility in using these. The automatic scaling feature is one feature that appears to support the determination of a window showing a global view of the function. In addition to having a mental image of the specific function type under consideration, the user also needs expertise in positioning any given view of the function as portrayed in the viewing window.

Affordance Bearers of Electronic Technologies for Learning Mathematics

Given the complexities of teaching and learning, we need to share knowledge of our use of technologies such as graphing calculators. There is more to this than knowledge of graphing calculator features. It is situating the use of these features in a learning environment with a particular goal in mind. If it is true that, given linear functions are being taught in Year 9, a teacher at a particular point in implementing a unit on linear functions can increase the importance of the numerical representation of a function by emphasising use of the TABLE on a graphing calculator, then the TABLE is at that point in time an “affordance bearer” (Scarantino, 2003, p. 958). In the remainder of this paper, I will consider a range of affordances of the electronic technologies for student learning of particular aspects of mathematics. Whilst these are presented using graphing calculators, most, if not all, are offered by other electronic technologies.
Storing Values: The Alpha Keys

The opportunity to store values for any given letter of the alphabet allows the user to perceive the role of the parameters in particular functions, as shown in Figure 3. Generally we present functions to students, or enter them on the graphing calculator, where values are already assigned to the parameters. In a sense this is presenting the function as an encapsulated object, whereas if we also present the function as a structure or template we are allowing students to perceive the function as composed of separate parts and begin to perceive the function as a gestalt, the perceived entity being more than the sum of its parts. Many conjectures could be suggested by students following the viewing of the four screens displayed in Figure 3, and the question posed as to what may happen as we edit the values of A or B. The conjectures proposed by students, in addition to the ‘desired’ ones, may also include those we as teachers are not expecting. Pursuing these often leads to students gaining a deeper understanding of what we are aiming for students to learn in this situation.

The Little Application (Apps): Transformation Graphing

The idea of storing values, leads us directly to a discussion of the little application (Apps): Transformation Graphing for the TI-83Plus. In the RITEMATHS project (HREF 1) the focus is on increasing student engagement in a technologically rich teaching and learning environment with the mathematics of change and variation being the focus. The Apps: Transformation Graphing has been explored as one way to do this for linear functions. Using this application the user is afforded the opportunity to observe the effect of changing the value of any one of up to four parameters of a function. This can be used for any function. In the example, shown in Figure 4, the algebraic representation of the function $y = Ax + B$ has been entered and the effect of varying $B$ is under investigation. In this example, $A$ is fixed at 0.5 and $B$ is increasing in increments of 5 beginning at -5. In the same way the effect of altering the value of $B$ can be explored. The algebraic representation of the function and the values of any parameters are displayed concurrently with the graphical representation of the function.

Figure 4 shows an example of what the user sees when the Apps: Transformation Graphing is used in play-pause mode. Now imagine this as a dynamic slide show (slow or fast versions available), with a smaller step size (i.e., more slides but the range of values for $B$ and the window viewing settings are the same). What would your students see? By using the Apps: Transformation Graphing students are able to establish a general sense of what is happening. By careful selection of parameter and increment values in conjunction with the WINDOW SETTINGS the opportunity is there for students to ‘see’ what we want students to ‘see’.
As teachers, we individually, and collectively, need to determine what it is we want our students to ‘see’ at any given time. For linear functions, I personally want my students to see that, for the function \( y = Ax + B \), as \( A \) changes so too does the steepness of the graph of the function. As \( A \) increases, the graph gets steeper and as \( A \) decreases the graph becomes less steep. In addition, if \( A \) is negative, the graph slopes down as we look from left to right along the graph. In addition, I want my students to see more specifically that if \( A \) doubles, then so too does the vertical distance between the graphs of the functions and the \( x \) axis at any given \( x \) ordinate. With the graphing calculator as part of the thinking of my students in this learning environment, they have the opportunity to become attuned to the effects of varying these parameters in these situations. The use of the Apps: Transformation Graphing provides the motion that enables all students to immediately see the dynamic effect of the parameter \( A \) as it takes on different (increasing or decreasing) values.

**Pause**

The ability to easily and efficiently produce multiple graphs on the same set of axes is another opportunity for students to focus more specifically on the effects of changing a parameter. Similarly, the simultaneous graphing feature and pausing the graphing at any time enable students to focus on the generalities rather than specifics of the situation. The screens in Figure 5, show five linear functions that have been entered and are graphed simultaneously rather than sequentially. Used in conjunction with the pause feature the windows provide views of the functions with the graphing halted at various values of the dependent variable. This provides an opportunity for students to focus on the change of the graphical representation relative to the \( x \) ordinate.

![Figure 5. Simultaneous graphing mode and the use of pause provide the opportunity to focus on changes relative to the \( x \) ordinate.](image)

**Window Settings**

Furthermore, I want my students to appreciate why we focus on the vertical change here when anyone can clearly see that the change is in the horizontal direction, or is the change in fact an oblique movement, as shown in the three views shown in Figure 6? The opportunity to quickly graph a family of functions and to edit the WINDOW SETTINGS affords the user different views of the same set of functions. These opportunities allow students to discuss descriptions of what is happening to a large range of functions. They can conjecture and test whether their ideas can be applied more widely.

![Figure 6. Various WINDOW SETTINGS and the apparent movement of the function.](image)
Typically, we teach about linear functions before any other functions. This is in part due to the by-hand only systems, of the past, of doing mathematics, whereby the algebraic and numerical calculations are relatively simple for linear functions. However, with more emphasis on the graphical representation, particularly comparing graphs and abstracting their relationships - linear functions are much harder to interpret than functions having more visually distinguishable shapes. Because we see only a portion of an infinite line and no discrete points are identified, the way a line appears to move (particularly to students with no previous experiences) will depend on the angle it makes with the window through which it is viewed (Goldenberg, 1991). When observing the three windows shown in Figure 6, a student is likely to infer movement in the horizontal direction, the vertical direction, and obliquely, respectively. Purposefully selected WINDOW SETTINGS, in conjunction with discussion and simultaneous viewing of the corresponding numerical representation of the functions, can support student understanding.

**Split Screen Views**

The horizontal split screen view affords the user opportunities to view two representations of a function simultaneously. For many users this dyad of visual opportunities affords links to be made directly between multiple representations of the same function, or families of functions as shown in Figure 7. In addition, the graph may be a statistical plot, which can be displayed in conjunction with the lists containing the numerical values from which the plot is constructed. The statistical plot could be overlaid with a graphical and algebraic representation as well thus showing all three representations. Numerical values of both the plot and the function can be displayed by selecting TRACE. This allows students to compare the two functions at a local level as well as more globally by comparing them graphically.

![Figure 7. Views showing the graphical representation in conjunction with the algebraic and numerical representations respectively.](image)

Similarly, use of the graph-table split screen view (Figure 8) affords the user opportunities to focus on the connections between two representations of a function simultaneously. The use of TRACE in this environment, whereby the coordinates of the point currently displayed are highlighted in the table view, provide the affordances for both local and global links between the numerical and algebraic representations of the functions. When TRACE is activated the algebraic representation of the selected function is displayed. The final two windows shown in Figure 10 allow students to focus on the fact that, as the graphical and numerical representations of \( y = (x+2)^2 \) and \( y = x^2 + 2 \) are not identical, then neither are their algebraic representations and hence \((x+2)^2 \neq x^2 + 2\).

![Figure 8. Affordances of the graph table view for linking the graphical and numerical representations.](image)
Composition of functions

The ability to link functions as shown in Figure 9 provides another opportunity for students to make links between changes in one representation and the resulting changes in a second representation. In this example, the changes to the algebraic representation are represented in the numerical representation. Discussion of the relationships apparent in the tables should lead to the development of individual, group or class conjectures which are then easily tested by editing the function entered in $y_1$ and confirming or modifying the conjecture. This can be followed by predicting and then checking the effect of editing the function entered in $y_2$, that is from $y_3 = y_1 + 10$ to $y_3 = y_1 + 8$ or $y_2 = y_1 - 10$ say, or from $y_2 = 3y_1$ to $y_2 = 0.5y_1$ or $y_2 = -y_1$ for example. We can give this idea a twist and ask students to identify A given the information shown in the last screen.

Figure 9. Considering the numerical effect of transforming functions.

Operations on functions

What do students expect to see given $y_1 = x + 3$, $y_2 = x + 1$, and $y_3$ the product of these two functions (see Figure 10)? Could this be part of an introduction to quadratic functions? Movshovitz-Hadar (1993) suggested this approach more than a decade ago.

Figure 10. Affordances of the functional features to perceive links between linear and quadratic functions, in three representations.

When the information provided by the algebraic representation of a function as to its possible global views and position relative to the origin is either unknown, ignored, or inadequate, what affordance bearers are available for students to support finding an appropriate viewing window? Do students perceive the affordances offered by TRACE and the TABLE (see Figure 11) that provide an immediate indication of possible changes needed to the current window settings in order to view the point or points indicated by the use of TRACE and or TABLE and hence view at least a portion of the graph of the function?

Figure 11. Affordances offered by TRACE and TABLE for identifying appropriate WINDOW SETTINGS.

Conclusion

In conclusion, there may be two distinct sets of teachers using electronic technology in their classroom. Those for whom using the technology in mathematics happened to them (e.g., assumed use of graphing
calculators in assessment) and those who are doing mathematics teaching with the technology. The former group may not perceive the breadth of affordances offered by the technology for any given learning situation as technology is treated as more of a supplement to how they always taught mathematics. For others a technologically rich learning environment is where teaching with technology is synonymous with a transformation of their thinking about their teaching and doing mathematics and what constitutes mathematical activity. Those doing with the technology are not adding on the technology—teaching with the technology has resulted in a reorganisation of their thinking as teachers of mathematics. These teachers have the additional responsibility of sharing their perceptions of the affordances of the technology in any given learning situation with colleagues and in turn their students.

References


HREF 1 http://extranet.edfac.unimelb.edu.au/DSME/RITEMATHS/